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Aspects of conditional symmetry and asymmetry of copulas

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Abstract

The assumption of conditional symmetry, which implies that the distribution of one random variable given another random variable has a symmetric form, plays a crucial role in various probability and statistics problems. This study aims to examine copula properties related to the conditional symmetry/asymmetry of two random variables. We investigate the possibility of ordering copulas based on their level of conditional asymmetry, similar to the concordance ordering for dependence. To quantify the degree of conditional asymmetry of a copula, we introduce measures that are monotonic for the proposed ordering. The characteristics of the proposed order and measures are elaborated upon. Several examples are included to demonstrate the results.

Keywords: Asymmetry, bivariate symmetry, conditional symmetry, copula, dependence.

1 Introduction

Finding a suitable model for multivariate data is a crucial component in many applications when departures from the assumption of normality can be found in data. Following Sklar's Theorem [26] a joint cumulative distribution function (CDF) can then be expressed in terms of a unique copula if the marginal distribution functions are continuous. Since copulas allow for the modeling of different dependence structures of data, modeling multivariate data using copulas has found many applications in different fields of research in the last decades; see, e.g, [6, 12, 16, 20] for a review and references therein.

Modeling the dependence structure of random variables in practice often involves fitting an appropriate copula to a given set of data. Various families of copulas with different dependence properties have been developed. However, selecting a suitable copula for a dataset from the numerous available options is a challenging task during the model-fitting process.

One approach to model selection involves initially testing copula structures, such as symmetry properties, to exclude copulas that lack the necessary symmetry properties to accurately depict the data. Thus it is necessary to know which symmetry property could be present in the data. In the univariate case, there is a single definition for the symmetry of a random variable X around the point a, and it is defined by $X - a \stackrel{d}{=} a - X$, where $\stackrel{d}{=}$ is the equality in distributions. In the bivariate and multivariate cases, there are several definitions for the symmetry of a random vector. Nelsen [19] examines several definitions of bivariate symmetry, including joint symmetry, radial symmetry, exchange (permutation) symmetry, and conditional symmetry. Since many symmetry characteristics of random variables can be studied by using copulas, they play an important role in modeling the dependence structure between random variables. Several copula-based measures of permutation asymmetry and statistical tests for permutation symmetry are defined and studied in the literature; see, e.g., [11, 15, 17, 21]. Measures of radial asymmetry and tests for radial symmetry could be seen in [1, 4, 7, 9, 14, 22] and references therein.

The assumption of conditional symmetry, which implies that the distribution of one random variable given another random variable has a symmetric form, plays an important role in many problems in probability and statistics. For instance, if the error terms of a time series model or regression model have some degree of conditional asymmetry, then

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conditional symmetric models are not adequate. The test of conditional symmetry of one variable about zero, given another variable, also known as the test of conditional symmetry of residuals, is a statistical test used to investigate the symmetry of the relationship between the residuals and the predictor variable. Since the conditional distribution is more difficult to estimate compared with the unconditional distribution, testing conditional symmetry is more complex than testing unconditional symmetry. Several existing methods use non-parametric smoothing techniques to estimate conditional distribution; see, e.g., [3, 5, 8, 13, 27].

Despite the widespread use of conditional symmetry, there is little work on the measures of conditional asymmetry and the copula properties of this concept of symmetry in the literature. Therefore, the central contribution of this paper is to investigate the copula properties of conditional symmetry. Following the notion of non-exchangeability order defined in [25], another objective of the present work is to explore the possibility of ordering copulas using their conditional asymmetry level and study its implications. This is motivated by providing a partial answer to this question: there are inequalities for conditional asymmetry of copulas, like those resulting from the concordance ordering [20] for dependence. To measure the degree of conditional asymmetry of a copula, we define measures of conditional asymmetry that are monotone for the proposed order.

The rest of this paper is organized as follows. Section 2 discussed the conditional symmetry concept in terms of copulas. An ordering of conditional asymmetry for copulas is discussed in Section 3. In Section 4, two conditional asymmetry measures that preserve the proposed order are examined. Section 5 concludes the paper.

2 Conditionally symmetric copulas

Let (X, Y) be a pair of continuous random variables with the joint distribution function $F(x, y) = P(X \leq x, Y \leq y)$, and univariate marginal distributions $F_X(x) = P(X \leq x)$, $F_Y(y) = P(Y \leq y)$, at each $x, y \in \mathbb{R}$. Following Sklar's Theorem [20] the joint cumulative distribution function of X and Y can then be expressed at each $x, y \in R$ as $F(x, y) = C(F_X(x), F_Y(y))$ in terms of a unique copula C, if F_X and F_Y are continuous. Consider the reflections associated with a copula C given by

$$C^*(u, v) = u - C(u, 1 - v), \quad C^{**}(u, v) = v - C(1 - u, v), \tag{1}$$

and $\widehat{C}(u, v) = u+v-1+C(1-u, 1-v)$. The copula C is radially symmetric if $C(u, v) = \widehat{C}(u, v)$ for all $u, v \in [0, 1]$. Note that the copula of the pair $(f(X), g(Y))$ is related to the copula of (X, Y) as follows (see Theorem 2.4.4 in [20]). The copula of $(f(X), g(Y))$ is given by C^* if f is strictly increasing and g is strictly decreasing; C^{**} if f strictly decreasing and g is strictly increasing. If both f and g are strictly decreasing, then the copula of $(f(X), g(Y))$ is given by $\widehat{C}$, which is the survival copula associated with C. The copula of the pair (Y, X) is given by $C^T(u, v) = C(v, u)$. A copula C is exchangeable (permutation symmetric), if $C(u, v) = C^T(u, v)$ [20]. The formal definition of conditional symmetry is given in the following.

Definition 2.1. *Let $F_{Y|X}(y|x) = P(Y \leq y|X = x)$ denote the conditional distribution of Y given $X = x$. The random variable Y is said to be conditionally symmetric (around zero) given $X = x$ if, $(Y|X = x)$ is symmetric; i.e, $F_{Y|X}(y|x) = F_{-Y|X}(y|x)$. We use the notation $(Y|X = x) \stackrel{d}{=} (-Y|X = x)$, for all x, where $\stackrel{d}{=}$ is the equality in distribution.*

Remark 2.2. *If distribution of Y given $X = x$ is continuous, then conditionally symmetric of Y given $X = x$ is equivalent to $F_{Y|X}(y|x) = 1 - F_{Y|X}(-y|x)$, for all x and y.*

The conditional symmetry of two random variables could be expressed in terms of their joint distribution function.

Proposition 2.3. *For a pair (X, Y) of absolutely continuous random variables, $(Y|X = x) \stackrel{d}{=} (-Y|X = x)$ for all x, if, and only if, $(X, -Y) \stackrel{d}{=} (X, Y)$.*

Proof. If $(Y|X=x) \stackrel{d}{=} (-Y|X=x)$ for all x, then

$$\begin{aligned} P(X \le x, Y \le y) &= \int_R P(X \le x, Y \le y|X=t)dF_X(t) \\ &= \int_{[t \le x]} P(Y \le y|X=t)dF_X(t) \\ &= \int_{[t \le x]} P(-Y \le y|X=t)dF_X(t) \\ &= \int_R P(X \le x, -Y \le y|X=t)dF_X(t) \\ &= P(X \le x, -Y \le y), \end{aligned}$$

that is $(X,Y) \stackrel{d}{=} (X,-Y)$. Conversely, if $(X,Y) \stackrel{d}{=} (X,-Y)$, then $F_{X,Y}(x,y) = F_{X,-Y}(x,y)$, for all x and y. If X has the density function $f_X(.)$, then

$$\frac{\partial}{\partial x}F_{X,Y}(x,y) = \frac{\partial}{\partial x}F_{X,-Y}(x,y),$$

or equivalently, $f_X(x)F_{Y|X}(y|x) = f_X(x)F_{-Y|X}(y|x)$, for all x and y, which is the required result. □

Remark 2.4. *Note that the conditional symmetry of random variables X and Y implies mean independence, i.e., $E(Y|X=x) = 0$, for all x, as long as the expectation exists. Conditionally symmetric random variables are also uncorrelated; i.e., $cov(X,Y) = 0$, as long as the second moments exist.*

If the pair (X,Y) has the copula C, then the copulas associated with $(X,-Y)$ and $(-X,Y)$, are given by C^* and C^{**}, define by (1). In terms of uniform (0,1) random variables, because of Sklar's Theorem, if the copula C is the joint distribution of (U,V), then C^* and C^{**} are the distribution of $(U,1-V)$ and $(1-U,V)$, respectively [20]. Because the condition for conditional symmetry involves the joint distribution of two pairs (X,Y) and $(X,-Y)$, it is natural to ask if their copulas play a role in conditional symmetry. The answer is provided by the next proposition.

Proposition 2.5. *Let (X,Y) be a pair of absolutely continuous random variables with associated copula C. $(Y|X=x) \stackrel{d}{=} (-Y|X=x)$, for all x if, and only if, $Y \stackrel{d}{=} -Y$ and $C(u,v) = C^*(u,v)$ for all $u,v \in [0,1]$, where C^* is the associated copula defined by (1).*

Proof. If $(Y|X=x) \stackrel{d}{=} (-Y|X=x)$, for all x, then from Proposition 2.3, we have that $(X,-Y) \stackrel{d}{=} (X,Y)$, that is $C(u,v) = C^*(u,v)$, for all u,v. Conversely, if $C(u,v) = C^*(u,v)$ for all u,v, then

$$P(V \le v|U=u) = \frac{\partial}{\partial u}C(u,v) = \frac{\partial}{\partial u}C^*(u,v) = P(1-V \le v|U=u),$$

(see, e.g., [20]). By using the transformations $F_X(x) = u$ and $F_Y(y) = v$, because of Sklar's Theorem, under the symmetry of Y, i.e., $1 - F_Y(y) = F_Y(-y)$ we have that

$$P(F_Y(Y) \le F_Y(y)|F_X(X) = F_X(x)) = P(F_Y(-Y) \le F_Y(y)|F_X(X) = F_X(x)),$$

which implies, $P(Y \le y|X=x) = P(-Y \le y|X=x)$, for all x,y. □

Note that we have a similar result for the conditional symmetry of X given $Y=y$. That is, $(X|Y=y) \stackrel{d}{=} (-X|Y=y)$, if, and only if, $X \stackrel{d}{=} -X$ and $C(u,v) = C^{**}(u,v)$ for all $u,v \in [0,1]$.

Remark 2.6. *Since for a pair (U,V) of uniform (0,1) random variables with the copula C, $P(1-U \le u|V=v) = \frac{\partial}{\partial v}C^{**}(u,v)$ and $P(1-V \le v|U=u) = \frac{\partial}{\partial u}C^*(u,v)$, we should define the conditional symmetry of a copula for the first and second variable.*

Definition 2.7. *We say that a copula C is conditionally symmetric for the first variable if, $C(u,v) = C^{**}(u,v)$, and for the second variable if, $C(u,v) = C^*(u,v)$, for all $u,v \in [0,1]$.*

Remark 2.8. *Note that a copula is conditionally symmetric is not the same as random variables being conditionally symmetric. The conditionally symmetric is a property of joint distribution functions. For marginally symmetric random variables, conditional symmetry is equivalent to the symmetry of their copula as expressed in Proposition 2.5. Since a copula C can also be considered as the joint distribution function of a pair (U, V) of uniform (0,1) random variables, conditional symmetry of C in the second variable, is corresponds to $(V|U = u) \stackrel{d}{=} (1 - V|U = u)$, for all u. But, $V \stackrel{d}{=} 1 - V$ always holds. Thus for uniform (0,1) random variables, the conditional symmetry of copula and random variables is equivalent.*

Note that the conditional symmetry of a copula for the first and second variables is not the same. If $C(u, v) = C^*(u, v) = C^{**}(u, v)$, then the copula C is called jointly symmetric. A copula may satisfies $C = C^*$ but $C \neq C^{**}$ as the following example shows.

Example 2.9. *For a given copula D, define*

$$C_1(u, v) = \frac{D(u, v) + D^*(u, v)}{2}, \quad C_2(u, v) = \frac{D(u, v) + D^{**}(u, v)}{2}, \tag{2}$$

for all $u, v \in [0, 1]$. Then, the copulas C_1 and C_2 satisfy $C_1(u, v) = C_1^*(u, v)$ and $C_2(u, v) = C_2^{**}(u, v)$, but $C_1^*(u, v) \neq C_1^{**}(u, v)$ and $C_2^*(u, v) \neq C_2^{**}(u, v)$. Figure 1 shows the contour plots for the density function of the conditionally symmetric versions of Ali-Mikhail-Haq (AMH) copula given by $D(u, v) = \frac{uv}{1-\theta(1-u)(1-v)}$, with the parameter $\theta \in [-1, 1]$ (see, [20]), constructed by (2). Note that, the AMH copula is absolutely continuous and has a density function; i.e., $\frac{\partial^2}{\partial u \partial v} D(u, v) > 0$.

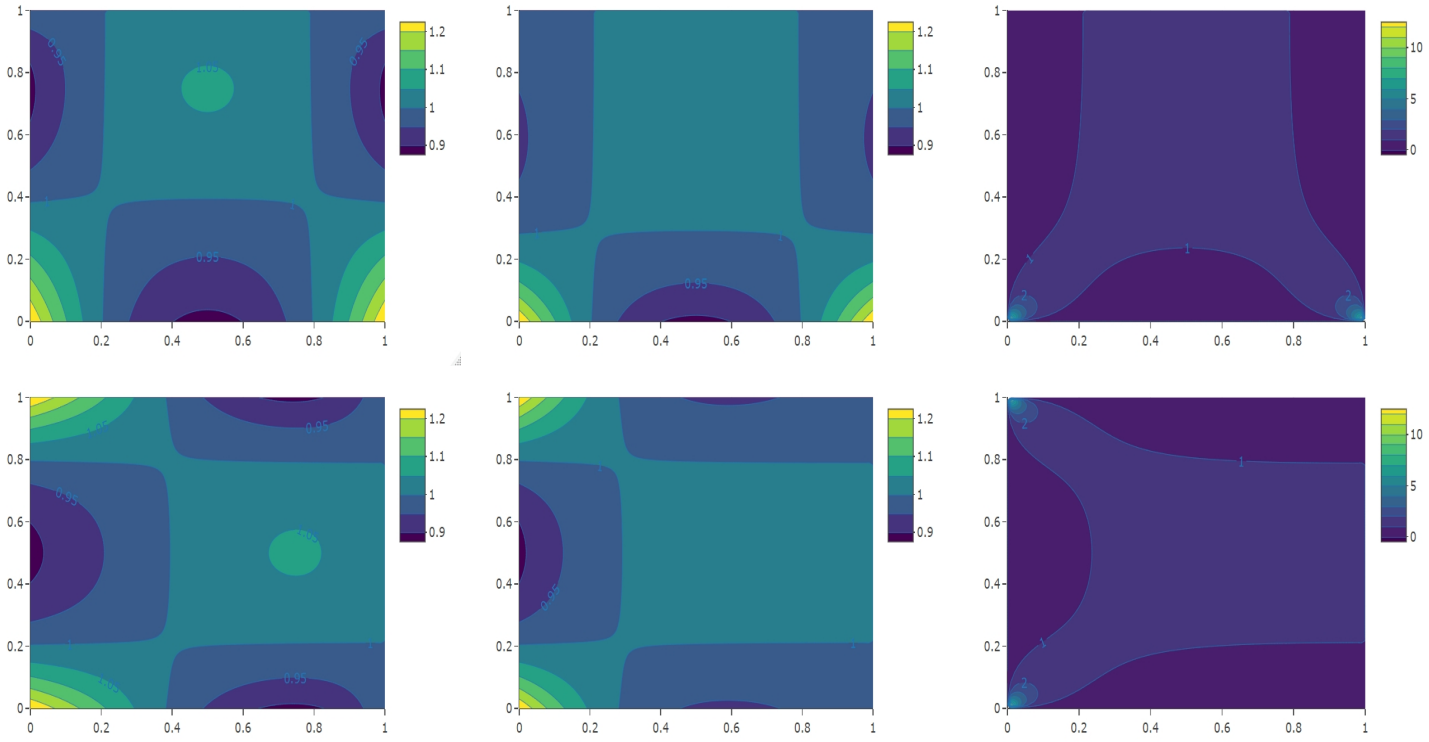


Figure 1: Contour plots for the density of the conditionally symmetric versions of the Ali-Mikhail-Haq copula defined in Example 2.9 by C_1 (top panels) and C_2 (bottom panels) with the parameters (left to right) $\theta = -1, \theta = 0.5$ and $\theta = 1$.

The following result provides a property for conditionally symmetric copulas.

Proposition 2.10. *A copula C is conditionally symmetric in the second variable, if, and only if*

$$\frac{\partial}{\partial u} C(u, \frac{1}{2} - v) + \frac{\partial}{\partial u} C(u, \frac{1}{2} + v) = 1, \tag{3}$$

for all $u \in [0,1]$ and $v \in [0, \frac{1}{2})$. A similar result holds for the conditional symmetry in the first variable.

Proof. Integrating both sides of (3) with respect to u gives $C(u, \frac{1}{2} - v) + C(u, \frac{1}{2} + v) = u$. By changing $\frac{1}{2} + v$ to v we have $C = C^*$. Conversely, if $C(u,v) = C^*(u,v)$, then by changing v to $v + \frac{1}{2}$, we have that $C(u, \frac{1}{2} + v) = u - C(u, \frac{1}{2} - v)$. Differentiating both sides for u gives (3). □

Corollary 2.11. *If the copula C is an absolutely continuous copula with the joint density function given by $\frac{\partial^2}{\partial u \partial v} C(u,v)$, then, C is conditionally symmetric in the second variable if, and only if, C satisfies*

$$\frac{\partial^2}{\partial u \partial v} C(u, \frac{1}{2} + v) = \frac{\partial^2}{\partial u \partial v} C(u, \frac{1}{2} - v),$$

and $C(u, \frac{1}{2}) = \frac{u}{2}$, for all $u \in [0,1]$ and $v \in [0, \frac{1}{2})$.

The following example provides a class of conditionally symmetric copulas.

Example 2.12. *Let C_θ be the copula defined by*

$$C_\theta(u,v) = uv + \theta \psi(u)\psi(v), \tag{4}$$

where $\theta \in [-1,1]$ and ψ is a function on $[0,1]$ with $\psi(0) = \psi(1) = 0$ and $|\psi(u) - \psi(v)| \leq |u - v|$ for all $u, v \in [0,1]$ [2]. If $\psi(u) + \psi(1-u) = 0$ for all $u \in [0,1]$, then $C_\theta = C_\theta^*$ and $C_\theta = C_\theta^{**}$. For instance, consider the function $\psi(u) = u(1-u)(1-2u)$, for all $u \in [0,1]$. Then C_θ is a member of the parametric family of copulas with cubic sections studied in [20].

3 Ordering of conditional asymmetry

Recall that for two pairs (X_1, Y_1) and (X_2, Y_2) with associated copulas C_1 and C_2, the pair (X_1, Y_1) is said to be less dependent than the pair (X_2, Y_2), in the sense of concordance ordering (denoted by $C_1 \prec_c C_2$), if $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in [0,1]^2$ [20]. Similar to this order of dependence on copulas, to compare their dependence, in this section, motivated by the idea in [25] for ordering of permutation asymmetry, we define conditional asymmetry ordering, on the class of copulas, by which copulas can be ordered based on their level of conditional asymmetry.

3.1 The proposed order

In practical problems such as regression analysis, $Y = f(x;\theta) + \epsilon$, the conditional symmetry of the response variable given the regression variable, i.e., $(Y|X = x)$, is of interest [13]. Therefore, in the following, we consider only the conditional symmetry in the second variable (given the first variable) and state the results. Similar results are obtained for the conditional symmetry for the first variable (given the second variable). A copula C is conditionally asymmetric if $C(u,v) \neq C^*(u,v)$, for some $u, v \in$[0,1]. Thus

$$\{(u,v) : C(u,v) - C^*(u,v) \neq 0\},$$

is the set of conditional asymmetry points of the copula C. A comparison of asymmetry sets of two copulas can be used to define an asymmetry order.

Definition 3.1. *For two copulas C_1 and C_2, the copula C_2 is said to be more conditionally asymmetric (CA) than the copula C_1 (denoted by $C_1 \prec_{CA} C_2$), if and only if, for all $u, v \in [0,1]$,*

$$|C_1(u,v) - C_1^*(u,v)| \leq |C_2(u,v) - C_2^*(u,v)|. \tag{5}$$

A totally ordered parametric family C_θ of copulas is positively ordered with respect to $\prec_{CA}$ if $C_{\theta_1} \prec_{CA} C_{\theta_2}$, whenever $\theta_1 \leq \theta_2$ and is negatively ordered with respect to $\prec_{CA}$ if $C_{\theta_1} \prec_{CA} C_{\theta_2}$, whenever $\theta_1 \geq \theta_2$. We call the ordering $\prec_{CA}$ as the conditional asymmetry order on the class of copulas denoted by $\mathcal{C}$.

Remark 3.2. *From Definition 3.1 the following properties hold:*
1. *For every copula $C \in \mathcal{C}$, it follows that $C \prec_{CA} C^*$ and $C^* \prec_{CA} C$.*
2. *The relation $\prec_{CA}$ is reflexive and transitive. But it is not antisymmetric. Since for two copulas $C_1, C_2 \in \mathcal{C}$, if $C_1 \prec_{CA} C_2$ and $C_2 \prec_{CA} C_1$, then $|C_1 - C_1^*| = |C_2 - C_2^*|$ which does not imply that $C_1 = C_2$ (consider, for instance, condionaly symmetric C_1 and C_2). Therefore, $\prec_{CA}$ is a preorder and not an order; nevertheless, we will use the term order of conditional asymmetry for $\prec_{CA}$.*
3. *The relation $\prec_{CA}$ is not total order, i.e., some copulas cannot be ordered by $\prec_{CA}$.*

In the following, we provide several examples for the proposed $\prec_{CA}$ order.

Example 3.3. *Consider the FGM family of copulas* [20] *given by*

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v), \quad u,v \in [0,1], \tag{6}$$

where $\theta \in [-1,1]$. *This copula satisfies* $C^*_\theta(u,v) = uv - \theta uv(1-u)(1-v) = C_{-\theta}(u,v)$ *and then* $|C_\theta(u,v) - C^*_\theta(u,v)| = 2uv(1-u)(1-v)|\theta|$. *Thus, for* $\theta \in [0,1]$*(resp,* $\theta \in [-1,0]$*) the copula* C_θ *is positively (resp, negatively) ordered concerning the* $\prec_{CA}$.

Example 3.4. *Let* $C_{\alpha,\beta}$ *be the two-parameter family of copulas given by* [20],

$$C_{\alpha,\beta}(u,v) = uv + uv(1-u)(1-v)(\alpha + (\beta-\alpha)v(1-u)), \quad u,v \in [0,1], \tag{7}$$

where $\alpha \in [-1,1]$ *and* $(\alpha - 3 - \sqrt{9+6\alpha-3\alpha^2})/2 \leq \beta \leq 1$. *Since* $|C_{\alpha,\beta}(u,v) - C^*_{\alpha,\beta}(u,v)| = uv(1-u)(1-v)|\alpha+\beta|$, *then* $C_{\alpha_1,\beta_1} \prec_{AC} C_{\alpha_2,\beta_2}$, *for every* α_1, α_2, β_1 *and* β_2 *such that* $|\alpha_1+\beta_1| \leq |\alpha_2+\beta_2|$.

Example 3.5. *Let* C_θ *be the copula defined by (4). We note that if* $\psi(u) + \psi(1-u) \neq 0$, *then* $C_\theta \neq C^*_\theta$. *Since* $|C_\theta(u,v) - C^*_\theta(u,v)| = |\theta||\psi(u)[\psi(v) - \psi(1-v)]|$, *then* C_θ *is positively (resp, negatively) ordered with respect to* $\prec_{CA}$ *for* $\theta > 0$ *(resp,* $\theta < 0$*).*

3.2 Properties of the proposed order

For dependence concepts, by using the fact that any copula C satisfies $W(u,v) \leq C(u,v) \leq M(u,v)$, for all $u,v \in [0,1]$, where $W(u,v) = \max(u+v-1,0)$ and $M(u,v) = \min(u,v)$, the maximal and minimal members of the class of copulas are M and W, respectively. The following result studies the minimal members of the class of conditionally asymmetric copulas.

Definition 3.6. *Consider a class of copulas* $\mathcal{C}$ *with the order* $\prec_{CA}$.
A copula $D \in \mathcal{C}$ *is called a maximal member if* $C \prec_{CA} D$ *for all* $C \in \mathcal{C}$ *which are comparable to* D, *i.e., which satisfy* $C \prec_{CA} D$ *or* $D \prec_{CA} C$. *A copula* $D \in \mathcal{C}$ *is called a greatest member if* $C \prec_{CA} D$ *for all* $C \in \mathcal{C}$. *It is clear that any greatest member is also maximal, but the converse, in general, is not true. Analogously, the minimal and the smallest members of* $\mathcal{C}$ *with respect to* $\prec_{CA}$ *could be defined.*

The following results state that the conditionally symmetric copulas are the least conditionally asymmetric copulas, and there is no unique maximally conditionally asymmetric copula. The following result shows that the copulas M and W are two maximal members in the class of all copulas.

Proposition 3.7. *If* $\mathcal{C}$ *is the class of all copulas then:*
(i) *Each conditionally symmetric copula is the smallest (hence a minimal) member of* $\mathcal{C}$ *for* $\prec_{CA}$ *and there are no other minimal elements.*
(ii) *the copulas* M *and* W *are two maximal member of* $\mathcal{C}$, *with respect to* $\prec_{CA}$, *that is, for every* $C \in \mathcal{C}$, $C \prec_{CA} M$ *and* $C \prec_{CA} W$.

Proof. Note that each conditionally symmetric copula C is comparable to any other copula D, and satisfies $C \prec_{CA} D$. Suppose B is the smallest member of $\mathcal{C}$. In that case, it must be comparable to all conditionally symmetric copulas, that is $B \prec_{CA} C$, i.e., $0 < |B(u,v) - B^*(u,v)| \leq |C(u,v) - C^*(u,v)| = 0$, for all $u,v \in [0,1]$ and thus any minimal member must be conditionally symmetric itself. To prove (ii), since each copula C satisfies the Frechét-Hoeffding bounds [20],

$$W(u,v) \leq C(u,v) \leq M(u,v), \quad for\ all\ u,v \in [0,1], \tag{8}$$

then for the copula C^*, $W(u,v) \leq C^*(u,v) \leq M(u,v)$, and we have that $|C(u,v) - C^*(u,v)| \leq M(u,v) - W(u,v)$, for all $u,v \in [0,1]$. Since $M^* = W$ and $W^* = M$, then

$$|C(u,v) - C^*(u,v)| \leq M(u,v) - M^*(u,v), \quad |C(u,v) - C^*(u,v)| \leq W(u,v) - W^*(u,v),$$

for all $u,v \in [0,1]$, which is the required result. □

We have demonstrated that the copulas M and W are the maximal elements within $\mathcal{C}$. In the following, we contend that they also represent the greatest members.

Proposition 3.8. *The copulas M and W are the largest members of $\mathcal{C}$ with respect to $\prec_{CA}$. Specifically, for every $D \in \mathcal{C}$, the condition $|D(u,v)-D^*(u,v)| = M(u,v)-W(u,v)$ holds for all $(u,v) \in [0,1]^2$ if and only if $D(u,v) = M(u,v)$ or $D(u,v) = W(u,v)$ for all $(u,v) \in [0,1]^2$.*

Proof. If $D(u,v) = M(u,v)$ (resp, $W(u,v)$) for all $(u,v) \in [0,1]^2$, then $D^*(u,v) = W(u,v)$ (resp, $M(u,v)$) for all $(u,v) \in [0,1]^2$. Consequently, the equality $|D(u,v) - D^*(u,v)| = M(u,v) - W(u,v)$ holds for all $(u,v) \in [0,1]^2$ trivially. To demonstrate necessity, define:

$$A_1 = \{(u,v) \in [0,1]^2 : D(u,v) > D^*(u,v)\},$$

$$A_2 = \{(u,v) \in [0,1]^2 : D(u,v) < D^*(u,v)\},$$

$$A_3 = \{(u,v) \in [0,1]^2 : D(u,v) = D^*(u,v)\}.$$

Note that $A_i \cap A_j = \emptyset$, $1 \le i < j \le 3$, and $A_1 \cup A_2 \cup A_3 = [0,1]^2$. For each $(u,v) \in A_1$, we have $D(u,v) - D^*(u,v) = M(u,v) - W(u,v)$, leading to:

$$W(u,v) - D^*(u,v) = M(u,v) - D(u,v) \ge 0.$$

From Frechét-Hoeffding bounds, since $W(u,v) \le D^*(u,v)$ for all $(u,v) \in [0,1]^2$, this equality holds only if for all $(u,v) \in A_1$, $D^*(u,v) = W(u,v)$ or $D(u,v) = M(u,v)$. Similarly, for $(u,v) \in A_2$:

$$D^*(u,v) - D(u,v) = M(u,v) - W(u,v),$$

which leads to:

$$W(u,v) - D(u,v) = M(u,v) - D^*(u,v) \ge 0.$$

Again, from Frechét-Hoeffding bounds, equality holds only, implying $D(u,v) = W(u,v)$ for all $(u,v) \in A_2$. For each $(u,v) \in A_3$, it must be that $D(u,v) = M(u,v) = W(u,v)$, which is satisfied only on the boundaries of $[0,1]^2$, thus $A_3 = \{(0,v),(1,v),(u,1),(u,0) : u,v \in [0,1]\}$. Next, we show that either $A_1 = \emptyset$ or $A_2 = \emptyset$. If both $A_1 \ne \emptyset$ and $A_2 \ne \emptyset$, it leads to a contradiction. Let $Q = \bar{A}_1 \cup \bar{A}_2$ where $\bar{A}_i$ is the closure of A_i, $i = 1,2$. Since $A_1 \cup A_2 = (0,1) \times (0,1)$, then for each boundary point $(u_0,v_0) \in Q \cap (0,1) \times (0,1)$, we have $M(u_0,v_0) \ne W(u_0,v_0)$. For every $r > 0$, consider the neighborhood $N(r,p_0) = \{(u,v) : (u-u_0)^2 + (v-v_0)^2 \le r^2\}$ around the point $p_0 = (u_0,v_0)$. Since p_0 is a boundary point, then part of $N(r,p_0)$ lies in A_1 and part in A_2, implying $N(r,p_0) \cap A_1 \ne \emptyset$ and $N(r,p_0) \cap A_2 \ne \emptyset$. For $\epsilon = (M(u_0,v_0) - W(u_0,v_0))/2$, we find that $|D(u,v) - D(u_0,v_0)| \ge M(u_0,v_0) - W(u_0,v_0) > \epsilon$. This implies that the copula D cannot be continuous at each boundary point (u_0,v_0), which contradicts the continuity property of copulas (see Theorem 2.2.4 in [20]). □

In the following, we provide several examples to illustrate the ordering property of conditionally asymmetric copulas.

Example 3.9. *Consider the Cuadras-Augé copula* [20],

$$C_\theta(u,v) = (uv)^\theta \min(u,v)^{1-\theta},$$

for all $u,v \in [0,1]$, $\theta \in [0,1]$ and the Clayton copula [20],

$$C_\theta(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}},$$

for all $u,v \in [0,1]$ and $\theta > 0$. For the Cuadras-Augé copula, $C_1(u,v) = \min(u,v)$ and for the Clayton copula $\lim_{\theta\to\infty} C_\theta(u,v) = \min(u,v)$. Figure 2 shows the plots of $|C(u,v) - C^*(u,v)|$ for the Cuadras-Augé copula and the Clayton copula with different parameters. As we see, for both copulas $|C_\theta(u,v) - C^*_\theta(u,v)| \to |M(u,v) - M^*(u,v)| = |M(u,v) - W(u,v)|$, as the parameter θ increases.

The following results provide an invariance property for $\prec_{CA}$ order. The invariance properties ensure that the conditional asymmetry order is not affected under permutation and strict monotone transformations of both variables.

Proposition 3.10. *If C_1 and C_2 are two copulas such that $C_1 \prec_{CA} C_2$ then $C_1^T \prec_{CA} C_2^T$ and $\widehat{C}_1 \prec_{CA} \widehat{C}_2$.*

Proof. Since $|C^T(u,v) - C^{T^*}(u,v)| = |C(v,u) - v + C(v,1-u)|$, letting $1-u = t$, we have that $|C(v,1-t) - v + C(v,t)| = |C(v,t) - C^*(v,t)|$ and thus $C_1^T \prec_{CA} C_2^T$. Similarly, for $|\widehat{C}(u,v) - \widehat{C}^*(u,v)| = |C(1-u,1-v) - (1-u-C(1-u,v))|$, letting $1-u = t$ and $1-v = s$, we have that $|C(t,s) - (t - C(t,1-s))| = |C(t,s) - C^*(t,s)|$, and then $\widehat{C}_1 \prec_{CA} \widehat{C}_2$. □

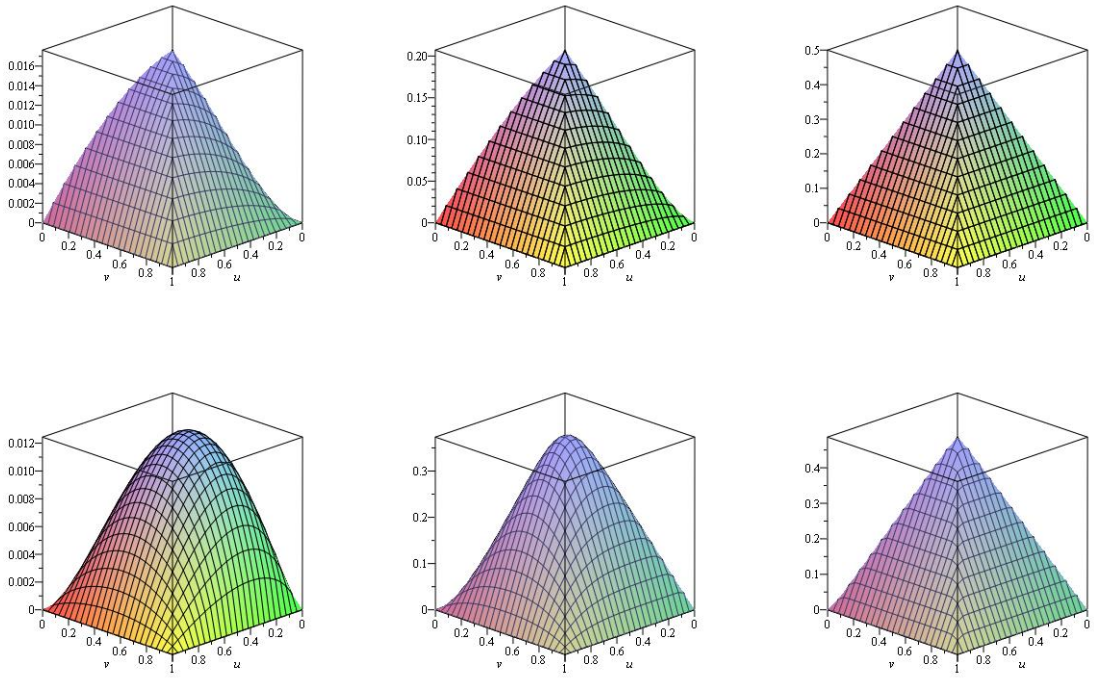


Figure 2: Plots of the asymmetry points of the Cuadras-Augé copula with the parameters $\theta = 0.05, \theta = 0.5$ and $\theta = 1$ (top panels left to right) and Clayton copula with the parameters $\theta = 0.05, \theta = 5$ and $\theta = 50$ (bottom panels left to right)

As is well known, for two copulas C_1 and C_2, their convex combination given by

$$C_\theta(u,v) = \theta C_1(u,v) + (1-\theta)C_2(u,v), \quad u,v \in [0,1]. \tag{9}$$

where $\theta \in [0,1]$, is also a copula [20]. The following proposition provides a result for $\prec_{CA}$ order of a convex combination of two copulas.

Proposition 3.11. *Let C_θ be the convex combination of C_1 and C_2 defined by (9). Then,*
(i) *If C_2 (resp, C_1) is conditionally symmetric, then C_θ is positively (resp, negatively) ordered for $\prec_{CA}$;*
(ii) *If $C_2 = C_1^*$ then C_θ is positively (resp, negatively) ordered for $\prec_{CA}$ for $\theta < \frac{1}{2}$ (resp, $\theta > \frac{1}{2}$).*

Proof. To see (i), Since $C_\theta^*(u,v) = \theta C_1^*(u,v) + (1-\theta)C_2^*(u,v)$ and

$$|C_\theta(u,v) - C_\theta^*(u,v)| = |\theta[C_1(u,v) - C_1^*(u,v)] + (1-\theta)[C_2(u,v) - C_2^*(u,v)]|,$$

then the result follows. To prove (ii), let $C_\theta(u,v) = \theta C_1(u,v) + (1-\theta)C_1^*(u,v)$. Since $|C_\theta(u,v) - C_\theta^*(u,v)| = |1 - 2\theta||C_1(u,v) - C_1^*(u,v)|$, then the result follows. We note that for $\theta = \frac{1}{2}$ the copula C_θ is conditionally symmetric. □

Example 3.12. *Since $M^* = W$, then the copula $C_\theta(u,v) = \theta W(u,v) + (1-\theta)M(u,v)$, satisfies Proposition 3.11(ii).*

Note that if the copula C is positive quadrant dependent (PQD), i.e., $C(u,v) \geq \Pi(u,v)$, for all $u,v \in$[0,1], where $\Pi(u,v) = uv$ is the independent copula, then C^* is negative quadrant dependent (NQD), i.e., $C^*(u,v) \leq \Pi(u,v)$, for all $u,v \in [0,1]$. If the vector (X,Y) satisfies some positive dependence property, then the vector $(X,-Y)$ satisfies the corresponding negative dependence property. Symmetric relationships hold between the copulas C, C^*, M, W and Π:

$$C(u,v) - \Pi(u,v) = \Pi(u,1-v) - C^*(u,1-v), \quad M(u,v) - C(u,v) = C^*(u,1-v) - W(u,1-v);$$

for all $u,v \in [0,1]$. The following result provides a connection between dependence ordering $\prec_c$ and asymmetry ordering $\prec_{CA}$.

Proposition 3.13. *Let C_1 and C_2 be two copulas such that $C_1 \prec_c C_2$. If C_1 is PQD then $C_1 \prec_{CA} C_2$, and if C_2 is NQD, then $C_2 \prec_{CA} C_1$.*

Proof. Since $\Pi \prec_c C_1 \prec_c C_2$ implies that $C_2^* \prec_c C_1^* \prec_c \Pi$, then $C_1(u,v) - C_1^*(u,v) \leq C_2(u,v) - C_2^*(u,v)$, for all $u, v \in [0,1]$, i.e., $C_1 \prec_{CA} C_2$. Similarly, if $C_1 \prec_c C_2 \prec_c \Pi$, then $C_1(u,v) - C_1^*(u,v) \leq C_2(u,v) - C_2^*(u,v) \leq 0$, i.e., $C_2 \prec_{CA} C_1$. □

4 Measuring conditional asymmetry of copulas

The order of conditional asymmetry expresses a qualitative relation between the asymmetry of two copulas. With an appropriate measure, one can assign a numerical value to a copula that quantitatively expresses its degree of asymmetry. In many problems, we may test whether conditional symmetry holds or not. The null and alternative hypotheses to be tested are: $H_0 : F_{Y|X} = F_{-Y|X}$ v.s $H_1 : F_{Y|X} \neq F_{-Y|X}$. Note that from Proposition 2.3, the null hypothesis is equivalent to the following: $H_0 : (X,Y) \stackrel{d}{=} (X,-Y)$, or equivalently, in terms of their copulas, $H_0 : C = C^*$. A measure of conditional asymmetry could be useful for testing this problem. If γ is a measure of conditional asymmetry, then the test for conditional symmetry could be done by testing $H_0 : \gamma = 0$ v.s $H_1 : \gamma \neq 0$.

4.1 Proposed conditional asymmetry measures

When a copula C is conditionally asymmetric, i.e., $C(u,v) \neq C^*(u,v)$, for some $u, v \in$[0,1], a suitable distance based on the quantity $C(u,v) - C^*(u,v)$ can be used to measure the amount of conditional asymmetry in C. From Proposition 3.7 and 3.10, mimicking what has been done in [7] for measures of radial asymmetry, we can present a set of axioms that are reasonable for measures of conditional asymmetry to satisfy:

Definition 4.1. *A function $\gamma : \mathcal{C} \to [0,1]$ on the set of bivariate copulas ($\mathcal{C}$) is a measure of conditional asymmetry if it satisfies the following conditions:*
(A1). *$\gamma(C)$ is well-defined for all $C \in \mathcal{C}$;*
(A2). *$\gamma(C) = 0$ if, and only if, C is conditionally symmetric;*
(A3). *γ is order-preserving, that is, for any $C_1, C_2 \in \mathcal{C}$, if $C_1 \prec_{CA} C_2$ then $\gamma(C_1) \leq \gamma(C_2)$;*
(A4). *If $C \in \mathcal{C}$ and $\{C_n\}_{n \in N}$ is a sequence of copulas such that $C_n \to C$ uniformly, then $\gamma(C_n) \to \gamma(C)$.*

In Definition 4.1, axiom (A1) has a clear meaning. Axiom (A2) shows the value of the measure for conditionally symmetric copulas. Axiom (A3) indicates that a measure of conditional asymmetry preserves the conditional asymmetry order of copulas. Axiom (A4) expresses the continuity property of the conditional asymmetry measures. Two natural such measures based on the classical distance between the copula C and C^* are given by

$$\gamma_1(C) = k_1 \int_0^1 \int_0^1 |C(u,v) - C^*(u,v)| du dv, \tag{10}$$

and

$$\gamma_2(C) = k_2 \sup_{(u,v) \in [0,1]^2} |C(u,v) - C^*(u,v)|, \tag{11}$$

where k_1 and k_2 are normalizing constant such that $\gamma_1, \gamma_2 \in [0,1]$. The following result provides the upper bound for γ_1 and γ_2.

Lemma 4.2. *For every copula C,*

$$\sup_{(u,v) \in [0,1]^2} |C(u,v) - C^*(u,v)| du dv \leq \frac{1}{2}, \tag{12}$$

and

$$\int_0^1 \int_0^1 |C(u,v) - C^*(u,v)| du dv \leq \frac{1}{6}. \tag{13}$$

Proof. Since $\sup_{(u,v) \in [0,1]^2} |M(u,v) - W(u,v)| = \frac{1}{2}$, then the bound (12) follows from part (ii) of Proposition 3.7. For (13) integrating both sides of the inequality $|C(u,v) - C^*(u,v)| \leq M(u,v) - W(u,v)$ gives

$$\begin{aligned} \int_0^1 \int_0^1 |C(u,v) - C^*(u,v)| du dv &\leq \int_0^1 \int_0^1 (M(u,v) - W(u,v)) du dv \\ &= \frac{1}{6}. \end{aligned}$$

□

Thus, two proposed measures of conditional asymmetry could be normalized as the following:

$$\gamma_1(C) = 6\int_0^1\int_0^1 |C(u,v) - C^*(u,v)|dudv, \tag{14}$$

and

$$\gamma_2(C) = 2 \sup_{(u,v)\in[0,1]^2} |C(u,v) - C^*(u,v)|. \tag{15}$$

Proposition 4.3. *For each copula $C \in \mathcal{C}$, the indices $\gamma_1(C)$ and $\gamma_2(C)$ are measures of conditional asymmetry satisfying properties given in Definition 4.1.*

Proof. The measures γ_1 and γ_2 are well-defined for each copula C. The Property A2 is trivial. Property A3 follows from the definition of the conditional asymmetry order. Since the point-wise convergence of a sequence of copulas implies uniform convergence, and every copula satisfies the Lipschitz condition on $[0,1]^2$ (see, e.g, Theorem 2.2.4 in [20]), using the dominated convergence theorem, the continuity property is satisfied. □

Remark 4.4. *We observe that $\gamma_i(C^{**}) = \gamma_i(C^*) = \gamma_i(\widehat{C}) = \gamma_i(C)$, $i = 1, 2$, for every copula C, as $(C^*)^*(u,v) = C(u,v)$, $(\widehat{C})^*(u,v) = C^{**}(u,v)$ and $(C^{**})^*(u,v) = \widehat{C}(u,v)$. For example,*

$$\begin{aligned}
\gamma_1(C^{**}) &= 6\int_0^1\int_0^1 |v - C(1-u,v) - u - v + 1 - C(1-u,1-v)|dudv \\
&= 6\int_0^1\int_0^1 |1 - u - C(1-u,1-v) - C(1-u,v)|dudv \\
&= 6\int_0^1\int_0^1 |t - C(t,1-v) - C(t,v)|dtdv \\
&= 6\int_0^1\int_0^1 |C^*(t,v) - C(t,v)|dtdv = \gamma_1(C).
\end{aligned}$$

It is indeed true for every asymmetry measure defined based on the L_p distance between the copula C and C^*.

Proposition 4.5. *Let $C_\theta(u,v) = \theta C_1(u,v) + (1-\theta)C_2(u,v)$ be the convex combination of two copulas C_1 and C_2. then, for conditional asymmetry measures γ_i, $i = 1, 2$, we have that*

$$\gamma_i(C) \le \theta\ \gamma_i(C_1) + (1-\theta)\gamma_i(C_2).$$

Proof. Since

$$\begin{aligned}
|C(u,v) - C^*(u,v)| &= |\theta(C_1(u,v) - C_1^*(u,v) + (1-\theta)(C_2(u,v) - C_2^*(u,v)| \\
&\le \theta|(C_1(u,v) - C_1^*(u,v) + (1-\theta)|C_2(u,v) - C_2^*(u,v)|,
\end{aligned}$$

the result follows. □

Example 4.6. *For $\theta \in [0,1]$, consider the copula $C_\theta(u,v) = \theta M(u,v) + (1-\theta)W(u,v)$. Then, $|C_\theta(u,v) - C_\theta^*(u,v)| = |2\theta - 1|(M(u,v) - W(u,v))$. For this copula we have*

$$\gamma_1(C_\theta) = \gamma_2(C_\theta) = |2\theta - 1|.$$

Since for the copula C_θ, we have $C_0 = W$ and $C_1 = M$, then,

$$\gamma_1(M) = \gamma_2(M) = \gamma_1(W) = \gamma_2(W) = 1.$$

Note also that $C_{\frac{1}{2}} = \frac{W+M}{2}$, is a conditionally symmetric copula with $\gamma_1(C_{\frac{1}{2}}) = \gamma_2(C_{\frac{1}{2}}) = 0$.

Remark 4.7. *Let D be a conditionally symmetric copula and consider the copula C defied by $C = \theta M + (1-\theta)D$. Then, $|C - C^*| = \theta|M - M^*|$, and hence $\gamma_i(C) = \theta\gamma_i(M) = \theta$. Thus, for each $\theta \in [0,1]$, there exists a copula C such that $\gamma_i(C) = \theta$, $i = 1, 2$.*

Example 4.8. *For the copula defined in Example 3.4,*

$$|C_{\alpha,\beta}(u,v) - C^*_{\alpha,\beta}(u,v)| = |\alpha + \beta|uv(1-u)(1-v),$$

and then

$$\gamma_1(C_{\alpha,\beta}) = \frac{|\alpha+\beta|}{6}, \quad \textit{and} \quad \gamma_2(C_{\alpha,\beta}) = \frac{|\alpha+\beta|}{8}.$$

For $\alpha \in [-1,1]$, $\beta = 1$, $\gamma_1(C_{\alpha,1}) \in [0, 1/3]$ and $\gamma_2(C_{\alpha,1}) \in [0, 1/4]$.

4.2 Conditional asymmetry and dependence measures

The most common copula-based measures of dependence between two continuous random variables X and Y are Kendall's tau and Spearman's rho given by [20]

$$\tau(C) = 4\int_0^1\int_0^1 C(u,v)\,dC(u,v) - 1, \quad \rho(C) = 12\int_0^1\int_0^1 C(u,v)dudv - 3. \tag{16}$$

These measures are known as measures of concordance since they satisfy a set of axioms due to Scarsini [23]. A fundamental property of any measure of concordance κ is that $\kappa(X,-Y) = -\kappa(X,Y)$. For a pair (X,Y) of random variables with a conditionally symmetric copula, any measure of concordance κ satisfies $\kappa(X,Y) = 0$. Hence, for conditionally symmetric copulas, measures of concordance cannot capture the dependence structure. Random variables having this property are also known as uncorrelated dependent. For constructing copula structures for uncorrelated dependent random variables, see, e.g., [10] and references therein. From Schweizer and Wolff [24] any suitably normalized measure of distance between the surfaces $z = C(u,v)$ and $z = \Pi(u,v) = uv$ should yield a measure of dependence. Two natural such measures based on the classical distance between the copula C and the independent copula Π are given by

$$\sigma_1(C) = 12\int_0^1\int_0^1 |C(u,v) - uv|dudv, \tag{17}$$

and

$$\sigma_2(C) = 4\sup_{(u,v)\in[0,1]^2} |C(u,v) - uv|. \tag{18}$$

Fundamental properties of such measure are studied in [24]. Note that, we have $\sigma_i(M) = \sigma_i(W) = 1$, $i = 1,2$. The following result compares the conditional asymmetry measure γ_i and the dependence measures σ_i, $i = 1,2$

Proposition 4.9. *For any copula C, $\gamma_i(C) \le \sigma_i(C)$, $i = 1,2$.*

Proof. Since every copula C satisfies $|C(u,v) - C^*(u,v)| \le |C(u,v) - uv| + |C^*(u,v) - uv|$, in view of the fact that $|C^*(u,v) - uv| = |u(1-v) - C(u,1-v)|$, we have that $\sup_{(u,v)\in[0,1]^2} |C^*(u,v) - uv| = \sup_{(u,v)\in[0,1]^2} |C(u,v) - uv|$, hence $\gamma_2(C) \le \sigma_2(C)$ is immediate. Integrating both sides and change of variable $1-v$ to v, gives the required result for $\gamma_1(C) \le \sigma_1(C)$. □

The following example compares the value of γ_i, σ_i $i = 1,2$, for the FGM family of copulas.

Example 4.10. *For the FGM copula C_θ given by (6), we have*

$$|C_\theta(u,v) - uv| = |\theta|uv(1-u)(1-v), \quad \textit{and} \quad |C_\theta(u,v) - C_\theta^*(u,v)| = 2|\theta|uv(1-u)(1-v);$$

and then

$$\gamma_1(C_\theta) = \sigma_1(C_\theta) = \frac{|\theta|}{3}, \quad \textit{and} \quad \gamma_2(C_\theta) = \sigma_2(C_\theta) = \frac{|\theta|}{4}.$$

Example 4.11. *For the copula $C_\theta(u,v) = \theta M(u,v) + (1-\theta)W(u,v)$, with $\theta \in [0,1]$, in Example 4.6 we obtained $\gamma_1(C_\theta) = |2\theta - 1|$. For this copula*

$$\sigma_1(C_\theta) = \begin{cases} 4\theta^4 - 2\theta + 1, & 0 \le \theta \le \frac{1}{2}, \\ 4\theta^4 - 16\theta^3 + 24\theta^2 - 14\theta + 3, & \frac{1}{2} \le \theta \le 1. \end{cases}$$

Figure 3 shows the plots of $\gamma_1(C_\theta)$ and $\sigma_1(C_\theta)$. We note that $\gamma_1(C_\theta) \le \sigma_1(C_\theta)$, for all $\theta \in [0,1]$. For $\theta \in \{0,1\}$, $\gamma_1(C_0) = \sigma_1(C_0) = \gamma_1(C_1) = \sigma_1(C_1) = 1$. Note also that, for $\theta = \frac{1}{2}$, we have that $C_{\frac{1}{2}} = (M+W)/2$, which is conditionally symmetric with $\gamma_1(C_{\frac{1}{2}}) = 0$, but $\sigma_1(C_{\frac{1}{2}}) = \frac{1}{4}$.

Remark 4.12. *The concept of conditional asymmetry is closely related to dependence. For a given $\alpha \in [0,1]$, we can construct a copula C such that, for instance, $\gamma_1(C) = \sigma_1(C) = \alpha$; that the copula C has the same degree of dependence and conditional asymmetry. If we define the copula C by $C = \alpha M + (1-\alpha)\Pi$, then $|C - \Pi| = \alpha|M - \Pi|$ and $|C - C^*| = \alpha|M - W|$, and hence $\sigma_1(C) = \alpha\sigma_1(M) = \alpha$ and $\gamma_1(C) = \alpha\gamma_1(M) = \alpha$.*

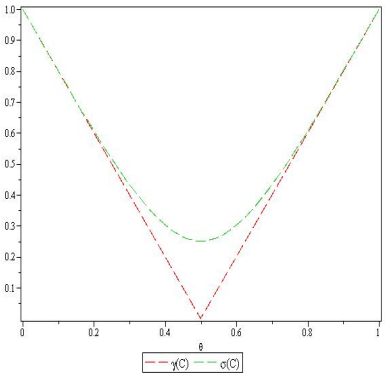


Figure 3: Plots of the asymmetry measure $\gamma_1(C_\theta)$ and the dependence measure $\sigma_1(C_\theta)$ for the copula $C_\theta = \theta M + (1-\theta)W$.

5 Conclusion

In this paper, we have studied the conditional symmetry/asymmetry of two random variables in terms of their copula. We have proposed an order of conditional asymmetry for copulas. Two measures of conditional asymmetry which are monotone for the proposed order, have been defined. In a similar way to the test of independence using measures of dependence, one application of the proposed conditional asymmetry measures is to use them for testing conditional symmetry. In many problems, we may test whether conditional symmetry holds or not. For instance, in a parametric regression model $y = g(X;\theta)+\epsilon$, where $g(.;.)$ is a known smooth function, θ is an unknown parameter and the error term ϵ satisfies that $E(\epsilon|X) = 0$. It is of interest to test whether the following hypothesis holds or not: $H_0 : F_{\epsilon|X} = F_{-\epsilon|X}$ v.s $H_1 : F_{\epsilon|X} \neq F_{-\epsilon|X}$; see, e.g, [13, 18]. A measure of conditional asymmetry could be useful for testing this problem. If γ is a measure of conditional asymmetry, then the test for conditional symmetry could be done by testing $H_0 : \gamma = 0$ v.s $H_1 : \gamma \neq 0$. As one of the reviewers pointed out, one line of research is to find the best-possible (upper and lower) bounds on the set of copulas with that given value of the conditional asymmetry measures, defined by (14) and (15). Although in this paper, we studied conditional symmetry/asymmetry of copulas for bivariate cases, the result can be checked for multivariate cases as well.

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