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A new matrix-based-algorithm for solving latticized linear programming subject to max-min-product fuzzy relation inequalities

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Abstract

This paper introduces a system of Fuzzy Relation Inequalities (FRIs) with the max-min-product composition operator. To determine the structure of solution set of the system, we firstly focus on a single inequality and study its solution set and properties. Then, the structure of solution set of the system is determined by the points. The necessary and sufficient conditions are proposed for its consistency. Some useful properties of the system of the max-min-product FRIs are presented to determine the structure of its minimal solutions. A latticized linear programming problem is proposed with constraints as FRIs using the max-min-product composition. It is shown that one of its optimal solutions can be given in terms of a closed form. Based on the closed form, a matrix-based-algorithm with a polynomial computational complexity is designed to find one of its optimal solutions. A practical example is presented to illustrate the system and the optimization problem in the area of data transmission mechanism.

Keywords: Fuzzy relation inequalities, max-min-product composition, latticized linear programming, matrix-based-algorithm, client-server system.

1 Introduction

In this paper, the latticized linear programming problem subject to a system of max-min-product fuzzy relation inequalities is formulated as follows:

$$\min \quad Z(x) = \bigvee_{i=1}^{n} x_i, \tag{1}$$

$$\text{s.t.} \quad \bigvee_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J = \{1, ..., m\}, \tag{2}$$

$$x_i \in [0, 1], \quad \forall i \in I = \{1, ..., n\}, \tag{3}$$

where $c_{ij}, a_{ij}, b_j \in [0, 1]$, for any $i \in I$ and $j \in J$.

The system of max-min Fuzzy Relation Equations (FREs) was first introduced by Sanchez [17]. Since then the fuzzy relation programming problems were appeared in many practical problems [1, 2, 3, 4, 5, 7, 6, 10, 24, 25, 32] and theoretical problems [15, 18, 19]. The fuzzy relation geometric programming problems are an important class of fuzzy relation programming problems [8, 11, 12, 13, 16, 20, 22, 23, 26, 27, 30, 33]. These problems have wide applications in the field of data transmission mechanism. Various models of geometric programming problems with the fuzzy relation constraints have been designed based on different management goals [16, 23, 26, 30, 31]. The first class of these problems was studied by Wang et al [20]. They considered the latticized linear programming problem with the constraints of the

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max-min FRIs as follows:

$$\begin{aligned} \min \quad & f = E \diamond x, && (4)\\ \text{s.t.} \quad & B \leq A \diamond x \leq D, && (5)\\ & x \in [0,1]^n, && (6) \end{aligned}$$

where A is an $m \times n$ matrix, E is an n-vector, B and D are m-vectors which all of their elements belong to $[0,1]$. The operator of $\diamond$ denotes the max-min composition. They used the conservative paths to find the optimal solution of the problem (4)-(6). The detection of conservative paths is an NP-hard problem. The problem (4)-(6) was investigated by Li and Wang [13]. They solved the problem based on computing the minimum value of the objective function among all minimal solutions of FRI constraints where the problem of finding all the minimal solutions is an NP-hard problem. They obtained its optimal solution set using the resolution of an FRE system which has an exponential time complexity. Yang and Cao [27] presented a fuzzy relation geometric programming problem with a monomial objective function and the max-min FRE constraints as follows:

$$\begin{aligned} \min \quad & f(x) = c\prod_{j=1}^{n} x_j^{r_j}, && (7)\\ \text{s.t.} \quad & A \diamond x = b, && (8)\\ & x \in [0,1]^n, && (9) \end{aligned}$$

where $c, r_j \in R$, for $j = 1, ..., n$, $A \in [0,1]^{m\times n}$, and $b \in [0,1]^{m\times 1}$. They proposed an algorithm based on finding all the minimal solutions of the system (8)-(9). Therefore, the algorithm has an exponential time complexity. Wu [22] studied the geometric programming problem with single-term exponents constrained by the max-min FREs as follows:

$$\begin{aligned} \min \quad & Z(x) = (c_1 \bigwedge x_1^{r_1}) \bigvee (c_2 \bigwedge x_2^{r_2}) \bigvee ... \bigvee (c_m \bigwedge x_m^{r_m}), && (10)\\ \text{s.t.} \quad & x \diamond A = b, && (11)\\ & x \in [0,1]^m, && (12) \end{aligned}$$

where $c_i \geq 0$, $r_i \in R$, for $i = 1, ..., m$, $b \in [0,1]^{1\times n}$, and $A \in [0,1]^{m\times n}$. The notations "$\bigvee$" and "$\bigwedge$" denote the maximum and minimum operator, respectively. To solve the problem (10)-(12), a simple value matrix was designed to capture the contribution of each variable in the objective function based on some useful properties. Wu developed an efficient procedure to solve the problem using the matrix without finding all the minimal solutions of the system (11)-(12). The following optimization problem was investigated by Hosseinyazdi [11] as follows:

$$\begin{aligned} \min \quad & Z = c \diamond x, && (13)\\ \text{s.t.} \quad & A \diamond x \geq b, && (14) \end{aligned}$$

where $A_{m\times n}$ and $b_{m\times 1}$ are fixed matrices and x is an unknown vector over a lattice. She presented two algorithms to solve the problem (13)-(14) over a bounded below distributive lattice L with "0" as the least element and on a Boolean lattice L with "0" as the least element, respectively. These algorithms need $O(n^2)$ computational complexity (see Steps 1-8 of (iii) in Algorithm 5.2 and Steps 1-6 of (iii) in Algorithm 5.7). In the worst case, we should check all the minimal solutions to find a minimal solution less than or equal to the "U" introduced in [11]. Finding all the minimal solutions is an NP-hard problem, in general case. Li and Fang [12] formulated the latticized linear programming on the unit interval as follows:

$$\begin{aligned} \min \quad & Z(x) = \bigvee_{j=1}^{n} (c_j \bigwedge x_j), && (15)\\ \text{s.t.} \quad & b^1 \leq A \diamond x \leq b^2, && (16)\\ & x \in [0,1]^n, && (17) \end{aligned}$$

where $A \in [0,1]^{m\times n}$ and $b^1, b^2 \in [0,1]^{m\times 1}$. They proposed an algorithm with $O(mn)$ computational complexity. However, if we convert the problem (1)-(3) to the problem (15)-(17) using the change-variable $c_{ij}.x_i = y_{ij}$, for each $i \in I$ and $j \in J$, then the number of variables increases to mn variables. Hence, the dimensions of matrix A in (16) become $m \times mn$. The time complexity of the algorithm increases to $O(m^2 n)$. When trying to find the values of x_i's after determining the values of y_{ij}'s, the other drawback of the method appears. For one fixed index $i \in I$ and any index

$j \in J$, we should solve $c_{ij}.x_i = y_{ij}$. From these equations, several distinct values may be derived for x_i, simultaneously, which is impossible. This difficulty is the common disadvantage of the above methods. Moreover, the algorithms in [11, 13, 20, 27] are based on finding the minimal solutions of FRE or FRI system which is an NP-hard problem. Zhou and Ahat [33] designed an algorithm to solve the problem (10)-(12) with the max-product composition. Their algorithm is based on the decomposition of the problem into two subproblems which are solved using the maximum solution and one of the minimal solutions of the system (11)-(12), respectively. In this algorithm, we need to check at most $|\Lambda_1| = |I'_1| \times ... \times |I'_n| (\leq m^n)$ paths to detect the optimal solution according to Steps 5-8 in [33]. This indicates that the time complexity is exponential. The constraints in [33] are as the FRE system. This model is a special case of the model (1)-(3). Yang et al [30] studied the single-variable term semi-latticized fuzzy relation programming problem where the objective function is as $Z(x) = \bigvee_{j=1}^{n} (\bigvee_{k=1}^{p_j} (c_{j_k}.x_j^{\gamma_{j_k}}))$ and the constraints are as the system (8)-(9) with the max-product composition. The constraints are a special case of FRIs and the given relations and properties in Yang et al [30] are not correct for the max-min FRI, in a general case. Yang [23] considered the recent problem with the objective function as $Z(x) = \bigvee_{k=1}^{p} (c_k. \prod_{j=1}^{n} (x_j^{\gamma_{k_j}}))$ along with the constraints as the system (8)-(9) which is a special case of the max-min FRIs. Yang et al [31] proposed the latticized linear programming problem subject to the max-product FRIs to optimize the management of the wireless communication emission base stations as follows:

$$\min \quad Z(x) = x_1 \bigvee x_2 \bigvee ... \bigvee x_n, \tag{18}$$

$$\text{s.t.} \quad A \circ x \geq b, \tag{19}$$

$$x \in [0,1]^n, \tag{20}$$

where $A \in [0,1]^{m \times n}$, $b \in [0,1]^{m \times 1}$, and the notation "$\circ$" denotes the max-product composition. This model is a special case of the proposed model (1)-(3). The Steps of algorithm in [31] cannot be applied to solve the proposed problem (1)-(3). The consistency conditions, the characteristic matrix, and the created sets by them are completely different from those in the model (1)-(3). The nonlinear generalization of latticized linear programming problems was presented by the objective function $\varphi(...(\varphi(\varphi(x_1, x_2), x_3)...)$ and the constraints as $A\psi x \geq b$ and $x \in [0,1]^n$ using the fuzzy max-ψ operator, where $\psi : [0,1]^2 \to [0,1]$ is an arbitrary continuous t-norm [8]. The algorithms 3.1, 4.1, and 5.1 in [8] are based on finding the minimal solutions of its feasible domain (see step 5 in Algorithm 3.1 and steps 5-6 in Algorithm 5.1) which is an NP-hard problem. Algorithm 4.1 has been designed for the case of $\varphi = maximum$ which its computational complexity is as $O(mn)$. As it was mentioned, the technique of the change-variable in the above models increases the computational complexity, considerably, and it may create inconsistencies in obtaining the values of variables of the original problem. Motivated by Li and Yang [14], Yang et al [28] used the addition-min FRIs to formulate the P2P file sharing system to find the variable-ordering solutions. In [29], the authors solved n subproblems to find the optimal solutions while Yang et al [28] obtained them in a direct method without solving n subproblems. Yang and his colleagues [24, 32] presented the min-max programming problem subject to the addition-min FRI constraints to reduce the network congestion. Sometimes, the random line faults take place between two terminals in a P2P network. Yang [25] suggested a system called random-term-absent addition-min FRIs to model the situation. Guu and Wu [10] proposed a multiple objective optimization problem which allows a system manager to control three objectives such as system congestion, cost, and penalty, simultaneously, in the P2P network. The system employed by Li and Yang [14], Yang et al [28, 32], and Guu and Wu [9] to describe the P2P file sharing system is as follows:

$$\sum_{j=1}^{n} (a_{ij} \bigwedge x_j) \geq b_i, \quad i = 1, ..., m. \tag{21}$$

where $a_{ij},\ x_j \in [0,1]$, $b_i > 0$, for any $1 \leq i \leq m$, $1 \leq j \leq n$. As it has been mentioned in Guu and Wu [10], the peer A_j may send out the file data of various quality levels to different peers A_i's. In the model (21), this difference has not been considered and the quality level was unified as x_j. To overcome this difference, Qiu and Yang [16] assumed that the peer A_j sends the file data to the peer A_i with the quality level $c_{ij}.x_j$, for $i = 1, ..., m$, $j = 1, ..., n$. Therefore, they modified the model (21) as follows:

$$\sum_{j=1}^{n} (c_{ij}.x_j \bigwedge a_{ij}) \geq b_i, \quad i = 1, ..., m. \tag{22}$$

where $c_{ij}, a_{ij}, x_j \in [0,1]$, $b_i > 0$, for any $1 \leq i \leq m$, $1 \leq j \leq n$. The feasible domain of the models in [9, 10, 14, 16, 24, 25, 28, 29, 32] is the addition-min FRI system. Its solution set is convex and the set of its minimal solutions is infinite in

the most cases. As it will be shown, the number of the minimal solutions in a consistent system of the max-min -product is finite and its solution set is non-convex, in a general case. Therefore, their properties are completely different from each other. Therefore, the algorithms in [9, 10, 14, 16, 24, 25, 28, 29, 32] cannot be applied to solve the problem (1)-(3). Yang [26] presented the problem of maximizing weighted objective function $Z(x) = \bigwedge_{j=1}^{n} w_j(x_j - p_j)$ with the constraints as $T \bigodot x \leq c$ and $p \leq x \leq q$, where $T = [t_{ij}]_{m\times n}$, $c = (c_1, ..., c_m)^T$, $p = (p_1, ..., p_n)$, and $q = (q_1, ..., q_n)$ are fuzzy matrices and $x \in [0, 1]^n$. The operator "$\bigodot$" denotes the min-addition composition. This model and its properties are completely different from the proposed model (1)-(3) due to the min-addition operator. The feasible index set-based algorithm in [26] cannot be applied to solve the problem (1)-(3). Wang et al [21] investigated the tri-composed FRI with a weighted-max-min composition, for the first time. They applied it to model the successful rate of the line connection within the P2P network system. To reduce network congestion in the system, the following model was proposed as follows:

$$\min \quad Z(x) = x_1 \bigvee x_2 \bigvee ... \bigvee x_n, \tag{23}$$

$$\text{s.t.} \quad b_i \leq (a_{i1} \bigwedge x_1)p_{i1} \bigvee (a_{i2} \bigwedge x_2)p_{i2} \bigvee ... \bigvee (a_{in} \bigwedge x_n)p_{in} \leq d_i, \quad for \quad i = 1, ..., m, \tag{24}$$

$$x_j \in [0, 1], \quad for \quad j = 1, ..., n, \tag{25}$$

A path-based algorithm was designed to solve the problem. The structure of the constraints in the model is completely different from the proposed model in this paper. The maximum solution of system (24)-(25) is completely different from that of the system (2)-(3). The steps 1,2,3, and 4 of the algorithm in [21] are based on the maximum solution. Hence, we cannot apply the algorithm in [21] to solve the problem (1)-(3). Therefore, we are motivated to study fuzzy relation inequalities with the max-min-product composition and its corresponding min-max programming problem due to its applications in the data transmission. As it was explained above in detail, the nature of this problem is completely different from the above problems. Hence, we firstly investigate the structure of its feasible domain. Some sufficient and necessary conditions are presented to check its consistency. Then, its solution set is completely determined. To solve the problem (1)-(3), we decompose it into some simple problems which can easily specify one of their optimal solutions. Using them, a closed form is presented to determine one of the optimal solutions of the proposed min-max programming problem. Also, a polynomial time algorithm is designed to find one of the optimal solutions of the problem (1)-(3). The main contributions are as follows:

(i) Some necessary and sufficient conditions are proposed to check the consistency of the system (2)-(3).

(ii) The complete solution set of system (2)-(3) is determined by the decomposition of the system (2)-(3) into $|J| = m$ inequalities. The complete solution set of the system can be found by the intersection of the complete solution set of $|J|$ inequalities.

(iii) The concept of FRI path is extended for the system (2)-(3). The complete solution set of system (2)-(3) is also specified using the FRI paths.

(iv) Based on the decomposition of the original problem into some simple problems, we present a closed form to specify one of its optimal solutions.

(v) Based on the closed form in (iv), a polynomial time algorithm is designed to solve the problem (1)-(3).

(vi) An application example is given to express the importance of the problem (1)-(3) to reduce the network congestion in the client-server networks.

The rest of this paper is organized as follows: Section 2 investigates the structure of solution set of max-min-product FRI system and presents some properties of its feasible solutions. Section 3 discusses about the resolution process of the min-max programming problem subject to the max-min-product FRI system and presents a matrix-based algorithm to solve the problem. Some numerical and application examples are also given to illustrate the algorithm in Section 4. Finally, conclusions are provided in Section 5.

2 The structure of feasible domain of the problem (1)-(3)

The solution set of the system (2)-(3) is to find the vectors $x = [x_i]_{n\times 1} \in [0, 1]^n$ such that

$$\bigvee_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J. \tag{26}$$

The feasible solution set of the system (2)-(3) is denoted by $S(C, A, b) = \{x \in [0, 1]^n | \bigvee_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \ \forall j \in J\}$. The system (26) is said to be consistent if $S(C, A, b) \neq \emptyset$. Otherwise, it is inconsistent. To study the structure of its

solution set, we require the necessary and sufficient conditions to check the consistency of the system (26).

Theorem 2.1. *A vector $x \in [0,1]^n$ is a solution for the system (26) if and only if the following condition is satisfied:* $\forall j \in J \quad \exists i_j \in I \quad s.t. \quad c_{i_j j}.x_{i_j} \bigwedge a_{i_j j} \geq b_j$.

Proof. It is obvious. □

The following lemma presents a special case of the system (26).

Lemma 2.2. *Let $S(C,A,b) \neq \emptyset$ and $J_0 = \{j \in J \mid \ b_j = 0\}$. Then the solution set of the system (26) is the same to the following system:*

$$\textit{System (I):} \begin{cases} \bigvee\limits_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, & \forall j \in J - J_0, \\ x \in [0,1]^n. \end{cases}$$

Proof. Assume that X_1 and X_2 are the solution sets of the systems (26) and (I), respectively. Obviously, $X_1 \subseteq X_2$. Now, suppose that $x \in X_2$. So, we have $\bigvee\limits_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J - J_0$, and $x \in [0,1]^n$. On the other hand, $\bigvee\limits_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq 0$, for each $j \in J_0$, due to $c_{ij}, x_i, a_{ij} \in [0,1]$, for each $i \in I$ and $j \in J$. Therefore, $x \in X_1$. Hence, $X_2 \subseteq X_1$. The proof is complete. □

Without loss of generality, we can omit the inequalities with $b_j = 0$ in the system (26). Therefore, $b_j > 0$ is supposed for any $j \in J$ throughout this article. We are now ready to specify the complete solution set of the system (26). First of all, some definitions are given that will help us in the processes.

Definition 2.3. *The order relation between two vectors $x = [x_i]_{i \in I}$ and $y = [y_i]_{i \in I} \in [0,1]^n$ is defined as $x \geq y$ if and only if $x_i \geq y_i$ for each $i \in I$. Also, $x > y$ is defined if and only if $x \geq y$ and $x \neq y$.*

The maximum solution and minimal solution are defined based on the order relation in the following definition.

Definition 2.4. (i) *A solution $\hat{x} \in S(C,A,b)$ is called a maximum solution of the system (26) if $\hat{x} \geq x$, for each $x \in S(C,A,b)$.*
(ii) *A solution $\check{x} \in S(C,A,b)$ is called a minimal solution of the system (26) if $x \leq \check{x}$ implies $x = \check{x}$ for each $x \in S(C,A,b)$.*

To determine the structure of the solution set of the system (26), the set $S(C,A,b)$ can be rewritten as follows:

$$S(C,A,b) = \bigcap_{j=1}^{m} S_j(C,A,b), \tag{27}$$

where $S_j(C,A,b) = \{x \in [0,1]^n \mid \ \bigvee\limits_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j\}$.
With regard to the relation (27), it is enough to find the solution set $S_j(C,A,b)$, for an arbitrary index $j \in J$. At first, the necessary and sufficient conditions are investigated for the consistency of the j^{th} inequality, i.e.,

$$\bigvee_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j. \tag{28}$$

or equivalently,

$$(c_{1j}x_1 \bigwedge a_{1j}) \bigvee (c_{2j}x_2 \bigwedge a_{2j}) \bigvee \ldots \bigvee (c_{nj}x_n \bigwedge a_{nj}) \geq b_j. \tag{29}$$

We now focus on the consistency of the inequality (28) (or (29)) and the structure of its solution set. Using the points, we will study the consistency and the structure of the solution set of the system (26).

Lemma 2.5. *The inequality of (28) is consistent if and only if $\bigvee\limits_{i=1}^{n} (c_{ij} \bigwedge a_{ij}) \geq b_j$.*

Proof. Suppose that the inequality (28) is consistent. Then there exists a solution $\bar{x} = [\bar{x}_i]_{i\in I} \in [0,1]^n$ such that $\bigvee_{i=1}^{n}(c_{ij} \bigwedge a_{ij}) \geq \bigvee_{i=1}^{n}(c_{ij}.\bar{x}_i \bigwedge a_{ij}) \geq b_j$. Therefore, $\bigvee_{i=1}^{n}(c_{ij} \bigwedge a_{ij}) \geq b_j$. Conversely, assume that $\bigvee_{i=1}^{n}(c_{ij} \bigwedge a_{ij}) \geq b_j$. This inequality shows that $\hat{x} = (1, ..., 1)$ is a solution for the inequality (28). So, the inequality (28) is consistent. □

The following corollary is a direct result of Lemma 2.5.

Corollary 2.6. *The inequality* (28) *is consistent if and only if* $\hat{x} = (1, ..., 1)$ *is a maximum solution of* $S_j(C, A, b)$.

The following lemma is concluded from the definition of the max-min-product operator and the order relation.

Lemma 2.7. *Suppose that* $\tilde{x} \in S_j(C, A, b)$. *If* $\tilde{x} \leq x \leq \hat{x}$, *then* $x \in S_j(C, A, b)$.

Proof. Since $\tilde{x} = [\tilde{x}_i]_{i\in I} \leq x = [x_i]_{i\in I}$, it is concluded that $\tilde{x}_i \leq x_i$, for each $i \in I$. So, $c_{ij}.\tilde{x}_i \leq c_{ij}.x_i$, for each $i \in I$ and $j \in J$. Hence, we have $c_{ij}.\tilde{x}_i \bigwedge a_{ij} \leq c_{ij}.x_i \bigwedge a_{ij}$, for each $i \in I$ and $j \in J$. Since $\tilde{x} \in S_j(C, A, b)$, we have $\bigvee_{i=1}^{n}(c_{ij}.x_i \bigwedge a_{ij}) \geq \bigvee_{i=1}^{n}(c_{ij}.\tilde{x}_i \bigwedge a_{ij}) \geq b_j$. This relation implies that $x \in S_j(C, A, b)$. □

Now, we focus on the structure of feasible solution set of the inequality (28). To do this, the characteristic matrix of the inequality (28) is defined. This matrix denotes the effect of components in holding the inequality (28).

Definition 2.8. (I) *The FRI characteristic matrix* $M = [m_{ij}]_{n\times 1}$ *for the inequality* (29) *is defined as follows:*

$$m_{ij} = \begin{cases} 1, & \text{if } c_{ij} \bigwedge a_{ij} \geq b_j, \\ 0, & \text{otherwise}, \end{cases} \quad \forall i \in I.$$

(II) *Define a series of index sets by* $I_j = \{i \in I \mid m_{ij} = 1\}$, *for* $j \in J$.

With regard to the part (I) of Definition 2.8, we can present a necessary and sufficient condition for the consistency of the inequality (29) based on the FRI characteristic matrix as follows.

Lemma 2.9. *The inequality* (29) *is consistent if and only if the matrix* M *has at least one non-zero component.*

Proof. It is a direct result of the part (I) of Definition 2.8 and Lemma 2.5. □

We are now ready to determine the structure of solution set of the inequality (29).

Theorem 2.10. *Suppose that* $S_j(C, A, b) \neq \emptyset$. *For each* $i \in I_j$, *let* $X_i^{(j)} = \prod_{t=1}^{n} P_{tj}^{(i)}$ *where* $P_{ij}^{(i)} = [\frac{b_j}{c_{ij}}, 1]$ *and* $P_{tj}^{(i)} = [0, 1]$, *for* $t \neq i$. *Then the complete solution set of the system* (29) *is as follows:*

$$S_j(C, A, b) = \bigcup_{i\in I_j} X_i^{(j)}. \tag{30}$$

Proof. If $x \in S_j(C, A, b)$, then there exists $i \in I$ such that $c_{ij}x_i \bigwedge a_{ij} \geq b_j$. This implies that $c_{ij}x_i \geq b_j$ and $a_{ij} \geq b_j$. Hence, we conclude that $x_i \geq \frac{b_j}{c_{ij}}$ and $a_{ij} \geq b_j$. It results that $x \in X_i^{(j)}$, for an $i \in I_j$. Therefore, $x \in \bigcup_{i\in I_j} X_i^{(j)}$. Hence, $S_j(C, A, b) \subseteq \bigcup_{i\in I_j} X_i^{(j)}$. Conversely, if $x \in \bigcup_{i\in I_j} X_i^{(j)}$, then there exists an $i \in I_j$ such that $x \in X_i^{(j)}$. This implies that $x_i \in [\frac{b_j}{c_{ij}}, 1]$, $x_t \in [0, 1]$, for each $t \neq i$, and $c_{ij} \bigwedge a_{ij} \geq b_j$. Two expressions $c_{ij}x_i \geq b_j$ and $a_{ij} \geq b_j$ show that $c_{ij}x_i \bigwedge a_{ij} \geq b_j$. So, $\bigvee_{i\in I}(c_{ij}x_i \bigwedge a_{ij}) \geq b_j$. This implies that $x \in S_j(C, A, b)$. So, $\bigcup_{i\in I_j} X_i^{(j)} \subseteq S_j(C, A, b)$. Therefore, we conclude that $S_j(C, A, b) = \bigcup_{i\in I_j} X_i^{(j)}$. □

We now explain Theorem 2.10 using a numerical example.

Example 2.11. *Consider the FRI as follows:*

$$(0.7x_1 \bigwedge 0.8) \bigvee (0.5x_2 \bigwedge 0.2) \bigvee (0.5x_3 \bigwedge 0.7) \bigvee (x_4 \bigwedge 0.9) \geq 0.6,$$

where $x = [x_1, x_2, x_3, x_4] \in [0,1]^4$, $C = [c_{11}, c_{21}, c_{31}, c_{41}] = [0.7, 0.5, 0.5, 1]$, $A = [a_{11}, a_{21}, a_{31}, a_{41}] = [0.8, 0.2, 0.7, 0.9]$, *and* $b = [b_1] = [0.6]$, *respectively. Obviously, the vector* $\hat{x} = (1,1,1,1)$ *is a maximum solution for the above inequality. Therefore, this system is consistent according to Corollary 2.6 and* $I_1 = \{1,4\}$. *According to Theorem 2.10, the complete solution set of the above inequality is as follows:*

$$S_1(C,A,b) = \bigcup_{i\in I_1} X_i^{(1)} = X_1^{(1)} \bigcup X_4^{(1)} = ([\frac{6}{7},1],[0,1],[0,1],[0,1]) \bigcup ([0,1],[0,1],[0,1],[\frac{6}{10},1]).$$

We now investigate the necessary and sufficient conditions for the consistency of the system (26) in the next lemma.

Lemma 2.12. *The following statements are equivalent:*
(i) *The system* (26) *is consistent.*
(ii) $\bigvee_{i=1}^{n} (c_{ij} \bigwedge a_{ij}) \geq b_j$, $\forall j \in J$.
(iii) $\hat{x} = (1,1,...,1)$ *is a maximum solution for the system* (26).

Proof. Assume that the statement (i) is true. Then, there exists a solution $\bar{x} = (\bar{x}_i)_{i\in I}$ such that $\bigvee_{i=1}^{n} (c_{ij} \bigwedge a_{ij}) \geq \bigvee_{i=1}^{n} (c_{ij}\bar{x}_i \bigwedge a_{ij}) \geq b_j$, for each $j \in J$. Therefore, the statement (ii) is concluded. Now, assume that the statement (ii) is true. Then, obviously, the vector $\hat{x} = (1,1,...,1)$ is a maximum solution for the system (26) and the part (iii) is concluded. Finally, suppose that the part (iii) is true. Then $\hat{x} \in S(C,A,b)$ and the system (26) is consistent. □

Using Theorem 2.10, we study the structure of the solution set of the system (26).

Theorem 2.13. *Suppose that the system* (26) *is consistent. Then its complete solution set is determined by the following relation:* $S(C,A,b) = \bigcap_{j\in J} S_j(C,A,b) = \bigcap_{j\in J} (\bigcup_{i\in I_j} X_i^{(j)})$.

Proof. It is a direct result of Theorem 2.10. □

We can now extend Definition 2.8 for an FRI characteristic matrix $M = [m_{ij}]_{n\times m}$ to check the consistency of the system (26).

Definition 2.14. *(I) The FRI characteristic matrix* $M = [m_{ij}]_{n\times m}$ *for the system* (26) *is defined as follows:*

$$m_{ij} = \begin{cases} 1, & \text{if } \ c_{ij} \bigwedge a_{ij} \geq b_j, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in I, j \in J.$$

(II) Define a series of index sets by $I_j = \{i \in I \mid m_{ij} = 1\}$, *for each* $j \in J$.

The necessary and sufficient conditions are proposed in terms of the FRI characteristic matrix M to check the consistency of the system (26) in the following lemma.

Lemma 2.15. *The system* (26) *is consistent if and only if for each* $j \in J$, *there exist(s) at least an index* $i \in I$ *such that* $m_{ij} = 1$.

Proof. Assume that the system (26) is consistent. Then there exist(s) at least a solution $\bar{x} = [\bar{x}_i]_{i\in I}$ for the system (26). Since $\bar{x} \in S(C,A,b)$, for each inequality $j \in J$, there exist(s) at least an index $i \in I$ such that $c_{ij}\bar{x}_i \bigwedge a_{ij} \geq b_j$. The inequality is concluded that $c_{ij} \bigwedge a_{ij} \geq c_{ij}\bar{x}_i \bigwedge a_{ij} \geq b_j$ which is equivalent to $m_{ij} = 1$. Now, we consider the converse of the lemma. Assume that for each $j \in J$, there exist(s) at least an index $i \in I$, such that $m_{ij} = 1$ which is equivalent to $c_{ij} \bigwedge a_{ij} \geq b_j$. This statement is concluded that $\hat{x} = (1,1,...,1)$ is a maximum solution for the system (26). Therefore, the system (26) is consistent. □

According to Theorem 2.13, we can easily determine the complete solution set of the system (26). This point is illustrated by the following example.

Example 2.16. *Consider the FRI system as follows:*

$$\begin{cases} E_1 : (0.5x_1 \bigwedge 0.4) \bigvee (0.6x_2 \bigwedge 0.7) \bigvee (0.9x_3 \bigwedge 0.5) \bigvee (0.1x_4 \bigwedge 0.2) \geq 0.5, \\ E_2 : (1x_1 \bigwedge 0.8) \bigvee (0.7x_2 \bigwedge 0.4) \bigvee (0.2x_3 \bigwedge 0.1) \bigvee (0.4x_4 \bigwedge 0.2) \geq 0.3, \\ E_3 : (0.8x_1 \bigwedge 0.6) \bigvee (0.6x_2 \bigwedge 1) \bigvee (0.4x_3 \bigwedge 0.2) \bigvee (0.6x_4 \bigwedge 0.3) \geq 0.4, \end{cases} \tag{31}$$

where $x \in [0,1]^4$. This system is consistent because $\hat{x} = (1,1,1,1)$ is a solution of this system according to Lemma 2.12. Now, we obtain the complete solution set of this system by Theorem 2.13 as follows. Let $S(C,A,b) = \bigcap_{j\in J} S_j(C,A,b)$, where $S_j(C,A,b)$, for $j \in J$, is as follows:

$$\begin{aligned} S_1(C,A,b) = \bigcup_{i\in I_1=\{2,3\}} X_i^{(1)} &= X_2^{(1)} \bigcup X_3^{(1)} = \prod_{t=1}^{4} P_{t1}^{(2)} \bigcup \prod_{t=1}^{4} P_{t1}^{(3)} \\ &= ([0,1],[\frac{5}{6},1],[0,1],[0,1]) \bigcup ([0,1],[0,1],[\frac{5}{9},1],[0,1]), \\ S_2(C,A,b) = \bigcup_{i\in I_2=\{1,2\}} X_i^{(2)} &= X_1^{(2)} \bigcup X_2^{(2)} = \prod_{t=1}^{4} P_{t2}^{(1)} \bigcup \prod_{t=1}^{4} P_{t2}^{(2)} \\ &= ([\frac{3}{10},1],[0,1],[0,1],[0,1]) \bigcup ([0,1],[\frac{3}{7},1],[0,1],[0,1]), \\ S_3(C,A,b) = \bigcup_{i\in I_3=\{1,2\}} X_i^{(3)} &= X_1^{(3)} \bigcup X_2^{(3)} = \prod_{t=1}^{4} P_{t3}^{(1)} \bigcup \prod_{t=1}^{4} P_{t3}^{(2)} \\ &= ([0.5,1],[0,1],[0,1],[0,1]) \bigcup ([0,1],[\frac{2}{3},1],[0,1],[0,1]). \end{aligned}$$

The following lemma presents a useful property to determine the complete solution set of the system (26).

Lemma 2.17. *Suppose that $\tilde{x} \in S(C,A,b)$. If $\tilde{x} \leq x \leq \hat{x}$, then $x \in S(C,A,b)$.*

Proof. Assume that $\tilde{x} = (\tilde{x}_i)_{i\in I} \leq x = (x_i)_{i\in I}$. According to Definition 2.3, we have $\tilde{x}_i \leq x_i$, for each $i \in I$. So, $c_{ij}\tilde{x}_i \leq c_{ij}x_i$, for each $i \in I$ and $j \in J$. Hence, $c_{ij}\tilde{x}_i \bigwedge a_{ij} \leq c_{ij}x_i \bigwedge a_{ij}$, for each $i \in I$ and $j \in J$. On the other hand, $\tilde{x} \in S(C,A,b)$ implies that $\bigvee_{i=1}^{n}(c_{ij}.x_i \bigwedge a_{ij}) \geq \bigvee_{i=1}^{n}(c_{ij}.\tilde{x}_i \bigwedge a_{ij}) \geq b_j$, for each $j \in J$. This relation shows that $x \in S(C,A,b)$. □

We are now ready to find the optimal solution of the problem (1)-(3), in the next section.

3 An algorithm for finding the optimal solution of the problem (1)-(3)

In this section, the latticized linear programming problem (1)-(3) is studied. First of all, we decompose the problem (1)-(3) to $|J| = m$ simple problems with an FRI constraint as follows:

$$\begin{aligned} P_j : \quad \min \quad & Z_j(x) = \bigvee_{i=1}^{n} x_i, \\ \text{s.t.} \quad & \bigvee_{i=1}^{n}(c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \\ & x \in [0,1]^n, \end{aligned}$$

where $j \in J$. We try to find the optimal solutions of the simple problems P_j, for $j \in J$.

Theorem 3.1. *Suppose that the problem P_j is feasible. Then the problem P_j has at least an optimal solution. One of the optimal solutions of the problem P_j is as $x^{(j)*} = [x_i^{(j)*}]_{i\in I}$ where*

$$x_i^{(j)*} = \begin{cases} \frac{b_j}{c_{ij}}, & \text{if } \ i \in arg \min_{t\in I}\{\frac{b_j}{c_{tj}} |\ m_{tj} = 1\}, \\ 0, & \text{otherwise}, \end{cases} \quad \forall i \in I. \tag{32}$$

Also, $Z_j^*(x^{(j)*}) = x_i^{(j)*}$ where $m_{ij} = 1$, for an index $i \in I$.

Proof. According to Theorem 2.10, the feasible domain of the problem P_j is a bounded and closed set. Since its objective function is continuous, the problem P_j has at least an optimal solution. We firstly claim that the vector $x^{(j)*}$ is feasible for the problem P_j. Since the problem P_j is feasible, there exist(s) at least one index $i_0 \in I$ such that $m_{i_0 j} = 1$. Therefore, $arg \min\limits_{i\in I}\{\frac{b_j}{c_{ij}} |\ m_{ij} = 1\} \neq \emptyset$. Hence, we can set $P = arg \min\limits_{i\in I}\{\frac{b_j}{c_{ij}} |\ m_{ij} = 1\} = \{i_1, i_2, ..., i_p\}$. So, we have

$$\begin{aligned}
\bigvee_{i=1}^{n}(c_{ij}x_i^{(j)*} \bigwedge a_{ij}) &= \bigvee_{i\in P}(c_{ij}x_i^{(j)*} \bigwedge a_{ij}) \bigvee \bigvee_{i\in I-P}(c_{ij}x_i^{(j)*} \bigwedge a_{ij}) \\
&= \bigvee_{i\in P}(b_j \bigwedge a_{ij}) \bigvee \bigvee_{i\in I-P}(0 \bigwedge a_{ij}) \\
&= \bigvee_{i\in P}(a_{ij} \bigwedge b_j) \bigvee \bigvee_{i\in I-P}(0) \\
&= \bigvee_{i\in P}(a_{ij} \bigwedge b_j).
\end{aligned}$$

Since $i \in P$, we have $m_{ij} = 1$. According to the definition of the matrix M, it is concluded that $c_{ij} \bigwedge a_{ij} \geq b_j$, for each $i \in P$. Hence, we obtain $a_{ij} \geq b_j$, for each $i \in P$. So, $\bigvee\limits_{i=1}^{n}(c_{ij}x_i^{(j)*} \bigwedge a_{ij}) = \bigvee\limits_{i\in P}(a_{ij} \bigwedge b_j) = b_j$. Therefore, the vector $x^{(j)*}$ is feasible for the problem P_j. Now, we show that the vector $x^{(j)*}$ is an optimal solution. The components of each feasible vector $x^{(j)}$ of the problem P_j should satisfy the relation (30) from Theorem 2.10. Therefore, $x_i^{(j)} \geq \frac{b_j}{c_{ij}}$ for an index $i \in I$ satisfying in $m_{ij} = 1$ and $x_i^{(j)} \geq 0$ for other indexes $i \in I$. Since $x_i^{(j)*} = \frac{b_j}{c_{ij}}$ for $i \in arg \min\limits_{t\in I}\{\frac{b_j}{c_{tj}} |\ m_{tj} = 1\}$ and $x_i^{(j)*} = 0$ for other indexes $i \in I$, obviously, the optimal objective value is $Z_j^*(x^{(j)*}) = x_i^{(j)*}$. The proof is completed. □

Now, we use the results of Theorem 3.1 to find the optimal solution of the original problem (1)-(3). First of all, we extend the definition of FRI path for the max-min-product FRI system (26).

Definition 3.2. *Let $p = (p_1, p_2, ..., p_m)$ such that $p_j \in I_j$, for each $j \in J$. The vector $x^p = (x_i^p)_{i\in I}$ is defined as follows: $x_i^p = \bigvee\limits_{j=1}^{m}\{\frac{b_j}{c_{ij}} \mid p_j = i\}$, where $\bigvee \emptyset = 0$. Also, let $P = \prod\limits_{j=1}^{m} I_j$.*

The following lemma determines the complete solution set of the max-min-product FRI system (26) using Definition 3.2.

Lemma 3.3. *Suppose that $S(C, A, b) \neq \emptyset$ and $\hat{x} = (1, 1, ..., 1)$. Then we have $S(C, A, b) = \bigcup\limits_{p\in P}\{x \mid x^p \leq x \leq \hat{x}\}$.*

Proof. Since $S(C, A, b) \neq \emptyset$, there exists at least an element $x \in S(C, A, b)$ such that $\bigvee\limits_{i=1}^{n}(c_{ij}x_i \bigwedge a_{ij}) \geq b_j$, for each $j \in J$. Therefore, for each $j \in J$, there exists at least one index $i \in I$ such that $(c_{ij}x_i \bigwedge a_{ij}) \geq b_j$. We now define I'_j as $I'_j = \{i \in I_j \mid (c_{ij}x_i \bigwedge a_{ij}) \geq b_j\}$ and $P' = \prod\limits_{j=1}^{m} I'_j$. Since $x_i \leq \hat{x}_i = 1$, for each $i \in I$, we have $I'_j \subseteq I_j$, for each $j \in J$, and $P' \subseteq P$. Let $p' = (p'_1, p'_2, ..., p'_m) \in P'$. Then, we have $x_i \geq \bigvee\limits_{j=1}^{m}\{\frac{b_j}{c_{ij}} \mid p'_j = i\} = x_i^{p'}$ due to the feasibility x. So, for each $x \in S(C, A, b)$, there exists $p \in P$ such that $x^p \leq x$. To complete the proof, it is enough to show that the vector x^p is feasible in the system (26). Since $S(C, A, b) \neq \emptyset$, by Lemma 2.15, for each $j \in J$, there exists $i \in I$, such that $p_j = i$. Hence, $x_i^p = \bigvee\limits_{j=1}^{m}\{\frac{b_j}{c_{ij}} \mid p_j = i\} \geq \frac{b_j}{c_{ij}}$ or $c_{ij}x_i^p \geq b_j$. On the other hand, $p_j = i \in I_j$ indicates that $c_{ij} \bigwedge a_{ij} \geq b_j$ which is equivalent to $c_{ij} \geq b_j$ and $a_{ij} \geq b_j$. Two statements $c_{ij}.x_i^p \geq b_j$ and $a_{ij} \geq b_j$ imply that $c_{ij}x_i^p \bigwedge a_{ij} \geq b_j$. Hence, $\bigvee\limits_{i=1}^{n}(c_{ij}x_i^p \bigwedge a_{ij}) \geq b_j$, for each $j \in J$. So, we conclude that $x^p \in S(C, A, b)$. □

The above lemma shows that in a consistent system with the max-min-product composition, the number of the minimal solutions is finite and its solution set is non-convex, in general. The following corollaries express two results of Lemma 3.3.

Corollary 3.4. *Suppose that $S(C,A,b) \neq \emptyset$ and $\check{S}(C,A,b)$ denotes the set of all minimal solutions of the system* (26)*. Then $\check{S}(C,A,b) \subseteq \{x^p \mid p \in P\}$.*

Proof. Assume that $\check{x} \in \check{S}(C,A,b)$. Then we have $\check{x} \in S(C,A,b)$. According to Lemma 3.3, there exists $p \in P$ such that $x^p \leq \check{x} \leq \hat{x}$. Since $\check{x}$ is a minimal solution, according to the part (ii) of Definition 2.4, it is concluded that $\check{x} = x^p$. Therefore, $\check{x} \in \{x^p \mid p \in P\}$. Hence, $\check{S}(C,A,b) \subseteq \{x^p \mid p \in P\}$. □

By pairwise comparison of the elements $\{x^p \mid p \in P\}$, we can find all the minimal solutions of the system (26).The following corollary determines the structure of solution set of the system (26) based on its maximum solution and all its minimal solutions.

Corollary 3.5. *Suppose that $S(C,A,b) \neq \emptyset$ and $\hat{x} = (1,1,...,1)$. Then $S(C,A,b) = \bigcup\limits_{\check{x} \in \check{S}(C,A,b)} \{x \mid \check{x} \leq x \leq \hat{x}\}$.*

Proof. Assume that $x \in S(C,A,b)$. Then there exist(s) $p \in P$ such that $x^p \leq x \leq \hat{x}$, according to Lemma 3.3. If there exist(s) $\check{x} \in \check{S}(C,A,b)$ such that $\check{x} = x^p$, then $\check{x} \leq x \leq \hat{x}$. Otherwise, for each $\check{x} \in \check{S}(C,A,b)$, we have $\check{x} \neq x^p$. Since $\check{S}(C,A,b)$ is the set of all the minimal solutions of the system (26) and $x^p \in S(C,A,b)$, there exists $\check{x} \in \check{S}(C,A,b)$ such that $\check{x} \leq x^p \leq x \leq \hat{x}$. Therefore, $x \in \bigcup\limits_{\check{x} \in \check{S}(C,A,b)} \{x \mid \check{x} \leq x \leq \hat{x}\}$. Hence, we have $S(C,A,b) \subseteq \bigcup\limits_{\check{x} \in \check{S}(C,A,b)} \{x \mid \check{x} \leq x \leq \hat{x}\}$. Conversely, let $\bar{x} \in \bigcup\limits_{\check{x} \in \check{S}(C,A,b)} \{x \mid \check{x} \leq x \leq \hat{x}\}$. Then there exists $\check{x} \in \check{S}(C,A,b)$ such that $\check{x} \leq \bar{x} \leq \hat{x}$. According to Corollary 3.4, there exists $p \in P$ such that $\check{x} = x^p$. Therefore, $x^p \leq \bar{x} \leq \hat{x}$. According to Lemma 3.3, we have $\bar{x} \in S(C,A,b)$. Therefore, it is concluded that $\bigcup\limits_{\check{x} \in \check{S}(C,A,b)} \{x \mid \check{x} \leq x \leq \hat{x}\} \subseteq S(C,A,b)$. The proof is completed. □

Now, we are ready to present a closed form of the optimal solution of the original problem (1)-(3) using Lemma 3.3.

Theorem 3.6. *Suppose that the problem* (1)-(3) *is feasible and $x^{(j)*} = [x_i^{(j)*}]_{i \in I}$ is the optimal solution of the problem P_j, for each $j \in J$. Then the problem* (1)-(3) *has at least an optimal solution. One of the optimal solutions of the problem* (1)-(3) *is as $x^* = [x_i^*]_{i \in I}$ where $x_i^* = \bigvee\limits_{j=1}^{m} \{x_i^{(j)*}\}$, for each $i \in I$. Moreover, $Z^*(x) = \bigvee\limits_{j=1}^{m} Z_j^*$ where Z_j^* is the optimal objective value of the problem P_j, for each $j \in J$.*

Proof. According to Theorems 2.10 and 2.13, the feasible domain of the problem (1)-(3) is a bounded and closed set. Since the objective function is continuous. So, the problem (1)-(3) has at least one optimal solution. First of all, we show that the vector x^* is a feasible vector of the problem (1)-(3). Now, two cases are considered as follows:
(I) If $x_i^* = 0$, for $i \in I$, then $x_i^{(j)*} = 0$, for each $j \in J$, and $i \notin arg \min\limits_{t \in I}\{\frac{b_j}{c_{tj}} \mid m_{tj} = 1\}$. Since $c_{ij} \geq 0$ and $a_{ij} \geq 0$, we have $c_{ij}x_i^* \bigwedge a_{ij} \geq 0$, for each $j \in J$ and $i \notin arg \min\limits_{t \in I}\{\frac{b_j}{c_{tj}} \mid m_{tj} = 1\}$.
(II) If $x_i^* > 0$, for $i \in I$, then there exists $j \in J$ such that $x_i^* = x_i^{(j)*} = \frac{b_j}{c_{ij}}$ where $m_{ij} = 1$. Hence, it is concluded that $c_{ij}x_i^* = b_j$ and $c_{ij} \bigwedge a_{ij} \geq b_j$. Two recent relations are obtained $c_{ij}x_i^* = b_j$ and $a_{ij} \geq b_j$. So, $c_{ij}x_i^* \bigwedge a_{ij} \geq b_j$.
With regard to above cases, for each $j \in J$, we have: $\bigvee\limits_{i=1}^{n}(c_{ij}x_i^* \bigwedge a_{ij}) = \bigvee\limits_{i \in CaseI}(c_{ij}x_i \bigwedge a_{ij}) \bigvee \bigvee\limits_{i \in CaseII}(c_{ij}x_i \bigwedge a_{ij}) \geq 0 \bigvee b_j = b_j$. Therefore, the vector x^* is a feasible vector for the problem (1)-(3). Now, we show that the vector x^* is the optimal solution of the problem (1)-(3). Suppose that $x \in S(C,A,b)$ is an arbitrary feasible solution of the problem (1)-(3). By Lemma 3.3, there exists $p \in P$ such that $x^p \leq x$, where $x_i^p = \bigvee\limits_{j=1}^{m}\{\frac{b_j}{c_{ij}} \mid p_j = i\}$, for each $i \in I$. Hence, we can write $x_i \geq x_i^p = \bigvee\limits_{j=1}^{m}\{\frac{b_j}{c_{ij}} \mid p_j = i\} \geq \frac{b_j}{c_{ij}}$ where $m_{ij} = 1$. So, $x_i \geq \frac{b_j}{c_{ij}}$, for each $i \in I$ and $j \in J$, such that $m_{ij} = 1$. For each $i \in I$ and $j \in J$, if $i \in arg \min\limits_{t \in I}\{\frac{b_j}{c_{tj}} \mid m_{tj} = 1\}$, then we conclude that $x_i \geq \frac{b_j}{c_{ij}} = \bigwedge\limits_{t \in I}\{\frac{b_j}{c_{tj}} \mid m_{tj} = 1\} = x_i^{(j)*}$. Otherwise, $x_i = 0$. It is concluded that $x_i \geq \bigvee\limits_{j=1}^{m}\{x_i^{(j)*}\}$. So, for each $x \in S(C,A,b)$, we have $x \geq x^*$. Therefore, x^* is the optimal solution of the problem (1)-(3). □

With regard to Theorems 3.1 and 3.6, we focus on the components which can be candidates for the optimality in the problem (1)-(3). To do this, a value matrix is defined based on Theorem 3.1.

Definition 3.7. *The matrix $V = [v_{ij}]_{n\times m}$ corresponding to the system* (26) *is defined as follows:*

$$v_{ij} = \begin{cases} \frac{b_j}{c_{ij}}, & \text{if } i \in arg \min\limits_{t\in I}\{\frac{b_j}{c_{tj}} |\ m_{tj} = 1\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in I, j \in J.$$

With regard to Definition 3.7, the value matrix V is unique for the given system (26). Now, we are ready to find the optimal solution of the problem (1)-(3) using Definition 3.7 and Theorem 3.6. The following corollary is a direct result of Theorem 3.6 and Definition 3.7.

Corollary 3.8. *Suppose that $S(C, A, b) \neq \emptyset$. Then there exists at least an optimal solution for the problem* (1)-(3). *One of its optimal solutions is as $x^* = [x_i^*]_{i\in I}$ where $x_i^* = \bigvee\limits_{j=1}^{m} v_{ij}$.*

With regard to the above points, we design a matrix-based algorithm to find one of the optimal solutions of the problem (1)-(3).

Algorithm 3.9. *An algorithm for finding one of the optimal solutions of the problem* (1)-(3).
Step 1. *Create the matrix $M = [m_{ij}]_{n\times m}$ using Definition 2.14. Check the consistency of the system* (26) *using Lemma 2.15. If $S(C, A, b) \neq \emptyset$, go to Step 2. Otherwise, Stop!*
Step 2. *Create the matrix $V = [v_{ij}]_{n\times m}$ by Definition 3.7.*
Step 3. *Produce the vector $x^* = [x_i^*]_{i\in I}$ by $x_i^* = \bigvee\limits_{j=1}^{m} v_{ij}$ according to Corollary 3.8.*
Step 4. *Stop!*

We now focus on the analysis of the computational complexity of Algorithm 3.9. To do this, the computational costs of each its step are computed.
Step 1. To create the matrix $M = [m_{ij}]_{n\times m}$, we need to compute $n \times m$ components of the matrix M using Definition 2.14. If the computational cost of each component m_{ij} is $O(1)$, then the required time for this work is $T' = O(mn)$. To check the consistency of the system (26) using Lemma 2.15, we require checking all the components of each row of the matrix M. This process takes $T'' = O(mn)$ time. The total time of Step 1 is as $T_1 = T' + T'' = 2O(mn) = O(mn)$.
Step 2. The required time for computing the matrix V is $T_2 = O(mn)$.
Step 3. For computing each x_i^*, we need to find $\bigvee\limits_{j=1}^{m} v_{ij}$ which its required time is $O(m-1)$. The total time for calculating the vector $x^* = [x_i^*]_{i\in I}$ is as $T_3 = n \times O(m-1) = O(mn)$.
With regard to the above points, the computational complexity of Algorithm 3.9 is as: $T = T_1 + T_2 + T_3 = 3O(mn) = O(mn)$.

Now, we firstly explain the importance of the problem (1)-(3) from a practical point of view. Then, two numerical examples are provided to clarify the steps of Algorithm 3.9, in the next section.

4 Application and numerical examples

We now present an application example of the problem (1)-(3) for the optimal management of data transmission in the client-server networks.

Example 4.1. *A three-tier client-server structure is studied as an efficient data transmission mechanism. The middle-tier receives data with various quality levels from the server and sends them to the clients. In this paper, it is assumed that there exist m clients, denoted as $A_1, A_2, ..., A_m$ (see Figure 1).*

The client A_j has access to the regional server X_i through the network connection A_{ij}. The regional server X_i sends the data to the clients A_j through the network connection A_{ij} with a bandwidth a_{ij}. If the data transmitted from X_i to A_j has a quality level x_i less than the bandwidth a_{ij}, the data will be transmitted with $c_{ij}.x_i$ quality level to A_j via the network connection A_{ij}, where $c_{ij} \in [0, 1]$. It means that the file data can be sent to the clients A_j by the regional server X_i with various quality levels. However, the client A_j needs at least one data provider with at least quality level

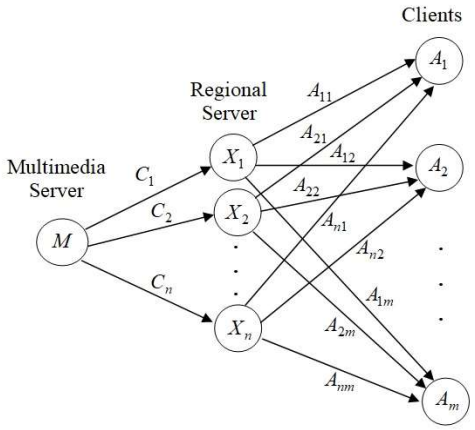

Figure 1: The structure of client-server networks

b_j. *These considerations can be formulated by the following FRIs using the max-min-product composition as follows:*

$$\begin{cases} (c_{11}x_1 \bigwedge a_{11}) \bigvee (c_{21}x_2 \bigwedge a_{21}) \bigvee ... \bigvee (c_{n1}x_n \bigwedge a_{n1}) \geq b_1, \\ (c_{12}x_1 \bigwedge a_{12}) \bigvee (c_{22}x_2 \bigwedge a_{22}) \bigvee ... \bigvee (c_{n2}x_n \bigwedge a_{n2}) \geq b_2, \\ \qquad\qquad\qquad \vdots \\ (c_{1m}x_1 \bigwedge a_{1m}) \bigvee (c_{2m}x_2 \bigwedge a_{2m}) \bigvee ... \bigvee (c_{nm}x_n \bigwedge a_{nm}) \geq b_m. \end{cases} \tag{33}$$

or equivalently,

$$\bigvee_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J = \{1, ..., m\}. \tag{34}$$

where $c_{ij}, a_{ij}, x_i, b_j \in [0, 1]$, for any $i \in I = \{1, ..., n\}$ and $j \in J$. The main disadvantage of a client-server model is the risk of system overload due to insufficient resources to serve all of the clients. If many different clients try to connect to the shared network at the same time, the connection may fail or become slow. In addition, if the internet connection goes down, no website or client in the world will be able to access the information. Large businesses may be at risk if they are unable to get the important information. Therefore, it is necessary to reduce network congestion and improve the stability of data transmission by minimizing the maximum quality level under the constraints (33) *(or* (34)*) as follows:*

$$\min \quad Z(x) = \bigvee_{i=1}^{n} x_i, \tag{35}$$

$$\text{s.t.} \quad \bigvee_{i=1}^{n} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J, \tag{36}$$

$$x_i \in [0, 1], \quad \forall i \in I, \tag{37}$$

where $c_{ij}, a_{ij}, b_j \in [0, 1]$, for any $i \in I$ and $j \in J$.

This section provides a numerical example from the above application to illustrate the steps of Algorithm 3.9.

Example 4.2. *Suppose that there exists a three-tier multimedia streaming architecture as Figure 1 with 8 clients $A_1, ..., A_8$ and 4 regional servers $X_1, ..., X_4$. The bandwidth between X_i and client A_j is given as a_{ij} in terms of the matrix A. The corresponding model to the system* (34) *for this example can be reduced to the following max-min-product fuzzy relation inequalities as follows:*

$$\bigvee_{i=1}^{4} (c_{ij}x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J = \{1, ..., 8\}$$

where $x = [x_i]_{4\times 1}$, $b = [b_j]_{8\times 1} = [0.45, 0.6, 0.5, 0.7, 0.8, 0.65, 0.85, 0.75]^T$,

$$A = [a_{ij}]_{4\times 8} = \begin{pmatrix} 0.5 & 0.73 & 0.6 & 0.9 & 0.9 & 1 & 0.15 & 0.8 \\ 0.6 & 0.35 & 0.7 & 0.8 & 0.8 & 0.75 & 0.55 & 0.4 \\ 0.45 & 0.5 & 0.8 & 0.45 & 0.35 & 0.3 & 0.85 & 0.2 \\ 0.7 & 0.65 & 0.65 & 0.5 & 0.9 & 0.5 & 0.9 & 0.9 \end{pmatrix}, \text{and}$$

$$C = [c_{ij}]_{4\times 8} = \begin{pmatrix} 0.85 & 0.7 & 0.55 & 0.8 & 0.85 & 0.7 & 0.3 & 0.85 \\ 0.7 & 0.9 & 0.6 & 0.75 & 0.95 & 0.8 & 0.5 & 0.55 \\ 0.6 & 0.6 & 0.3 & 0.35 & 0.25 & 0.9 & 0.9 & 0.35 \\ 0.35 & 0.75 & 0.8 & 0.4 & 1 & 0.4 & 0.95 & 0.8 \end{pmatrix}.$$

Here, the coefficient c_{ij} *denotes the weight of the* j^{th} *user downloading data from the* i^{th} *regional server. Now, we intend to find an optimal solution for the following programming problem with the max-min-product fuzzy relation inequalities as:*

$$\min \quad Z(x) = \bigvee_{i=1}^{4} x_i, \tag{38}$$

$$\text{s.t.} \quad \bigvee_{i=1}^{4} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J = \{1, ..., 8\}, \tag{39}$$

$$x_i \in [0, 1], \quad \forall i \in I = \{1, ..., 4\}. \tag{40}$$

We now apply Algorithm 3.9 to find the optimal solution of the problem (38)-(40).
Step 1. *We create the matrix* $M = [m_{ij}]_{4\times 8}$ *using Definition 2.14 as follows:*

$$M = [m_{ij}]_{4\times 8} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Since, for each $j \in J = \{1, ..., 8\}$, *there exists at least one* $i \in I$ *such that* $m_{ij} = 1$, *the system* (39)-(40) *is consistent, according to Lemma 2.15, and* $S(C, A, b) \neq \emptyset$. *Go to Step 2.*
Step 2. *The matrix* $V = [v_{ij}]_{4\times 8}$, *with regard to Definition 3.7, is as follows:*

$$V = [v_{ij}]_{4\times 8} = \begin{pmatrix} \frac{9}{17} & 0 & 0 & \frac{7}{8} & 0 & 0 & 0 & \frac{15}{17} \\ 0 & 0 & 0 & 0 & 0 & \frac{13}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{5}{8} & 0 & \frac{4}{5} & 0 & \frac{17}{19} & 0 \end{pmatrix}.$$

Step 3. *The optimal solution* $x^* = [x_i^*]_{i\in I}$, *according to Corollary 3.8, is as follows:* $x^* = (x_1^*, x_2^*, x_3^*, x_4^*) = (\frac{15}{17}, \frac{13}{16}, 0, \frac{17}{19})$ *with the optimal objective value* $Z^* = \frac{17}{19}$.

Remark 4.3. *If we apply the change-variable* $c_{ij}.x_i = y_{ij}$, *for each* $i \in I$ *and* $j \in J$, *in Example 4.2, then we will have* $|I| \times |J| = 4 \times 8 = 32$ *new variables. Therefore, a problem with four variables is converted to a new problem with 32 variables. If we apply the methods in* [8, 11, 12, 13, 20, 22, 23, 27] *to find the 32 variables, then the rate of computations increases by increasing the number of variables, considerably. After finding the values* y_{ij}*'s, we should solve the following equations* $c_{ij}.x_i = y_{ij}$, *for each* $i \in I$ *and* $j \in J$. *For example, if* $i = 1$, *then we should solve eight equations to obtain the value* x_1 *as follows:* $c_{1j}.x_1 = y_{1j}$, *for each* $j \in J$, *which can create distinct values for* x_1, *simultaneously. Therefore, the processes may cause inconsistency in obtaining a value for* x_1.

The following example shows that the optimal solution of the problem (1)-(3) is not unique, in a general case.

Example 4.4. *Consider the problem* (1)-(3) *as follows:*

$$\min \quad Z(x) = \bigvee_{i=1}^{4} x_i, \tag{41}$$

$$\text{s.t.} \quad \bigvee_{i=1}^{4} (c_{ij}.x_i \bigwedge a_{ij}) \geq b_j, \quad \forall j \in J = \{1,2,3,4,5\}, \tag{42}$$

$$x_i \in [0,1], \quad \forall i \in I = \{1,2,3,4\}, \tag{43}$$

where $x = [x_i]_{4\times 1}$, $b = [b_j]_{5\times 1} = [0.3, 0.4, 0.6, 0.45, 0.5]^T$,

$$A = [a_{ij}]_{4\times 5} = \begin{pmatrix} 0.4 & 0.46 & 0.5 & 1 & 0.85 \\ 0.45 & 0.55 & 0.4 & 0.6 & 0.8 \\ 0.5 & 0.45 & 0.66 & 0.55 & 0.9 \\ 0.45 & 0.64 & 0.8 & 0.65 & 0.6 \end{pmatrix}, \text{and}$$

$$C = [c_{ij}]_{4\times 5} = \begin{pmatrix} 0.5 & 0.8 & 0.6 & 0.9 & 0.9 \\ 0.5 & 0.8 & 0.8 & 1 & 0.85 \\ 0.5 & 0.8 & 0.4 & 0.75 & 0.4 \\ 0.5 & 0.8 & 1 & 0.95 & 1 \end{pmatrix}.$$

Algorithm 3.9 is applied to find the optimal solution of the problem (41)-(43).
Step 1. *According to Definition 2.14, the matrix* $M = [m_{ij}]_{4\times 5}$ *is as follows:*

$$M = [m_{ij}]_{4\times 5} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since, for each $j \in J$, *there exists at least one* $i \in I$ *such that* $m_{ij} = 1$, *the system* (42)-(43) *is consistent, according to Lemma 2.15, and* $S(C, A, b) \neq \emptyset$. *Go to Step 2.*
Step 2. *According to Definition 3.7, the matrix* $V = [v_{ij}]_{4\times 5}$ *is as follows:*

$$V = [v_{ij}]_{4\times 5} = \begin{pmatrix} 0.6 & 0.5 & 0 & 0.5 & 0.556 \\ 0.6 & 0.5 & 0 & 0.45 & 0.588 \\ 0.6 & 0.5 & 0 & 0.6 & 0 \\ 0.6 & 0.5 & 0.6 & 0.474 & 0.5 \end{pmatrix}.$$

Step 3. *The optimal solution* x^*, *according to Corollary 3.8, is as follows:* $x^* = (x_1^*, x_2^*, x_3^*, x_4^*) = (0.6, 0.6, 0.6, 0.6)$ *with the optimal objective value* $Z^* = 0.6$. *In this example, the optimal solution is not unique. For example,* $x^{**} = (0.6, 0.5, 0.5, 0.6)$ *is the other optimal solution with* $Z^{**} = 0.6$.

5 Conclusions and future researches

In this paper, a system of fuzzy relation inequalities was introduced using the max-min-product operator. The structure of its solution set was investigated based on the maximum solution and minimal solutions of the system. Some necessary and sufficient conditions were given to check the consistency of the system based on the FRI characteristic matrix. Then, the latticized linear programming problem provided to the system of max-min-product FRIs was studied and decomposed to some simple problems. The optimal solution of the original problem was obtained based on the optimal solutions of the simple problems and a closed form was proposed to compute its optimal solution. Using the above points, we designed a matrix-based-algorithm with a polynomial time complexity as $O(mn)$. Finally, an application of the problem was given to decrease the congestion of client-server networks. In future research directions, we will focus on finding all the optimal solutions for the problem (1)-(3). Moreover, we will study the system (2)-(3) with different management objectives and focus on their solving algorithms.

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