Iranian Journal of Fuzzy Systems

Volume 21, Number 5, (2024), pp. 71-88

Original paper

University of Sistan and Baluchestan

A dual measure of fuzzy entropy: Fuzzy extropy

S. Kar [1], B. K. Roy [2] and M. K. Mohanty [3]

[1,2] *Department of Mathematics, National Institution of Technology Durgapur, West Bengal, India*
[3] *Department of Computer Science and Engineering, National Institution of Technology Durgapur, West Bengal, India*

skar.maths@nitdgp.ac.in, bikashroy235@gmail.com, munkun41@gmail.com

Abstract

In this paper, we introduce a new concept of fuzzy extropy measure, which can help to measure any system's uncertainty or fuzziness in parallel to fuzzy entropy. Fuzzy extropy can be considered as a dual concept of fuzzy entropy. Additionally, we provide ideas for higher-order fuzzy extropy and hybrid fuzzy extropy. Then, we introduce concepts of credibilistic extropy for both discrete and continuous variables. Furthermore, we have provided examples of these concepts and compared the extropy results with existing fuzzy entropy results. Finally, the validation through simulation studies offers a visual representation of extropy parameters and density distributions, providing empirical insights into the proposed measures' behavior and enhancing their credibility and understanding.

Keywords: Uncertainty measure, fuzzy extropy, fuzzy higher-order extropy, fuzzy hybrid extropy, credibilistic extropy.

1 Introduction

Measuring the uncertainty of any set is always a challenging job for researchers. Many researchers provided many tools and techniques for measuring uncertainty. Shannon (1948) [33] first introduced the probabilistic entropy measure to measure the uncertainty of any system. After Shannon in last few decades many researchers further proposed different generalisations of Shannon's entropy to measure uncertainties of information system. Few of them are Renyi entropy, Tsallis entropy, residual entropy, cumulative entropy, weighted entropy etc. Renyi entropy was first proposed by Renyi in 1961, [31] which was one of the generalisations of Shannon's entropy as for different values of its parameter α, it turns into different entropies. Especially for $\alpha \to 1$, it dissolve's to Shannon's entropy. To measure uncertainty of extensibility of any system Tsallis (1988) [36] proposed Tsallis entropy. Ebrahimi (1996) [12] first proposed the concept of residual entropy using Shannon entropy. Song (2001) [35] introduced varentropy which measures the variability of information from its entropy simply by replacing x by –log(f(X)) in the formula of Var(x). Ubriaco (2009) [37] defined the concept of fractionl entropy using fractional calculus. A generalization of probability theory is Dempster-Shafer theory (DST) also known as evidence theory was developed by Dempster (2008) [7] and Shafer (1976) [32]. In DST the mass function also known as basic probability assignment (BPA) was introduced to construct models of uncertainty. Deng (2016, 2020) [8, 9] used this concept and proposed Deng entropy which is also a generalisation of Shannon's entropy. Pan & Deng (2018) [29] used Deng entropy to present a new entropy called belief entropy where they replaced mass function of Deng entropy with belief function and plausibility function. Kazemi et. al. (2021) [15] defined fractional Deng entropy. In recent years also many new researches on entropy took place. For example Yu & Deng (2024) [40] reinvestigated Renyi entropy and replaced its parameter by a scaled one and further investigated different properties of scaled Renyi entropy. In this paper they also presented an interesting idea in which they constructed a scaled Renyi entropy using the definition of Deng extropy proposed by Buono et. al. (2020) [4]. Chen & Deng (2024) [5] proposed entropy of random permutation set which is compatible with both probability theory and DST.

Zadeh (1965) [41] first introduced the concept of fuzzy set and introduced the idea of fuzzy entropy. However, questions arose regarding the applicability of Zadeh's fuzzy entropy to capture the full spectrum of system fuzziness.

Corresponding Author: S. Kar
Received: July 2024; Revised: September 2024; Accepted: November 2024.
https://doi.org/10.22111/ijfs.2024.49275.8693

Watching this flaw and gap of fuzziness measure, many further definitions of fuzzy entropy came into the picture. Kaufmann (1975) [14] defined fuzzy entropy, but the main fault was that it used the relative value of membership grade instead of absolute value. Deluca and Termini (1972) [6]provided one fuzzy entropy measure, which was based on Shannon's probabilistic entropy measure. Further, some researchers provided different fuzzy entropy measures, which we would like to consider as the measure proposed by Pal and Pal (1989) [27]. The basic difference between the above two fuzzy entropy measures is Deluca and Termini (1972) [6] used a logarithmic function to define fuzzy entropy, while Pal and Pal (1989) [27] used an exponential function to define fuzzy entropy. Building on this foundation, Pal and Pal (1992) [28] provided the concept of higher-order and hybrid fuzzy entropy and developed a few properties related to these concepts. In parallel developments, Zadeh (1978) [43] pioneered to propose the idea of possibility measure, which he used as a tool to measure fuzzy restrictions of any set. Many subsequent researchers like Nahmias (1978) [22], Dubois and Parade (1982, 1985) [10, 11] etc., further strengthened Possibility theory with their research.

Liu et al. (2002) [19] introduced a new measure and named it as credibility measure. The benefit of credibility measure over possibility measure was the duality property of credibility measure. Credibility measure is helpful to measure the chance of occurrence of any arbitrary fuzzy event. Lad et al. (2015) [17]took a significant step forward by defining a complementary dual measure of probabilistic entropy, named probabilistic extropy, and established a few properties related to the extropy measure. After the establishment of idea of extropy as a dual measure of entropy, it attracted accelerated interests of many researchers over last few years both in theoretical and application point of view. Qiu (2017) [30] defined residual extropy. Kamari et. al. (2020) [13] and Krishnan et al. (2020) [16] studied the past extropy analogously. Balakrishnan et. al. (2022) [3] proposed Tsallis extropy and applied practically in pattern recognition. Xue & Deng (2023) [39] studied a different form of Tsallis extropy. Balkrishnan et. al. ([3]) also investigated weighted extropy, bivariate extropy, bivariate weighted extropy, weighted residual and past extropy along with some theorems and examples in their paper. Buono et. al. (2020) [4] investigated Deng extropy and Kazemi et. al. (2021) [15] investigated fractional Deng extropy. Though it is very easy to calculate information volume of probability distribution using Shannon's entropy but it was not that easy for mass function. Liu and Xiao (2022) [20] proposed an algorithm to find information volume of mass function using the value of extropy. Liu et. al. (2023) [21] proposed Renyi extropy. Recently Zhou & Deng (2023) [45] studied belief extropy. Vaselabadi et. al. (2021) [38] studied varextopy and Zamini et. al. [44] further extended the literature by studing varextropy of doubly truncated random variable.

In recent years, there has been significant progress in extending classical approximation methods to the fuzzy function space, particularly in the context of Bernstein-type operators. Ozkan (2022) [23] made a notable contribution by introducing fuzzy post-quantum Bernstein polynomials and investigating their approximation properties via interval-valued fuzzy numbers. This work not only established the existence of these fuzzy polynomials but also developed critical approximation results. Ozkan obtained inequalities estimating the rate of fuzzy convergence, using tools like the fuzzy modulus of continuity and Lipschitz-type fuzzy functions, which provide precise control over the approximation error in fuzzy environments. Furthering this work, Ozkan and Hazarika (2023) [25]expanded on the application of fuzzy Bernstein-type rational functions. They extend the approximation results of these rational functions from real function spaces to fuzzy function spaces by utilizing the properties of interval analysis. Like the earlier study, they also investigated Korovkin-type approximation results, which are crucial for ensuring that the fuzzy functions converge appropriately. The authors also presented a Voronovskaja-type result for these rational functions, showing how they asymptotically expand, thereby contributing to the growing body of knowledge on the relationship between rational functions and fuzzy approximation techniques. In a parallel effort, Ozkan (2022) [24]investigated the approximation properties of fuzzy (p,q)-Bernstein-Chlodowsky operators. This study extends the approximation results of Bernstein-Chlodowsky operators from real function spaces to fuzzy function spaces, following a similar line of inquiry as in the previous two works. This study further reinforced the role of fuzzy operators in generalizing classical approximation techniques, and it too presented estimates for fuzzy asymptotic expansions, contributing to the broader field of fuzzy approximation theory.

The article, Pal and Pal (1993) [28], discusses various image segmentation techniques, highlighting their strengths and limitations, particularly in dealing with noise. It reviews methods based on gray-level histograms, spatial details, fuzzy set theory, and the Markov Random Field (MRF) model, noting that while MRF is noise-robust, it is computationally intensive. The article also compares segmentation techniques for gray-tone and color images, with a focus on less explored areas such as color, range images, and magnetic resonance images. The concept of entropy is relevant in this context as it can be used to measure the uncertainty or information content during image analysis and segmentation, aiding in the evaluation of segmentation quality.

Since 2015, the concept of fuzzy extropy has been largely unexplored. Though a dual measure to fuzzy entropy measure named as fuzzy knowledge measure was studied by Arya & Kumar (2021) [1] in their paper and they also applied their concept in decision making problem. In this paper, we propose the concept of fuzzy extropy, which is a dual measure of fuzzy entropy. Following the insights of Pal and Pal (1992) [28], we further define higher-order fuzzy

extropy and fuzzy hybrid extropy. Also, the best part of fuzzy hybrid extropy is that at certain conditions, it coincides with the probabilistic extropy defined by Lad et al. (2015) [17]. The remainder of the paper is organized as follows: In Section 2, we introduce preliminary definitions of probabilistic and fuzzy entropy measures. Section 3 covers the concept of fuzzy extropy, including fuzzy extropy, higher-order fuzzy extropy, and hybrid fuzzy extropy, with a comparative analysis against various types of fuzzy entropy. Section 4 explores the extropy of credibility measures for discrete and continuous fuzzy variables. A practical application is presented in Section 5, where the proposed extropy measures are applied on the provided data. Finally, Section 6 concludes the paper, summarizing key findings and contributions of the introduced fuzzy extropy measures.

2 Preliminaries

This section presents some fundamental definitions of probabilistic and fuzzy entropy measures, probabilistic extropy measure and other related concepts. This section explores various measures for quantifying uncertainty in fuzzy sets, including formulations by Zadeh (1978) [43], Kaufmann (1975) [14], and Pal and Pal (1989, 1992) [27, 28]. Examining Zadeh's possibility and necessity measures provides insights into constraints within fuzzy systems. Additionally, Liu et al.'s (2002) [19] credibility measure, derived from possibility and necessity measures, offered a comprehensive evaluation of arbitrary fuzzy events. Extending these concepts, Li et al. (2008) [18] contributed expected value computations and entropy measures for discrete and continuous fuzzy variables. Lad et al. (2015) [17] introduced probabilistic extropy as a complementary measure to probabilistic entropy, enhancing our understanding of uncertainty in probabilistic settings.

Definition 2.1. [41] *Zadeh introduced the concept of a fuzzy set in comparison with a classical set where an element takes a value of either* 0 *or* 1*, the elements of a fuzzy set can take any value between* 0 *and* 1*. A fuzzy set A can be defined as*

$$A = \{(x_i, \mu_A(x_i)) : x_i \in X, i = 1, 2, 3, \cdots, n\}.$$

Where X is the universe of elements and $\mu_A(x_i)$ is the membership function of the element x_i in A.

Definition 2.2. *The idea of probabilistic entropy was first introduced by Shannon (1948). Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ and $P(X = x_i) = p_i, i = 1, 2, 3, \ldots, n$ are probability mass functions of every element of X. Then the entropy measure of X is defined as $H(X) = -k\sum_{i=1}^{n} p_i log(p_i)$, where k is any normalizing constant.*

Definition 2.3. [27] *Pal and Pal proposed a new definition of probabilistic entropy and pointed out the advantages of their definition over Shannon's entropy. According to their definition, if $X = \{x_1, x_2, x_3, \ldots, x_n\}$ is the given set and if $P(X = x_i) = p_i, i = 1, 2, 3, \ldots, n$ are probability mass functions of elements of X, then the entropy measure of X is defined as*

$$H(X) = \sum_{i=1}^{n} p_i exp(1 - p_i).$$

Based on Shannon's concept of probabilistic entropy, many researchers proposed the idea of fuzzy entropy. In case of Shannon's entropy where he used information gain as $log(1/x)$ is unbounded when $x \to 0$ and the value of information gain tends to 0 as $x \to 1$. But in practical situation the value of information gain of an event must lie between two finite limits, whether highly probable or highly unlikely. So it is far better to use exponential function instead of logarithmic function so that information gain function must remain bounded. Watching this advantage we choose Pal and Pal's version to define our extropy function as they were pioneer to introduce entropy function in terms of exponential information gain.

Definition 2.4. [42] *Zadeh defined the entropy of fuzzy sets $A = \{x_1, x_2, x_3, \cdots, x_n\}$ with probability distributions $\{p_1, p_2, p_3, cdots, p_n\}$ as*

$$H(A) = -\sum_{i=1}^{n} \mu_A(x_i)\, p_i \log(p_i),$$

where $\mu_A(x_i)$ is the membership function of the element x_i in A and p_i denotes the probability of occurrence of the element x_i.

Definition 2.5. [14] *Kaufmann defined the entropy of fuzzy set A with n elements as*

$$H(A) = -\left\{\frac{1}{\log(n)}\right\}\sum_{i=1}^{n} \varphi_i \log(\varphi_i),$$

where $\varphi_i = \frac{\mu_i}{\sum_{i=1}^{n}\mu_i}, i = 1, 2, 3, \ldots, n.$

Definition 2.6. *Deluca and Termini (1972) [6] defined the entropy of a fuzzy set A with n elements as*

$$H(A) = -k\sum_{i=1}^{n}\{\mu_i \log(\mu_i) + (1-\mu_i)\log(1-\mu_i)\},$$

where k is any normalizing constant.

Definition 2.7. [27] *Pal and Pal also proposed a definition of fuzzy entropy based on their probabilistic entropy definition as*

$$H(A) = k\sum_{i=1}^{n}\{\mu_i \exp(1-\mu_i) + (1-\mu_i)\exp(\mu_i)\},$$

where k is any normalizing constant.

Definition 2.8. [28] *Pal and Pal proposed the idea of higher order fuzzy entropy in their paper. According to their definition, A is a fuzzy set with n elements i.e.*

$$A = \{(x_i, \mu_A(x_i)) : x_i \in X, i = 1, 2, 3, \cdots, n\},$$

where $\mu_A(x_i)$ is the membership function of the element x_i in A. From these n elements, consider a collection of r elements. Let i^{th} such combination is denoted by S_i^r and $\mu(S_i^r)$ is the membership degree of S_i^r. Then the r^{th} order entropy is defined as

$$H^r = \left(\frac{1}{\binom{n}{r}}\right)\sum_{i=1}^{\binom{n}{r}}\{\mu(S_i^r)\exp(1-\mu(S_i^r)) + (1-\mu(S_i^r))\exp(\mu(S_i^r))\}.$$

Pal and Pal (1992) [28] *also proposed the definition of higher order entropy in logarithmic version as*

$$H^r = \left(\frac{1}{\binom{n}{r}}\right)\sum_{i=1}^{\binom{n}{r}}\{\mu(S_i^r)\log(1-\mu(S_i^r)) + (1-\mu(S_i^r))\log(\mu(S_i^r))\}.$$

Definition 2.9. [28] *Pal and Pal proposed the idea of hybrid fuzzy entropy, which is defined as follows:*

$H_{hy} = p_0\exp(1-E_0) + p_1\exp(1-E_1)$ *(exponential form),*

$H_{hy} = -p_0\log(E_0) - p_1\log(E_1)$ *(logarithmic form),*

where $E_0 = \left(\frac{1}{n}\right)\sum_{i=1}^{n}(1-\mu_i)exp(\mu_i)$, $E_1 = \left(\frac{1}{n}\right)\sum_{i=1}^{n}\mu_i exp(1-\mu_i)$ and $p_0 \wedge p_1$ denotes the probabilities of occurrence of 0 and 1 respectively.

Definition 2.10. [43] *Zadeh first proposed the definition of possibility measure. Let ξ be a fuzzy variable and let the possibility distribution function of ξ be $\mu : R \to [0,1]$. Let r be any real number. Then for the event $\{\xi \geq r\}$ possibility measure is defined as*

$$Pos\{\xi \geq r\} = \underset{x\geq r}{Sup}\,\mu(x).$$

Definition 2.11. *From the idea of possibility measure we can generate the definition of necessity measure. Let ξ be a fuzzy variable and let the possibility distribution function of ξ be $\mu : R \to [0,1]$. Let r be any real number. Then for the event $\{\xi \geq r\}$ necessity measure is defined as*

$$Nec\{\xi \geq r\} = 1 - Pos\{\xi < r\} = 1 - \underset{x<r}{Sup}\,\mu(x).$$

Definition 2.12. *The idea of credibility measure was first proposed by Liu et al. (2002)* [19]. *Credibility measure corresponds to a unique pair of possibility and necessity measures defined as*

$$Cr\{\xi \in A\} = \frac{1}{2}\left(\underset{x\in A}{Sup}\,\mu(x) + 1 - \underset{x\in A^c}{Sup}\,\mu(x)\right),$$

where ξ is a fuzzy variable with membership function μ and A is any set of real numbers.

Definition 2.13. *: The idea of expected value using credibility measure was first proposed by Liu et al. (2002)* [19]*. If ξ be a fuzzy variable then expected value of ξ is defined as*

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}\, dr - \int_{-\infty}^0 Cr\{\xi \leq r\}\, dr.$$

Definition 2.14. *: Entropy of a fuzzy variable using the idea of credibility value was first proposed by Li et al. (2008)* [18]*. If ξ be a discrete fuzzy variable then the entropy value of its credibility distribution can be defined as*

$$H(\xi) = \sum_{i=1}^{n} S\left(Cr\{\xi = x_i\}\right),$$

Where $\{x_1, x_2, \cdots\cdots, x_n\}$ are different values taken by the variable ξ and $S(t) = -tln(t) - (1-t)ln(1-t)$.

Similarly if ξ be a continuous fuzzy variable then the entropy value of its credibility distribution can be defined as

$$H(\xi) = \int_{-\infty}^{\infty} S\left(Cr\{\xi = r\}\right) dr,$$

where $S(t) = -tln(t) - (1-t)ln(1-t)$.

Definition 2.15. [17] *Lad et al. first proposed the concept of probabilistic extropy as a dual of probabilistic entropy. Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ and $P(X = x_i) = p_i, i = 1, 2, 3, \ldots, n$ are the probability mass functions of every element of X. Then the extropy measure of X is defined as*

$$J(X) = -\sum_{i=1}^{n} (1 - p_i)\, log\ \ (1 - p_i)\,.$$

3 Fuzzy extropy

Inspired by the work of Pal and Pal (1989, 1992) [27, 28] and Lad et al. (2015) [17], in this section, we propose a new definition of fuzzy extropy, higher-order fuzzy extropy and hybrid fuzzy extropy. We also introduce the basic properties of fuzzy extropy and some results along with the proofs. The results are compared with different types of fuzzy entropy using the same example used by Pal and Pal (1992).

Definition 3.1. *Let $A = \{(x_i, \mu_A(x_i)) : x_i \in X, i = 1, 2, 3, \cdots, n\}$, where $\mu_i = \mu_A(x_i)$ is the membership function of the element x_i in A. Then fuzzy extropy can be defined as*

$$J(A) = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \left\{\mu_i \exp(\mu_i) + (1 - \mu_i) \exp(1 - \mu_i)\right\}.$$

It must be noted that $J(A)$ is a dual measure of $H(A)$ (Fuzzy entropy, Pal and Pal (1989)) [27] *as $J(A)$ shows dual nature to $H(A)$ but $J(A)$ must not be considered as the complement of $H(A)$. One can easily verify this using any arbitrary example.*

Next, we will give a few general properties which must be satisfied by any measure to be a fuzzy extropy measure, and these properties must have a dual nature towards the properties of fuzzy entropy measure.

3.1 Properties

Let $A = \{(x_i,\ \mu_A(x_i))\ : x_i \in X,\ i\ = 1,\ 2,\ 3, \cdots, n\}$, where $\mu_A(x_i)$ is the membership function of the element x_i in A and $J(A)$ be the fuzzy extropy measure of the set A. Then $J(A)$ must satisfy the following properties.

(i) $J(A)\ = 1$, when A is a crisp set.

(ii) $J(A)$ is minimum when A is a maximum fuzzy set.

(iii) $J(A^*) \geq\ J(A)$, where A^* is a sharpened version of A.

(iv) $J(A) = J(\overline{A}\ )$, where $\overline{A}$ denotes the complement of A.

We now show that our proposed extropy measure satisfies the above mentioned properties.

Proof of (i): If A is a crisp set then $\mu_i = 0$ or 1.

For $\mu_i = 0$, $J(A) = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{0 + 1 * e\} = 1$,

For $\mu_i = 1$, $J(A) = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{1 * e + 0\} = 1$.

Proof of (ii): Differentiating $J(A)$ with respect to μ_i we get

$$\begin{aligned}\frac{\partial J(A)}{\partial \mu_i} &= \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{\exp(\mu_i) + \mu_i \exp(\mu_i) - \ exp(1-\mu_i) - (1-\mu_i)\, exp(1- \ \mu_i)\} \\ &= \frac{1}{ne} \sum_{i=1}^{n} \{(1+\mu_i)\exp(\mu_i) + (-2+\mu_i)\exp(1-\mu_i)\}.\end{aligned}$$

For $\mu_i = \frac{1}{2}$,

$$\tfrac{\partial J(A)}{\partial \mu_i} = \left(\tfrac{1}{ne}\right) \sum_{i=1}^{n} \left\{\tfrac{3}{2}\exp(1/2) - \tfrac{3}{2} exp(1/2)\right\} = 0.$$

Now

$$\tfrac{\partial^2 J(A)}{\partial {\mu_i}^2} = \left(\tfrac{1}{ne}\right) \sum_{i=1}^{n} \{(2+\mu_i) exp(\mu_i) + (3-\mu_i)\, exp(1- \ \mu_i)\}.$$

For

$$\mu_i = \tfrac{1}{2},\ \tfrac{\partial^2 J(A)}{\partial {\mu_i}^2} = \left(\tfrac{1}{ne}\right) 5exp(1/2) \ > \ 0.$$

Therefore $J(A)$ is minimum if $\mu_i = \frac{1}{2}$.

Proof of (iii): Let A^* be a sharpened version of A. If $\mu_i \geq 0.5$, then $\mu_i \leq \widetilde{\mu}_i$ and if $\mu_i \leq 0.5$, then $\mu_i \geq \widetilde{\mu}_i$ where $\widetilde{\mu}_i = \mu_{A^*}(x_i)$ is the membership function of the element x_i of A^*.

Now from (1) $\frac{\partial J(A)}{\partial \mu_i} = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{(1+\mu_i) exp(\mu_i) + (-2+\mu_i)\, exp(1- \ \mu_i)\}$.

Now for $\mu_i = 0$, $\frac{\partial J(A)}{\partial \mu_i} = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{1 - 2e\} < 0$.

For $\mu_i = \frac{1}{2}$, $\frac{\partial J(A)}{\partial \mu_i} = 0$ and $\mu_i = 1$, $\frac{\partial J(A)}{\partial \mu_i} = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{2e - 1\} > 0$.

Since $J(A)$ is decreasing function of μ_i in $[0,\ 0.5)$ and increasing function of μ_i in $(0.5,\ 1]$, therefore $\mu_i \leq \widetilde{\mu}_i$ implies $J(A^*) \geq J(A)$ in $(0.5,\ 1]$ and $\mu_i \geq \widetilde{\mu}_i$ implies $J(A^*) \geq J(A)$ in $[0,\ 0.5)$.

Therefore $J(A^*) \geq J(A)$, where A^* is a sharpened version of A.

Proof of (iv): $J(\overline{A}) = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{\overline{\mu_i} \exp(\overline{\mu_i}) + (1-\overline{\mu_i}) \exp(1-\overline{\mu_i})\}$, where $\overline{\mu_i} = \mu_{\overline{A}}(x_i)$ is the membership function of the element x_i in $\overline{A}$ and $\overline{\mu_i} = 1 - \mu_i$. Therefore

$$\begin{aligned}J(\overline{A}) &= \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{(1-\mu_i)\exp(1-\mu_i) + \ (1-1+\ \mu_i)\exp(1-1+\mu_i)\} \\ &= \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \{\mu_i \exp(\mu_i) + \ (1- \ \mu_i)\exp(1-\mu_i)\} = J(A).\end{aligned}$$

Now here we define the higher order fuzzy extropy in the light of concept of higher order fuzzy entropy defined by Pal and Pal (1992) [28].

Definition 3.2. *Let A be a fuzzy set with n elements i.e. $A = \{(x_i,\ \mu_A(x_i)) \ : x_i \in X,\ i \ = 1,\ 2,\ 3, \cdots, n\}$, where $\mu_A(x_i)$ is the membership function of the element x_i of A. From these n elements, we consider a collection of r elements. Let i^{th} such combination is denoted by S_i^r and $\mu(S_i^r)$ be the membership degree of S_i^r. Then the r^{th} order extropy is defined as*

$J^r(A) = \left(\frac{1}{e\binom{n}{r}}\right) \sum_{i=1}^{\binom{n}{r}} \{\mu(S_i^r)\exp(\mu(S_i^r)) + \ (1- \ \mu(S_i^r))\exp(1-\mu(S_i^r))\}$.

$J^r(A)$ satisfies the following properties of fuzzy extropy.

(i) *$J^r(A) = 1$ when A is a crisp set.*

(ii) *$J^r(A)$ is minimum when A is a maximum fuzzy set.*

(iii) *$J^r(A^*) \geq J^r(A)$, where A^* is a sharpened version of A.*

(iv) $J^r(A) = J^r(\overline{A})$ *not holds in general where* $\overline{A}$ *denotes the complement of* A.

(v) $J^{r+1}(A) \geq J^r(A)$ *when* $\mu_i \in [0,\ 0.5)$ *and* $J^{r+1}(A) \leq J^r(A)$ *when* $\mu_i \in (0.5,\ 1]$.

Properties (i) – (iii) can be proved similarly to what we previously proved for fuzzy extropy. Property (iv) is generally invalid in higher-order fuzzy extropy like higher order fuzzy entropy. Property (v), in general, shows that $J^r(A)$ is a monotonically non-decreasing function in $[0,\ 0.5)$ and a monotonically non-increasing function in $(0.5,\ 1]$ (while considering a minimization operator to calculate the group membership value).

We now prove property (v).

Proof of (v): Let the fuzzy set for i^{th} combination S_i^r be denoted by A_i^r where

$$A_i^r = \left\{ \left(x_j^i,\ \mu_j^i\right) :\ x_j^i \in X, j = 1,\ 2,\ 3, \ldots, r,\ i\ =\ 1,\ 2,\ 3, \cdots,\ \binom{n}{r} \right\},$$

Where $x_j^i \in S_i^r$ and $\mu_j^i = \mu_j^i(x_j^i)$ is the degree of membership of the j^{th} element in the ith combination. Now let us consider $\mu(S_i^r) = \min\{\mu_j^i,\ j = 1,\ 2,\ 3, \ldots, r\}$. There exists $(n-r)$ subsets of A of size $(r+1)$ so that $S_i^r \subset S_i^{r+1}$. Let us denote these subsets by $S_{i,k}^{r+1}$, $k = 1,\ 2,\ \cdots,\ n-r$, i.e., $S_{i,k}^{r+1}$ is the k^{th} subset of A of size $(r+1)$ such that

$$S_i^r \subset S_{i,k}^{r+1}.$$

Now $S_i^r \subset S_{i,k}^{r+1} \Rightarrow \min\{\mu_1^i, \mu_2^i,\ \mu_3^i, \cdots,\ \mu_r^i\} \geq \min\{\mu_1^{i,k}, \mu_2^{i,k}, \mu_3^{i,k}, \cdots,\ \mu_r^{i,k}\} \Rightarrow \mu\left(S_i^r\right) \geq \mu\left(S_{i,k}^{r+1}\right),\ k = 1,\ 2,\ 3, \cdots, n-r.$

Thus, for every S_i^r, $i = 1,\ 2,\ \cdots,\ \binom{n}{r}$, there are $(n-r)$ subsets of size $(r+1)$ such that the above equation is true. Thus for every $\mu(S_i^r)$, there are $(n-r)$ times $\mu(S_i^{r+1})$ such that

$\mu(S_i^r) \geq \mu(S_{i,k}^{r+1})$, $k = 1,\ 2,\ 3, \cdots, n-r$.

Since from a set of size $(r+1)$ we can generate exactly $(r+1)$ distinct set of size r, we can infer that in the sequence $\mu\left(S_{i,k}^{r+1}\right),\ i = 1,\ 2,\ 3, \cdots,\ \binom{n}{r}$, $k = 1,\ 2,\ \cdots,\ n-r$ each of $\mu(S_i^{r+1})$, $i =\ 1,\ 2,\ \cdots,\ \binom{n}{r+1}$ has occurred exactly $(r+1)$ times.

Thus $(n-r)\sum_{i=1}^{\binom{n}{r}} \mu(S_i^r)\ \geq\ (r+1)\ \sum_{i=1}^{\binom{n}{r+1}} \mu(S_i^{r+1})$

$$\Rightarrow \frac{(n-r)\sum_{i=1}^{\binom{n}{r}} \mu(S_i^r)}{\binom{n}{r}} \geq \frac{(r+1)\ \sum_{i=1}^{\binom{n}{r+1}} \mu(S_i^{r+1})}{\binom{n}{r}}$$

$$\Rightarrow (n-r)\overline{\mu(S_i^r)}\ \geq (r+1)\frac{\sum_{i=1}^{\binom{n}{r+1}} \mu(S_i^{r+1})}{\binom{n}{r}}$$

$$\Rightarrow \overline{\mu(S_i^r)} \geq \frac{\sum_{i=1}^{\binom{n}{r+1}} \mu(S_i^{r+1})}{\binom{n}{r+1}}$$

$$\Rightarrow \overline{\mu(S_i^r)} \geq \overline{\mu(S_i^{r+1})}.$$

Thus, the average value of all $\mu(S_i^r)$ is not less than that of $\mu(S_i^{r+1})$.

Now

$$J(A)\ =\ \left(\frac{1}{ne}\right)\sum_{i=1}^{n}\{\mu_i \exp(\mu_i) +\ (1-\ \mu_i)\exp(1-\mu_i)\} =\ \left(\frac{1}{ne}\right)\sum_{i=1}^{n} f(\mu_i),$$

where $f(\mu_i) = \mu_i \exp(\mu_i) +\ (1-\ \mu_i)\exp(1-\mu_i)$. Now, differentiating $f(\mu_i)$ with respect to μ_i, we get

$$\frac{df(\mu_i)}{d\mu_i} = (1\ +\ \mu_i)\exp(\mu_i) + (-2+\mu_i)\exp(1-\ \mu_i)$$
$$\Rightarrow \frac{df(\mu_i)}{d\mu_i} < 0 \text{ for } 0 \leq \mu_i < 0.5,$$
$$> 0 \text{ for } 0.5 < \mu_i \leq 1,$$
$$= 0 \text{ for } \mu_i = 0.5.$$

Thus, $f(\mu_i)$ decreases monotonically with μ_i over $[0,\ 0.5)$, attains a minimum at 0.5, and then monotonically increases over $(0.5, 1]$. This fact and the above inequality establish the property (v).

Here we define hybrid fuzzy extropy in the light of the concept of hybrid fuzzy entropy (Pal and Pal (1992)) [28].

Definition 3.3. *Let A be a fuzzy set with n elements i.e., $A = \{(x_i,\ \mu_A(x_i))\ : x_i \in X,\ i\ = 1,\ 2,\ 3, \cdots, n\}$, where $\mu_A(x_i)$ is the membership function of the element x_i of A. Then hybrid fuzzy extropy can be defined as*

$$J_{hy}(A) = -\left(1 - p_0\right) \log\left(1 - E_0\right) - \left(1 - p_1\right) \log\left(1 - E_1\right), \tag{1}$$

where $E_0 = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \left(1 - \ \mu_i\right) exp(1 - \mu_i)$, $E_1 = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \mu_i exp(\mu_i)$ and p_0 and p_1 denote the probabilities of occurrence of 0 and 1 respectively.

Here we replicate those properties of hybrid fuzzy entropy (Pal and Pal (1992)) for hybrid fuzzy extropy.

Property 1: In the absence of fuzziness, when np_0 elements take the value zero (0) and np_1 elements take the value (1), then E_0 and E_1 reduce to p_0 and p_1, respectively, and Equation (2) boils down to classical probabilistic extropy which was defined by Lad et al. (2015). Thus we see that $J_{hy}(A)$ boils down to classical probabilistic extropy when a proper defuzzification process is applied to detect (restore) correctly the incoming symbols.

Property 2: If $\mu_i = 0.5$ for all $i = 1,\ 2,\ \cdots,\ n$, then $E_0 = E_1$ and Equation (2) takes the form

$$\begin{aligned} J_{hy}(A) &= -\left(1 - p_0\right) \log\left(1 - \left(\frac{1}{ne}\right) \sum_{i=1}^{n} 0.5 \exp(0.5)\right) - \left(1 - p_1\right) \log\left(1 - \left(\frac{1}{ne}\right) \sum_{i=1}^{n} 0.5 \exp(0.5)\right) \\ &= -\ log\left(1 - \ \left(\frac{1}{ne}\right) \sum_{i=1}^{n} 0.5 exp(0.5)\right). \end{aligned}$$

i.e., $J_{hy}(A)$ takes a constant value and becomes independent of p_0 and p_1.

Property 3: E_0 and E_1 cannot take any arbitrary values. They satisfy the following conditions:
$0.5\ exp(-0.5)p_1 \leq\ E_1 \leq 0.5\ exp(-0.5) + (1 - 0.5\ exp(-0.5))\ p_1$, and

$$0.5\ exp(-0.5)\ \ (1 - p_1) \leq\ \ E_0 \leq\ \ (1 -\ \ 0.5p_1)\ \ exp(-0.5).$$

Proof. $E_1 = \left(\frac{1}{ne}\right) \sum_{i=1}^{n} \mu_i exp(\mu_i)$, and $E_{1max} = \left(\frac{1}{ne}\right) [np_0\ 0.5\ exp(0.5)\ +\ np_1 e]$. Since E_1 is found to be monotonically increasing function of μ_i and the expected number of symbol 0 (1) is np_0 (np_1) and each of them can take a maximum value of 0.5 (1).

$$E_{1\max} = p_1 + \left(1 - p_1\right) 0.5 \exp(-0.5) = 0.5 \exp(-0.5) + \left(1 - 0.5 \exp(-0.5)\right) p_1, \tag{i}$$

and

$$E_{1\min} = \left(\frac{1}{ne}\right) [np_0 \cdot 0 + np_1 \cdot 0.5 \cdot \exp(-0.5)] = p_1 \cdot 0.5 \cdot \exp(-0.5). \tag{ii}$$

Combining (i) and (ii) we get

$$p_1 \cdot 0.5 \cdot exp(-0.5) \leq E_1 \leq 0.5\ exp(-0.5) + (1 - 0.5\ exp(-0.5))p_1.$$

Similarly

$$\begin{aligned} E_{0\max} &= \left(\frac{1}{ne}\right) [np_1 \cdot 1 + np_1 \cdot 0.5 \cdot \exp(-0.5)] \\ &= (1 - p_1) \cdot \exp(-0.5) + p_1 \cdot 0.5 \cdot \exp(-0.5) \\ &= \exp(-0.5) - 0.5 \cdot p_1 \cdot \exp(-0.5) \\ &= (1 - 0.5 \cdot p_1) \cdot \exp(-0.5). \end{aligned} \tag{iii}$$

and

$$E_{0\min} = \left(\frac{1}{ne}\right) [np_0 \cdot 0.5 \cdot \exp(-0.5) + np_1 \cdot 0] = p_0 \cdot 0.5 \cdot \exp(-0.5) = (1 - p_1) \cdot 0.5 \cdot \exp(-0.5). \tag{iv}$$

Combining (iii) and (iv), we get

$$0.5 \cdot exp(-0.5)\ (1 -\ \ p_1) \leq E_0 \leq (1 -\ \ 0.5p_1)\ \ exp(-0.5).$$

□

Property 4: $E_0 + E_1$ attains the minimum value of $exp(0.5)$ when all μ_i assume a value of 0.5.
Thus $E_0 + E_1 \geq\ exp(-0.5)$.

4 Extropy of credibility measure

In this section, we will define extropy of credibility measure both for discrete and continuous variables along with that we will try to state and prove few properties related to credibility extropy whose idea we received from Li et al. (2008) [18], where they proved few properties regarding credibilistic entropy.

4.1 Extropy of discrete credibility distribution

If ξ be a discrete fuzzy variable then the extropy value of its credibility distribution can be defined as

$$J(\xi) = \sum_{i=1}^{n} S\left(Cr\left\{\xi = x_i\right\}\right).$$

Where $\{x_1, x_2, \cdots, x_n\}$ are different values taken by the variable ξ and $S(t) = texp(t) + (1-t)\exp(1-t)$.

We now state and prove few properties related to extropy of credibility value.

Property 1

Let ξ and η be two discrete fuzzy variables that takes values in $\{x_1, x_2, \cdots, x_n\}$ with possibilities $\{\mu_1, \mu_2, \cdots, \mu_n\}$, and $\{\nu_1, \nu_2, \cdots, \nu_n\}$, respectively, where $\{\nu_1, \nu_2, \cdots, \nu_n\}$ is a rearrangement of the sequence $\{\mu_1, \mu_2, \cdots, \mu_n\}$. Then, we have $J(\xi) = J(\eta)$.

Proof. The proof is obvious. □

We now state a lemma without proof which is necessary to establish our next property.

Lemma 4.1. *Let ξ be a discrete fuzzy variable taking values in $\{x_1, x_2, \cdots, x_n\}$. If there exists an index p such that $Cr\left\{\xi = x_p\right\} = 1$, then we have $Cr\left\{\xi = x_i\right\} = 0,\ \forall i \neq p$*

Property 2

Let ξ be a discrete fuzzy variable that takes values in $\{x_1, x_2, \cdots\cdots, x_n\}$. Then

$$0 < J(\xi) \leq ne,$$

and equality holds if and only if there exists an index p such that the membership function of ξ is

$$\mu\left(x_i\right) = \begin{cases} 1, & \text{if } i = p \\ 0, & \text{otherwise} \end{cases}$$

i.e. ξ is obviously a crisp number.

Proof: It is obvious from the definition of $J(\xi)$ that the value of $J(\xi)$ is always strictly greater than 0. Also from definition it is true that $S(t)$ attains its maximum value e at $t = 0$ *or* 1, thus we have

$$J(\xi) = \sum_{i=1}^{n} S\left(Cr\left\{\xi = x_i\right\}\right) \leq ne,$$

and equality holds if and only if $Cr\left\{\xi = x_i\right\} = 0$ *or* 1 for each i. Now by above lemma there exists only one index p such that $Cr\left\{\xi = x_i\right\} = 1$ i.e. the membership function of ξ is

$$\mu\left(x_i\right) = \begin{cases} 1, & \text{if } i = p \\ 0, & \text{otherwise} \end{cases}$$

Conversely if ξ is a crisp number, it has the membership function as the previous form. It is easy to verify that the extropy $J(\xi) = ne$.

Property 3

Let ξ be a discrete fuzzy variable that takes values in $\{x_1, x_2, \cdots\cdots, x_n\}$ with possibilities$\{\mu_1, \mu_2, \cdots\cdots, \mu_n\}$, and let η be a discrete fuzzy variable taking values in $\{x_1, x_2, \cdots\cdots, x_n, x_{n+1}\}$ with possibilities $\{\mu_1, \mu_2, \cdots\cdots, \mu_n, \mu_{n+1}\}$. Then

$$J(\xi) < J(\eta).$$

Proof. The property directly follows from the definition of extropy function. □

Property 4

Let ξ and η be two discrete fuzzy variables that takes values in $\{x_1, x_2, \cdots\cdots, x_n\}$ with possibilities $\{\mu_1, \mu_2, \cdots\cdots, \mu_n\}$, and $\{\nu_1, \nu_2, \cdots\cdots, \nu_n\}$, respectively. Then if $\mu_i \leq \nu_i$ *for all* $i = 1, 2, \ldots\ldots, n$, we have

$$J(\xi) \geq J(\eta).$$

Furthermore, if there exists an index p such that $\mu_p < \nu_p$, then we have

$$J(\xi) > J(\eta).$$

Proof. Let $c_i = Cr\{\xi = x_i\} = \frac{1}{2}(\mu_i + 1 - \max_{j\neq i}\mu_j)$ and $c_i^{'} = Cr\{\eta = x_i\} = \frac{1}{2}(\nu_i + 1 - \max_{j\neq i}\nu_j)$ for $i = 1, 2, \ldots\ldots, n$ respectively. We may have the following two cases.

Case 1. If $\mu_i = 1$. In this case we have $\nu_i = 1$ and $\max_{j\neq i}\mu_j \leq \max_{j\neq i}\nu_j$. Then $1 - \max_{j\neq i}\mu_j \geq 1 - \max_{j\neq i}\nu_j$ and $0.5 \leq \frac{1}{2}(\nu_i + 1 - \max_{j\neq i}\nu_j) \leq \frac{1}{2}(\mu_i + 1 - \max_{j\neq i}\mu_j)$. That is, $0.5 \leq c_i^{'} \leq c_i$, then $S(c_i) \geq S(c_i^{'})$ holds.

Case 2. If $\mu_i < 1$. In this case there exists an index $p \neq i$ such that $\mu_p = 1$ and $\nu_p = 1$ due to normalization, i.e. $\max_{1\leq i\leq n}\mu_i = \max_{1\leq i\leq n}\nu_i = 1$. Then we have $c_i = \frac{\mu_i}{2}$ and $c_i^{'} = \frac{\nu_i}{2}$. Also $c_i \leq c_i^{'} \leq 0.5$ and then the inequality $S(c_i) \geq S(c_i^{'})$ holds. Now we have $S(c_i) \geq S(c_i^{'})$ for all $i = 1, 2, \ldots\ldots, n$ and thus $J(\xi) = \sum_{i=1}^{n} S(c_i) \geq \sum_{i=1}^{n} S\left(c_i^{'}\right) = J(\eta)$.

Furthermore, if there exists an index p such that $\mu_p < \nu_p$ holds then from the above two cases we must have $S(c_i) \geq S(c_i^{'})$ and thus $J(\xi) > J(\eta)$. □

Property 5

Let ξ be a discrete fuzzy variable that takes values in $\{x_1, x_2, \cdots\cdots, x_n\}$ with possibilities $\{\mu_1, \mu_2, \cdots\cdots, \mu_n\}$, and let η be a discrete fuzzy variable taking values in $\{x_1, x_2, \cdots\cdots, x_n\}$ with possibilities $\{\nu_1, \nu_2, \cdots\cdots, \nu_n\}$. Then

$$\lambda J(\xi) + (1-\lambda)J(\eta) < \sum_{i=1}^{n} S\left(\lambda c_i + (1-\lambda)c_i^{'}\right) \ \forall\ \lambda \in (0,1),$$

where $c_i = Cr\{\xi = x_i\}$ and $c_i^{'} = Cr\{\eta = x_i\}$ *for* $i = 1, 2, \ldots\ldots, n$ respectively.

Proof. Since S(t) is a strictly convex function the proof of above theorem is obvious. □

4.1 Example 1

Let us consider the weight of Amit is almost 70 kg and the term "almost 70 kg" in this case can be written in terms of fuzzy set as {(64, 0.1), (66, 0.5), (68, 0.8), (70, 1), (72, 0.8), (74, 0.5), (76, 0.1)}, where the integer value represents the possible value of the occurrence and the compatibility degree between the weight 64 kg and the term "almostt 70 kg" is 0.1. Let ξ be the possible weight of Amit. Thus, ξ is a simple fuzzy variable which takes values in {64, 66, 68, 70, 72, 74, 76}. Computing the credibility distribution of ξ , we get the result {0.05, 0.25, 0.4, 0.6, 0.4, 0.25, 0.05}. Thus the extropy J(ξ) = 13.9055(approx). Since the maximum extropy of a credibility distribution with seven elements is 7e. We can normalize the result as

$$J^*(\xi) = \frac{J(\xi)}{7e} = .7308.$$

Here e is the Euler's number lying between 2 and 3. Now we will define the extropy of continuous credibility distribution and will state and prove few properties.

4.2 Extropy of continuous credibility distribution

There may be some instances where the membership function of fuzzy variable is continuous function over the set of real numbers. To deal with these types of situations we need the idea of extropy of continuous credibility distributions. If ξ be a continuous fuzzy variable then the extropy value of its credibility distribution can be defined as

$$J(\xi) = \int_{-\infty}^{\infty} S(Cr\{\xi = r\})\, dr.$$

Where $S(t) = texp(t) + (1-t)\exp(1-t)$. We now state and prove few properties related to extropy of continuous credibility distributions.

Property 1

Let ξ be a continuous fuzzy variable that takes values in the interval [a, b]. Then

$$0 < J(\xi) \leq (b-a)e.$$

Proof. The proof is obvious. □

Property 2

Let ξ and η be two continuous fuzzy variables whose membership functions are $\mu(x)$ and $\nu(x)$ respectively. If $\mu(x) \leq \nu(x)\ \forall\ x \in \mathbb{R}$, then

$$J(\xi) \geq J(\eta).$$

Furthermore, if there exists finite extropy values of the variables then there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) < \nu(x_0)$, we have $J(\xi) > J(\eta)$.

Proof. As $\mu(x) \leq \nu(x)\ \forall\ x \in \mathbb{R}$, then $Cr\{\xi = x\} \leq Cr\{\eta = x\} \leq 0.5$. Therefore $S(Cr\{\xi = x\}) \geq S(Cr\{\eta = x\})$ this implies that

$$J(\xi) \geq J(\eta).$$

Further if $J(\xi)$ *and* $J(\eta)$ are finite and there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) < \nu(x_0)$, then there must be an open interval (a, b) that contains x_0 such that $\mu(x) < \nu(x)\ \forall\ x \in (a,\ b)$. Therefore $S(Cr\{\xi = x\}) > S(Cr\{\eta = x\}),\ \forall\ x \in (a,\ b)$ this implies that $\int_a^b S(Cr\{\xi = x\})\,dx > \int_a^b S(Cr\{\eta = x\})\,dx$ This implies $J(\xi) > J(\eta)$. □

5 Illustrative example

The primary focus of this section is to demonstrate the behavior of extropy-based measures in fuzzy environments. The examples, provided here, are crucial for showing how extropy and its fuzzy extensions, such as fuzzy hybrid extropy and higher order fuzzy extropy, capture the uncertainty and variability of a system in ways that traditional entropy measures might not. For instance, the fluctuations observed in the measure $J(A)$ reveal a better understanding of the system's unpredictability compared to standard entropy measures like $H(A)$. By analyzing different measures such as E_0, E_1, $J(A)$, and $J_{hy}(A)$, we also illustrate how the fuzzy extropy measures behave in response to noise, offering a clearer perspective of the behaviour of these measures. By these examples, we emphasize the relevance of extropy in capturing uncertainties within fuzzy systems and lay the foundation for further studies on its application across other areas, such as fuzzy approximation and decision-making models.

5.1 Example of higher order extropy

To demonstrate higher order extropy we considered the example of Pal and Pal (1992) [28]. In this example, they considered a group of 10 players and the problem is to find the stability of a team of 5 players from the group of 10 players in terms of fuzzy uncertainty. They calculated the uncertainty in terms of fuzzy entropy and higher order fuzzy entropy. We calculated the uncertainty in terms of fuzzy extropy and higher order fuzzy extropy, which we presented in Table 1. Based on Table 1, the following observations can be made.

The values of J^1 to J^6 are the same when all the μ_i values are the same. Also, it is very clear that the values of J^r are dependent on the values of μ_i as in case 1 of Table 1, when all $\mu_i$values are 1, then J^r values are highest and in case 2 when all μ_i values are 0.5 then J^r values are the smallest which is just opposite to that of entropy results described by Pal and Pal (1992) [28].

In case 3 of Table 1, the difference of J^1 and J^2 is very high, which shows that singleton of fuzzy sets are highly dissimilar with respect to some fuzzy characteristics, whereas in case 4 of Table 1, difference of J^1 and J^2 is much less which shows that singleton of fuzzy sets are more similar with respect to the same fuzzy characteristics which is similar to that of the results described by Pal and Pal (1992).

Depending on the above observation we can define extropy similarity index $S' = \frac{J^1}{J^2}$ (when $J^1 = J^2 = 0$ we will consider S' as 1) when $\mu_i \in [0, 0.5]$ so that S' lies in [0, 1] but when $\mu_i \in [0.5, 1]$ we define $S' = \frac{J^2}{J^1}$ as in [0.5, 1] $J^r \geq J^{r+1}$ and due to this adjustment S' will lie in [0, 1].

In case 5 and case 6 of Table 1, J^1 values are exactly the same, whereas the extropy similarity index is higher for case 6 than that of case 5 which confirms that case 6 has more similar elements than case 5.

Table 1: Higher order extropy

Case	μ	J^1	J^2	J^3	J^4	J^5	J^6	$S^{'}$
1	1, 1, 1, 1, 1, 1, 1, 1, 1, 1	1	1	1	1	1	1	1
2	0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5	0.6065	0.6065	0.6065	0.6065	0.6065	0.6065	1
3	1, 1, 1, 1, 1, 0.5, 0.5, 0.5, 0.5, 0.5	0.8033	0.6940	0.6393	0.6159	0.6081	0.6065	0.8639
4	0.5, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.6, 0.6, 0.6	0.6141	0.6099	0.6078	0.6069	0.6066	0.6065	0.9932
5	0.6, 0.6, 0.65, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.915	0.7889	0.7336	0.6931	0.6638	0.6438	0.6314	0.9299
6	0.8, 0.8, 0.8, 0.8, 0.8, 0.8, 0.9, 0.9, 0.9, 0.9	0.7889	0.7595	0.7485	0.7454	0.7449	0.7449	0.9627
7	0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.9, 0.9, 0.9, 0.9	0.7059	0.6397	0.6148	0.6077	0.6065	0.6065	0.9062
8	0.7, 0.7, 0.7, 0.7, 0.7, 0.8, 0.8, 0.8, 0.8, 0.8	0.7062	0.6847	0.6740	0.6694	0.6679	0.6675	0.9696

In case 7 and case 8 the values of J^1 are almost similar but the values are lesser than the values of J^1 for case 5 and case 6. So we can consider that the fuzzy sets of case 7 and 8 shows less fuzziness than case 5 and 6 in [0.5, 1]. Also the extropy similarity index of case 8 is more than case 7 so we can also conclude that the elements of the fuzzy set of case 8 are more alike than the elements of the fuzzy set of case 7. This result also matches the results of Pal and Pal (1992).

From the above discussions, we can conclude that fuzzy extropy can only indicate the fuzziness of any set. It can be concluded whether the fuzziness is high or not. Besides that, if we incorporate the values of J^r, then it will help us to measure the similarities between the elements of any given fuzzy set. Also, the extropy similarity index will be helpful to measure intraclass and interclass ambiguity of any image processing and pattern recognition problems similar to Pal and Pal (1992).

5.2 5.2 Example of hybrid extropy

We examine the instance given in Pal and Pal (1992) [28], where they used hybrid entropy in image processing. Assuming the same probability of white pixel and dark pixel and using the same fuzzy set, we calculated values of E_0, E_1, $J_{hy}(A)$, $J(A)$, $|J(A) - J_{hy}(A)|$ and presented in Table 2.

Table 2: Hybrid fuzzy extropy with $p_0 = 0.3$ and $p_1 = 0.7$

Case	μ_A	E_0	E_1	$J_{hy}(A)$	J(A)	$\lvert J(A) - J_{hy}(A)\rvert$
1	(0.9, 0.8, 0.8, 0.8, 0.8, 0.7, 0.6, 0.45, 0.4, 0.3)	0.204008	0.498318	0.366653	0.702327	0.335673
2	(0.99, 0.9, 0.9, 0.9, 0.8, 0.79, 0.7, 0.2, 0.1, 0.05)	0.283284	0.538699	0.465264	0.821983	0.356719
3	(0.999, 0.99, 0.99, 0.99, 0.98, 0.9, 0.9, 0.1, 0.05, 0)	0.281836	0.658775	0.554303	0.94061	0.386307
4	(1, 1, 1, 1, 1, 1, 1, 0.0004, 0, 0)	0.29992	0.700015	0.610799	0.999935	0.389136
5	(1, 1, 1, 1, 1, 1, 1, 0, 0, 0)	0.3	0.7	0.610864	1	0.389136

Table 2 shows that E_0 approaches 0.3 and E_1 approaches 0.7, as Pal and Pal (1992) [28]. Also, we can observe that both $J_{hy}(A)$ and $J(A)$ increase monotonically and $|J(A)- J_{hy}(A)|$ becomes an increasing function opposite to $|H - H_{hy}|$, which also establishes the duality nature of fuzzy extropy concerning fuzzy entropy. In Table 3 we presented the hybrid extropy values of wrong defuzzification and this example was also present in the paper of Pal and Pal (1992) [28]. From both Table 2 and 3 we can conclude that $|J(A) - J_{hy}(A)|$ can be considered as objective criterion for image contrast enhancement as this value is always non-zero in any situation which matches the conclusion of Pal and Pal (1992) [28].

Table 3: Some wrong defuzzifications ($P_0 = 0.3, P_1 = 0.7$)

Case	μ	E_0	E_1	$J(A)$	$J_{hy}(A)$	$\lvert J(A) - J_{hy}(A)\rvert$
1	0.9, 0.9, 0.8, 0.8, 0.7, 0.6, 0.5, 0.5, 0.4, 0.3	0.2157	0.4834	0.6991	0.3682	0.3309
2	0.999, 0.999, 0.9, 0.8, 0.7, 0.7, 0.3, 0.3, 0.2, 0.1	0.2936	0.4931	0.7867	0.4471	0.3396
3	1, 1, 1, 0.99, 0.9, 0.9, 0.1, 0.1, 0, 0	0.3714	0.5690	0.9404	0.5775	0.3629
4	1, 1, 1, 1, 1, 1, 0, 0, 0, 0	0.4	0.6	1	0.6325	0.3675
5	0.99, 0.99, 0.1, 0.1, 0.9, 0.8, 0.7, 0.2, 0.1, 0.1	0.4199	0.4201	0.8400	0.5447	0.2953
6	1, 1, 0, 0, 1, 1, 1, 0, 0, 0	0.5	0.5	1	0.6931	0.3069

5.3 Simulation study

The input instances provided in the article by Pal and Pal (1992) [28] are utilized to simulate the proposed work. The input instances from the same article are augmented through (i) the addition of 10% Gaussian noise and (ii) a rotation and scaling method. Both methods yielded 50 instances each, resulting in a total dataset of 100 instances. By augmenting their instances with noise, rotation, and scaling, the simulation aims to analyse the behaviour of the proposed extropy measures in real-world conditions (such as image processing with noise). This highlights the adaptability and practicality of these measures for real-world applications, especially in fields like image processing, where noisy or altered data is present. The coding is executed in a Python environment, specifically using Python 3.11.2, on a machine equipped with an Intel i7 processor and 16 GB of RAM. The open-source software libraries Numpy and Pandas, dedicated to machine learning and artificial intelligence, are employed for implementing the proposed model.

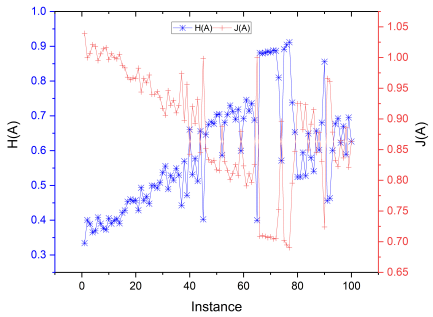


Figure 1: $H(A)$ and $J(A)$

Figure 1 compares two parameters, $H(A)$ and $J(A)$ across various instances. The $H(A)$ demonstrates a steady increase from 0.33 to approximately 0.7 at the 40^{th} instance. After the 50^{th} instance, $H(A)$ shows moderate fluctuations between 0.7 and 1. This indicates that the entropy represented by $H(A)$ increases significantly in the initial phase and then stabilizes with moderate variability. In contrast, the $J(A)$ remains relatively high during the initial 20 instances. Post the 40^{th} instance, $J(A)$ exhibits pronounced variability, oscillating more broadly between approximately 0.7 and 0.9. After the 80^{th} instance, the values of $J(A)$ once again become higher. This suggests that $J(A)$ maintains higher values when $H(A)$ is either on the lower or higher side. This comparative analysis provides an essential understanding of how extropy behaves differently than entropy. While $H(A)$ demonstrates an increase in its value, $J(A)$ decreases with less variability across instances. This fluctuation of $J(A)$ reveals the system's uncertainty and behaviour in a way that differ from those provided by $H(A)$. The extropy analysis thus adds depth to our understanding of uncertainty in fuzzy systems. In summary, the comparative analysis provides valuable insights into the behavior of the measures represented by $H(A)$ and $J(A)$.

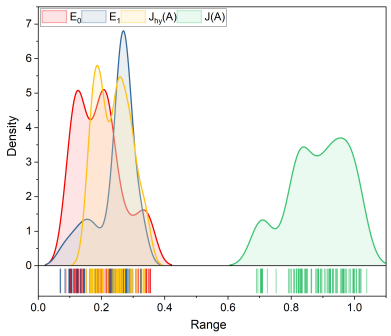


Figure 2: Density Distribution

Figure 2 illustrates the density distributions of four different variables: E_0, E_1, $J_{hy}(A)$, and $J(A)$. These variables are related to the concepts of fuzzy extropy and fuzzy hybrid extropy. Both E_0 and E_1 demonstrate similar variations but with distinct peaks. E_1, in particular, exhibits a greater mean compared to that of E_0, indicating that E_1 consistently

maintains higher values than those observed in E_0. Furthermore, E_1 features a higher peak than that of E_0, suggesting a more concentrated distribution of points. Consequently, E_1 signifies a higher level of consistency than E_0. $J_{hy}(A)$ has a lower peak density compared to E_1 but follows a similar pattern. This might suggest that the hybrid fuzzy extropy, as represented by $J_{hy}(A)$, exhibits a certain level of predictability, with a lesser range of values. $J(A)$ exhibits a distinct behavior with its peak shifted towards the higher end of the range and has a slightly broader distribution indicating more variability in the data points. This could suggest that the system's uncertainty as measured by $J(A)$ is more variable and less predictable. Figure 2 reveals that while E_0 and E_1 show similar distributions, the extropy measures ($J(A)$ and $J_{hy}(A)$) capture distinct behaviors, with $J(A)$ showing greater variability. The variability of $J(A)$ makes it an alternative measure for capturing the uncertainty differently in complex systems, especially when considering fuzzy information measures. This highlights the relevance of extropy in systems where the entropy may not capture the full variability and unpredictability present in a system. In summary, the analysis of these density distributions provides valuable insights into the behavior of the system's uncertainty as measured by fuzzy extropy and hybrid fuzzy extropy.

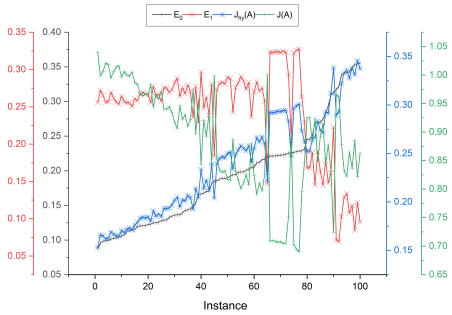


Figure 3: E_0, E_1, $J_{hy}(A)$, and $J(A)$

Figure 3 illustrates the density distributions of four different variables: E_0, E_1, $J_{hy}(A)$, and $J(A)$. The E_0 line (black) initiates at approximately 0.05 and exhibits an upward trend, reaching a value of 0.35. This suggests a potential increase in the E_0 value over instances. The E_1 line (red) fluctuates between approximately 0.25 and 0.35 until instance number 60, where it experiences a sharp drop to 0.05 in subsequent instances. This indicates that the E_1 value is variable and somewhat inversely related to E_0. The $J_{hy}(A)$ line (blue) steadily increases with the growing number of instances. This may suggest lower volatility in the $J_{hy}(A)$ values within the range of 0.08 to 0.35. The $J(A)$ line (green) starts from a value of 1. Around instance number 60, it exhibits high fluctuations before stabilizing at 0.8 towards instance number 100. This implies that $J(A)$ values remain within a range from 0.7 to 1. Figure 3 shows that extropy and hybrid extropy behave differently from E_0 and E_1. Notably, $J_{hy}(A)$ increases steadily, indicating less volatility compared to the higher fluctuations in $J(A)$. This demonstrates that hybrid extropy showing more stability than extropy alone, making these measures suitable for analyzing behaviour of systems having different volatility. These extropy measures provide the insights that traditional measures like E_0 and E_1 do not depict, which makes the extropy measures valuable for capturing uncertainty in dynamic, fuzzy environments.

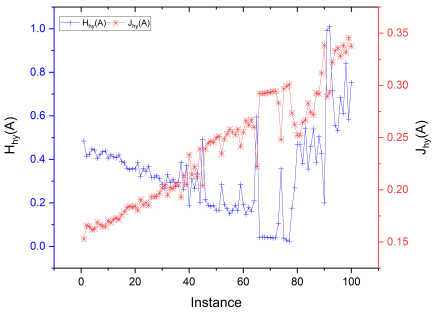


Figure 4: $H_{hy}(A)$ and $J_{hy}(A)$

Figure 4 illustrates two measures, $H_{hy}(A)$ and $J_{hy}(A)$, represented by blue squares and red asterisks respectively. The $H_{hy}(A)$ dataset shows a decreasing trend from 0.5 to approximately 0 as the instance increases from 0 to around 70. After this point, there is an increase with significant fluctuations reaching up to about 0.8 at instance 100. The $J_{hy}(A)$ dataset starts at a lower value of around 0.15 and increases sharply with lesser fluctuations, reaching its peak of approximately 0.35 at instance 100. Both measures exhibit significant variability, indicating specific underlying characteristics of the measures. This analysis of hybrid entropy and extropy measures further underscores their relevance. The behavior of $H_{hy}(A)$ and $J_{hy}(A)$ reflects changes over the instances, with significant variability observed in both measures. This suggests that hybrid extropy, as represented by $J_{hy}(A)$, offers an alternative way to understand the variability and uncertainty in systems. The analysis provided by these examples helps establish the dual nature of fuzzy entropy and extropy, revealing their complementary roles in measuring uncertainty in fuzzy systems.

The proposed fuzzy extropy measure has numerous real-world applications across diverse domains due to its ability to measure uncertainty and information content. In machine learning, it can be used for feature selection and calculating information gain in decision trees, helping improve classification and regression tasks. In clustering, fuzzy extropy can also evaluates the homogeneity of clusters. It is essential in image processing tasks like segmentation, as low fuzzy extropy regions can indicate complexity or noise. In anomaly detection, fuzzy extropy helps identify outliers by detecting irregularities in data distributions. In cryptography, fuzzy extropy can be used to assess the randomness and security of data. In compression algorithms like Huffman coding, fuzzy extropy optimizes data storage by assigning shorter codes to frequent elements. It's also applied in financial markets to measure volatility and market unpredictability. In natural language processing, fuzzy extropy helps gauge text complexity and word sequence predictability. Additionally, this measure plays a critical role in bioinformatics by analyzing the complexity of biological sequences like Deoxyribonucleic Acid (DNA). The versatility of the proposed measure extends to fields like network traffic analysis, speech recognition, and even quantum computing for assessing quantum state complexity.

6 Conclusion

The fuzzy extropy measure and higher order fuzzy extropy measure are defined in this paper. Since the classical set is a particular case of the fuzzy set so we also define hybrid fuzzy extropy measure that takes into account also both probability of occurrence of an event and the membership degree of that event and in some circumstances this measure coincides with probabilistic extropy measure defined by Lad et al. (2015) [17]. We also intend to define credibility based extropy measure and state few important properties. From the higher order extropy, we can conclude that instead of using only fuzzy extropy it will be a far better approach to use fuzzy extropy and extropy similarity index to measure the uncertainty of any system. Using only fuzzy extropy may present a challenge that in case of two different types of fuzzy set may have same (cases 5 and 6 of Table 1) or almost similar extropy (cases 7 and 8 of Table 1) and to create better decision in these instances we may use extropy similarity index also. In the illustrated example of hybrid extropy measure, we see that the value increases steadily whenever we consider sharpened versions. The simulation study conducted further supports our findings, demonstrating the effectiveness of fuzzy extropy and hybrid extropy measures. Finally, both converge towards their probability values. Also from Tables 2 and 3 it is also very clear that $|J(A) - J_{hy}(A)|$ or $J_{hy}(A)$ can be used as an objective measure for defuzzification. In the future, many further studies can be conducted using fuzzy extropy measure. This measure has a very broad range of applications, including reliability theory, multicriteria decision making, portfolio optimization and many more. In addition, study areas that employ the notion of fuzzy entropy may also use the concept of fuzzy extropy. The applicability of our measures can be extended to include the field of fuzzy approximation operators. It emphasizes the versatility of the proposed fuzzy extropy measure in broader mathematical contexts. This shows the practical implications of our work for future research It can create research opportunities in many different areas, also in the development of new operators informed by fuzzy extropy. Thus, the proposed measure enhances the measure's relevance to mathematical and applied disciplines. This expanded perspective not only validates the importance of our proposed work but also paves the way for future studies in both theoretical and applied fuzzy system research.

Acknowledgement

We sincerely appreciate the suggestions for improvements made by the referees. It was a pleasure to contribute to the refinement of the paper. We are also glad that these suggestions were helpful in enhancing its quality.

References

[1] V. Arya, S. Kumar, *Knowledge measure and entropy: A complementary concept in fuzzy theory*, Granular Computing, **6**(3) (2021), 631-643. https://doi.org/10.1007/s41066-020-00221-7

[2] N. Balakrishnan, F. Buono, M. Longobardi, *On weighted extropies*, Communications in Statistics Theory and Methods, **51**(18) (2022), 6250-6267. https://doi.org/10.1080/03610926.2020.1860222

[3] N. Balakrishnan, F. Buono, M. Longobardi, *On Tsallis extropy with an application to pattern recognition*, Statistics and Probability Letters, **180** (2022), 109241. https://doi.org/10.1016/j.spl.2021.109241

[4] F. Buono, M. Longobardi, *A dual measure of uncertainty: The deng extropy*, Entropy, **22** (2020), 582. https://doi.org/10.3390/e22050582

[5] L. Chen, Y. Deng, *Entropy of random permutation set*, Communications in Statistics-Theory and Methods, **53**(11) (2024), 4127-4146. https://doi.org/10.1080/03610926.2023.2173975

[6] A. Deluca, S. Tennini, *A definition of nonprobabilistic entropy in the setting of fuzzy sets theory*, Information and Control, **20** (1972), 301-312. https://doi.org/10.1016/B978-1-4832-1450-4.50020-1

[7] A. P. Dempster, *Upper and lower probabilities induced by a multivalued mapping*, In Classic Works of the Dempster-Shafer Theory of Belief Functions, (2008), 57-72. https://doi.org/10.1007/978-3-540-44792-4_3.

[8] Y. Deng, *Deng entropy*, Chaos, Solitons and Fractals, **91** (2016), 549-53. https://doi.org/10.1016/j.chaos.2016.07.014

[9] Y. Deng, *Information volume of mass function*, International Journal of Computers Communications and Control, **15**(6) (2020), 3983. https://doi.org/10.15837/ijccc.2020.6.3983

[10] D. Dubois, H. Prade, *A class of fuzzy measures based on triangular norms*, International Journal of General Systems, **8** (1982), 43-61. https://doi.org/10.1080/03081078208934833

[11] D. Dubois, H. Prade, *Théorie des Possibilités*, Paris, France: Masson, (1985). https://hal.science/hal-04058264

[12] N. Ebrahimi, *How to measure uncertainty in the residual lifetime distribution*, Sankhyá: The Indian Journal of Statistics, **58** (1996), 48-56. https://www.jstor.org/stable/i25051076

[13] O. Kamari, F. Buono, *On extropy of past lifetime distribution*, Ricerche di Matematica, **70** (2021), 505-515. https://doi.org/10.1007/s11587-020-00488-7

[14] A. Kaufmann, *Introduction to the theory of fuzzy subsets-fundamental theoretical elements*, Academic, New York, **1** (1975). https://archive.org/details/introductiontoth0000kauf

[15] M. R. Kazemi, S. Tahmasebi, F. Buono, M. Longobardi, *Fractional deng entropy and extropy and some applications*, Entropy, **23**(5) (2021), 623. https://doi.org/10.3390/e23050623

[16] A. S. Krishnan, S. M. Sunoj, N. U. Nair, *Some reliability properties of extropy for residual and past lifetime random variables*, Journal of the Korean Statistical Society, **49** (2020), 457-474. https://doi.org/10.1007/s42952-019-00023-x

[17] F. Lad, G. Sanfilippo, G. Agró, *Extropy: Complementary dual of entropy*, Statistical Science, **30**(1) (2015), 40-58. https://doi.org/10.1214/14-STS430

[18] P. Li, B. Liu, *Entropy of credibility distributions for fuzzy variables*, IEEE Transactions on Fuzzy Systems, **16** (2008), 123-129. https://doi.org/10.1109/TFUZZ.2007.894975

[19] B. Liu, Y. K. Liu, *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Transactions on Fuzzy Systems, **10**(4) (2002), 445-450. https://doi.org/10.1109/TFUZZ.2002.800692

[20] J. Liu, F. Xiao, *Information volume of mass function based on extropy*, Soft Computing, **26**(5) (2022), 2409-2418. https://doi.org/10.1007/s00500-021-06410-z

[21] J. Liu, F. Xiao, *Rényi extropy*, Communications in Statistics-Theory and Methods, **52**(16) (2023), 5836-5847. https://www.tandfonline.com/toc/lsta20/52/16

[22] S. Nahmias, *Fuzzy variables*, Fuzzy Sets and Systems, **1** (1978), 97-110. https://doi.org/10.1016/0165-0114(78)90011-8

[23] E. Y. Ozkan, *Inequalities for approximation of new defined fuzzy post-quantum Bernstein polynomials via interval-valued fuzzy numbers*, Symmetry, **14** (2022), 696. https://doi.org/10.3390/sym14040696

[24] E. Y. Ozkan, *Approximation by fuzzy* (p, q)*-Bernstein-Chlodowsky operators*, Sahand Communications in Mathematical Analysis, **19**(2) (2022), 113-132. https://doi.org/10.22130/scma.2022.524506.910

[25] E. Y. Ozkan, B. Hazarika, *Approximation results by fuzzy Bernstein type rational functions via intervalvalued fuzzy number*, Soft Computing, **27** (2023), 6893-6901. https://doi.org/10.1007/s00500-023-08013-2

[26] S. K. Pal, B. Chakraborty, *Fuzzy set theoretic measure for automatic feature evaluation*, IEEE Transactions on Systems, Man, and Cybernetics, SMC, **16**(5) (1986), 754-760. http://hdl.handle.net/10263/921

[27] N. R. Pal, S. K. Pal, *Object background segmentation using a new definition of entropy*, IEE Proceedings E (Computers and Digital Techniques), **136** (1989), 284-295. https://repository.ias.ac.in/26062/

[28] N. R. Pal, S. K. Pal, *Higher order fuzzy entropy and hybrid entropy of a set*, Information Sciences, **61** (1992), 211-231. https://doi.org/10.1016/0020-0255(92)90051-9

[29] L. Pan, Y. Deng, *A new belief entropy to measure uncertainty of basic probability assignments based on belief function and plausibility function*, Entropy, **20**(11) (2018), 842. https://doi.org/10.3390/e20110842

[30] G. Qiu, *The extropy of order statistics and record values*, Statistics and Probability Letters, **120** (2017), 52-60. https://doi.org/10.1016/j.spl.2016.09.016

[31] A. Rényi, *On measures of entropy and information*, In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Contributions to the Theory of Statistics, **1** (1961), 547-561. http://l.academicdirect.org/Horticulture/GAs/Refs/Renyi_1961.pdf

[32] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, **42** (1976). https://www.ams.org/journals/bull/1977-83-04/S0002-9904-1977-14338-3/S0002-9904-1977-14338-3.pdf

[33] C. E. Shannon, *A mathematical theory of communication*, The Bell System Technical Journal, **27** (1948), 379-423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x

[34] C. E. Shannon, W. Weaver, *The mathematical theory of communication*, University of Illinois Press, **27** (1949), 379-423, 623-656. https://psycnet.apa.org/record/1950-04584-000

[35] K. S. Song, *Rényi information*, Loglikelihood and an Intrinsic Distribution Measure. Journal of Statistical Planning and Inference, **93**(1-2) (2001), 51-69. https://doi.org/10.1016/S0378-3758(00)00169-5

[36] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, Journal of Statistical Physics, **52** (1988), 479-487. http://dx.doi.org/10.1007/BF01016429

[37] M. R. Ubriaco, *Entropies based on fractional calculus*, Physics Letters A, **373** (2009), 2516-2519. https://doi.org/10.1016/j.physleta.2009.05.026

[38] N. M. Vaselabadi, S. Tahmasebi, M. R. Kazemi, F. Buono, *Results on varextropy measure of random variables*, Entropy, **23**(3) (2021), 356. https://doi.org/10.3390/e23030356

[39] Y. Xue, Y. Deng, *Tsallis extropy*, Communications in Statistics-Theory and Methods, **52**(3) (2023), 751-762. https://doi.org/10.1080/03610926.2021.1921804

[40] P. Yu, Y. Deng, *On the scaled Rényi entropy and application*, Communications in Statistics Theory and Methods, **1** (2024). https://doi.org/10.1080/03610926.2024.2301986

[41] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[42] L. A. Zadeh, *Probability measures of fuzzy events*, Journal of Mathematical Analysis and Applications, **2** (1968), 421-427. https://doi.org/10.1016/0022-247X(68)90078-4

[43] L. A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems, **1** (1978), 3-28. https://doi.org/10.1016/0165-0114(78)90029-5

[44] R. Zamini, S. Ghafouri, F. Goodarzi, *Varextropy of doubly truncated random variable*, arXiv preprint-arXiv:2407.21423, (2024). https://doi.org/10.48550/arXiv.2407.21423

[45] Q. Zhou, Y. Deng, *Belief extropy: Measure uncertainty from negation*, Communications in Statistics-Theory and Methods, **52**(11) (2023), 3825-3847. https://doi.org/10.1080/03610926.2021.1980049