

## $L$ -convex convergence relation and $L$ -convex ideal-convergence structure

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### Abstract

In the framework of  $L$ -convex spaces, this paper is aim to study fuzzy convergence structures of  $L$ -convex space in a viewpoint of fuzzy relations. For this purpose,  $L$ -directed relation is introduced and some of its examples are proposed. Based on this, notions of  $L$ -convergence relation and  $L$ -convex convergence relation are introduced. It is proved that there is a Galois's connection between the category of  $L$ -convergence relation spaces and the category of  $L$ -convex enclosed relation spaces. In particular,  $L$ -convex convergence relation spaces and  $L$ -convex enclosed relation spaces are categorically isomorphic. Also, notions of  $L$ -ideal-convergence structure and  $L$ -convex ideal-convergence structure are introduced. It is proved that  $L$ -convergence relation spaces and  $L$ -ideal-convergence spaces are categorically isomorphic. In particular,  $L$ -convex convergence relation spaces and  $L$ -convex ideal-convergence spaces are also categorically isomorphic.

*Keywords:*  $L$ -convex space,  $L$ -directed relation,  $L$ -convergence relation,  $L$ -convex convergence relation,  $L$ -convex ideal-convergence structure,  $L$ -ideal-convergence structure.

## 1 Introduction

The concept of convexity is originally inspired by some elementary geometric problems such as the shapes of circles and polytopes in Euclidean spaces [1]. Later, this concept has being found to match with the same property of many different mathematical structures. Eventually, it was generalized and abstracted as the concept of convex structure. Specifically, a convex structure on a nonempty set is a family of subsets containing the empty set and the largest set and is closed under arbitrary intersections and nested unions. Its theory is called the convex structure theory which is a new branch of mathematics and involves many mathematical structures such as graph [2], poset [4], median algebra [11], metric space [20], lattice [30] and vector space [31].

Convex structure as well as its theory has been extended to different fuzzy settings. Maruyama introduced  $L$ -convex structure [10]. Many subsequent studies have been done [16, 17, 18, 21, 34, 40]. Shi and Xiu introduced  $M$ -fuzzifying convex structures [28]. Its theory has being studied by many scholars [24, 32, 42, 41]. Also, Shi and Xiu introduced  $(L, M)$ -fuzzy convex structure which is a unified form of  $L$ -convex structure and  $M$ -fuzzifying convex structure [29]. Many studies have been done recently [15, 34, 35]. These fuzzy forms of convex structures have being applied to many mathematical structures such as fuzzy topology [9, 22, 25, 32, 39, 35], fuzzy convergence [5, 8, 12, 13, 19, 14, 45, 46] and fuzzy matroid [33, 43].

Relation is a useful tool to study fuzzy mathematical structures. In the  $L$ -setting, Shi and Shi introduced  $L$ -topological internal relation and  $L$ -topological enclosed relation by which they characterized  $L$ -topologies [26]. Liao and Wu introduced  $L$ -convex enclosed relation and characterized  $L$ -convex structures [9]. Wu et al. introduced several derived relations such as  $L$ -topological derived enclosed relations,  $L$ -topological derived internal relations and  $L$ -topological derived remoteness relations which are used to characterize  $L$ -topologies [36, 37]. In  $(L, M)$ -fuzzy setting, Shi and Shi introduced  $(L, M)$ -fuzzy topological internal relation and  $(L, M)$ -fuzzy topological enclosed relation

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which are used to characterize  $(L, M)$ -fuzzy topologies [27]. Wu and Shi introduced  $(L, M)$ -fuzzy topological derived enclosed relations and characterized  $(L, M)$ -fuzzy topologies [38].

Convergence theory is frequently used in interpreting topological structures and convex structures. In the framework of topological theory, Min introduced fuzzy limit structures by prefilters. Höhle and Šostak constructed fuzzy convergence spaces by introducing stratified  $L$ -filters [7]. Fang introduced stratified  $L$ -ordered convergence structures [3]. Yao introduced  $L$ -fuzzifying convergence structures and study its categorical relations with  $L$ -fuzzifying topology [44]. Pang introduced  $(L, M)$ -fuzzy convergence space by which he characterized  $(L, M)$ -fuzzy topological space [12]. Later, Han and Pang investigated convergence structures in  $L$ -concave spaces [6]. All above mentioned convergence structures are constructed to interpret topology. However, in the framework of convex theory, Pang studied convergent structures in  $M$ -fuzzifying convex spaces [14]. Xiu et al. investigated fuzzy convergence structures in the framework of  $L$ -convex spaces via  $L$ -convex ideals, where  $L$  is a completely distributive lattice with an inverse involution [40]. Its main notions and results depend inevitably on the inverse involution. As we know that not all completely distributive lattices are suitable for adopting inverse involutions, it may encounter some limitations in applying these results to related topics in  $L$ -fuzzy theories where  $L$  has no inverse involution. Being inspired by this limitation, we adopt a new approach to discuss relationships between convergence structure and convex structure in the  $L$ -setting, where  $L$  is not required to be equipped with an inverse involution.

The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results related to  $L$ -convex space. In Section 3, we introduce  $L$ -directed relations,  $L$ -convergence relations and  $L$ -convex convergence relations. We prove that the category of  $L$ -convex enclosed relation spaces can be embedded into the category of  $L$ -convergence relation spaces as a subcategory. We further prove that the category of  $L$ -convex enclosed relation spaces isomorphic to the category of  $L$ -convex convergence relation spaces. In Section 4, we introduce  $L$ -ideal convergence structures and  $L$ -convex ideal convergence structures. We prove that the category of  $L$ -convergence relation spaces is isomorphic to the category of  $L$ -ideal-convergence spaces. We also prove that the category of  $L$ -convex convergence relation spaces is isomorphic to the category of  $L$ -convex ideal-convergence spaces.

## 2 Preliminaries

In this paper,  $X$  and  $Y$  are nonempty sets. The power set of  $X$  is denoted by  $2^X$ .  $L$  is a completely distributive lattice. The smallest (resp. largest) element in  $L$  is denoted by  $\perp$  (resp.  $\top$ ). An element  $a \in L$  is called a co-prime element, if for all  $b, c \in L$ ,  $a \leq b \vee c$  implies  $a \leq b$  or  $a \leq c$ . The set of all co-prime elements in  $L \setminus \{\perp\}$  is denoted by  $J(L)$ . For any  $a \in L$ , there is an  $L_1 \subseteq J(L)$  such that  $a = \bigvee_{b \in L_1} b$ . A binary relation  $\prec$  on  $L$  is defined by  $a \prec b$  iff for each  $L_1 \subseteq L$ ,  $b \leq \bigvee L_1$  implies a  $d \in L_1$  with  $a \leq d$ . The mapping  $\beta : L \rightarrow 2^L$ , defined by  $\beta(a) = \{b : b \prec a\}$ , satisfies  $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$  for any  $\{a_i\}_{i \in I} \subseteq L$ . For any  $a \in L$ , we denote  $\beta^*(a) = \beta(a) \cap J(L)$ . It is proved that  $a = \bigvee \beta(a) = \bigvee \beta^*(a)$ ,  $\beta(a) = \bigcup_{b \in \beta^*(a)} \beta(b)$  and  $\beta^*(a) = \bigcup_{b \in \beta^*(a)} \beta^*(b)$  [23].

An  $L$ -fuzzy set on  $X$  is a mapping  $A : X \rightarrow L$ . The set of all  $L$ -fuzzy sets on  $X$  is denoted by  $L^X$ . The smallest (resp. largest) element in  $L^X$  is denoted by  $\underline{\perp}$  (resp.  $\underline{\top}$ ). An  $L$ -fuzzy point  $x_\lambda$  ( $\lambda \in L \setminus \{\perp\}$ ) is defined by  $x_\lambda(x) = \lambda$  and  $x_\lambda(y) = \perp$  for any  $y \in X \setminus \{x\}$ . The set of all  $L$ -fuzzy points on  $L^X$  is denoted by  $Pt(L^X)$ . In addition, we denote  $J(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in J(L)\}$  and  $\beta^*(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in \beta^*(L)\}$ . A family  $\{A_i\}_{i \in I} \subseteq L^X$  is said to be directed, denoted by  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$ , if for all  $i, j \in I$  there is a  $k \in I$  such that  $A_i \vee A_j \leq A_k$ . In this case, we further denote  $\bigvee_{i \in I}^{dir} A_i$  for  $\bigvee_{i \in I} A_i$ .

For a mapping  $f : X \rightarrow Y$ , the  $L$ -fuzzy mapping  $f_L^{\rightarrow} : L^X \rightarrow L^Y$  is defined by  $f_L^{\rightarrow}(A)(y) = \bigvee \{A(x) : f(x) = y\}$  for all  $A \in L^X$  and  $y \in Y$ , and the mapping  $f_L^{\leftarrow} : L^Y \rightarrow L^X$  is defined by  $f_L^{\leftarrow}(B)(x) = B(f(x))$  for  $B \in L^Y$  and  $x \in X$  [23].

Next, we recall some basic notions and results related to  $L$ -convex spaces.

**Definition 2.1.** [10] *A subset  $\mathcal{C} \subseteq L^X$  is called an  $L$ -convex structure on  $X$  and the pair  $(X, \mathcal{C})$  is called an  $L$ -convex space if*

- (LC1)  $\underline{\top}, \underline{\perp} \in \mathcal{C}$ ;
- (LC2)  $\forall \{A_i\}_{i \in I} \subseteq \mathcal{C}, \bigwedge_{i \in I} A_i \in \mathcal{C}$ ;
- (LC3)  $\forall \{A_i\}_{i \in I}^{dir} \subseteq \mathcal{C}, \bigvee_{i \in I}^{dir} A_i \in \mathcal{C}$ .

Let  $(X, \mathcal{C}_X)$  and  $(Y, \mathcal{C}_Y)$  be  $L$ -convex spaces. A mapping  $f : X \rightarrow Y$  is called an  $L$ -convexity preserving mapping, if  $f_L^{\leftarrow}(A) \in \mathcal{C}_X$  for any  $A \in \mathcal{C}_Y$ . The category of  $L$ -convex spaces and  $L$ -convexity preserving mappings is denoted by  $L\text{-CS}$  [18].

**Definition 2.2.** [18] *An operator  $co : L^X \rightarrow L^X$  is called an  $L$ -convex hull operator on  $L^X$  and the pair  $(X, co)$  is called an  $L$ -convex hull space, if  $co$  satisfies*

- (LCO1)  $co(\perp) = \perp$ ;
- (LCO2)  $A \leq co(A)$ ;
- (LCO3)  $co(co(A)) = co(A)$ ;
- (LCO4)  $co(\bigvee_{i \in I}^{dir} A_i) = \bigvee_{i \in I}^{dir} co(A_i)$ .

Let  $(X, co_X)$  and  $(Y, co_Y)$  be  $L$ -convex hull spaces. A mapping  $f : X \rightarrow Y$  is called an  $L$ -convex hull preserving mapping, if  $co_X(A)(x) \leq co_Y(f(A))(f(x))$  for all  $A \in L^X$  and  $x \in X$ . The category of  $L$ -convex hull spaces and  $L$ -convex hull preserving mappings is denoted by  $L\text{-CHS}$  [18].

**Proposition 2.3.** [18] (1) *Let  $(X, C)$  be an  $L$ -convex space. The operator  $co_C : L^X \rightarrow L^X$ , defined by  $co_C(A) = \bigwedge \{B \in C : A \leq B\}$  for any  $A \in L^X$ , is an  $L$ -convex hull operator.*

- (2) *If  $(X, co)$  is an  $L$ -convex hull space, then  $C_{co} = \{A \in L^X : co(A) = A\}$  is an  $L$ -convex structure.*
- (3)  *$L\text{-CS}$  is isomorphic to  $L\text{-CHS}$ .*

**Definition 2.4.** [35] *A binary relation  $\preceq$  is called an  $L$ -convex enclosed relation on  $L^X$  and the pair  $(X, \preceq)$  is called an  $L$ -convex enclosed relation space, if  $\preceq$  satisfies*

- (LCER1)  $\perp \preceq \perp$ ;
- (LCER2)  $A \preceq B$  implies  $A \leq B$ ;
- (LCER3)  $A \preceq \bigwedge_{i \in I} B_i$  iff  $A \preceq B_i$  for all  $i \in I$ ;
- (LCER4)  $A \preceq B$  implies a  $C \in L^X$  with  $A \preceq C \preceq B$ ;
- (LCER5)  $\bigvee_{i \in I}^{dir} A_i \preceq C$  iff  $A_i \preceq C$  for any  $i \in I$ .

Let  $(X, \preceq_X)$  and  $(Y, \preceq_Y)$  be  $L$ -convex enclosed relation spaces. A mapping  $f : X \rightarrow Y$  is called an  $L$ -convex enclosed relation preserving mapping, if  $f_L^+(A) \preceq_X f_L^+(B)$  for all  $A, B \in L^Y$  with  $A \preceq_Y B$ . The category of  $L$ -convex enclosed relation spaces and  $L$ -convex enclosed relation preserving mappings is denoted by  $L\text{-CERS}$  [35].

**Theorem 2.5.** [35] (1) *Let  $(X, \preceq)$  be an  $L$ -convex enclosed relation space. An operator  $co_{\preceq} : L^X \rightarrow L^X$  defined by  $co_{\preceq}(A) = \bigwedge \{B \in L^X : A \preceq B\}$  for any  $A \in L^X$ , is the  $L$ -convex hull operator of an  $L$ -convex structure  $C_{\preceq}$  on  $L^X$ .*

- (2) *Let  $(X, C)$  be an  $L$ -convex space. A binary operator  $\preceq_C$  on  $L^X$ , defined by  $A \preceq_C B$  iff  $co_C(A) \leq B$  for all  $A, B \in L^X$ , is an  $L$ -convex enclosed relation on  $L^X$ .*
- (3)  *$L\text{-CS}$  is isomorphic to  $L\text{-CERS}$ .*

**Lemma 2.6.** [36] *Let  $x_\lambda \in \beta^*(L^X)$ . We denote  $\beta_\lambda^*(L) = \{\eta \in \beta^*(L) : \lambda \in \beta^*(\eta)\}$  and*

$$\psi_{x_\lambda}(L^X) = \{A \in L^X : \forall \eta \in \beta_\lambda^*(L), x_\eta \not\leq A\}.$$

*For all  $B \in L^X$  and  $\{A_i\}_{i \in I} \subseteq L^X$ , the following results hold.*

- (1)  $\perp \in \psi_{x_\lambda}(L^X)$  and  $\top \notin \psi_{x_\lambda}(L^X)$ .
- (2)  $A \leq B \in \psi_{x_\lambda}(L^X)$  implies  $A \in \psi_{x_\lambda}(L^X)$ .
- (3)  $\bigvee_{i \in I} A_i \in \psi_{x_\lambda}(L^X)$  iff  $A_i \in \psi_{x_\lambda}(L^X)$  for any  $i \in I$ .
- (4)  $\psi_{x_\lambda}(L^X) = \bigcap_{\eta \in \beta_\lambda^*(L)} \psi_{x_\eta}(L^X)$ .

**Definition 2.7.** [36] *A family  $\mathcal{R} = \{\mathcal{R}_{x_\lambda} \subseteq \psi_{x_\lambda}(L^X) : x_\lambda \in \beta^*(L^X)\} \subseteq 2^{L^X}$  is called an  $L$ -convex  $\beta^*$ -remoteness system on  $L^X$  and the pair  $(X, \mathcal{R})$  is called an  $L$ -convex  $\beta^*$ -remoteness space, if  $\mathcal{R}_{x_\lambda} \subseteq L^X$  satisfies*

- (LCBRH1)  $\perp \in \mathcal{R}_{x_\lambda}$ ;
- (LCBRH2)  $A \in \mathcal{R}_{x_\lambda}$  iff each  $\eta \in \beta_\lambda^*(L)$  implies a  $B \in \psi_{x_\eta}(L^X)$  such that  $A \leq B \in \mathcal{R}_{y_\mu}$  for any  $y_\mu \in \beta^*(L^X)$  with  $B \in \psi_{y_\mu}(L^X)$ ;
- (LCBRH3)  $\bigvee_{i \in I}^{dir} A_i \in \mathcal{R}_{x_\lambda}$  iff  $A_i \in \mathcal{R}_{x_\lambda}$  for any  $i \in I$ .

Let  $(X, \mathcal{R}_X)$  and  $(Y, \mathcal{R}_Y)$  be  $L$ -convex  $\beta^*$ -remoteness spaces. A mapping  $f : X \rightarrow Y$  is called an  $L$ -convex  $\beta^*$ -remoteness preserving mapping, if  $B \in (\mathcal{R}_Y)_{f_L^+(x_\lambda)}$  implies  $f_L^+(B) \in (\mathcal{R}_X)_{x_\lambda}$  for  $x_\lambda \in \beta^*(L^X)$  and  $B \in L^Y$ . The category of  $L$ -convex  $\beta^*$ -remoteness spaces and  $L$ -convex  $\beta^*$ -remoteness preserving mappings is denoted by  $L\text{-CBRHS}$  [36].

**Theorem 2.8.** [36] (1) *Let  $(X, \mathcal{R})$  be an  $L$ -convex  $\beta^*$ -remoteness space. Then*

$$C_{\mathcal{R}} = \{A \in L^X : \forall x_\lambda \in \beta^*(L^X), A \in \psi_{x_\lambda}(L^X) \text{ implies } A \in \mathcal{R}_{x_\lambda}\},$$

*is an  $L$ -convex structure on  $L^X$ .*

(2) Let  $(X, \mathcal{C})$  be an  $L$ -convex space. For any  $x_\lambda \in \beta^*(L^X)$ , define a set

$$\mathcal{R}_{x_\lambda}^{\mathcal{C}} = \{A \in L^X : \exists B \in \mathcal{C} \cap \psi_{x_\lambda}(L^X), A \leq B\}.$$

Then  $\mathcal{R}_{\mathcal{C}} = \{\mathcal{R}_{x_\lambda}^{\mathcal{C}} : x_\lambda \in \beta^*(L^X)\}$  is an  $L$ -convex  $\beta^*$ -remotehood system on  $L^X$ .

(3)  $L$ -CBRHS is isomorphic to  $L$ -CS.

**Definition 2.9.** [40] A subset  $\mathcal{I} \subseteq L^X$  is called an  $L$ -convex ideal on  $L^X$  and the pair  $(X, \mathcal{I})$  is called an  $L$ -convex ideal space, if

- (LCI1)  $\perp \in \mathcal{I}$  and  $\top \notin \mathcal{I}$ ;
- (LCI2)  $A \leq B \in \mathcal{I}$  implies  $A \in \mathcal{I}$ ;
- (LCI3)  $\{A_i\}_{i \in I}^{dir} \subseteq \mathcal{I}$  implies  $\bigvee_{i \in I}^{dir} \in \mathcal{I}$ .

The set of all  $L$ -convex ideals on  $L^X$  is denoted by  $\mathfrak{J}(L^X)$ .

**Definition 2.10.** [40] Assume that  $L$  has an inverse involution  $'$ . An  $L$ -convergence structure on  $X$  is a mapping  $\lim : \mathfrak{J}(L^X) \rightarrow 2^{J(L^X)}$  satisfying

- (LCV1)  $x_\lambda \in \lim([x_\lambda])$ , where  $[x_\lambda] = \{A \in L^X : x_\lambda \leq A'\}$ ;
- (LCV2)  $\mathcal{I}_1 \subseteq \mathcal{I}_2$  implies  $\lim(\mathcal{I}_1) \subseteq \lim(\mathcal{I}_2)$ ;
- (LCV3)  $\mathcal{R}_{x_\lambda}^{\lim} = \bigcap_{\mu < \lambda} \mathcal{R}_{x_\mu}^{\lim}$ , where  $\mathcal{R}_{x_\lambda}^{\lim} = \bigcap_{x_\lambda \in \lim(\mathcal{I})} \mathcal{I}$ .

### 3 $L$ -directed relation space and $L$ -convex convergence relation space

In this section, we define the notion of  $L$ -directed relation and present some related examples. Based on this, we introduce notions of  $L$ -convergence relation space and  $L$ -convex convergence relation space. Then we further study their relations with  $L$ -convex enclosed relation space.

**Definition 3.1.** A binary operator  $\leq$  on  $L^X$  is called an  $L$ -directed relation and the pair  $(X, \leq)$  is called an  $L$ -directed relation space, if for all  $A, B, C \in L^X$  and  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$ ,

- (LDR1)  $\perp \leq \perp$  and  $\top \not\leq \top$ ;
- (LDR2)  $A \leq B$  iff  $A \leq A \leq B$ ;
- (LDR3)  $\bigvee_{i \in I}^{dir} A_i \leq C$  iff  $A_i \leq C$  for all  $i \in I$ .

In the sequel, the set of all  $L$ -directed relations on  $L^X$  will be denoted by  $\mathfrak{R}(L^X)$ . For  $\leq_1, \leq_2 \in \mathfrak{R}(L^X)$ , we say that  $\leq_1$  is coarser than  $\leq_2$ , denoted by  $\leq_1 \leq \leq_2$ , provided that  $A \leq_1 B$  implies  $A \leq_2 B$  for all  $A, B \in L^X$ .

**Lemma 3.2.** Let  $(X, \leq)$  be an  $L$ -directed relation space. For all  $A, B, C, D \in L^X$ ,

- (1)  $A \leq C \leq D \leq B$  implies  $A \leq B$ ;
- (2)  $A \leq B$  implies  $A \wedge C \leq B \wedge C$ .

*Proof.* (1) Since  $C \leq D \leq B$ , it is clear that  $C \leq C \leq D \leq B$  by (LDR2). Thus  $C \leq B$ . Since the set  $\{A, C\}$  is directed and  $A \vee C = C \leq B$ , it follows from (LDR3) that  $A \leq B$ .

(2) Since  $A \wedge C \leq A \leq B$ , it is clear that  $A \wedge C \leq B$  by (1). Thus  $A \wedge C \leq A \wedge C \leq B \wedge C$  by (LDR2). Hence  $A \wedge C \leq B \wedge C$ .  $\square$

We present some examples of  $L$ -directed relations by the following proposition.

**Proposition 3.3.** Let  $x_\lambda \in \beta^*(L^X)$ .

(1) Define a binary relation  $\leq_{x_\lambda}$  on  $L^X$  by

$$\forall A, B \in L^X, A \leq_{x_\lambda} B \iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \leq B.$$

Then  $\leq_{x_\lambda}$  is an  $L$ -directed relation.

(2) Let  $(X, \leq)$  be an  $L$ -convex enclosed relation space. Define a binary relation  $\leq_{x_\lambda}^{\leq}$  by

$$\forall A, B \in L^X, A \leq_{x_\lambda}^{\leq} B \iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \leq D.$$

Then  $\leq_{x_\lambda}^{\leq}$  is an  $L$ -directed relation satisfying

(i)  $\leq_{x_\lambda}^{\leq} \leq \leq_{x_\lambda} \leq \leq_{x_\lambda}^{\leq_{fin}}$ , where  $\leq_{x_\lambda}^{\leq_{fin}}$  is the finest  $L$ -convex enclosed relation on  $L^X$ , defined by  $A \leq_{x_\lambda}^{\leq_{fin}} B$  iff  $A \leq B$  for all  $A, B \in L^X$ ;

- (ii)  $A \ll_{x_\lambda}^{\mathcal{C}} B$  iff  $A \ll_{x_\eta}^{\mathcal{C}} B$  for any  $\eta \in \beta_\lambda^*(L)$ .
- (3) Let  $(X, \mathcal{C})$  be an  $L$ -convex space. Define a binary relation  $\ll_{x_\lambda}^{\mathcal{C}}$  on  $L^X$  by

$$\forall A, B \in L^X, \quad A \ll_{x_\lambda}^{\mathcal{C}} B \iff \exists C \in \mathcal{R}_{x_\lambda}^{\mathcal{C}} \text{ s.t. } A \leq C \leq B.$$

Then  $\ll_{x_\lambda}^{\mathcal{C}}$  is an  $L$ -directed relation satisfying  $\ll_{x_\lambda}^{\mathcal{C}} \leq \ll_{x_\lambda} = \ll_{x_\lambda}^{\mathcal{C}_{fin}}$ , where  $\mathcal{C}_{fin} = L^X$  is the finest  $L$ -convex structure on  $L^X$ .

*Proof.* (1) It is easy to check that  $A \leq C \ll_{x_\lambda} D \leq B$  implies  $A \ll_{x_\lambda} B$ . We check that  $\ll_{x_\lambda}$  satisfies (LDR1)–(LDR3).  
 (LDR1) Since  $\perp \in \psi_{x_\lambda}(L^X)$  and  $\top \notin \psi_{x_\lambda}(L^X)$ , it is clear that  $\perp \ll_{x_\lambda} \perp$  and  $\top \not\ll_{x_\lambda} \top$ .  
 (LDR2) For all  $A, B \in L^X$ , we have

$$\begin{aligned} A \ll_{x_\lambda} B &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \leq B \\ &\iff A \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq B \\ &\iff A \ll_{x_\lambda} A \leq B. \end{aligned}$$

(LDR3) Let  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$  and  $B \in L^X$ . If  $\bigvee_{i \in I}^{dir} A_i \ll_{x_\lambda} B$  then it is clear that  $A_i \ll_{x_\lambda} B$  for any  $i \in I$ . Conversely, assume that  $A_i \ll_{x_\lambda} B$  for any  $i \in I$ . Then there is  $D_i \in \psi_{x_\lambda}(L^X)$  such that  $A_i \leq D_i \leq B$  for any  $i \in I$ . Thus  $\bigvee_{i \in I}^{dir} A_i \leq \bigvee_{i \in I} D_i \leq B$ . It follows from Lemma 2.6(2) that  $\bigvee_{i \in I} D_i \in \psi_{x_\lambda}(L^X)$ . Therefore  $\bigvee_{i \in I}^{dir} A_i \ll_{x_\lambda} B$ .

- (2) It is clear that  $A \leq C \ll_{x_\lambda}^{\mathcal{C}} D \leq B$  implies  $A \ll_{x_\lambda}^{\mathcal{C}} B$ . Next, we check that  $\ll_{x_\lambda}^{\mathcal{C}}$  satisfies (LDR1)–(LDR3).  
 (LDR1) Since  $\top \notin \psi_{x_\lambda}(L^X)$  and  $\perp \in \psi_{x_\lambda}(L^X)$ , it is clear that  $\top \not\ll_{x_\lambda}^{\mathcal{C}} \top$  and  $\perp \ll_{x_\lambda}^{\mathcal{C}} \perp$ .  
 (LDR2) For all  $A, B \in L^X$ ,

$$\begin{aligned} A \ll_{x_\lambda}^{\mathcal{C}} B &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \preceq D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge A \leq D \preceq D \text{ and } A \leq B \\ &\iff A \ll_{x_\lambda}^{\mathcal{C}} A \leq B. \end{aligned}$$

(LDR3) Let  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$  and  $B \in L^X$ . Then  $\bigvee_{i \in I}^{dir} A_i \ll_{x_\lambda}^{\mathcal{C}} B$  clearly implies  $A_i \ll_{x_\lambda}^{\mathcal{C}} B$  for any  $i \in I$ . Conversely, assume that  $A_i \ll_{x_\lambda}^{\mathcal{C}} B$  for any  $i \in I$ . There is  $D_i \in \psi_{x_\lambda}(L^X)$  such that  $A_i \leq D_i \wedge B \leq D_i \preceq D_i$  for any  $i \in I$ . Let

$$\varphi_i = \{C_i \in \psi_{x_\lambda}(L^X) : A_i \leq C_i \wedge B \leq C_i \preceq C_i\}.$$

Let  $E_i = \bigwedge \varphi_i$  for any  $i \in I$ . Then  $E_i \leq D_i$ . Next, we check that  $\{E_i\}_{i \in I}^{dir} \subseteq L^X$  and  $E_i \in \varphi_i$  for any  $i \in I$ .

Let  $i, j \in I$ . Since  $\{A_i\}_{i \in I}$  is directed, there is an index  $k \in I$  such that  $A_i \vee A_j \leq A_k$ . For any  $C_k \in \varphi_k$ , we have  $C_k \preceq C_k \geq C_k \wedge B \geq A_k \geq A_i \vee A_j$ . Thus  $C_k \in \varphi_i \cap \varphi_j$  which implies that  $E_i \vee E_j \leq E_k$ . So  $\{E_i\}_{i \in I}$  is directed. For any  $i \in I$  and any  $C_i \in \varphi_i$ , we have  $E_i \leq C_i \preceq C_i$ . Thus  $E_i \preceq C_i$  and  $E_i \preceq \bigwedge_{C_i \in \varphi_i} C_i = E_i$  by (LCER3). Hence  $A_i \leq E_i \wedge B \leq E_i \preceq E_i$  which implies  $E_i \in \varphi_i$ .

Let  $E = \bigvee_{i \in I}^{dir} E_i$ . Then  $E_i \preceq E$  for any  $i \in I$ . Thus  $E \preceq E$  by (LCER5). Further, since  $E_i \in \psi_{x_\lambda}(L^X)$  for any  $i \in I$ , it follows from Lemma 2.6(2) that  $E \in \psi_{x_\lambda}(L^X)$ . Hence  $\bigvee_{i \in I}^{dir} A_i \leq E \wedge B \leq E \preceq E$ . Therefore  $\bigvee_{i \in I}^{dir} A_i \ll_{x_\lambda}^{\mathcal{C}} B$ .

Next, we prove other results.

- (i) In order to prove that  $\ll_{x_\lambda}^{\mathcal{C}} \leq \ll_{x_\lambda}$ , let  $A, B \in L^X$ . Then

$$\begin{aligned} A \ll_{x_\lambda}^{\mathcal{C}} B &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \preceq D \\ &\implies A \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq B \\ &\iff A \ll_{x_\lambda} B. \end{aligned}$$

Thus  $\ll_{x_\lambda}^{\mathcal{C}} \leq \ll_{x_\lambda}$ .

It is clear that  $\preceq_{fin}$  is an  $L$ -convex enclosed relation. So  $\ll_{x_\lambda}^{\mathcal{C}_{fin}} \leq \ll_{x_\lambda}$ . To check that  $\ll_{x_\lambda} \leq \ll_{x_\lambda}^{\mathcal{C}_{fin}}$ , let  $A \ll_{x_\lambda} B$ . There is  $C \in \psi_{x_\lambda}(L^X)$  such that  $A \leq C \leq B$ . Thus  $A \leq C \wedge B \leq C \preceq_{fin} C$  which implies that  $A \ll_{x_\lambda}^{\mathcal{C}_{fin}} B$ . Hence  $\ll_{x_\lambda} \leq \ll_{x_\lambda}^{\mathcal{C}_{fin}}$ . Therefore  $\ll_{x_\lambda}^{\mathcal{C}_{fin}} = \ll_{x_\lambda}$ .

- (ii) It directly follows from Lemma 2.6(3).

- (3) (LDR1) Since  $\perp \in \mathcal{R}_{x_\lambda}^{\mathcal{C}}$  and  $\top \notin \mathcal{R}_{x_\lambda}^{\mathcal{C}}$ , it is clear that  $\perp \ll_{x_\lambda}^{\mathcal{C}} \perp$  and  $\top \not\ll_{x_\lambda}^{\mathcal{C}} \top$ .

(LDR2) For all  $A, B \in L^X$ , it is clear that

$$A \ll_{x_\lambda}^{\mathcal{C}} B \iff \exists E \in \mathcal{R}_{x_\lambda}^{\mathcal{C}} \text{ s.t. } A \leq E \leq B \iff A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}} \text{ s.t. } A \leq B \iff A \ll_{x_\lambda}^{\mathcal{C}} A \leq B.$$

(LDR3). Let  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$  and  $B \in L^X$ . If  $\bigvee_{i \in I}^{dir} A_i \leq_{x_\lambda}^C B$  then it is clear that  $A_i \leq_{x_\lambda}^C B$  for any  $i \in I$ . Conversely, let  $A_i \leq_{x_\lambda}^C B$  for any  $i \in I$ . Then there is  $D_i \in \mathcal{R}_{x_\lambda}^C$  such that  $A_i \leq D_i \leq B$  for any  $i \in I$ .

Let  $\varphi_i = \{C_i \in \mathcal{R}_{x_\lambda}^C : A_i \leq C_i \leq B\}$  for any  $i \in I$ . Then  $\bigwedge \varphi_i \in \mathcal{R}_{x_\lambda}^C$ . In addition, since  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$ , it is easy to check that the set  $\{\bigwedge \varphi_i\}_{i \in I}$  is directed. Thus  $\bigvee_{i \in I}^{dir} \bigwedge \varphi_i \in \mathcal{R}_{x_\lambda}^C$  and  $\bigvee_{i \in I}^{dir} A_i \leq \bigvee_{i \in I}^{dir} \bigwedge \varphi_i \leq B$ . Hence  $\bigvee_{i \in I}^{dir} A_i \leq_{x_\lambda}^C B$ .

Therefore  $\leq_{x_\lambda}^C$  is an  $L$ -directed relation.

Let  $A, B \in L^X$ . If  $A \leq_{x_\lambda}^C B$  then there is a  $C \in \mathcal{R}_{x_\lambda}^C \subseteq \psi_{x_\lambda}(L^X)$  such that  $A \leq C \leq B$ . This shows that  $A \leq_{x_\lambda} B$ . Thus  $\leq_{x_\lambda}^C \leq_{x_\lambda} B$ .

Since  $\mathcal{R}_{x_\lambda}^{Cfin} = \psi_{x_\lambda}(L^X)$ , it is easy to check that  $A \leq_{x_\lambda} B$  iff  $A \leq_{x_\lambda}^{Cfin} B$  for all  $A, B \in L^X$ . Therefore  $\leq_{x_\lambda} = \leq_{x_\lambda}^{Cfin}$ .  $\square$

**Proposition 3.4.** Let  $f : X \rightarrow Y$  be a mapping. For any  $\leq_X \in \mathfrak{R}(L^X)$ , we define a binary relation  $(\leq_X)_{f(X)}$  (simply,  $\leq_{f(X)}$ ) on  $L^Y$  by

$$\forall A, B \in L^Y, \quad A \leq_{f(X)} B \iff A \leq B \text{ and } f_L^+(A) \leq_X f_L^-(B).$$

Then  $\leq_{f(X)} \in \mathfrak{R}(L^Y)$ .

*Proof.* Its proof is direct.  $\square$

Based on examples of  $L$ -directed relations presented in Proposition 3.3, we introduce  $L$ -convergence relation spaces and study its relation with  $L$ -convex enclosed relation spaces.

**Definition 3.5.** A relation  $\ll$  on  $\beta^*(L^X) \times \mathfrak{R}(L^X)$  is called an  $L$ -pre-convergence relation on  $L^X$  if for all  $x_\lambda \in \beta^*(L^X)$  and  $\leq_1, \leq_2 \in \mathfrak{R}(L^X)$ ,

(LCR1)  $x_\lambda \ll \leq_{x_\lambda}$ ;

(LCR2)  $x_\lambda \ll \leq_1$  and  $\leq_1 \leq \leq_2$  imply  $x_\lambda \ll \leq_2$ .

**Lemma 3.6.** Let  $\ll$  be an  $L$ -pre-convergence relation on  $L^X$  and  $x_\lambda \in \beta^*(L^X)$ . Define a binary relation  $\ll_{x_\lambda}$  on  $L^X$  by

$$\forall A, B \in L^X, \quad A \ll_{x_\lambda} B \iff \forall \leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \quad A \leq B,$$

where  $\mathfrak{R}_{x_\lambda}^{\ll}(L^X) = \{\leq \in \mathfrak{R}(L^X) : x_\lambda \ll \leq\}$ . Then  $\ll_{x_\lambda}$  is an  $L$ -directed relation on  $L^X$ .

*Proof.* (LDR1) Since  $x_\lambda \ll \leq_{x_\lambda}$  by (LCR1) and  $\perp \not\leq_{x_\lambda} \perp$ , it is clear that  $\perp \not\ll_{x_\lambda} \perp$ . For any  $\leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$ , it is trivial that  $\perp \leq \perp$  by (LDR1). Thus  $\perp \ll_{x_\lambda} \perp$  holds.

(LDR2) For all  $A, B \in L^X$ ,

$$\begin{aligned} A \ll_{x_\lambda} B &\iff \forall \leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \quad A \leq B \\ &\iff \forall \leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \quad A \leq A \leq B \\ &\iff A \ll_{x_\lambda} A \leq B. \end{aligned}$$

(LDR3) For all  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$  and  $B \in L^X$ ,

$$\begin{aligned} \bigvee_{i \in I}^{dir} A_i \ll_{x_\lambda} B &\iff \forall \leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \quad \bigvee_{i \in I}^{dir} A_i \leq B \\ &\iff \forall \leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \quad \forall i \in I, \quad A_i \leq B \\ &\iff \forall i \in I, \quad A_i \ll_{x_\lambda} B. \end{aligned}$$

Therefore  $\ll_{x_\lambda}$  is an  $L$ -directed relation.  $\square$

**Definition 3.7.** An  $L$ -pre-convergence relation  $\ll$  on  $L^X$  is called an  $L$ -convergence relation and the pair  $(X, \ll)$  is called an  $L$ -convergence relation space if  $\ll_{x_\lambda}$  further satisfies

(LCR3)  $\forall A, B \in L^X, \quad A \ll_{x_\lambda} B \iff A \ll_{x_\eta} B$  for any  $\eta \in \beta_\lambda^*(L)$ .

Let  $\ll_1$  and  $\ll_2$  be  $L$ -convergence relations on  $L^X$ .  $\ll_1$  is said to be coarser than  $\ll_2$ , denoted by  $\ll_1 \leq \ll_2$ , provided that  $x_\lambda \ll_1 \leq$  implies  $x_\lambda \ll_2 \leq$  for all  $x_\lambda \in \beta^*(L^X)$  and  $\leq \in \mathfrak{R}(L^X)$ .

**Definition 3.8.** Let  $(X, \ll_X)$  and  $(Y, \ll_Y)$  be  $L$ -convergence relation spaces. A mapping  $f : X \rightarrow Y$  is called an  $L$ -convergence relation preserving mapping if  $x_\lambda \ll_X \leq_X$  implies  $f_L^+(x_\lambda) \ll_Y \leq_{f(X)}$  for all  $x_\lambda \in \beta^*(L^X)$  and  $\leq_X \in \mathfrak{R}(L^X)$ .

**Example 3.9.** Let  $X = \{x\}$  and  $L = \{\perp, a, b, c, d, \top\}$ , where  $c = a \wedge b$ ,  $d = a \vee b$  and  $a, b$  are incomparable. Then  $\beta^*(L^X) = J(L^X) = \{x_a, x_b, x_c\}$  and  $\mathfrak{R}(L^X) = \{\leq_w : w \in L \setminus \{\top\}\}$ , where  $\leq_w$  is defined by  $A \leq_w B$  iff  $A \leq x_w \wedge B$  for all  $A, B \in L^X$  and  $w \in L \setminus \{\top\}$ . It is easy to check that  $\leq_\perp = \leq_{x_c}$ ,  $\leq_a = \leq_{x_b}$  and  $\leq_b = \leq_{x_a}$ .

Let  $\ll$  and  $\ll_1$  be relations on  $\beta^*(L^X) \times \mathfrak{R}(L^X)$  defined by Tables 1 and 2 as follows.

	$\leq_\perp$	$\leq_c$	$\leq_a$	$\leq_b$	$\leq_d$
$x_c$	$\ll$	$\ll$	$\ll$	$\ll$	$\ll$
$x_a$				$\ll$	$\ll$
$x_b$			$\ll$		$\ll$

Table 1: The definition of  $\ll$

	$\leq_\perp$	$\leq_c$	$\leq_a$	$\leq_b$	$\leq_d$
$x_c$	$\ll$	$\ll$	$\ll$	$\ll$	$\ll$
$x_a$			$\ll$	$\ll$	$\ll$
$x_b$			$\ll$	$\ll$	$\ll$

Table 2: The definition of  $\ll_1$

It is easy to check that both  $\ll$  and  $\ll_1$  are *L-convergence structures*.

The category of *L-convergence relation spaces* and *L-convergence relation preserving mappings* is denoted by **LCRS**. We now study relationships between *L-convergence relation spaces* and *L-convex enclosed relation spaces*.

**Proposition 3.10.** Let  $(X, \leq)$  be an *L-convex enclosed relation space*. Define a relation  $\ll_{\leq}$  on  $\beta^*(L^X) \times \mathfrak{R}(L^X)$  by

$$\forall x_\lambda \in \beta^*(L^X), \forall \leq \in \mathfrak{R}(L^X), \quad x_\lambda \ll_{\leq} \leq \iff \leq_{x_\lambda} \leq \leq.$$

Then  $\ll_{\leq}$  is an *L-convergence relation* satisfying  $\ll_{x_\lambda}^{\ll_{\leq}} = \leq_{x_\lambda}^{\leq}$ .

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $\leq_1, \leq_2 \in \mathfrak{R}(L^X)$ .

(LCR1) It is clear that  $\leq_{x_\lambda}^{\leq} \leq \leq_{x_\lambda}$  by Proposition 3.3(2). Thus  $x_\lambda \ll_{\leq} \leq_{x_\lambda}$ .

(LCR2) Clearly,  $x_\lambda \ll_{\leq} \leq_1$  and  $\leq_1 \leq \leq_2$  imply  $\leq_{x_\lambda}^{\leq} \leq \leq_1 \leq \leq_2$ . That is,  $x_\lambda \ll_{\leq} \leq_2$ .

(LCR3) For all  $A, B \in L^X$ , it is clear that

$$A \ll_{x_\lambda}^{\ll_{\leq}} B \iff \forall x_\lambda \ll_{\leq} \leq, A \leq B \iff \forall \leq_{x_\lambda}^{\leq} \leq \leq, A \leq B \iff A \ll_{x_\lambda}^{\leq} B.$$

This shows that  $\ll_{x_\lambda}^{\ll_{\leq}} = \ll_{x_\lambda}^{\leq}$ . Further, it follows from Proposition 3.3(2) that

$$A \ll_{x_\lambda}^{\ll_{\leq}} B \iff A \ll_{x_\lambda}^{\leq} B \iff \forall \eta \in \beta^*(L), A \ll_{x_\lambda}^{\leq} B \iff \forall \eta \in \beta^*(L), A \ll_{x_\eta}^{\ll_{\leq}} B.$$

This shows that (LCR3) holds for  $\ll_{\leq}$ .

Therefore  $\ll_{\leq}$  is an *L-convergence relation* satisfying  $\ll_{x_\lambda}^{\ll_{\leq}} = \ll_{x_\lambda}^{\leq}$ . □

**Theorem 3.11.** Let  $(X, \leq_X)$  and  $(Y, \leq_Y)$  be *L-convex enclosed relation spaces*. If  $f : X \rightarrow Y$  is an *L-convex enclosed relation preserving mapping*, then  $f : (X, \ll_{\leq_X}) \rightarrow (Y, \ll_{\leq_Y})$  is an *L-convergence relation preserving mapping*.

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $\leq_X \in \mathfrak{R}(L^X)$  with  $x_\lambda \ll_{\leq_X} \leq_X$ . Then  $\leq_{x_\lambda}^{\leq_X} \leq \leq_X$ . In order to prove that  $f_L^\rightarrow(x_\lambda) \ll_{\leq_Y} \leq_{f(X)}$ , it is sufficient to prove that  $\leq_{f_L^\rightarrow(x_\lambda)}^{\leq_Y} \leq \leq_{f(X)}$ . If  $D \in \psi_{f_L^\rightarrow(x_\lambda)}(L^Y)$  then  $f_L^\leftarrow(D) \in \psi_{x_\lambda}(L^X)$ . Thus, for all  $A, B \in L^X$ ,

$$\begin{aligned} A \ll_{f_L^\rightarrow(x_\lambda)}^{\leq_Y} B &\iff \exists D \in \psi_{f_L^\rightarrow(x_\lambda)}(L^Y) \text{ s.t. } A \leq D \wedge B \leq D \leq_Y D \\ &\implies A \leq B \text{ and } \exists f_L^\leftarrow(D) \in \psi_{x_\lambda}(L^X) \text{ s.t.} \\ &\quad f_L^\leftarrow(A) \leq f_L^\leftarrow(D) \wedge f_L^\leftarrow(B) \leq f_L^\leftarrow(D) \leq_X f_L^\leftarrow(D) \\ &\implies A \leq B \text{ and } f_L^\leftarrow(A) \ll_{x_\lambda}^{\leq_X} f_L^\leftarrow(B) \\ &\implies A \leq B \text{ and } f_L^\leftarrow(A) \leq_X f_L^\leftarrow(B) \\ &\implies A \leq_{f(X)} B. \end{aligned}$$

Thus  $\leq_{f_L^\rightarrow(x_\lambda)}^{\leq_Y} \leq \leq_{f(X)}$  which shows that  $f_L^\rightarrow(x_\lambda) \ll_{\leq_Y} \leq_{f(X)}$ . Hence  $f$  is an *L-convergence relation preserving mapping* from  $(X, \ll_{\leq_X})$  to  $(Y, \ll_{\leq_Y})$ . □

**Proposition 3.12.** Let  $(X, \ll)$  be an *L-convergence relation space*. Let  $\leq_{\ll}$  be defined by

$$\begin{aligned} \forall A, B \in L^X, A \leq_{\ll} B &\iff \exists A \leq D \leq B \text{ s.t. } \forall x_\lambda \in \beta^*(L^X), \forall \leq \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \\ &\quad x_\lambda \not\leq D \text{ implies } D \leq B. \end{aligned}$$

Then  $\leq_{\ll}$  is an *L-convex enclosed relation* on  $L^X$ .

*Proof.* It is clear that  $A \leq C \preceq_{\ll} D \leq B$  implies  $A \preceq_{\ll} B$  for all  $A, B, C, D \in L^X$ . Next, we check that  $\preceq_{\ll}$  satisfies (LCER1)–(LCER5).

(LCER1) It is trivial that  $\perp \preceq_{\ll} \perp$ .

(LCER2) If  $A \preceq_{\ll} B$  then it is clear that  $A \leq B$ .

(LCER3) If  $A \preceq_{\ll} \bigwedge_{i \in I} B_i$  then it is clear that  $A \preceq_{\ll} B_i$  for any  $i \in I$ . Conversely, let  $A \preceq_{\ll} B_i$  for any  $i \in I$ . By  $A \preceq_{\ll} B_i$ , there is a  $D_i \in L^X$  such that  $A \leq D_i \leq B_i$  and for all  $x_\lambda \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$ ,  $x_\lambda \not\leq D_i$  implies  $D_i \leq B_i$ .

Let  $D = \bigwedge_{i \in I} D_i$ . Then  $A \leq D \leq \bigwedge_{i \in I} B_i$ . For any  $z_\theta \in \beta^*(L^X)$  and any  $\ll \in \mathfrak{R}_{z_\theta}^{\ll}(L^X)$  with  $z_\theta \not\leq D$ , there is  $j \in I$  such that  $z_\theta \not\leq D_j$ . Thus  $D_j \leq B_j$  by  $A \preceq_{\ll} B_j$ . Hence

$$D \leq D_j \wedge \bigwedge_{i \in I} B_i \leq B_j \wedge \bigwedge_{i \in I} B_i = \bigwedge_{i \in I} B_i.$$

Hence  $D \leq \bigwedge_{i \in I} B_i$ . Therefore  $A \preceq_{\ll} \bigwedge_{i \in I} B_i$  as desired.

(LCER4) Let  $A \preceq_{\ll} B$ . We need to find a set  $D \in L^X$  such that  $A \preceq_{\ll} D \preceq_{\ll} B$ .

By  $A \preceq_{\ll} B$ , there is a  $D \in L^X$  such that  $A \leq D \leq B$  and all  $x_\lambda \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$  with  $x_\lambda \not\leq D$  imply  $D \leq B$ .

We say that  $D \preceq_{\ll} B$ . Indeed,  $D \leq D \leq B$ . For all  $y_\mu \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}_{y_\mu}^{\ll}(L^X)$  with  $y_\mu \not\leq D$ , we have  $D \leq B$  by  $A \preceq_{\ll} B$ . Thus  $D \preceq_{\ll} B$ . We also say that  $A \preceq_{\ll} D$ . Indeed,  $A \leq D \leq D$ . For all  $z_\eta \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}_{z_\eta}^{\ll}(L^X)$  with  $z_\eta \not\leq D$ , we have  $D \leq B$  by  $A \preceq_{\ll} B$ . Thus  $D \preceq_{\ll} D$  by (LDR2). Hence  $A \preceq_{\ll} D$ . Therefore  $A \preceq_{\ll} D \preceq_{\ll} B$  as desired.

(LCER5) If  $\bigvee_{i \in I}^{dir} A_i \preceq_{\ll} B$  then it is clear that  $A_i \preceq_{\ll} B$  for any  $i \in I$ . Conversely, assume that  $A_i \preceq_{\ll} B$  for any  $i \in I$ . Thus, for any  $i \in I$ , there is a set  $D_i \in L^X$  such that  $A_i \leq D_i \leq B$  and  $D_i \leq B$  for any  $x_\lambda \in \beta^*(L^X)$  and any  $\ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$  with  $x_\lambda \not\leq D_i$ . Let  $E_i = \bigwedge \varphi_i$  for any  $i \in I$ , where

$$\varphi_i = \{D_i \in L^X : A_i \leq D_i \leq B \text{ and } \forall x_\lambda \in \beta^*(L^X), \forall \ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), x_\lambda \not\leq D_i \text{ implies } D_i \leq B\}.$$

Then  $A_i \leq E_i \in \varphi_i$  and  $\{E_i\}_{i \in I}$  is directed since  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$ . Thus  $\bigvee_{i \in I}^{dir} A_i \leq \bigvee_{i \in I}^{dir} E_i \leq B$ .

In order to prove that  $\bigvee_{i \in I}^{dir} A_i \preceq_{\ll} B$ , let  $x_\lambda \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$  with  $x_\lambda \not\leq \bigvee_{i \in I}^{dir} E_i$ . Since  $\ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$  and  $x_\lambda \not\leq E_i$  for any  $i \in I$ , it follows that  $E_i \leq D_i \leq B$  by  $A_i \preceq_{\ll} B$ . Thus  $E_i \leq B$ . Hence  $\bigvee_{i \in I}^{dir} E_i \leq B$  by (LDR3). Therefore  $\bigvee_{i \in I}^{dir} A_i \preceq_{\ll} B$ .  $\square$

**Proposition 3.13.** *Let  $(X, \ll_X)$  and  $(Y, \ll_Y)$  be  $L$ -convergence relation spaces. If  $f : X \rightarrow Y$  is an  $L$ -convergence relation preserving mapping from  $(X, \ll_X)$  to  $(Y, \ll_Y)$ , then  $f : (X, \preceq_{\ll_X}) \rightarrow (Y, \preceq_{\ll_Y})$  is an  $L$ -convex enclosed relation preserving mapping.*

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $\ll_X \in \mathfrak{R}_{x_\lambda}^{\ll_X}(L^X)$ . Since  $f : X \rightarrow Y$  is an  $L$ -convergence relation preserving mapping, it is clear that  $\ll_{f(X)} \in \mathfrak{R}_{f_L^{\rightarrow}(x_\lambda)}^{\ll_Y}(L^Y)$ . For all  $A, B \in L^Y$ , it follows that

$$\begin{aligned} A \preceq_{\ll_Y} B &\iff \exists A \leq D \leq B \text{ s.t. } \forall y_\mu \in \beta^*(L^Y), \forall \ll_Y \in \mathfrak{R}_{y_\mu}^{\ll_Y}(L^Y), \\ &\quad y_\mu \not\leq D \text{ implies } D \leq_Y B \\ &\implies \exists A \leq D \leq B \text{ s.t. } \forall x_\lambda \in \beta^*(L^X), \forall \ll_X \in \mathfrak{R}_{x_\lambda}^{\ll_X}(L^X), \\ &\quad f_L^{\rightarrow}(x_\lambda) \not\leq D \text{ implies } D \leq_{f(X)} B \\ &\implies \exists f_L^{\leftarrow}(A) \leq H \leq f_L^{\leftarrow}(B) \text{ s.t. } \forall x_\lambda \in \beta^*(L^X), \forall \ll_X \in \mathfrak{R}_{x_\lambda}^{\ll_X}(L^X), \\ &\quad x_\lambda \not\leq H \text{ implies } H \leq_X f_L^{\leftarrow}(B) \\ &\implies f_L^{\leftarrow}(A) \preceq_{\ll_X} f_L^{\leftarrow}(B). \end{aligned}$$

Therefore  $f$  is an  $L$ -convex enclosed relation preserving mapping.  $\square$

**Lemma 3.14.** *Let  $(X, \ll)$  be an  $L$ -convergence relation space. Then  $\ll_{x_\lambda}^{\ll} \leq \ll_{x_\lambda}^{\ll} \leq \ll_{x_\lambda}$  for  $x_\lambda \in \beta^*(L^X)$ .*

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $A, B \in L^X$ . If  $A \ll_{x_\lambda}^{\ll} B$  then there is a  $D \in \psi_{x_\lambda}(L^X)$  such that  $A \leq D \wedge B \leq D \preceq_{\ll} D$ .

It follows from (LCR3) of  $\ll$ , Lemma 3.2(2) and (LDR2) of  $\ll_{x_\lambda}$  that

$$\begin{aligned}
 D \prec_{\ll} D &\iff \forall y_\theta \in \beta^*(L^X), \forall \ll \in \mathfrak{R}_{y_\theta}^{\ll x}(L^X), y_\theta \not\leq D \text{ implies } D \leq D \\
 &\implies \forall \eta \in \beta_\lambda^*(L), \forall \ll \in \mathfrak{R}_{x_\eta}^{\ll x}(L^X), x_\eta \not\leq D \text{ implies } D \leq D \\
 &\implies \forall \eta \in \beta_\lambda^*(L), D \leq_{\ll_{x_\eta}} D \\
 &\iff D \leq_{\ll_{x_\lambda}} D \\
 &\implies A \leq_{\ll_{x_\lambda}} A \leq B \\
 &\iff A \leq_{\ll_{x_\lambda}} B.
 \end{aligned}$$

Thus  $\ll_{x_\lambda} \leq \ll_{x_\lambda} \leq \ll_{x_\lambda}$ .

Since  $\ll_{x_\lambda} \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$  by (LCR1),  $A \leq_{\ll_{x_\lambda}} B$  implies  $A \leq_{x_\lambda} B$ . Therefore  $\ll_{x_\lambda} \leq \ll_{x_\lambda}$ .  $\square$

**Proposition 3.15.** *If  $(X, \ll)$  is an L-convergence relation space, then  $\ll \leq \ll_{\ll} \leq \ll$ .*

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}(L^X)$ . If  $x_\lambda \ll \ll$  then  $\ll_{x_\lambda} \leq \ll \leq \ll_{x_\lambda}$  by Lemmas 3.6 and 3.14. This implies that  $x_\lambda \ll_{\ll} \ll$  by Proposition 3.10. This shows that  $\ll \leq \ll_{\ll} \leq \ll$ .  $\square$

**Proposition 3.16.** *If  $(X, \preceq)$  is an L-convex enclosed relation space, then  $\preceq_{\ll} = \preceq$ .*

*Proof.* It follows from Proposition 3.10 that  $\ll_{x_\lambda} \preceq_{\ll} = \preceq_{x_\lambda}$ . For all  $A, B \in L^X$ ,

$$\begin{aligned}
 A \preceq_{\ll} B &\iff \exists A \leq D \leq B \text{ s.t. } \forall x_\lambda \in \beta^*(L^X), \forall \ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), \\
 &\quad x_\lambda \not\leq D \text{ implies } D \leq B \\
 &\iff \exists A \leq D \leq B \text{ s.t. } \forall x_\lambda \in \beta^*(L^X), x_\lambda \not\leq D \text{ implies } D \leq_{\ll_{x_\lambda}} B.
 \end{aligned}$$

Thus  $A \preceq_{\ll} B$  implies a set  $D \in L^X$  such that  $A \leq D \leq B$  and  $D \leq_{\ll_{x_\lambda}} B$  for any  $x_\lambda \in \beta^*(L^X)$  with  $x_\lambda \not\leq D$ . Further, by  $D \leq_{\ll_{x_\lambda}} B$ , there is a set  $E_{x_\lambda} \in \psi_{x_\lambda}(L^X)$  such that  $D \leq E_{x_\lambda} \wedge B \leq E_{x_\lambda} \preceq E_{x_\lambda}$ .

Let  $E = \bigwedge_{x_\lambda \not\leq D} E_{x_\lambda}$ . Then  $E \preceq E$  by (LCER3) of  $\preceq$ . We say that  $E = D$ . Indeed, it is clear that  $D \leq E$ . Suppose that  $E \not\leq D$ . Then there is  $y_\mu \in \beta^*(E)$  such that  $y_\mu \not\leq D$ . By  $y_\mu \in \beta^*(E)$ , there is  $y_\eta \in \beta^*(E)$  such that  $y_\mu \in \beta^*(y_\eta)$ . But  $y_\eta \prec E \leq E_{y_\mu} \in \psi_{y_\mu}(L^X)$  which is a contradiction. Thus  $E \leq D$ . Hence  $E = D$ . As a result,  $A \leq E \preceq E \leq B$  which shows that  $A \preceq B$ . In conclusion,  $A \preceq_{\ll} B$  implies  $A \preceq B$ . Therefore  $\preceq_{\ll} \leq \preceq$ .

Conversely, let  $A \preceq B$ . (LCER4) yields a set  $F \in L^X$  such that  $A \preceq F \preceq B$ . Let  $D = \bigwedge \{G \in L^X : A \preceq G\}$ . Then  $A \leq D \preceq D \leq B$  by (LCER3). This shows that  $A \leq D \leq B$ .

In order to prove that  $A \preceq_{\ll} B$ , let  $x_\lambda \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X)$  with  $x_\lambda \not\leq D$ . Then  $D \in \psi_{x_\lambda}(L^X)$  and  $\ll_{x_\lambda} \leq \ll$ . Since  $D \leq D \wedge B \leq D \preceq D$ , it is follows that  $D \leq_{\ll_{x_\lambda}} B$ . Thus  $D \leq B$ . This shows that  $A \preceq_{\ll} B$ . Therefore  $\preceq \leq \preceq_{\ll}$ .

In conclusion,  $\preceq_{\ll} = \preceq$  as desired.  $\square$

Based on Propositions 3.10 and 3.11, we get a functor  $\mathbb{U} : L\text{-CERS} \rightarrow L\text{-CRS}$  defined by

$$\mathbb{U} : \begin{cases} L\text{-CERS} & \longrightarrow & L\text{-CRS}; \\ (X, \preceq) & \longmapsto & (X, \ll_{\ll}); \\ f & \longmapsto & f. \end{cases}$$

Based on Propositions 3.12 and 3.13, we get a functor  $\mathbb{V} : L\text{-CRS} \rightarrow L\text{-CERS}$  defined by

$$\mathbb{V} : \begin{cases} L\text{-CRS} & \longrightarrow & L\text{-CERS}; \\ (X, \ll) & \longmapsto & (X, \preceq_{\ll}); \\ f & \longmapsto & f. \end{cases}$$

Based on Propositions 3.10–3.16, we have the following conclusion.

**Theorem 3.17.**  *$(\mathbb{U}, \mathbb{V})$  is a Galois's connection, where  $\mathbb{V}$  is a left inverse of  $\mathbb{U}$ .*

In order to study further relationships between L-convergence relations and L-convex enclosed relations, we introduce L-convex convergence relations as follows.

**Definition 3.18.** An  $L$ -convergence relation  $\ll$  on  $L^X$  is called an  $L$ -convex convergence relation and the pair  $(X, \ll)$  is called an  $L$ -convex convergence relation space, if for all  $x_\lambda \in \beta^*(L^X)$  and  $A, B \in L^X$ ,

$$(LCCR1) \quad x_\lambda \ll_{x_\lambda} \ll_{x_\lambda};$$

$$(LCCR2) \quad A \ll_{x_\lambda} B \iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq B \wedge D \leq D \ll_{y_\mu} D \text{ for all } y_\mu \not\leq D.$$

The category of  $L$ -convex convergence relation spaces and  $L$ -convergence relation preserving mappings is denoted by  $L\text{-CCRS}$ . Next, we discuss the relationships between  $L\text{-CERS}$  and  $L\text{-CCRS}$ .

**Remark 3.19.** The  $L$ -convergence relation  $\ll$  defined in Example 3.9 is an  $L$ -convex convergence relation on  $L^X$ , while the  $L$ -convergence relation  $\ll_1$  is not an  $L$ -convex convergence relation. In fact,  $\ll_{x_a}^1 = \ll_{x_b}^1 = \ll_c$  while  $x_a \not\ll_c = \ll_{x_a}^1$ . That is, (LCCR1) fails for  $\ll_1$ . Therefore  $\ll_1$  is not an  $L$ -convex convergence relation.

**Proposition 3.20.** If  $(X, \preceq)$  is an  $L$ -convex enclosed relation space, then  $\ll_{\preceq}$  is an  $L$ -convex convergence relation on  $L^X$ .

*Proof.* It is sufficient to prove that  $\ll_{\preceq}$  satisfies (LCCR1) and (LCCR2).

(LCCR1) Let  $x_\lambda \in \beta^*(L^X)$ . We have  $x_\lambda \ll_{x_\lambda} \ll_{x_\lambda}$  by  $\ll_{x_\lambda} \leq \ll_{x_\lambda}$ . Thus  $x_\lambda \ll_{x_\lambda} \ll_{x_\lambda}$  by Proposition 3.10.

(LCCR2). For any  $D \in L^X$ , it is clear that  $D \preceq D$  imply  $D \ll_{y_\mu} D$  for any  $y_\mu \not\leq D$ . On one hand, it follows from Proposition 3.10 that

$$\begin{aligned} A \ll_{x_\lambda} B &\iff A \ll_{x_\lambda} B \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \preceq D \\ &\implies \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq D \wedge B \leq D \ll_{y_\mu} D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq D \wedge B \leq D \ll_{y_\mu} D. \end{aligned}$$

On the other hand, by (LCR3) of  $\ll_{x_\lambda}$ , Lemma 3.2(2) and Proposition 3.10, it follows that

$$\begin{aligned} &\exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq D \wedge B \leq D \ll_{y_\mu} D \\ \implies &\exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall \eta \in \beta_\lambda^*(L), A \leq D \wedge B \leq D \ll_{x_\eta} D \\ \implies &\exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \ll_{x_\lambda} D \quad (\text{by (LCR3)}) \\ \iff &\exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \ll_{x_\lambda} D \quad (\text{by Proposition 3.10}) \\ \implies &A \ll_{x_\lambda} A \leq B \quad (\text{by Lemma 3.2(2)}) \\ \iff &A \ll_{x_\lambda} A \leq B \quad (\text{by Proposition 3.10}) \\ \iff &A \ll_{x_\lambda} B. \end{aligned}$$

Thus (LCCR2) holds for  $\ll_{\preceq}$ .

Therefore  $\ll_{\preceq}$  is an  $L$ -convex convergence relation. □

**Lemma 3.21.** If  $(X, \ll)$  is an  $L$ -convex convergence relation space, then  $\ll_{x_\lambda} \ll_{x_\lambda} = \ll_{x_\lambda}$  for  $x_\lambda \in \beta^*(L^X)$ .

*Proof.* Let  $A, B \in L^X$ . We have

$$\begin{aligned} A \ll_{x_\lambda} B &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \ll_{x_\lambda} D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, \forall \ll \in \mathfrak{R}_{y_\mu}^\ll(L^X), A \leq D \wedge B \leq D \ll D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq D \wedge B \leq D \ll_{y_\mu} D \\ &\iff A \ll_{x_\lambda} B. \end{aligned}$$

Therefore  $\ll_{x_\lambda} \ll_{x_\lambda} = \ll_{x_\lambda}$ . □

**Proposition 3.22.** Let  $(X, \ll)$  be an  $L$ -convex convergence relation space. Then  $\ll_{\preceq} = \ll$ .

*Proof.* It follows from Proposition 3.15 that  $\ll \leq \ll_{\preceq}$ . In order to prove that  $\ll_{\preceq} \leq \ll$ , let  $x_\lambda \in \beta^*(L^X)$  and  $\ll \in \mathfrak{R}(L^X)$  with  $x_\lambda \ll_{\preceq} \ll$ . Then  $\ll_{x_\lambda} \ll_{x_\lambda} = \ll$ . Thus  $x_\lambda \ll_{x_\lambda} \ll_{x_\lambda} = \ll_{x_\lambda} \ll_{x_\lambda} \leq \ll$  by (LCCR1) and Lemma 3.21. Hence  $x_\lambda \ll_{\preceq} \ll$  by (LCR2). Therefore  $\ll_{\preceq} \leq \ll$ . □

Based on Propositions 3.10–3.13, 3.15, 3.20 and 3.22, we have the following conclusion.

**Theorem 3.23.**  $L\text{-CERS}$  is isomorphic to  $L\text{-CCRS}$  which is a bireflective subcategory of  $L\text{-CRS}$ .

## 4 *L*-ideal-convergence spaces and *L*-convex ideal-convergence spaces

In this section, we introduce *L*-ideal-convergence spaces and *L*-convex ideal-convergence spaces. Then we investigate their relations with *L*-convergence relations spaces and *L*-convex convergence relation spaces.

**Proposition 4.1.** *Let  $(X, \mathcal{R})$  be an *L*-convex  $\beta^*$ -remotehood space and  $(X, \mathcal{C})$  be an *L*-convex space. We denote  $\mathcal{I}_{x_\lambda} = \psi_{x_\lambda}(L^X)$ ,  $\mathcal{I}_{x_\lambda}^{\mathcal{R}} = \mathcal{R}_{x_\lambda}$  and  $\mathcal{I}_{x_\lambda}^{\mathcal{C}} = \mathcal{R}_{x_\lambda}^{\mathcal{C}}$  for any  $x_\lambda \in \beta^*(\mathbb{I})$ . Then  $\mathcal{I}_{x_\lambda}, \mathcal{I}_{x_\lambda}^{\mathcal{R}}, \mathcal{I}_{x_\lambda}^{\mathcal{C}}$  are *L*-convex ideals on  $L^X$ .*

*Proof.* They are direct results of Lemma 2.6, Definition 2.7 and Theorem 2.8. □

**Proposition 4.2.** *Let  $(X, \preceq)$  be an *L*-directed relation space. Then  $\mathcal{I}_{\preceq} = \{B \in L^X : B \preceq B\} \in \mathfrak{J}(L^X)$ .*

*Proof.* (LCI1) It directly follows from (LDR1) that  $\perp \in \mathcal{I}_{\preceq}$  and  $\top \notin \mathcal{I}_{\preceq}$ .

(LCI2) If  $A \leq B \in \mathcal{I}_{\preceq}$  then  $A \leq B \preceq B$ . Thus  $A \preceq A \preceq B$  by Lemma 3.2(1) and (LDR2). So  $A \in \mathcal{I}_{\preceq}$ .

(LCI3) If  $\{A_i\}_{i \in I}^{dir} \subseteq \mathcal{I}_{\preceq}$  then  $A_i \preceq A_i \leq \bigvee_{i \in I}^{dir} A_i$  for any  $i \in I$ . Thus  $\bigvee_{i \in I}^{dir} A_i \preceq \bigvee_{i \in I}^{dir} A_i$  by (LDR3). Therefore  $\bigvee_{i \in I}^{dir} A_i \in \mathcal{I}_{\preceq}$ . □

**Proposition 4.3.** *Let  $(X, \mathcal{I})$  be an *L*-convex ideal spaces. Define a binary relation  $\preceq_{\mathcal{I}}$  by*

$$\forall A, B \in L^X, \quad A \preceq_{\mathcal{I}} B \iff A \in \mathcal{I} \text{ and } A \leq B.$$

*Then  $\preceq_{\mathcal{I}}$  is an *L*-directed relation on  $L^X$ .*

*Proof.* (LDR1) It directly follows from (LCI1) that  $\perp \preceq_{\mathcal{I}} \perp$  and  $\top \not\preceq_{\mathcal{I}} \top$ .

(LDR2) For all  $A, B \in L^X$ , we have

$$A \preceq_{\mathcal{I}} B \iff A \in \mathcal{I} \text{ and } A \leq A \leq B \iff A \preceq_{\mathcal{I}} A \leq B.$$

(LDR3) Let  $\{A_i\}_{i \in I}^{dir} \subseteq L^X$  and  $B \in L^X$ . If  $\bigvee_{i \in I}^{dir} A_i \preceq_{\mathcal{I}} B$  then  $A_i \preceq_{\mathcal{I}} B$  for any  $i \in I$ . Conversely, if  $A_i \preceq_{\mathcal{I}} B$  for any  $i \in I$  then  $A_i \in \mathcal{I}$  and  $A_i \leq B$  for each  $i \in I$ . Thus  $\bigvee_{i \in I}^{dir} A_i \leq B$  and  $\bigvee_{i \in I}^{dir} A_i \in \mathcal{I}$ . This shows that  $\bigvee_{i \in I}^{dir} A_i \preceq_{\mathcal{I}} B$ . □

**Proposition 4.4.** (1) *If  $(X, \preceq)$  is an *L*-directed relation space, then  $\preceq = \preceq_{\mathcal{I}_{\preceq}}$ .*

(2) *If  $(X, \mathcal{I})$  is an *L*-convex ideal space, then  $\mathcal{I}_{\preceq_{\mathcal{I}}} = \mathcal{I}$ .*

*Proof.* (1) For all  $A, B \in L^X$ , (LDR2) implies that

$$A \preceq B \iff A \preceq A \leq B \iff A \in \mathcal{I}_{\preceq} \text{ and } A \leq B \iff A \preceq_{\mathcal{I}_{\preceq}} A \leq B \iff A \preceq_{\mathcal{I}_{\preceq}} B.$$

So  $\preceq = \preceq_{\mathcal{I}_{\preceq}}$ .

(2) For any  $A \in L^X$ , we have  $A \in \mathcal{I}_{\preceq_{\mathcal{I}}} \iff A \preceq_{\mathcal{I}} A \iff A \in \mathcal{I}$ . Therefore  $\mathcal{I}_{\preceq_{\mathcal{I}}} = \mathcal{I}$ . □

**Lemma 4.5.** *For any  $x_\lambda \in \beta^*(L^X)$ , the following statements are valid.*

(1)  $\mathcal{I}_{x_\lambda} = \mathcal{I}_{\preceq_{x_\lambda}}$  and  $\preceq_{x_\lambda} = \preceq_{\mathcal{I}_{x_\lambda}}$ .

(2) If  $(X, \mathcal{C})$  is an *L*-convex space, then  $\mathcal{I}_{\preceq_{x_\lambda}^{\mathcal{C}}} = \mathcal{R}_{x_\lambda}^{\mathcal{C}}$  and  $\preceq_{x_\lambda}^{\mathcal{C}} = \preceq_{\mathcal{R}_{x_\lambda}^{\mathcal{C}}}$ .

(3) If  $\{\mathcal{I}_i\}_{i \in I} \subseteq \mathfrak{J}(L^X)$  is nonempty then  $\bigcap_{i \in I} \mathcal{I}_i \in \mathfrak{J}(L^X)$ .

*Proof.* (1) For any  $A \in L^X$ , we have

$$A \in \mathcal{I}_{x_\lambda} \iff A \in \psi_{x_\lambda}(L^X) \iff A \preceq_{x_\lambda} A \iff A \in \mathcal{I}_{\preceq_{x_\lambda}}.$$

Thus  $\mathcal{I}_{x_\lambda} = \mathcal{I}_{\preceq_{x_\lambda}}$ . Also, for all  $A, B \in L^X$ , we have

$$A \preceq_{x_\lambda} B \iff A \in \psi_{x_\lambda}(L^X) \text{ and } A \leq B \iff A \preceq_{\mathcal{I}_{x_\lambda}} A \leq B \iff A \preceq_{\mathcal{I}_{x_\lambda}} B.$$

Therefore  $\preceq_{x_\lambda} = \preceq_{\mathcal{I}_{x_\lambda}}$ .

(2) Let  $A, B \in L^X$ . Since  $A \preceq A$  iff  $A \in \mathcal{C}_{\preceq}$ , we have

$$A \in \mathcal{I}_{\preceq_{x_\lambda}^{\mathcal{C}}} \iff A \preceq A \in \psi_{x_\lambda}(L^X) \iff A \in \mathcal{C}_{\preceq} \cap \psi_{x_\lambda}(L^X) \iff A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}}.$$

Thus  $\mathcal{I}_{\ll_{x_\lambda}} = \mathcal{R}_{x_\lambda}^{\mathcal{C}}$ . Also, since  $A \in \mathcal{C}$  iff  $A \preceq_{\mathcal{C}} A$ , we have

$$\begin{aligned} A \ll_{x_\lambda}^{\mathcal{C}} B &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \preceq_{\mathcal{C}} D \\ &\iff \exists D \in \mathcal{C} \cap \psi_{x_\lambda}(L^X) \text{ s.t. } A \leq D \wedge B \leq D \\ &\iff A \in \mathcal{R}_{x_\lambda}^{\mathcal{C}} \text{ and } A \leq B \\ &\iff A \ll_{x_\lambda}^{\mathcal{C}} B. \end{aligned}$$

Therefore  $\ll_{x_\lambda}^{\mathcal{C}} = \ll_{x_\lambda}^{\mathcal{C}}$ .

(3) Its proof is direct.  $\square$

**Proposition 4.6.** *Let  $f : X \rightarrow Y$  be a mapping,  $\mathcal{I}_X \in \mathfrak{J}(L^X)$  and  $\ll_X \in \mathfrak{R}(L^X)$ . Then*

- (1)  $\mathcal{I}_{f(X)} = \{G \in L^Y : f_L^+(G) \in \mathcal{I}_X\} \in \mathfrak{R}(L^Y)$  satisfying  $(\ll_{\mathcal{I}_X})_{f(X)} = \ll_{\mathcal{I}_{f(X)}}$ ;
- (2)  $(\mathcal{I}_{\ll_X})_{f(X)} = \mathcal{I}_{\ll_{f(X)}}$ .

*Proof.* (1) It is easy to check that  $\mathcal{I}_{f(X)}$  is an  $L$ -convex ideal on  $L^Y$ . For  $A, B \in L^Y$ ,

$$\begin{aligned} A(\ll_{\mathcal{I}_X})_{f(X)} B &\iff A \leq B \text{ and } f_L^+(A) \ll_{\mathcal{I}_X} f_L^+(B) \\ &\iff A \leq B \text{ and } f_L^+(A) \in \mathcal{I}_X \\ &\iff A \leq B \text{ and } A \in \mathcal{I}_{f(X)} \\ &\iff A \ll_{\mathcal{I}_{f(X)}} A \leq B \\ &\iff A \ll_{\mathcal{I}_{f(X)}} B. \end{aligned}$$

So  $(\ll_{\mathcal{I}_X})_{f(X)} = \ll_{\mathcal{I}_{f(X)}}$ .

(2) For all  $A \in L^Y$ ,

$$A \in (\mathcal{I}_{\ll_X})_{f(X)} \iff f_L^+(A) \in \mathcal{I}_{\ll_X} \iff A \ll_{f(X)} A \iff A \in \mathcal{I}_{\ll_{f(X)}}.$$

Therefore  $(\mathcal{I}_{\ll_X})_{f(X)} = \mathcal{I}_{\ll_{f(X)}}$ .  $\square$

By Proposition 4.4, there is a one-to-one correspondence between  $L$ -convex ideals and  $L$ -directed relations. We next introduce  $L$ -ideal-convergence structure and  $L$ -convex ideal convergence structure and discuss their relationships with  $L$ -directed relations and  $L$ -convex convergence relations.

**Definition 4.7.** *Let  $\mathfrak{J}(L^X)$  be the set of all  $L$ -convex ideals on  $L^X$ . A mapping  $\lim : \mathfrak{J}(L^X) \rightarrow 2^{\beta^*(L^X)}$  is called an  $L$ -ideal-convergence and the pair  $(X, \lim)$  is called an  $L$ -ideal-convergence space if  $\lim$  satisfies*

- (LIC1)  $x_\lambda \in \lim(\mathcal{I}_{x_\lambda})$ ;
- (LIC2)  $\forall \mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{J}(L^X)$ ,  $\mathcal{I}_1 \subseteq \mathcal{I}_2$  implies  $\lim(\mathcal{I}_1) \subseteq \lim(\mathcal{I}_2)$ ;
- (LIC3)  $\mathcal{I}_{x_\lambda}^{\lim} = \bigcap_{\eta \in \beta_\lambda^*(L)} \mathcal{I}_{x_\eta}^{\lim}$ , where  $\mathcal{I}_{x_\lambda}^{\lim} = \bigcap_{x_\lambda \in \lim(\mathcal{I})} \mathcal{I}$ .

**Definition 4.8.** *Let  $(X, \lim_X)$  and  $(Y, \lim_Y)$  be  $L$ -ideal-convergence spaces. A mapping  $f : X \rightarrow Y$  is called an  $L$ -ideal-convergence preserving mapping, if  $x_\lambda \in \lim_X(\mathcal{I})$  implies  $f_L^+(x_\lambda) \in \lim_Y(\mathcal{I}_{f(X)})$  for all  $x_\lambda \in \beta^*(L^X)$  and  $\mathcal{I} \in \mathfrak{J}(L^X)$ .*

The category consisting of  $L$ -ideal-convergence spaces as objects and  $L$ -ideal-convergence preserving mappings as morphisms is denoted by  $L$ -ICS.

**Example 4.9.** *Let  $X = \{x\}$  and  $L = \{\perp, a, b, c, d, \top\}$ , where  $c = a \wedge b$ ,  $d = a \vee b$  and  $a, b$  are incomparable. Then  $\beta^*(L^X) = J(L^X) = \{x_a, x_b, x_c\}$  and  $\mathfrak{J}(L^X) = \{\mathcal{I}_w : w \in L \setminus \{\top\}\} \cup \{\mathcal{I}_{a,b}\}$ , where  $\mathcal{I}_{a,b} = \mathcal{I}_a \cup \mathcal{I}_b$  and  $\mathcal{I}_w = \{x_v \in L^X : v \leq w\}$  for any  $w \in L \setminus \{\top\}$ . In addition,  $\mathcal{I}_\perp = \mathcal{I}_{x_c}$ ,  $\mathcal{I}_a = \mathcal{I}_{x_b}$  and  $\mathcal{I}_b = \mathcal{I}_{x_a}$ .*

(1) Let  $\lim : \mathfrak{J}(L^X) \rightarrow 2^{\beta^*(L^X)}$  be defined by  $\lim(\mathcal{I}_\perp) = \lim(\mathcal{I}_c) = \{x_c\}$ ,  $\lim(\mathcal{I}_a) = \{x_c, x_b\}$ ,  $\lim(\mathcal{I}_b) = \{x_c, x_a\}$  and  $\lim(\mathcal{I}_{a,b}) = \lim(\mathcal{I}_d) = \beta^*(L^X)$ . It is easy to check that  $\lim$  is an  $L$ -ideal-convergence structure.

(2) Let  $\lim_1 : \mathfrak{J}(L^X) \rightarrow 2^{\beta^*(L^X)}$  be defined by  $\lim_1(\mathcal{I}_\perp) = \lim_1(\mathcal{I}_c) = \{x_c\}$  and  $\lim_1(\mathcal{I}_a) = \lim_1(\mathcal{I}_b) = \lim_1(\mathcal{I}_{a,b}) = \lim_1(\mathcal{I}_d) = \beta^*(L^X)$ . Then  $\lim_1$  is an  $L$ -ideal-convergence structure.

**Lemma 4.10.** *Let  $(X, \ll)$  be an  $L$ -convergence relation space. Define  $\lim_{\ll} : \mathfrak{J}(L^X) \rightarrow 2^{\beta^*(L^X)}$  by*

$$\forall \mathcal{I} \in \mathfrak{J}(L^X), \quad \lim_{\ll}(\mathcal{I}) = \{x_\lambda \in \beta^*(L^X) : x_\lambda \ll \ll_{\mathcal{I}}\}.$$

Then  $\mathcal{I}_{x_\lambda}^{\lim_{\ll}} = \{A \in L^X : A \ll_{x_\lambda}^{\ll} A\}$  for any  $x_\lambda \in \beta^*(L^X)$ .

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $A \in L^X$ . Proposition 4.4 yields that

$$\begin{aligned}
 A \in \mathcal{I}_{x_\lambda}^{\lim\ll} &\iff \forall \mathcal{I} \in \mathfrak{J}(L^X), x_\lambda \in \lim_{\ll}(\mathcal{I}) \text{ implies } A \in \mathcal{I} \\
 &\iff \forall \mathcal{I} \in \mathfrak{J}(L^X), x_\lambda \ll\ll_{\mathcal{I}} \text{ implies } A \ll_{\mathcal{I}} A \\
 &\iff \forall \ll \in \mathfrak{R}(L^X), x_\lambda \ll\ll_{\ll} \text{ implies } A \in \mathcal{I}_{\ll} \\
 &\iff \forall \ll \in \mathfrak{R}(L^X), x_\lambda \ll\ll \text{ implies } A \ll A \\
 &\iff \forall \ll \in \mathfrak{R}_{x_\lambda}^{\ll}(L^X), A \ll A \\
 &\iff A \ll_{x_\lambda}^{\ll} A.
 \end{aligned}$$

Thus  $\mathcal{I}_{x_\lambda}^{\lim\ll} = \{A \in L^X : A \ll_{x_\lambda}^{\ll} A\}$ . □

**Proposition 4.11.** *If  $(X, \ll)$  is an L-convergence relation space, then  $\lim_{\ll}$  is an L-ideal-convergence structure on  $L^X$ .*

*Proof.* (LIC1) Let  $x_\lambda \in \beta^*(L^X)$ . We have  $x_\lambda \ll\ll_{x_\lambda} = \ll_{\mathcal{I}_{x_\lambda}}$  by (LCR1) and Lemma 4.5(1). Thus  $x_\lambda \in \lim_{\ll}(\mathcal{I}_{x_\lambda})$ .

(LIC2) If  $\mathcal{I}_1 \subseteq \mathcal{I}_2$  and  $x_\lambda \in \lim_{\ll}(\mathcal{I}_1)$  then  $x_\lambda \ll\ll_{\mathcal{I}_1} \leq \ll_{\mathcal{I}_2}$ . So  $x_\lambda \ll\ll_{\mathcal{I}_2}$  by (LCR2). This implies that  $x_\lambda \in \lim_{\ll}(\mathcal{I}_2)$ . Thus  $\lim_{\ll}(\mathcal{I}_1) \subseteq \lim_{\ll}(\mathcal{I}_2)$ .

(LIC3) Let  $x_\lambda \in \beta^*(L^X)$  and  $A \in L^X$ . It follows from Lemma 4.10 and (LCR3) that

$$\begin{aligned}
 A \in \mathcal{I}_{x_\lambda}^{\lim\ll} &\iff A \ll_{x_\lambda}^{\ll} A \\
 &\iff \forall \eta \in \beta_\lambda^*(L), A \ll_{x_\eta}^{\ll} F \\
 &\iff \forall \eta \in \beta_\lambda^*(L), A \in \mathcal{I}_{x_\eta}^{\lim\ll} \\
 &\iff A \in \bigcap_{\eta \in \beta_\lambda^*(L)} \mathcal{I}_{x_\eta}^{\lim\ll}.
 \end{aligned}$$

Therefore  $\mathcal{I}_{x_\lambda}^{\lim\ll} = \bigcap_{\mu \in \beta^*(\lambda)} \mathcal{I}_{x_\mu}^{\lim\ll}$ .

Therefore  $\lim_{\ll}$  is an L-ideal-convergence structure. □

**Proposition 4.12.** *Let  $(X, \ll_X)$  and  $(Y, \ll_Y)$  be L-convergence relation spaces. If  $f : X \rightarrow Y$  is an L-directed convergence relation preserving mapping from  $(X, \ll_X)$  to  $(Y, \ll_Y)$ , then  $f : (X, \lim_{\ll_X}) \rightarrow (Y, \lim_{\ll_Y})$  is an L-ideal-convergence preserving mapping.*

*Proof.* Let  $\mathcal{I}_X \in \mathfrak{J}(L^X)$  and let  $x_\lambda \in \beta^*(L^X)$ . Since  $f$  is an L-convergence relation preserving mapping, it follows from Proposition 4.6 that

$$\begin{aligned}
 x_\lambda \in \lim_{\ll_X}(\mathcal{I}_X) &\iff x_\lambda \ll_X \ll_{\mathcal{I}_X} \\
 &\implies f_L^\rightarrow(x_\lambda) \ll_Y \ll_{\mathcal{I}_f(X)} \\
 &\iff f_L^\rightarrow(x_\lambda) \in \lim_{\ll_Y}(\mathcal{I}_f(X)).
 \end{aligned}$$

Therefore  $f$  is an L-ideal-convergence relation preserving mapping. □

**Proposition 4.13.** *Let  $(X, \lim)$  be an L-ideal-convergence space. Define a relation  $\ll_{\lim}$  on  $\beta^*(L^X) \times \mathfrak{R}(L^X)$  by*

$$\forall x_\lambda \in \beta^*(L^X), \forall \ll \in \mathfrak{R}(L^X), x_\lambda \ll_{\lim} \ll \iff x_\lambda \in \lim(\mathcal{I}_{\ll}).$$

*Then  $\ll_{\lim}$  is an L-convergence relation satisfying  $\ll_{x_\lambda}^{\ll_{\lim}} = \ll_{\mathcal{I}_{x_\lambda}^{\lim}}$ .*

*Proof.* (LCR1) Since  $x_\lambda \in \lim(\mathcal{I}_{x_\lambda}) = \lim(\mathcal{I}_{\ll_{x_\lambda}})$  by (LIC1) and Lemma 4.5(1), it follows that  $x_\lambda \ll_{\lim} \ll_{x_\lambda}$ .

(LCR2) Let  $\ll_1, \ll_2 \in \mathfrak{R}(L^X)$  with  $\ll_1 \leq \ll_2$  and  $x_\lambda \ll_{\lim} \ll_1$ . Then  $\mathcal{I}_{\ll_1} \subseteq \mathcal{I}_{\ll_2}$  and  $x_\lambda \in \lim(\mathcal{I}_{\ll_1}) \subseteq \lim(\mathcal{I}_{\ll_2})$ . Thus  $x_\lambda \ll_{\lim} \ll_2$ .

(LCR3) Let  $x_\lambda \in \beta^*(L^X)$ . We check that  $\ll_{x_\lambda}^{\ll_{\lim}} = \ll_{\mathcal{I}_{x_\lambda}^{\lim}}$ . For all  $A, B \in L^X$ , Proposition 4.4 yields that

$$\begin{aligned}
 A \ll_{x_\lambda}^{\ll_{\lim}} B &\iff \forall \ll \in \mathfrak{R}_{x_\lambda}^{\ll_{\lim}}(L^X), A \ll B \\
 &\iff \forall \ll \in \mathfrak{R}(L^X), x_\lambda \in \lim(\mathcal{I}_{\ll}) \text{ implies } A \ll B \\
 &\iff \forall \ll \in \mathfrak{R}(L^X), x_\lambda \in \lim(\mathcal{I}_{\ll}) \text{ implies } A \in \mathcal{I}_{\ll} \text{ and } A \leq B \\
 &\iff \forall \mathcal{I} \in \mathfrak{J}(L^X), x_\lambda \in \lim(\mathcal{I}_{\ll_{\mathcal{I}}}) \text{ implies } A \in \mathcal{I}_{\ll_{\mathcal{I}}} \text{ and } A \leq B \\
 &\iff \forall \mathcal{I} \in \mathfrak{J}(L^X), x_\lambda \in \lim(\mathcal{I}) \text{ implies } A \in \mathcal{I} \text{ and } A \leq B \\
 &\iff A \in \mathcal{I}_{x_\lambda}^{\lim} \text{ and } A \leq B \\
 &\iff A \ll_{\mathcal{I}_{x_\lambda}^{\lim}} B.
 \end{aligned}$$

Thus  $\ll_{x_\lambda}^{\lim} = \ll_{\mathcal{I}_{x_\lambda}^{\lim}}$ . Hence (LIC3) implies that

$$\begin{aligned} A \ll_{x_\lambda}^{\lim} B &\iff A \ll_{\mathcal{I}_{x_\lambda}^{\lim}} B \\ &\iff A \ll_{\bigcap_{\mu \in \beta_\lambda^*(L)} \mathcal{I}_{x_\mu}^{\lim}} B \\ &\iff \forall \mu \in \beta_\lambda^*(L), A \ll_{\mathcal{I}_{x_\mu}^{\lim}} B \\ &\iff \forall \mu \in \beta_\lambda^*(L), A \ll_{x_\mu}^{\lim} B. \end{aligned}$$

Hence (LCR3) holds for  $\ll_{\lim}$ .

Therefore  $\ll_{\lim}$  is an  $L$ -convergence relation satisfying  $\ll_{x_\lambda}^{\lim} = \ll_{\mathcal{I}_{x_\lambda}^{\lim}}$ .  $\square$

**Proposition 4.14.** *Let  $(X, \lim_X)$  and  $(Y, \lim_Y)$  be  $L$ -ideal-convergence spaces. If  $f : X \rightarrow Y$  is an  $L$ -ideal-convergence preserving mapping from  $(X, \lim_X)$  to  $(Y, \lim_Y)$  then  $f : (X, \ll_{\lim_X}) \rightarrow (Y, \ll_{\lim_Y})$  is an  $L$ -convergence relation preserving mapping.*

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$ . If  $\ll_{x_\lambda} \in \mathfrak{R}_{x_\lambda}^{\ll_{\lim_X}}(L^X)$  then  $x_\lambda \in \lim_X(\mathcal{I}_{\ll_{x_\lambda}})$  and  $f_L^\rightarrow(x_\lambda) \in \lim_Y((\mathcal{I}_{\ll_{x_\lambda}})_{f(X)}) = \lim_Y(\mathcal{I}_{\ll_{f(x_\lambda)}})$ . Thus  $f_L^\rightarrow(x_\lambda) \ll_{\lim_Y} \ll_{f(X)}$ . So  $f$  is an  $L$ -convergence relation preserving mapping.  $\square$

**Proposition 4.15.** *If  $(X, \ll)$  is an  $L$ -convergence relation space, then  $\ll_{\lim_{\ll}} = \ll$ .*

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$ . It follows from Proposition 4.4 that

$$x_\lambda \ll_{\lim_{\ll}} \ll \iff x_\lambda \in \lim_{\ll}(\mathcal{I}_{\ll}) \iff x_\lambda \ll_{\ll_{\ll}} \iff x_\lambda \ll \ll.$$

This shows that  $\ll_{\lim_{\ll}} = \ll$ .  $\square$

**Proposition 4.16.** *Let  $(X, \lim)$  be an  $L$ -ideal-convergence space. Then  $\lim_{\ll_{\lim}} = \lim$ .*

*Proof.* Let  $x_\lambda \in \beta^*(L^X)$  and  $\mathcal{I} \in \mathfrak{J}(L^X)$ . It follows from Proposition 4.4 that

$$x_\lambda \in \lim_{\ll_{\lim}}(\mathcal{I}) \iff x_\lambda \ll_{\lim} \ll_{\mathcal{I}} \iff x_\lambda \in \lim(\mathcal{I}_{\ll_{\mathcal{I}}}) \iff x_\lambda \in \lim(\mathcal{I}).$$

Thus  $\lim_{\ll_{\lim}} = \lim$ .  $\square$

Based on Propositions 4.11 and 4.12, we obtain a functor  $\mathbb{T} : L\text{-CRS} \rightarrow L\text{-FCS}$  by

$$\mathbb{T} : \begin{cases} L\text{-CRS} & \rightarrow & L\text{-FCS}; \\ (X, \ll) & \mapsto & (X, \lim_{\ll}); \\ f & \mapsto & f. \end{cases}$$

Based on Propositions 4.11–4.16,  $\mathbb{T}$  is isomorphic. Thus we have the following result.

**Theorem 4.17.**  *$L\text{-CRS}$  is isomorphic to  $L\text{-ICS}$ .*

**Definition 4.18.** *An  $L$ -ideal-convergence  $\lim : \mathfrak{J}(L^X) \rightarrow 2^{\beta^*(L^X)}$  is called an  $L$ -convex ideal-convergence structure and the pair  $(X, \lim)$  is called an  $L$ -convex ideal-convergence space if  $\lim$  further satisfies*

(LCIC1)  $x_\lambda \in \lim(\mathcal{I}_{x_\lambda}^{\lim})$ ;

(LCIC2)  $A \in \mathcal{I}_{x_\lambda}^{\lim}$  iff there is  $B \in \psi_{x_\lambda}(L^X)$  such that  $A \leq B \in \mathcal{I}_{y_\mu}^{\lim}$  for all  $y_\mu \not\leq B$ .

The category of all  $L$ -convex ideal-convergence spaces and  $L$ -ideal-convergence preserving mappings is denoted by  $L\text{-CICS}$ . Clearly, it is a bireflective subcategory of  $L\text{-ICS}$ .

**Remark 4.19.** *The  $L$ -ideal-convergence structure  $\lim$  defined in Example 4.9(1) is an  $L$ -convex ideal-convergence structure on  $L^X$ , while the  $L$ -ideal-convergence structure  $\lim_1$  defined in Example 4.9(2) is not an  $L$ -convex ideal-convergence structure. In fact,  $\mathcal{I}_{x_a}^{\lim_1} = \mathcal{I}_{x_b}^{\lim_1} = \mathcal{I}_c$  and  $x_a \notin \lim_1(\mathcal{I}_{x_a}^{\lim_1}) = \lim_1(\mathcal{I}_c) = \{x_c\}$ . So  $x_a \notin \lim_1(\mathcal{I}_{x_a}^{\lim_1})$  which shows that (LCIC1) fails for  $\lim_1$ . So  $\lim_1$  is not an  $L$ -convex ideal-convergence convergence structure.*

**Proposition 4.20.** *If  $(X, \ll)$  is an  $L$ -convex ideal-convergence relation space, then  $\lim_{\ll}$  is an  $L$ -convex ideal-convergence structure.*

*Proof.* It follows from Proposition 4.11 that  $\lim_{\ll}$  is an  $L$ -ideal-convergence structure. It is sufficient to verify that (LCIC1) and (LCIC2) hold for  $\lim_{\ll}$ .

(LCIC1) Let  $x_\lambda \in \beta^*(L^X)$ . It follows from Propositions 4.13 and 4.15 that  $\ll_{x_\lambda}^{\ll} = \ll_{x_\lambda}^{\ll \lim_{\ll}} = \ll_{\mathcal{T}_{x_\lambda}^{\lim_{\ll}}}$ .

Since  $x_\lambda \ll \ll_{x_\lambda}^{\ll}$  by (LCCR1), it is clear that  $x_\lambda \ll \ll_{\mathcal{T}_{x_\lambda}^{\lim_{\ll}}}$ . That is,  $x_\lambda \in \lim_{\ll}(\mathcal{T}_{x_\lambda}^{\lim_{\ll}})$ .

(LCIC2) Let  $A \in L^X$ . It follows from Lemma 4.10 and (LCCR2) of  $\ll$  that

$$\begin{aligned} A \in \mathcal{T}_{x_\lambda}^{\lim_{\ll}} &\iff A \ll_{x_\lambda}^{\ll} A \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq D \ll_{y_\mu}^{\ll} D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq D \in \mathcal{T}_{y_\mu}^{\lim_{\ll}}. \end{aligned}$$

So (LCIC2) holds for  $\lim_{\ll}$ . □

**Proposition 4.21.** *If  $(X, \lim)$  is an  $L$ -convex ideal-convergence space, then  $\ll_{\lim}$  is an  $L$ -convex convergence relation on  $L^X$ .*

*Proof.* It follows from Proposition 4.13 that  $\ll_{\lim}$  is an  $L$ -convergence relation. It is sufficient to verify that (LCCR1) and (LCCR2) hold for  $\ll_{\lim}$ .

(LCCR1) Let  $x_\lambda \in \beta^*(L^X)$ . Then  $\ll_{x_\lambda}^{\ll \lim} = \ll_{\mathcal{T}_{x_\lambda}^{\lim}}$  by Proposition 4.13. Since  $x_\lambda \in \lim(\mathcal{T}_{x_\lambda}^{\lim}) = \lim_{\ll_{\lim}}(\mathcal{T}_{x_\lambda}^{\lim})$  by (LCIC1) and Proposition 4.16, it follows that  $x_\lambda \ll_{\lim} \ll_{\mathcal{T}_{x_\lambda}^{\lim}}$ . Therefore  $x_\lambda \ll_{\lim} \ll_{x_\lambda}^{\ll \lim}$ .

(LCCR2) Let  $A, B \in L^X$ . It follows from Proposition 4.13 and (LDR2) that

$$A \ll_{x_\lambda}^{\ll \lim} B \iff A \ll_{\mathcal{T}_{x_\lambda}^{\lim}} B \iff A \ll_{\mathcal{T}_{x_\lambda}^{\lim}} A \leq B \iff A \in \mathcal{T}_{x_\lambda}^{\lim} \text{ and } A \leq B.$$

Hence, (LCIC2) of  $\lim$  and Theorem 4.13 yield that

$$\begin{aligned} A \ll_{x_\lambda}^{\ll \lim} B &\iff A \leq B \text{ and } A \in \mathcal{T}_{x_\lambda}^{\lim} \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq B \wedge D \leq D \in \mathcal{T}_{y_\mu}^{\lim} \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq B \wedge D \leq D \ll_{y_\mu}^{\lim} D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq B \wedge D \leq D \ll_{y_\mu}^{\ll \lim} D \\ &\iff \exists D \in \psi_{x_\lambda}(L^X) \text{ s.t. } \forall y_\mu \not\leq D, A \leq B \wedge D \leq D \ll_{y_\mu}^{\ll \lim} D. \end{aligned}$$

Thus (LCCR2) holds for  $\ll_{\lim}$ . □

Based on Propositions 4.20 and 4.21 and Theorem 4.17, we have the following result.

**Theorem 4.22.**  *$L\text{-CCRS}$  is isomorphic to  $L\text{-CICS}$  which is a bireflective subcategory of  $L\text{-ICS}$ .*

Also, relationships among  $L$ -convex ideal-convergence space,  $L$ -convex enclosed relation space and  $L$ -convex space are presented as follows.

**Remark 4.23.** *Relations between  $L\text{-CICS}$  and  $L\text{-CERS}$  are presented as follows.*

(1) Let  $(X, \lim)$  be an  $L$ -convex ideal-convergence space. Define a binary relation  $\preceq_{\lim}$  on  $L^X$  by

$$\forall A, B \in L^X, \quad A \preceq_{\lim} B \iff \exists A \leq D \leq B \text{ s.t. } \forall x_\lambda \not\leq D, D \in \mathcal{T}_{x_\lambda}^{\lim}.$$

Then  $\preceq_{\lim}$  is an  $L$ -convex enclosed relation on  $L^X$ .

(2) Let  $(X, \preceq)$  be an  $L$ -convex enclosed relation space. Define a mapping  $\lim_{\preceq} : \mathfrak{J}(L^X) \longrightarrow 2^{\beta^*(L^X)}$  by

$$\forall \mathcal{I} \in \mathfrak{J}(L^X), \quad \lim_{\preceq}(\mathcal{I}) = \{x_\lambda \in \beta^*(L^X) : \ll_{x_\lambda}^{\preceq} \leq \mathcal{I}\}.$$

Then  $\lim_{\preceq}$  is an  $L$ -convex ideal-convergence structure on  $L^X$ .

(3)  $\preceq_{\lim_{\preceq}} = \preceq$  and  $\lim_{\preceq_{\lim}} = \lim$ .

(4)  $L\text{-CICS}$  is isomorphic to  $L\text{-CERS}$ .

**Remark 4.24.** Relationships between *L-CICS* and *L-CS* are presented as follows.

(1) Let  $(X, \lim)$  be an *L-convex ideal-convergence space*. Then the set  $\mathcal{C}_{\lim} = \{A \in L^X : \forall x_\lambda \preceq A, A \in \mathcal{I}_{x_\lambda}^{\lim}\}$  is an *L-convex structure on  $L^X$* .

(2) Let  $(X, \mathcal{C})$  be an *L-convex space*. Define a mapping  $\lim_{\mathcal{C}} : \mathfrak{I}(L^X) \rightarrow 2^{\beta^*(L^X)}$  by  $\lim_{\mathcal{C}}(\mathcal{I}) = \{x_\lambda \in \beta^*(L^X) : \mathcal{I}_{x_\lambda}^{\mathcal{C}} \subseteq \mathcal{I}\}$  for any  $\mathcal{I} \in \mathfrak{I}(L^X)$ . Then  $\lim_{\mathcal{C}}$  is an *L-convex ideal-convergence structure on  $L^X$* .

(3)  $\mathcal{C}_{\lim_{\mathcal{C}}} = \mathcal{C}$  and  $\lim_{\mathcal{C}_{\lim}} = \lim$ .

(4) *L-CICS* is isomorphic to *L-CS*.

We present the following diagram to show relationships among the categories mentioned in this paper.

$$\begin{array}{ccccc}
 L\text{-CERS} & \xleftrightarrow{\text{iso}} & L\text{-CCRS} & \xrightarrow{\text{biref}} & L\text{-CRS} \\
 \updownarrow \text{iso} & & \updownarrow \text{iso} & & \updownarrow \text{iso} \\
 L\text{-CS} & \xleftrightarrow{\text{iso}} & L\text{-CICS} & \xrightarrow{\text{biref}} & L\text{-ICS}
 \end{array}$$

## 5 Conclusions

In [40], Xiu et al. studied *L-convex ideal* and *L-convergence structure* in the framework of *L-setting*, where *L* is a completely distributive lattice with an inverse involution. Its main notions and results depended inevitably on the inverse involution. Being inspired by this limitation, we adopt *L-convex  $\beta^*$ -remoteness* to construct *L-convergence* (resp. *L-convex convergence*) relation and *L-ideal-convergence* (resp. *L-convex ideal-convergence*) structure, and establish their relationships with *L-convex enclosed relation* and *L-convex structure*, where *L* is not required to be equipped with an inverse involution.

Indeed, if *L* is a complete lattice with an inverse involution, then *L-convex filter* is a dual concept of *L-convex ideal*. Similarly, *L-convex ideal-convergence structure* is a dual concept of *L-convergence structure*. However, *L-convex ideal-convergence structure* can adapt to a more general environment where the complete lattice *L* has no inverse involution.

(1) Even if the lattice *L* has an inverse involution  $'$ , *L-convex ideal-convergence structure* and *L-convergence structure* in the sense of Xiu et al. [40] (alternatively, Definition 2.10) are independent concepts. We have the following examples.

We define an inverse involution  $'$  on *L* in Example 4.9 by  $\perp' = \top$ ,  $a' = b$  and  $c' = d$ .

(i) The *L-convex ideal-convergence structure*  $\lim : \mathfrak{I}(L^X) \rightarrow 2^{\beta^*(L^X)}$  defined in Example 4.9 is not an *L-convergence structure* in the sense of Xiu. In fact,  $\mathcal{I}_{x_a}^{\lim} = \mathcal{I}_b$  and  $\bigcap_{w \prec a} \mathcal{I}_{x_w}^{\lim} = \mathcal{I}_{x_c}^{\lim} \cap \mathcal{I}_{x_a}^{\lim} = \mathcal{I}_\perp \cap \mathcal{I}_b = \mathcal{I}_\perp$ . So  $\mathcal{I}_{x_a}^{\lim} \neq \bigcap_{\mu \prec a} \mathcal{I}_{x_\mu}^{\lim}$  (i.e., (LCV3) in Definition 2.10 fails). Thus  $\lim$  is not an *L-convergence structure*.

(ii) Define a mapping  $\lim_1 : \mathfrak{I}(L^X) \rightarrow 2^{J(L^X)}$  by  $\lim_1(\mathcal{I}_\perp) = \lim_1(\mathcal{I}_c) = \emptyset$ ,  $\lim_1(\mathcal{I}_a) = \{x_b\}$ ,  $\lim_1(\mathcal{I}_b) = \{x_a\}$  and  $\lim_1(\mathcal{I}_d) = \lim_1(\mathcal{I}_{a,b}) = J(L^X)$ . It is easy to check that  $\lim_1$  is an *L-convergence structure* in the sense of Xiu et al. [40]. However, we have  $\mathcal{I}_{x_c}^{\lim_1} = \mathcal{I}_{a,b}$  and  $\bigcap_{\eta \in \beta_c^*(L)} \mathcal{I}_{x_\eta}^{\lim_1}$ . So  $\mathcal{I}_{x_c}^{\lim_1} \neq \bigcap_{\eta \in \beta_c^*(L)} \mathcal{I}_{x_\eta}^{\lim_1}$ . That is, (LIC3) fails for  $\lim_1$ . Therefore  $\lim_1$  is not an *L-ideal-convergence structure*.

In recent years, fuzzy relations and fuzzy convergence structures have been applied to many fuzzy mathematical structures such as *L-topological spaces*,  $(L, M)$ -fuzzy topological spaces, *M-fuzzifying convex spaces* and  $(L, M)$ -fuzzy convex spaces. Fuzzy relations and fuzzy ideal-convergence structures discussed in this paper may provide an alternative way to study related fuzzy mathematical structures in the future.

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