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REPRESENTATION THEOREMS OF $L-$SUBSETS AND $L-$FAMILIES ON COMPLETE RESIDUATED LATTICE

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Abstract. In this paper, our purpose is twofold. Firstly, the tensor and residuum operations on $L-$nested systems are introduced under the condition of complete residuated lattice. Then we show that $L-$nested systems form a complete residuated lattice, which is precisely the classical isomorphic object of complete residuated power set lattice. Thus the new representation theorem of $L-$subsets on complete residuated lattice is obtained. Secondly, we introduce the concepts of $L-$family and the system of $L-$subsets, then with the tool of the system of $L-$subsets, we obtain the representation theorem of intersection-preserving $L-$families on complete residuated lattice.

1. Introduction

Since Zadeh proposed $L-$subset theory in 1965, many scholars worked on the connection between $L-$subsets and classical sets. Representation theorem is a main form to establish this connection, whose essence is to search the classical isomorphic object of $L-$power set lattice. Luo [5] first proposed the concept of nested systems, and established a representation theorem of $L-$subsets with them (in his case, $L = [0, 1]$). Then Zhang [13], Shi [7], R. Bělohlávek [1] further studied representation theorems of $L-$subsets based on different forms of nested systems. It is of value to note Xiong [12], Fang and Han [2] studied the representation theorems of $L-$subsets with different tools on the condition that L is only a complete lattice. Recently, a number of related work are constantly in progress, among which literatures [3, 8-10] are newer results.

Recently, scholars usually use complete residuated lattice as the membership degree value lattice of $L-$subsets, e.g. R. Bělohlávek [1]. As a matter of fact, we can prove that L^X is indeed a complete residuated lattice w.r.t the tensor and residuum operations induced from L, hence we call it complete residuated power set lattice. One aim of this paper is to give the classical isomorphic object of complete residuated power set lattice, thus establishing a new representation theorem of $L-$subsets on complete residuated lattice.

In scholars' investigations, there are some special $L-$subsets which are maps from L^X to L, such as many-valued filters, lattice-valued convergence structures and so on. In this paper, we call these special $L-$subsets $L-$families. It should be pointed out that $L-$families also have some kind of "levels", and they can

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be described with certain "level structures" as well, e.g. G.Jäger [4] discussed the case of lattice-valued uniform convergence spaces and lattice-valued uniform spaces. Hence, it makes sense to find the relations between $L-$family and its "levels". For this purpose, we introduce the concept of the system of $L-$subsets. Moreover, we prove that there is a one-to-one correspondence between intersection-preserving $L-$families and the systems of $L-$subsets. That is the representation theorem of intersection-preserving $L-$families on complete residuated lattice.

2. Preliminaries

In this paper, we consider L a complete lattice, 0 and 1 the smallest and the greatest elements of L respectively, and X a nonempty set. An $L-$subset in X is a map $A : X \to L$. The set of all $L-$subsets in X will be denoted by L^X. Let 0_X and 1_X denote the smallest and the greatest elements of L^X. Denote the set of all subsets of X by $\mathcal{P}(X)$. In this paper, we do not distinguish between subsets of X and their characteristic functions. For each $a \in L$ and $A \in L^X$, we denote the cut set of A by $A_a = \{x \in X \mid A(x) \geq a\}$. $L-$subsets $(a \wedge A) : X \to L$ and $(a \vee A) : X \to L$ mean $(a \wedge A)(x) = a \wedge A(x)$ and $(a \vee A)(x) = a \vee A(x)$ for each $x \in X$.

Generally, when L is a complete lattice, L^X is also a complete lattice. In the following, some basic facts needed in the sequel are presented.

Theorem 2.1. [13, 7] *Let X be a nonempty set and L be a complete lattice. Then for each $A \in L^X$, we have $A = \bigvee_{a \in L}(a \wedge A_a)$.*

Definition 2.2. [1] A map $H : L \to \mathcal{P}(X)$ subjects to the conditions

(LH1) For $a, b \in L$, $a \leq b$ implies $H(b) \subseteq H(a)$,

(LH2) For each $x \in X$, the subset $\{ a \mid x \in H(a) \}$ of L is nonempty and has a greatest element,

is called an $L-$nested system. The family of all $L-$nested systems on X will be denoted by $H_L(X)$.

Let $H, G \in H_L(X)$. We define a partial order " $\leq$ " on $H_L(X)$ as follows:

$$H \leq G \Leftrightarrow \forall a \in L,\ H(a) \subseteq G(a).$$

Then $H_L(X)$ has a smallest element $H^0 : L \to \mathcal{P}(X)$ defined as follows:

$$H^0(a) = \begin{cases} X, & a = 0, \\ \emptyset, & a \neq 0. \end{cases}$$

By the above definition, we can prove that the partially ordered set $(H_L(X)\,, \leq)$ is a complete lattice. That is the following proposition.

Proposition 2.3. *Let X be a nonempty set and L be a complete lattice, then $(H_L(X)\,, \leq)$ is a complete lattice.*

Let $\{ H_t \mid t \in T \} \subseteq H_L(X)$. By Proposition 2.3, the infimum and supremum of $H_L(X)$ is defined as:

$$\Big(\bigwedge_{t \in T} H_t\Big)(a) = \bigcap_{t \in T} H_t(a),$$

$$\Big(\bigvee_{t \in T} H_t\Big) = \bigwedge\{H \in H_L(X) \mid \ \forall t \in T,\ H \geq H_t\}.$$

Definition 2.4. [1] Let $(L\,,\,\leq)$ be a complete lattice. If there are binary operations $\otimes$ and $\rightarrow$ on L that satisfy:

(R1) $(L\,,\otimes\,,1)$ is a commutative monoid, i.e., $\otimes$ is commutative, associative, and the identity $x\otimes 1=x$ holds for each $x\in X$,

(R2) adjointness property, i.e. $x\otimes y\leq z\Leftrightarrow x\leq y\rightarrow z$ holds for all $x,y,z\in L$, then L is called a complete residuated lattice with respect to $\otimes$ and $\rightarrow$. Operations $\otimes$ and $\rightarrow$ are called tensor and residuum on L respectively.

In the following proposition, we give a list of some properties of operations $\otimes$ and $\rightarrow$.

Proposition 2.5. [1] *Let L be a complete residuated lattice. Then the following holds for all $x,y,z\in L$ and $\{y_i\}_{i\in I}\subseteq L$:*

(a) $x\rightarrow x=1$,
(b) $(x\otimes y)\rightarrow z=x\rightarrow(y\rightarrow z)$,
(c) $x\leq(x\rightarrow y)\rightarrow y$,
(d) If $y\leq z$, *then* $x\rightarrow y\leq x\rightarrow z$,
(e) If $x\leq y$, *then* $y\rightarrow z\leq x\rightarrow z$,
(f) $x\rightarrow\bigwedge_{i\in I}y_i=\bigwedge_{i\in I}(x\rightarrow y_i)$,
(g) $\bigvee_{i\in I}y_i\rightarrow y=\bigwedge_{i\in I}(y_i\rightarrow y)$.

3. Representation Theorem of $L-$subsets on Complete Residuated Lattice

In this section, L is always assumed to be a complete residuated lattice, we shall define tensor and residuum operations on $H_L(X)$ and show that $H_L(X)$ forms a complete residuated lattice with respect to these two operations. Then the classical isomorphic object of complete residuated power set lattice will be given, and the new representation theorem of $L-$subsets is obtained.

Definition 3.1. Let $H\in H_L(X)$. An $L-$subset $\theta_H:X\rightarrow L$ defined by

$$\forall x\in X,\ \theta_H(x)=\bigvee\{a\in L\mid x\in H(a)\}$$

is called an $L-$subset induced by H.

By the above definition, the tensor and residuum operations on $H_L(X)$ can be defined in a natural way. For this purpose, we need two lemmas to show rationality of the definition.

Lemma 3.2. *Let $H,G\in H_L(X)$. Define a map $H\otimes G:L\rightarrow\mathcal{P}(X)$ as follows:*

$$\forall a\in L,\ (H\otimes G)(a)=\{x\mid\theta_H(x)\otimes\theta_G(x)\geq a\},$$

then $H\otimes G\in H_L(X)$, i.e., $H\otimes G$ is an $L-$nested system.

Proof. By Definition 2.2, we check that $H\otimes G$ satisfies (LH1) and (LH2) as follows.

(LH1) For $a,b\in L$, if $a\leq b$, then

$$\begin{aligned}(H\otimes G)(b)&=\{x\mid\theta_H(x)\otimes\theta_G(x)\geq b\}\\&\subseteq\{x\mid\theta_H(x)\otimes\theta_G(x)\geq a\}\\&=(H\otimes G)(a).\end{aligned}$$

(LH2) For each $x \in X$, since $\{\ a \mid x \in (H \otimes G)(a)\} = \{\ a \mid \theta_H(x) \otimes \theta_G(x) \geq a\}$, $\theta_H(x) \otimes \theta_G(x)$ is the greatest element of the subset $\{\ a \mid x \in (H \otimes G)(a)\}$ of L. $\square$

Lemma 3.3. *Let $H, G \in H_L(X)$. Define a map $H \to G : L \to \mathcal{P}(X)$ as follows:*

$$\forall a \in L,\ (H \to G)(a) = \{\ x \mid \theta_H(x) \to \theta_G(x) \geq a\},$$

then $H \to G \in H_L(X)$, i.e., $H \to G$ is an $L-$nested system.

By Lemmas 3.2 and 3.3, for $H, G \in H_L(X)$, $H \otimes G$ and $H \to G$ are both $L-$nested systems. The defined operations $\otimes$ and $\to$ are called tensor and residuum on $H_L(X)$ respectively.

Based on the operations on $H_L(X)$ defined above, we obtain the following important theorem.

Theorem 3.4. *The complete lattice $H_L(X)$ forms a complete residuated lattice with respect to $(\otimes, \to)$.*

Proof. (1) By Proposition 2.3, we know that $(H_L(X), \leq)$ is a complete lattice, and its greatest element H^1 is defined by: $\forall a \in L,\ H^1(a) = X$.

(2) We prove that $(H_L(X), \otimes, H^1)$ is a commutative monoid, i.e. (R1) holds.

Firstly, we need to prove that $\otimes$ satisfies commutative law. For each $H,\ G \in H_L(X)$ and each $a \in L$, we have

$$\begin{aligned}(H \otimes G)(a) &= \{\ x \mid \theta_H(x) \otimes \theta_G(x) \geq a\}\\ &= \{\ x \mid \theta_G(x) \otimes \theta_H(x) \geq a\}\\ &= (G \otimes H)(a).\end{aligned}$$

By the arbitrariness of a, we obtain $H \otimes G = G \otimes H$.

Secondly, we prove that $\otimes$ has a unit element. For each $H \in H_L(X)$ and each $a \in L$, the following holds:

$$\begin{aligned}(H \otimes H^1)(a) &= \{\ x \mid \theta_H(x) \otimes \theta_{H^1}(x) \geq a\}\\ &= \{x \mid \theta_H(x) \geq a\}\\ &= H(a).\end{aligned}$$

By the arbitrariness of a, we know that $H \otimes H^1 = H$, which means H^1 is the unit element.

Thirdly, we prove $\otimes$ satisfies the associative law. Let $H,\ G,\ M \in H_L(X)$ and $a \in L$. For all $x \in X$, since

$$\begin{aligned}\theta_{H \otimes G}(x) &= \bigvee\{\ a \in L \mid x \in (H \otimes G)(a)\}\\ &= \bigvee\{\ a \in L \mid \theta_H(x) \otimes \theta_G(x) \geq a\}\\ &= \theta_H(x) \otimes \theta_G(x),\end{aligned}$$

it is observed that

$$\begin{aligned}((H \otimes G) \otimes M)(a) &= \{\ x \mid \theta_{H \otimes G}(x) \otimes \theta_M(x) \geq a\}\\ &= \{\ x \mid \theta_H(x) \otimes \theta_G(x) \otimes \theta_M(x) \geq a\}\\ &= \{\ x \mid \theta_H(x) \otimes \theta_{G \otimes M}(x) \geq a\}\\ &= (H \otimes (G \otimes M))(a).\end{aligned}$$

By the arbitrariness of a, $(H \otimes G) \otimes M = H \otimes (G \otimes M)$ holds.

(3) We need to prove that $(\otimes\,,\rightarrow)$ satisfies (R2). For each $H,\ G,\ M \in H_L(X)$, it remains to prove $H \otimes G \leq M \Leftrightarrow H \leq G \rightarrow M$. In fact, this can be proved by the following equations:

$$\begin{aligned} (H \otimes G) \leq M &\Leftrightarrow \forall x \in X,\ \theta_{H\otimes G}(x) \leq \theta_M(x) \\ &\Leftrightarrow \forall x \in X,\ \theta_H(x) \otimes \theta_G(x) \leq \theta_M(x) \\ &\Leftrightarrow \forall x \in X,\ \theta_H(x) \leq \theta_G(x) \rightarrow \theta_M(x) \\ &\Leftrightarrow \forall x \in X,\ \theta_H(x) \leq \theta_{G\rightarrow M}(x) \\ &\Leftrightarrow H \leq G \rightarrow M. \end{aligned}$$

Finally, it follows from the above (1)-(3) and Definition 2.4 that $(H_L(X)\,,\leq)$ is a complete residuated lattice w.r.t the operations $\otimes$ and $\rightarrow$ defined in Lemmas 3.2 and 3.3. □

In order to obtain the new representation theorem of $L-$subsets on complete residuated lattice, we need the following lemma for preparation.

Lemma 3.5. *Let M, L be complete residuated lattices. If $f : M \rightarrow L$ is an isomorphism between complete lattices, then f preserves the tensor operation iff f preserves the residuum operation.*

The classical isomorphic object of complete residuated power set lattice is obtained from the following theorem.

Theorem 3.6. *Let X be a nonempty set and L be a complete residuated lattice. Then $(H_L(X)\,,\vee\,,\wedge\,,\otimes\,,\rightarrow) \cong (L^X\,,\vee\,,\wedge\,,\otimes\,,\rightarrow)$.*

Proof. Define a map $f : H_L(X) \rightarrow L^X$ by $f(H) = \bigvee_{a\in L}(a \wedge H(a))$ for each $H \in H_L(X)$. As it is known that f is a bijection and preserves intersection and union operations, we only need to prove f preserves tensor and residuum operations. For each $x \in X$ and $H,\ G \in H_L(X)$, the following equations hold:

$$\begin{aligned} f(H \otimes G)(x) &= \bigvee_{a\in L}(a \wedge (H \otimes G)(a)(x)) \\ &= \bigvee\{a \in L \mid x \in (H \otimes G)(a)\} \\ &= \bigvee\{a \mid \theta_H(x) \otimes \theta_G(x) \geq a\} \\ &= \theta_H(x) \otimes \theta_G(x) \\ &= \bigvee\{a \in L \mid x \in H(a)\} \otimes \bigvee\{a \in L \mid x \in G(a)\} \\ &= \bigvee_{a\in L}(a \wedge H(a)(x)) \otimes \bigvee_{a\in L}(a \wedge G(a)(x)) \\ &= f(H)(x) \otimes f(G)(x) \\ &= (f(H) \otimes f(G))(x). \end{aligned}$$

By the arbitrariness of x we have $f(H \otimes G) = f(H) \otimes f(G)$, i.e., f preserves tensor operation. By Lemma 3.5 we obtain that f also preserves residuum operation. Therefore, f is an isomorphism between $(H_L(X)\,,\vee\,,\wedge\,,\otimes\,,\rightarrow)$ and $(L^X\,,\vee\,,\wedge\,,\otimes\,,\rightarrow)$, i.e., $(H_L(X)\,,\vee\,,\wedge\,,\otimes\,,\rightarrow) \cong (L^X\,,\vee\,,\wedge\,,\otimes\,,\rightarrow)$, as desired. □

Consequently, Theorem 3.6 approves that $H_L(X)$ is the classical isomorphic object of $L-$power set lattice L^X under the condition that L is a complete residuated lattice. That is the new representation theorem of $L-$subsets on complete residuated lattice.

4. The Level $L-$subsets of $L-$families and Their Representation

In this section, we introduce the concept of $L-$families and define the system of $L-$subsets on the condition that L is a complete residuated lattice. It is proved that there is a one-to-one correspondence between all intersection-preserving $L-$families and all systems of $L-$subsets, that is the representation theorem of intersection-preserving $L-$families on complete residuated lattice.

In the following, we introduce the concept of $L-$families, then discuss the union and intersection operations of $L-$families w.r.t $L-$partial order $\mathcal{S}(-,-)$.

Definition 4.1. Let L be a complete residuated lattice, X be a nonempty set. An $L-$family in X is a map $\mathscr{A} : L^X \to L$. For each $B \in L^X$, $\mathscr{A}(B)$ is called the membership degree of B in $\mathscr{A}$.

Example 4.2. Let L be a complete residuated lattice, X be a nonempty set. If $F : L^X \to L$ satisfies:

(F1) $F(1_X) = 1,\ F(0_X) = 0,$
(F2) $F(A) \wedge F(B) \leq F(A \wedge B),$
(F3) $A \leq B \Rightarrow F(A) \leq F(B),$

for all $A,\ B \in L^X$. Then F is an $L-$family in X.

For each $A, B \in L^X$, define $\mathcal{S} : L^X \times L^X \to L$ as follows (which can be seen in [1]):

$$\mathcal{S}(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)),$$

Then, $\mathcal{S}$ is a binary $L-$relation on L^X. $\mathcal{S}(A, B)$ is called subsethood degree of A in B.

It is easily checked that $\mathcal{S}(-,-)$ is an $L-$partial order on L^X, which means that $\mathcal{S}(-,-)$ fulfills:

(P1) $\mathcal{S}(A, A) = 1$;
(P2) If $\mathcal{S}(A, B) = 1$ and $\mathcal{S}(B, A) = 1$, then $A = B$;
(P3) $\mathcal{S}(A, B) \otimes \mathcal{S}(B, C) \leq \mathcal{S}(A, C)$.

The pair $(L^X, \mathcal{S}(-,-))$ forms an $L-$partially ordered set.

Definition 4.3 gives the union and intersection of $L-$family $\mathscr{A}$ w.r.t $L-$partial order $\mathcal{S}(-,-)$.

Definition 4.3. [1] Let $\mathscr{A} : L^X \to L$ be an $L-$family in X. Define $L-$subsets $\sup \mathscr{A}$, $\inf \mathscr{A} \in L^X$ such that for all $x \in X$,

$$\sup \mathscr{A}(x) = \bigvee_{B \in L^X} (\mathscr{A}(B) \otimes B(x)),$$

$$\inf \mathscr{A}(x) = \bigwedge_{C \in L^X} (\mathscr{A}(C) \to C(x)).$$

Then $\sup \mathscr{A}$ is called the union of $\mathscr{A}$ w.r.t $\mathcal{S}(-,-)$ and $\inf \mathscr{A}$ is called the intersection of $\mathscr{A}$ w.r.t $\mathcal{S}(-,-)$.

Based on the union and intersection of an $L-$family, we have the following result.

Proposition 4.4. *Let $\mathscr{A} : L^X \to L$ be an $L-$family in X. Then for all $B \in L^X$, the following equations hold:*

$$\mathcal{S}(\sup \mathscr{A}, B) = \bigwedge_{C \in L^X} (\mathscr{A}(C) \to \mathcal{S}(C, B)),$$

$$\mathcal{S}(B, \inf \mathscr{A}) = \bigwedge_{C \in L^X} (\mathscr{A}(C) \to \mathcal{S}(B, C)).$$

Next, we give the concept of level $L-$subsets of an $L-$family.

Definition 4.5. Let $\mathscr{A} : L^X \to L$ be an $L-$family. An $L-$subset $\mathcal{A}_\alpha$ defined by $\mathcal{A}_\alpha = \bigwedge\{ C \mid \mathscr{A}(C) \geq \alpha\}$ is called a level $L-$subset of $\mathscr{A}$, where $\alpha \in L$. $\{\mathcal{A}_\alpha\}_{\alpha \in L}$ is called the system of level $L-$subsets of $\mathscr{A}$.

In the following, we introduce the concept of the system of $L-$subsets, with which we establish the representation theorem of intersection-preserving $L-$families on complete residuated lattice.

Definition 4.6. Let $\{\mathcal{H}_\alpha\}_{\alpha \in L} \subseteq L^X$ satisfy:

(C1) If $M \subseteq L$, then $\mathcal{H}_{\vee M} = \bigvee_{\alpha \in M} \mathcal{H}_\alpha$,

(C2) $\mathcal{H}_0 = 0_X$,

then $\{\mathcal{H}_\alpha\}_{\alpha \in L}$ is called the system of $L-$subsets.

The following example shows that the system of level $L-$subsets of an intersection-preserving $L-$family is the system of $L-$subsets, that is to say the system of $L-$subsets defined above exists.

Example 4.7. Let $\mathscr{A} : L^X \to L$ be an $L-$family which preserves arbitrary intersections, $\{\mathcal{A}_\alpha\}_{\alpha \in L}$ be the system of level $L-$subsets of $\mathscr{A}$. Then $\{\mathcal{A}_\alpha\}_{\alpha \in L}$ is the system of $L-$subsets.

Proof. That $\{\mathcal{A}_\alpha\}_{\alpha \in L}$ satisfies (C2) in Definition 4.6 is obvious. We verify (C1) as follows:

(1) For all $\alpha \in M$, we have $\mathcal{A}_\alpha = \bigwedge_{\mathscr{A}(C) \geq \alpha} C \leq \bigwedge_{\mathscr{A}(C) \geq \vee M} C = \mathcal{A}_{\vee M}$, thus $\bigvee_{\alpha \in M} \mathcal{A}_\alpha \leq \mathcal{A}_{\vee M}$.

(2) Under the condition that $\mathscr{A}$ is an intersection-preserving map, we have $\mathscr{A}\Big(\bigvee_{\alpha \in M} \mathcal{A}_\alpha\Big) \geq \mathscr{A}(\mathcal{A}_\alpha) = \mathscr{A}\Big(\bigwedge_{\mathscr{A}(C) \geq \alpha} C\Big) = \bigwedge_{\mathscr{A}(C) \geq \alpha} \mathscr{A}(C) \geq \alpha$ for all $\alpha \in M$. Hence, $\mathscr{A}\Big(\bigvee_{\alpha \in M} \mathcal{A}_\alpha\Big) \geq \vee M$, which implies $\mathcal{A}_{\vee M} \leq \bigvee_{\alpha \in M} \mathcal{A}_\alpha$.

To sum up, we have $\mathcal{A}_{\vee M} = \bigvee_{\alpha \in M} \mathcal{A}_\alpha$, as desired. □

The following Lemma 4.8 shows the sufficient and necessary condition that an $L-$family can be represented by its system of level $L-$subsets.

Before this, we introduce a term first. Let $\mathscr{A} : L^X \to L$ be an $L-$family, $\{\mathcal{B}_\alpha\}_{\alpha\in L}$ be a family of $L-$subsets. Then we say that $\mathscr{A}$ can be represented by $\{\mathcal{B}_\alpha\}_{\alpha\in L}$ if it satisfies $\mathscr{A}(C) = \bigvee\{\ \alpha \mid C \geq \mathcal{B}_\alpha\}$ for all $C \in L^X$.

A map $\mathscr{A} : L^X \to L$ is called intersection-preserving iff $\mathscr{A}\Big(\bigwedge_{j\in J} B_j\Big) = \bigwedge_{j\in J} \mathscr{A}(B_j)$ holds for all $\{B_j\}_{j\in J} \subseteq L^X$.

Lemma 4.8. *Let $\mathscr{A} : L^X \to L$ be an $L-$family, $\{\mathcal{A}_\alpha\}_{\alpha\in L}$ be the system of level $L-$subsets of $\mathscr{A}$. Then $\mathscr{A}$ can be represented by $\{\mathcal{A}_\alpha\}_{\alpha\in L}$ iff $\mathscr{A}$ is an intersection-preserving map.*

Proof. Necessity. For $\{B_j\}_{j\in J} \subseteq L^X$, put $\alpha = \bigwedge_{j\in J} \mathscr{A}(B_j)$. Then $B_j \geq \mathcal{A}_\alpha$ holds for all $j \in J$. This shows $\bigwedge_{j\in J} B_j \geq \mathcal{A}_\alpha$. Since $\mathscr{A} : L^X \to L$ can be represented by $\{\mathcal{A}_\alpha\}_{\alpha\in L}$, it follows that

$$\mathscr{A}\Big(\bigwedge_{j\in J} B_j\Big) = \bigvee\{\ \beta \mid \bigwedge B_j \geq \mathcal{A}_\beta\} \geq \alpha = \bigwedge_{j\in J} \mathscr{A}(B_j).$$

On the other hand, for each $C, D \in L^X$ with $C \leq D$, we have $\mathscr{A}(C) \leq \mathscr{A}(D)$. This implies $\mathscr{A}\Big(\bigwedge_{j\in J} B_j\Big) \leq \bigwedge_{j\in J} \mathscr{A}(B_j)$. Therefore, $\mathscr{A}\Big(\bigwedge_{j\in J} B_j\Big) = \bigwedge_{j\in J} \mathscr{A}(B_j)$.

sufficiency. On one hand, for all $\alpha \in L$, we have

$$\mathscr{A}(\mathcal{A}_\alpha) = \mathscr{A}\Big(\bigwedge_{\mathscr{A}(C)\geq\alpha} C\Big) = \bigwedge_{\mathscr{A}(C)\geq\alpha} \mathscr{A}(C) \geq \alpha.$$

By this, for each $D \in L^X$, with $D \geq \mathcal{A}_\alpha$, $\mathscr{A}(D) \geq \alpha$ always hold. So $\mathscr{A}(D) \geq \bigvee\{\ \alpha \mid D \geq \mathcal{A}_\alpha\}$ holds for each $D \in L^X$.

On the other hand, for each $D \in L^X$, put $\alpha_D = \mathscr{A}(D)$, then we have $D \geq \mathscr{A}_{\alpha_D}$. Thus $\mathscr{A}(D) = \alpha_D \leq \bigvee\{\ \alpha \mid D \geq \mathcal{A}_\alpha\}$. Hence, $\mathscr{A}(D) = \bigvee\{\ \alpha \mid D \geq \mathcal{A}_\alpha\}$ holds for every $D \in L^X$, which means $\mathscr{A}$ can be represented by $\{\mathcal{A}_\alpha\}_{\alpha\in L}$. □

We collect our main results in the following two theorems.

The following theorem shows that when $L-$family $\mathscr{A}$ preserves arbitrary intersections, any $L-$family which can be represented by the system of level $L-$subsets of $\mathscr{A}$ is equal to $\mathscr{A}$.

Theorem 4.9. *Let $\mathscr{A} : L^X \to L$ be an $L-$family which preserves arbitrary intersections, $\{\mathcal{A}_\alpha\}_{\alpha\in L}$ be the system of level $L-$subsets of $\mathscr{A}$. If $\mathscr{B} : L^X \to L$ is a map that can be represented by $\{\mathcal{A}_\alpha\}_{\alpha\in L}$, then*

(1) $\mathscr{B}$ is an $L-$family that preserves arbitrary intersections,

(2) $\mathscr{B} = \mathscr{A}$.

Proof. (1) In order to show $\mathscr{B}$ is an intersection-preserving map, we need to check $\mathscr{B}\Big(\bigwedge_{j\in J} D_j\Big) = \bigwedge_{j\in J} \mathscr{B}(D_j)$ for all $\{D_j\}_{j\in J} \subseteq L^X$. First of all, since $\mathscr{B}$ can be represented by $\{\mathcal{A}_\alpha\}_{\alpha\in L}$, we have $\mathscr{B}(C) \leq \mathscr{B}(D)$ whenever $C, D \in L^X$ with $C \leq D$.

On one hand, put $\alpha = \bigwedge_{j\in J} \mathscr{B}(D_j)$, then $\alpha \leq \mathscr{B}(D_j)$ holds for all $j \in J$. Put $\mathscr{B}(D_j) = \bigvee\{\ \beta \mid D_j \geq \mathcal{A}_\beta\} = \gamma$, as $\{\mathcal{A}_\alpha\}_{\alpha\in L}$ is the system of $L-$subsets it satisfies (C1) of Definition 4.6, we have $D_j \geq \mathcal{A}_\gamma$. Actually, that is because: If we put $M = \{\ \beta \mid D_j \geq \mathcal{A}_\beta\}$, then $\gamma = \bigvee M$. We have $\mathcal{A}_\gamma = \mathcal{A}_{\bigvee M} = \bigvee_{\beta\in M} \mathcal{A}_\beta \leq$

$\bigvee_{\beta\in M} D_j = D_j$. Thus $D_j \geq \mathcal{A}_\alpha$ for all $j \in J$. Furthermore, $\bigwedge_{j\in J} D_j \geq \mathcal{A}_\alpha$, which implies $\mathscr{B}\Big(\bigwedge_{j\in J} D_j\Big) = \bigvee\{\ \beta \mid \bigwedge_{j\in J} D_j \geq \mathcal{A}_\beta\} \geq \alpha = \bigwedge_{j\in J} \mathscr{B}(D_j)$.

On the other hand, since $\bigwedge_{j\in J} D_j \leq D_j$ holds for all $j \in J$, $\mathscr{B}\Big(\bigwedge_{j\in J} D_j\Big) \leq \mathscr{B}(D_j)$ holds for each $j \in J$. We have with this $\mathscr{B}\Big(\bigwedge_{j\in J} D_j\Big) \leq \bigwedge_{j\in J} \mathscr{B}(D_j)$.

By the above proof, $\mathscr{B}\Big(\bigwedge_{j\in J} D_j\Big) = \bigwedge_{j\in J} \mathscr{B}(D_j)$ holds.

(2) By the fact that $\mathscr{A}$ is an $L-$family which preserves arbitrary intersections together with Lemma 4.8, we obtain $\mathscr{B}(C) = \bigvee\{\ \alpha \mid C \geq \mathcal{A}_\alpha\} = \mathscr{A}(C)$ for every $C \in L^X$. Therefore, $\mathscr{B} = \mathscr{A}$. □

Let $\{\mathcal{H}_\alpha\}_{\alpha\in L}$ be the system of $L-$subsets. If $\mathscr{A}$ is an $L-$family that can be represented by $\{\mathcal{H}_\alpha\}_{\alpha\in L}$, then there is a question whether the system of level $L-$subsets of $\mathscr{A}$ is precisely $\{\mathcal{H}_\alpha\}_{\alpha\in L}$ or not. The following Theorem 4.10 gives the answer.

Theorem 4.10. *Let $\{\mathcal{H}_\alpha\}_{\alpha\in L}$ be the system of $L-$subsets. If $\mathscr{A} : L^X \to L$ is an $L-$family that can be represented by $\{\mathcal{H}_\alpha\}_{\alpha\in L}$, $\{\mathcal{A}_\alpha\}_{\alpha\in L}$ is the system of level $L-$subsets of $\mathscr{A}$, then we have*

(1) $\mathscr{A}$ is an $L-$family which preserves arbitrary intersections,

(2) $\mathcal{H}_\alpha = \mathcal{A}_\alpha$ holds for all $\alpha \in L$.

Proof. (1) It is similar to the proof of Theorem 4.9 (1).

(2) As $\mathscr{A}$ can be represented by $\{\mathcal{H}_\alpha\}_{\alpha\in L}$, for every $\alpha \in L$, $\mathscr{A}(\mathcal{H}_\alpha) = \bigvee\{\ \beta \mid \mathcal{H}_\alpha \geq \mathcal{H}_\beta\}$ holds. Since $\mathcal{H}_\alpha \geq \mathcal{H}_\alpha$, we have $\mathscr{A}(\mathcal{H}_\alpha) \geq \alpha$. So $\mathcal{A}_\alpha \leq \mathcal{H}_\alpha$ holds since $\{\mathcal{A}_\alpha\}_{\alpha\in L}$ is the system of level $L-$subsets of $\mathscr{A}$. In the following we prove $\mathcal{H}_\alpha \leq \mathcal{A}_\alpha$ for all $\alpha \in L$.

It is sufficient to verify for a given $\alpha \in L$ and each $D \in L^X, that\ D \geq \mathcal{A}_\alpha \Rightarrow D \geq \mathcal{H}_\alpha$. For $D \geq \mathcal{A}_\alpha$, $\mathscr{A}(D) \geq \mathscr{A}(\mathcal{A}_\alpha)$ holds since $\mathscr{A}$ is an order-preserving map. It follows from the property of $\mathscr{A}$ being an intersection-preserving map that $\mathscr{A}(\mathcal{A}_\alpha) = \mathscr{A}\Big(\bigwedge_{\mathscr{A}(C)\geq\alpha} C\Big) = \bigwedge_{\mathscr{A}(C)\geq\alpha} \mathscr{A}(C) \geq \alpha$, which means $\mathscr{A}(D) \geq \alpha$. Letting $\xi := \mathscr{A}(D) = \bigvee\{\ \beta \mid D \geq \mathcal{H}_\beta\}$, it follows from $\{\mathcal{H}_\alpha\}_{\alpha\in L}$ satisfying (C1) that $D \geq \mathcal{H}_\xi$. In fact, if we denote $M = \{\ \beta \mid D \geq \mathcal{H}_\beta\}$, then $\xi = \bigvee M$. Therefore, $\mathcal{H}_\xi = \mathcal{H}_{\bigvee M} = \bigvee_{\beta\in M} \mathcal{H}_\beta \leq \bigvee_{\beta\in M} D = D$. Since $\xi \geq \alpha$, we have $D \geq \mathcal{H}_\xi \geq \mathcal{H}_\alpha$. Hence, $\mathcal{H}_\alpha \leq \mathcal{A}_\alpha$ holds for all $\alpha \in L$. □

The above Theorems 4.9 and 4.10 establish the one-to-one correspondence between all intersection-preserving $L-$families and all systems of $L-$subsets. That is so called representation theorem of intersection-preserving $L-$families.

Next, we present a kind of $L-$families which satisfy some additional conditions and we call them principal $L-$families.

Definition 4.11. Let $\mathscr{A} : L^X \to L$ be an $L-$family in X. If there exists some $B \in L^X$ such that $\mathscr{A}(C) = \mathcal{S}(B, C)$ for all $C \in L^X$, then $\mathscr{A}$ is called a principal $L-$family.

The following theorem describes the property of a principal $L-$family.

Theorem 4.12. *A map $\mathscr{A} : L^X \to L$ is a principal $L-$family iff $\mathscr{A}(C) = \mathcal{S}(\inf \mathscr{A}, C)$ holds for all $C \in L^X$.*

Proof. The sufficiency is easily proved by Definition 4.11, we prove the necessity as follows.

Suppose there exists some $B \in L^X$ such that $\mathscr{A}(C) = \mathcal{S}(B, C)$ for each $C \in L^X$, it remains to prove $B = \inf \mathscr{A}$.

First, we have

$$\begin{aligned} \mathcal{S}(B, \inf \mathscr{A}) &= \bigwedge_{C \in L^X} (\mathscr{A}(C) \to \mathcal{S}(B, C)) \\ &= \bigwedge_{C \in L^X} (\mathscr{A}(C) \to \mathscr{A}(C)) \\ &= 1. \end{aligned}$$

Then, by $\mathscr{A}(B) = \mathcal{S}(B, B) = 1$ the following holds:

$$\begin{aligned} \mathcal{S}(\inf \mathscr{A}, B) &= \bigwedge_{x \in X} (\inf \mathscr{A}(x) \to B(x)) \\ &= \bigwedge_{x \in X} \Big(\bigwedge_{C \in L^X} (\mathscr{A}(C) \to C(x)) \to B(x) \Big) \\ &\geq \bigwedge_{x \in X} ((\mathscr{A}(B) \to B(x)) \to B(x)) \\ &\geq \mathscr{A}(B) \\ &= 1. \end{aligned}$$

To sum up, it follows from $\mathcal{S}(\inf \mathscr{A}, B) = 1$ and $\mathcal{S}(B, \inf \mathscr{A}) = 1$ that $B = \inf \mathscr{A}$. □

Corollary 4.13. *If $\mathscr{A} : L^X \to L$ is a principal $L-$family, then $\mathscr{A}$ is an intersection-preserving map, that is for $\{B_j \mid j \in J\} \subseteq L^X$,*

$$\mathscr{A}\Big(\bigwedge_{j \in J} B_j \Big) = \bigwedge_{j \in J} \mathscr{A}(B_j).$$

.

Proof. Let $\{B_j\}_{j \in J} \subseteq L^X$. By Theorem 4.12, we have

$$\begin{aligned} \mathscr{A}\Big(\bigwedge_{j \in J} B_j \Big) &= \mathcal{S}\Big(\inf \mathscr{A}, \bigwedge_{j \in J} B_j \Big) \\ &= \bigwedge_{x \in X} \Big(\inf \mathscr{A}(x) \to \bigwedge_{j \in J} B_j(x) \Big) \\ &= \bigwedge_{x \in X} \bigwedge_{j \in J} (\inf \mathscr{A}(x) \to B_j(x)) \\ &= \bigwedge_{j \in J} \mathcal{S}(\inf \mathscr{A}, B_j) \\ &= \bigwedge_{j \in J} \mathscr{A}(B_j), \end{aligned}$$

which means that $\mathscr{A}$ is an intersection-preserving map. □

The next proposition shows when $\mathscr{A}$ is a principal $L-$family, $\inf \mathscr{A}$ can be represented by the system of level $L-$subsets of $\mathscr{A}$ as the following form.

Proposition 4.14. *Let $\mathscr{A} : L^X \to L$ be a principal $L-$family, $\{\mathcal{A}_\alpha\}_{\alpha \in L}$ be the system of level $L-$subsets of $\mathscr{A}$. Then $\inf \mathscr{A} = \bigwedge_{\alpha \in L} (\alpha \to \mathcal{A}_\alpha)$ holds.*

Proof. (1) Recall that $\inf \mathscr{A} = \bigwedge_{C\in L^X}(\mathscr{A}(C) \to C)$, and let $\alpha_c = \mathscr{A}(C)$ for each $C \in L^X$. Then $C \geq \mathcal{A}_{\alpha_c}$. Thus for each $x \in X$,

$$\begin{aligned}\inf \mathscr{A}(x) &= \bigwedge_{C\in L^X}(\mathscr{A}(C) \to C(x)) \\ &\geq \bigwedge_{C\in L^X}(\alpha_c \to \mathcal{A}_{\alpha_c}(x)) \\ &\geq \bigwedge_{\alpha\in L}(\alpha \to \mathcal{A}_\alpha(x)).\end{aligned}$$

(2) Moreover, for all $\alpha \in L$, the following equations hold:

$$\begin{aligned}\mathscr{A}(\mathcal{A}_\alpha) &= \mathcal{S}(\inf \mathscr{A}, \mathcal{A}_\alpha) \\ &= \bigwedge_{x\in X}(\inf \mathscr{A}(x) \to \mathcal{A}_\alpha(x)) \\ &= \bigwedge_{x\in X}\Big(\inf \mathscr{A}(x) \to \bigwedge_{\mathscr{A}(C)\geq\alpha} C(x)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{\mathscr{A}(C)\geq\alpha}(\inf \mathscr{A}(x) \to C(x)) \\ &= \bigwedge_{\mathscr{A}(C)\geq\alpha}\mathcal{S}(\inf \mathscr{A}, C) \\ &= \bigwedge_{\mathscr{A}(C)\geq\alpha}\mathscr{A}(C) \\ &\geq \alpha,\end{aligned}$$

i.e. $\mathscr{A}(\mathcal{A}_\alpha) \geq \alpha$ for all $\alpha \in L$. From this, we obtain for each $x \in X$,

$$\begin{aligned}\inf \mathscr{A}(x) &= \bigwedge_{C\in L^X}(\mathscr{A}(C) \to C(x)) \\ &\leq \bigwedge_{\alpha\in L}(\mathscr{A}(\mathcal{A}_\alpha) \to \mathcal{A}_\alpha(x)) \\ &\leq \bigwedge_{\alpha\in L}(\alpha \to \mathcal{A}_\alpha(x)).\end{aligned}$$

From the above proof, it follows by the arbitrariness of x in X that $\inf \mathscr{A} = \bigwedge_{\alpha\in L}(\alpha \to \mathcal{A}_\alpha)$. □

5. **Conclusion**

In this paper, we establish representation theorems of $L-$subsets and $L-$families on complete residuated lattice. The new representation theorem of $L-$subsets shows that not only $L-$power set lattice but also its classical isomorphic object namely $L-$nested systems form complete residuated lattices, and they are still isomorphic.

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