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Original paper

Some construction methods of interval-valued implications on bounded lattices

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Abstract

In recent years, implication operators have been studied intensively from many perspectives. These include the study of construction methods of implication operators. Additionally, it has been investigated whether some aggregation operators such as t-norm in interval-valued L-fuzzy set theory are representable. In this paper, we present some construction methods to obtain interval-valued implications, which are i-representable or not depending on the lattice and fuzzy logic operators, via some fuzzy logic operators, order preserving and/or order reversing functions. Moreover, many illustrative examples are included.

Keywords: Implications, lattice of closed intervals, interval-valued fuzzy set, bounded lattice, construction method, order preserving function, order reversing function.

1 Introduction

Fuzzy set theory extends the notion of classical set theory by permitting the gradual membership of elements in a set. It has been shown that fuzzy set theory is a useful tool for describing situations when the data is vague or imprecise. It manages such situations by assigning a degree to which a certain object belongs to a set. Interval-valued fuzzy set theory, which is an extension of ordinary set theory and denoted by L^I, is an important tool especially when it is impossible to determine an exact degree of membership because of vague information or lack of knowledge, etc [11].

Fuzzy implications generalize the classical implication on $\{0,1\}$ to fuzzy logic in which truth values belong to the unit interval. Since fuzzy implications have a great number of applications such as fuzzy control, approximate reasoning, and decision support systems and fuzzy control [3, 18, 19, 22], they have been studied extensively recent years [1, 12, 13, 16, 21, 25]. In the literature, besides the studies on well-known classes of implications such as R-, (S,N)-, and QL-implications, construction methods for implications from given implications or other fuzzy connectives on more general structures than the unit interval $[0,1]$ such as bounded lattices have been studied [14, 15]. After some important fuzzy conjunctions were defined and studied on L^I such as triangular norms (conorms), uninorms etc. [7, 8, 10], it is quite expected to study implications on L^I and [9] introduced a concept of representability for interval-valued fuzzy implications by fuzzy implication operators. For interval-valued fuzzy implication operators by means of given fuzzy implications and aggregation functions, some construction methods are offered in the paper [2]. However, a study of construction methods for interval-valued implications on L^I is not still available. Aim of this paper is to find construction methods of interval-valued implication operators. In order to this, we use implications and some logic operators on a bounded lattice L. Moreover, we reveal various new construction methods of interval-valued implications using order-preserving or order-reversing functions on a bounded lattice L. The construction methods we obtained can be applied on $[0,1]^I$. Thus, that construction methods play an important role for interval-valued fuzzy sets which have many application areas; soft computing, approximate reasoning, optimization and etc. [6, 23, 24].

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This paper is organized as follows. The next Section 2 is a preliminary that includes the basic concepts and definitions required in this paper. In Section 3, some construction methods of interval-valued implications have been introduced by means of some fuzzy connectives (t-norms, t-conorms, implication operators, etc.), negation on L and some arbitrary elements of L. Also, some different construction methods for interval-valued implication operators are obtained from order-preserving (order-reversing) functions as well as some fuzzy connectives, negations on L and some arbitrary of L. Additionally, we add some examples including illustrative figures. Finally, our work ends with concluding remarks Section 4.

2 Preliminaries

In this section, we list some basic notions and results which will be use in the paper.

A set L with a binary relation $\leq$ is called a partially ordered set if for $x, y, z \in L$ it fulfills the following axioms:

- $x \leq x$ (reflexive).
- $x \leq y$ and $y \leq x$ imply $x = y$ (antisymetric).
- $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitive).

We say that an element $a \in L$ is an upper bound of a subset $A \subseteq L$ if $x \leq a$ for all $x \in A$. Similarly, we mention that an element $b \in L$ is a lower bound of a subset $A \subseteq L$ if $b \leq x$ for all $x \in A$. Also, notation $\overline{A}$ denotes the set of the upper bounds of A with respect to $\leq$. Dually $\underline{A}$ denotes the set of the lower bounds of A with respect to $\leq$. Element $s \in A$ (if exists) is called the supremum of A if s is the smallest element of the set $\overline{A}$. Dually, element $i \in A$ (if exists) is called the infumum of A if i is the greatest element of the set $\underline{A}$. It is immediately observed that if the supremum and the infumum exist, they are unique. Specially if $A = \{a, b\}$ is taken, the supremum of A is denoted by $a \vee b$ and infumum of A is denoted by $a \wedge b$. In lattice L there exists $a \vee b$ and $a \wedge b$ for all $a, b \in L$. A lattice L is called bounded if it is a partially ordered set with a smallest element denoted by 0 and a greatest element denoted by 1. A bounded lattice L with $\leq$ is represented by $(L, \leq, 0, 1)$. A lattice L is called complete if supremum and infimum exist for any subset $H \subseteq L$. Readers can refer to [5] for more information.

Definition 2.1. [5] *Let $(L, \leq, 0, 1)$ be a bounded lattice. The elements x and y are called comparable if $x \leq y$ or $y \leq x$. Otherwise, x and y are called incomparable. In this situation, the notation $x||y$ are used. If any two elements of L are comparable, L is called a chain.*

Definition 2.2. [5] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. The sublattice $[a, b]$ is defined as*

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, $(a, b] = \{x \in L \mid a < x \leq b\}$, $[a, b) = \{x \in L \mid a \leq x < b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$ can be defined.

Definition 2.3. [5] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A function $\beta : L \to L$ is called order preserving or isotone if $x \leq y$ implies $\beta(x) \leq \beta(y)$ for any $x, y \in L$. Similarly, a function $\theta : L \to L$ is called order reversing or antitone if $x \leq y$ implies $\theta(x) \geq \theta(y)$ for any $x, y \in L$.*

Definition 2.4. [17] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A function $T : L^2 \longrightarrow L$ is a t-norm if it satisfies the following conditions for any $x, y \in L$.*

(T1) $T(x, y) = T(y, x)$. *(commutavity)*
(T2) $T(x, 1) = x$. *(neutral element)*
(T3) *If $y \leq z$, then $T(x, y) \leq T(x, z)$.* *(monotonicity)*
(T4) $T(x, T(y, z)) = T(T(x, y), z)$. *(associativity)*

Remark 2.5. *Directly from Definition 2.4 we can deduce that a t-norm T satisfies the following additional conditions for any $x, y \in L$.*

(i) $T(x, 0) = T(0, x) = 0$.
(ii) $T(1, x) = x$.
(iii) $T(x, y) \leq x$ *and* $T(x, y) \leq y$.

Definition 2.6. [17] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A function $S : L^2 \longrightarrow L$ is a t-conorm if it satisfies the following conditions for any $x, y \in L$.*

(S1) $S(x, y) = S(y, x)$. *(commutavity)*
(S2) $S(x, 0) = x$. *(neutral element)*
(S3) *If* $y \leq z$, *then* $S(x, y) \leq S(x, z)$. *(monotonicity)*
(S4) $S(x, S(y, z)) = S(S(x, y), z)$. *(associativity)*

Remark 2.7. *Directly from Definition 2.6 we can deduce that a t-conorm S satisfies the following additional contions for any $x, y \in L$.*

(i) $S(x, 1) = S(1, x) = 1$.
(ii) $S(0, x) = x$.
(iii) $x \leq S(x, y)$ *and* $y \leq S(x, y)$.

Definition 2.8. [3, 15] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$.*

Definition 2.9. [3, 15] *A function $I : L^2 \to L$ on a bounded lattice $(L, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:*

(I1) *I is a decreasing operation on the first variable, that is, for every $x, z \in L$ with $x \leq z$, $I(z, y) \leq I(x, y)$ for all $y \in L$.*

(I2) *I is an increasing operation on the second variable, that is, for every $y, z \in L$ with $y \leq z$, $I(x, y) \leq I(x, z)$ for all $x \in L$.*

(I3) $I(0, 0) = 1$.
(I4) $I(1, 1) = 1$.
(I5) $I(1, 0) = 0$.

Theorem 2.10. [13] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L$. Then the function $I_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$I_a(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y, \\ a, & \text{otherwise,} \end{cases} \tag{1}$$

is an implication.

Theorem 2.11. [13] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm and $N : L \to L$ be a negation. Then the function $I : L^2 \to L$ defined by, for all $x, y \in L$,*

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y, \\ S(N(x), y), & \text{otherwise,} \end{cases} \tag{2}$$

is an implication.

Theorem 2.12. [14] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications and $a \in L$. The function $TS_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$TS_a(x,y) = T(S(a, I(x,y)), J(x,y)), \tag{3}$$

is an implication.

Theorem 2.13. [14] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications and $a \in L$. The function $ST_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$ST_a(x,y) = S(T(a, I(x,y)), J(x,y)), \tag{4}$$

is an implication.

Theorem 2.14. [16] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications, $N : L \to L$ be a negation and $a \in L$. The function $K^{I,J}_{a,T,S,N} : L^2 \to L$ defined by, for all $x, y \in L$,*

$$K^{I,J}_{a,T,S,N} = S(T(a, I(x, y)), T(N(a), J(x, y))), \tag{5}$$

is an implication if and only if $S(a, N(a)) = 1$.

Definition 2.15. [7] *Let $(L, \leq, 0, 1)$ be a bounded lattice. We define $(L^I, \leq_{L^I})$ by*

$$L^I = \left\{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leqslant {}_L x_2\right\},$$
$$[x_1, x_2] \leqslant_{L^I} [y_1, y_2] \Longleftrightarrow x_1 \leqslant y_1 \text{ and } x_2 \leqslant y_2 \quad \text{for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I.$$

Throughout this paper we will denote the element $[x_1, x_2] \in L^I$ by $\mathbf{x}$. It is easily verified that $\mathbf{x} \vee \mathbf{y} = [x_1 \vee y_1, x_2 \vee y_2]$ and $\mathbf{x} \wedge \mathbf{y} = [x_1 \wedge y_1, x_2 \wedge y_2]$ for $\mathbf{x}, \mathbf{y} \in L^I$. The smallest and the greatest element of L^I are denoted by $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$ respectively, where 0 and 1 are the extremal elements of L. Note that the order on L^I defined above is not the same as the inclusion order for sublattices.

Definition 2.16. [4, 9] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $\mathcal{I} : (L^I)^2 \to L^I$ is called an implication on L^I if it satisfies the following conditions:*

(I1) *$\mathcal{I}$ is a decreasing operation on the first variable, that is, for every $\mathbf{x}, \mathbf{z} \in L^I$ with $\mathbf{x} \leq_{L^I} \mathbf{z}$, $\mathcal{I}(\mathbf{z}, \mathbf{y}) \leq_{L^I} \mathcal{I}(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in L^I$.*

(I2) *$\mathcal{I}$ is an increasing operation on the second variable, that is, for every $\mathbf{y}, \mathbf{z} \in L^I$ with $\mathbf{y} \leq_{L^I} \mathbf{z}$, $\mathcal{I}(\mathbf{x}, \mathbf{y}) \leq_{L^I} \mathcal{I}(\mathbf{x}, \mathbf{z})$ for all $\mathbf{x} \in L^I$.*

(I3) $\mathcal{I}(\mathbf{0}, \mathbf{0}) = \mathbf{1}$.

(I4) $\mathcal{I}(\mathbf{1}, \mathbf{1}) = \mathbf{1}$.

(I5) $\mathcal{I}(\mathbf{1}, \mathbf{0}) = \mathbf{0}$.

Theorem 2.17. [9] *Let $(L, \leq, 0, 1)$ be a bounded lattice. Given two implications I_1 and I_2 on L. The mapping $\mathcal{I}^r_{I_1,I_2} : (L^I)^2 \to L^I$ defined by, for $\mathbf{x}, \mathbf{y} \in L^I$,*

$$\mathcal{I}^r_{I_1,I_2}(\mathbf{x}, \mathbf{y}) = [I_1(x_2, y_1), I_2(x_1, y_2)],$$

is an implication on L^I.

Definition 2.18. [9] *The implication $\mathcal{I}^r_{I_1,I_2}$ given by Theorem 2.17 is called the i-representable implication on L^I with representatives I_1 and I_2.*

Remark 2.19. *For any i-represantable implication $\mathcal{I}$ on L^I with representatives I_1 and I_2, it holds the following conditions:*

- $\mathcal{I}([0, 1], [0, 1]) = [0, 1]$.
- $\mathcal{I}([0, 1], [0, 0]) = [0, 1]$.
- $\mathcal{I}([1, 1], [0, 1]) = [0, 1]$.

3 Some construction methods of implication on L^I

In this section, we present several construction methods for implication operators on L^I. To do that, we will use some logic operators, special functions and arbitrary fix elements on bounded lattice. First, in Theorems 3.1 and 3.4 we present construction methods for implication operators and also use some logic operators to build them.

Theorem 3.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice, T be a t-norm, S be a t-conorm and I be an implication on L, c be an arbitrary element of the bounded lattice L and $k, l \in \{1, 2\}$. Then, the mapping $\mathcal{I}^{kl}_{ISTc} : (L^I)^2 \to L^I$ defined by, for all $\mathbf{x}, \mathbf{y} \in L^I$,*

$$\mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{y}) = \Big[I(x_k, y_1), S\Big(S\Big(T\big(c, I(x_2, y_2)\big), I(x_1, y_2)\Big), I(x_2, y_l)\Big)\Big], \tag{6}$$

is an implication on L^I.

Proof. Let $\mathbf{x}, \mathbf{y} \in L^I$ and $k, l \in \{1, 2\}$. It follows that:

$$\begin{aligned} I(x_k, y_1) &\leq I(x_1, y_1) \\ &\leq I(x_1, y_2) \\ &\leq S\big(T\big(c, I(x_2, y_2)\big), I(x_1, y_2)\big) \\ &\leq S\Big(S\Big(T\big(c, I(x_2, y_2)\big), I(x_1, y_2)\Big), I(x_2, y_l)\Big), \end{aligned}$$

thus $\mathcal{I}^{kl}_{ISTc}(x, y) \in L^I$.

(I1) Let us show that $\mathcal{I}^{kl}_{ISTc}(\mathbf{z}, \mathbf{y}) \leq_{L^I} \mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{y})$ for every elements $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L^I$ with $\mathbf{x} \leq_{L^I} \mathbf{z}$. It follows from $x_k \leq z_k$ that

$$\begin{aligned} \mathcal{I}^{kl}_{ISTc}(\mathbf{z}, \mathbf{y}) &= \Big[I(z_k, y_1), S\Big(S\Big(T\big(c, I(z_2, y_2)\big), I(z_1, y_2)\Big), I(z_2, y_l)\Big)\Big] \\ &\leq_{L^I} \Big[I(x_k, y_1), S\Big(S\Big(T\big(c, I(x_2, y_2)\big), I(x_1, y_2)\Big), I(x_2, y_l)\Big)\Big] = \mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{y}), \end{aligned}$$

for $k, l \in \{1, 2\}$.

(I2) Let us show that $\mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{y}) \leq_{L^I} \mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{z})$ for every elements $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L^I$ with $\mathbf{y} \leq_{L^I} \mathbf{z}$. It follows from $y_k \leq z_k$ that

$$\begin{aligned} \mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{y}) &= \Big[I(x_k, y_1), S\Big(S\Big(T\big(c, I(x_2, y_2)\big), I(x_1, y_2)\Big), I(x_2, y_l)\Big)\Big] \\ &\leq_{L^I} \Big[I(x_k, z_1), S\Big(S\Big(T\big(c, I(x_2, z_2)\big), I(x_1, z_2)\Big), I(x_2, z_l)\Big)\Big] = \mathcal{I}^{kl}_{ISTc}(\mathbf{x}, \mathbf{z}), \end{aligned}$$

for $k, l \in \{1, 2\}$.

(I3)

$$\begin{aligned} \mathcal{I}^{kl}_{ISTc}(\mathbf{0}, \mathbf{0}) &= \Big[I(0, 0), S\Big(S\Big(T\big(c, I(0, 0)\big), I(0, 0)\Big), I(0, 0)\Big)\Big] \\ &= \big[1, S\big(S\big(T(c, 1), 1\big), 1\big)\big] \\ &= [1, 1]. \end{aligned}$$

(I4)

$$\begin{aligned} \mathcal{I}^{kl}_{ISTc}(\mathbf{1}, \mathbf{1}) &= \Big[I(1, 1), S\Big(S\Big(T\big(c, I(1, 1)\big), I(1, 1)\Big), I(1, 1)\Big)\Big] \\ &= \big[1, S\big(S\big(T(c, 1), 1\big), 1\big)\big] \\ &= [1, 1]. \end{aligned}$$

(I5)

$$\begin{aligned} \mathcal{I}^{kl}_{ISTc}(\mathbf{1}, \mathbf{0}) &= \Big[I(1, 0), S\Big(S\Big(T\big(c, I(1, 0)\big), I(1, 0)\Big), I(1, 0)\Big)\Big] \\ &= \big[0, S\big(S\big(T(c, 0), 0\big), 0\big)\big] \\ &= \big[0, S\big(S(0, 0), 0\big)\big] \\ &= [0, S(0, 0)] \\ &= [0, 0]. \end{aligned}$$

□

Remark 3.2. *In Theorem 3.1, the following equality is satisfied.*

$$\begin{aligned} \mathcal{I}^{11}_{ISTc}([0, 1], [0, 1]) &= \Big[I(0, 0), S\Big(S\Big(T\big(c, I(1, 1)\big), I(0, 1)\Big), I(1, 0)\Big)\Big] \\ &= \Big[1, S\Big(S\Big(T(c, 1), 1\Big), 0\Big)\Big] \\ &= [1, S(S(c, 1), 0)] \\ &= [1, S(1, 0)] \\ &= [1, 1]. \end{aligned}$$

Then $\mathcal{I}^{kl}_{ISTc}$ is an implication operator which is not an i-repsentable on L^I.

Example 3.3. *Consider the bounded lattice* $(L = \{0, a, b, 1\}, \leq, 0, 1)$ *characterized by the Hasse diagram in Figure 1.*

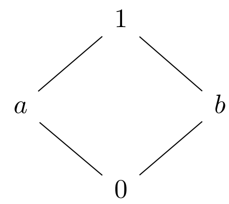

Figure 1: The lattice L.

It is easy to obtain the lattice $(L^I = \{[0,0],[0,a],[0,b],[a,a],[b,b],[0,1],[a,1],[b,1],[1,1]\}, \leq_{L^I}, [0,0],[1,1])$ *as in Figure 2.*

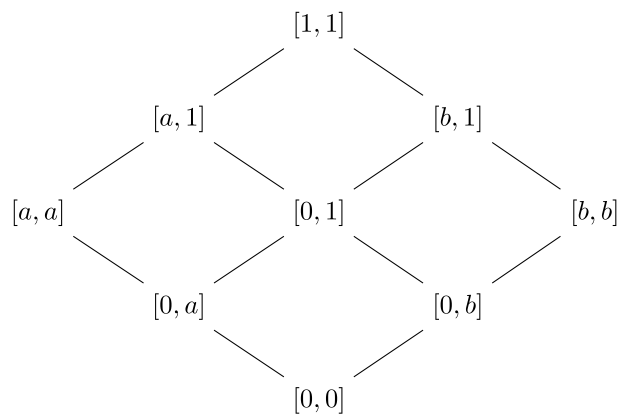

Figure 2: The lattice L^I.

Take the implicaton I, the t-norm T and the t-conorm S, respectively, as follows:

I	0	a	b	1
0	1	1	1	1
a	0	1	a	1
b	0	b	1	1
1	0	0	0	1

Table 1: The implication I on L,

T	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	0	b
1	0	a	b	1

Table 2: The t-norm T on L,

S	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	1	1
1	1	1	1	1

Table 3: The t-conorm S on L.

By applying the formula (6) in Theorem 3.1 with $c = b$, $k = 2$ and $l = 1$, the implication $\mathcal{I}^{21}_{ISTb}$ can be obtained as in Table 4.

$\mathcal{I}^{21}_{ISTb}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$	$[a,1]$	$[0,1]$	$[1,1]$	$[a,1]$	$[1,1]$
$[0,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[b,1]$	$[1,1]$	$[0,1]$	$[b,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[0,0]$	$[0,1]$	$[0,a]$	$[1,1]$	$[a,a]$	$[0,1]$	$[1,1]$	$[a,1]$	$[1,1]$
$[b,b]$	$[0,0]$	$[0,b]$	$[0,1]$	$[b,1]$	$[1,1]$	$[0,1]$	$[b,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[a,1]$	$[0,0]$	$[0,1]$	$[0,a]$	$[0,1]$	$[0,a]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[b,1]$	$[0,0]$	$[0,b]$	$[0,1]$	$[0,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$

Table 4: The implication $\mathcal{I}^{21}_{ISTb}$ on L^I.

Theorem 3.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice, T be a t-norm and S be a t-conorm on L, N be a negation on L and $k, t \in \{1, 2\}$ with $k \leq t$. Then, the mapping $\mathcal{I}^{kt}_{STN} : (L^I)^2 \to L^I$ defined by, for all $\mathbf{x}, \mathbf{y} \in L^I$,*

$$\mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{y}) = \big[S\big(T(N(x_2), N(x_1)), y_k\big), S\big(T(N(x_2), N(x_1)), y_t\big)\big], \tag{7}$$

is an implication on L^I.

Proof. From $S\big(T(N(x_2), N(x_1)), y_k\big) \leq S\big(T(N(x_2), N(x_1)), y_t\big)$ for $k, t \in \{1, 2\}$ with $k \leq t$, it follows that $\mathcal{I}^{kt}_{STN}(x, y) \in L^I$.

(I1) Let us show that $\mathcal{I}^{kt}_{STN}(\mathbf{z}, \mathbf{y}) \leq_{L^I} \mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{y})$ for every elements $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L^I$ with $\mathbf{x} \leq_{L^I} \mathbf{z}$. It follows from $x_k \leq z_k$ that

$$\begin{aligned} \mathcal{I}^{kt}_{STN}(\mathbf{z}, \mathbf{y}) &= \big[S\big(T(N(z_2), N(z_1)), y_k\big), S\big(T(N(z_2), N(z_1)), y_t\big)\big] \\ &\leq_{L^I} \big[S\big(T(N(x_2), N(x_1)), y_k\big), S\big(T(N(x_2), N(x_1)), y_t\big)\big] = \mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{y}), \end{aligned}$$

for $k, t \in \{1, 2\}$ with $k \leq t$.

(I2) Let us show that $\mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{y}) \leq_{L^I} \mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{z})$ for every elements $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L^I$ with $\mathbf{y} \leq_{L^I} \mathbf{z}$. It follows from $y_k \leq z_k$ that

$$\begin{aligned} \mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{y}) &= \big[S\big(T(N(x_2), N(x_1)), y_k\big), S\big(T(N(x_2), N(x_1)), y_t\big)\big] \\ &\leq_{L^I} \big[S\big(T(N(x_2), N(x_1)), z_k\big), S\big(T(N(x_2), N(x_1)), z_t\big)\big] = \mathcal{I}^{kt}_{STN}(\mathbf{x}, \mathbf{z}), \end{aligned}$$

for $k, t \in \{1, 2\}$ with $k \leq t$.

(I3)

$$\begin{aligned} \mathcal{I}^{kt}_{STN}(\mathbf{0}, \mathbf{0}) &= \big[S\big(T(N(0), N(0)), 0\big), S\big(T(N(0), N(0)), 0\big)\big] \\ &= \big[S\big(T(1, 1), 0\big), S\big(T(1, 1), 0\big)\big] \\ &= [S(1, 0), S(1, 0)] \\ &= [1, 1]. \end{aligned}$$

(I4)

$$\begin{aligned}\mathcal{I}^{kt}_{STN}(\mathbf{1},\mathbf{1}) &= \big[S\big(T(N(1),N(1)),1\big),S\big(T(N(1),N(1)),1\big)\big]\\ &= [1,1].\end{aligned}$$

(I5)

$$\begin{aligned}\mathcal{I}^{kt}_{STN}(\mathbf{1},\mathbf{0}) &= \big[S\big(T(N(1),N(1)),0\big),S\big(T(N(1),N(1)),0\big)\big]\\ &= \big[S\big(T(0,0),0\big),S\big(T(0,0),0\big)\big]\\ &= [S(0,0),S(0,0)]\\ &= [0,0].\end{aligned}$$

□

Remark 3.5. *In Theorem 3.4, the following equality is satisfied.*

$$\begin{aligned}\mathcal{I}^{kt}_{STN}([0,1],[0,0]) &= \big[S\big(T(N(1),N(0)),0\big),S\big(T(N(1),N(0)),0\big)\big]\\ &= \big[S\big(T(0,1),0\big),S\big(T(0,1),0\big)\big]\\ &= [S(0,0),S(0,0)]\\ &= [0,0].\end{aligned}$$

Then $\mathcal{I}^{kt}_{STN}$ is an implication operator which is not an i-repsentable on L^I.

Example 3.6. *Consider the lattice L^I as given in Figure 2, the t-norm T and t-conorm S as given in Tables 2 and 3. Let the negation N be the following.*

$$N(x) = \begin{cases} 1, & \text{if } x = 0,\\ b, & \text{if } x = a,\\ a, & \text{if } x = b,\\ 0, & \text{if } x = 1.\end{cases}$$

By applying the formula (7) in Theorem 3.4 with $k = 1$ and $t = 2$, the implication $\mathcal{I}^{12}_{STN}$ can be obtained as in Table 5.

$\mathcal{I}^{12}_{STN}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[b,b]$	$[b,1]$	$[b,1]$	$[1,1]$	$[1,1]$	$[b,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,b]$	$[a,a]$	$[a,a]$	$[a,1]$	$[a,a]$	$[1,1]$	$[a,1]$	$[a,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[b,b]$	$[a,a]$	$[a,a]$	$[a,1]$	$[a,a]$	$[1,1]$	$[a,1]$	$[a,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[a,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[b,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$

Table 5: The implication $\mathcal{I}^{12}_{STN}$ on L^I.

In the following theorem, we offer a method to construct interval-valued implication operators from given two implications on L.

Theorem 3.7. *Let $(L,\leq,0,1)$ be a bounded lattice, J_1, J_2 be two implications on L and $k,t\in\{1,2\}$. Then, the mapping $\mathcal{I}^{kt}_{J_1J_2} : (L^I)^2 \to L^I$ defined by, for all $\mathbf{x},\mathbf{y}\in L^I$,*

$$\mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y}) = \big[J_1\big(x_2,J_2(x_k,y_t)\big),J_1\big(x_1,J_2(x_k,y_2)\big)\big], \tag{8}$$

is an implication on L^I.

Proof. Since

$$J_1\big(x_2, J_2(x_k, y_t)\big) \leq J_1\big(x_1, J_2(x_k, y_t)\big) \\ \leq J_1\big(x_1, J_2(x_k, y_2)\big),$$

for $k,t \in \{1,2\}$ it follows that $\mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y}) = \big[J_1\big(x_2, J_2(x_k, y_t)\big), J_1\big(x_1, J_2(x_k, y_2)\big)\big] \in L^I$.

(I1) Let us show that $\mathcal{I}^{kt}_{J_1J_2}(\mathbf{z},\mathbf{y}) \leq_{L^I} \mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y},\mathbf{z} \in L^I$ with $\mathbf{x} \leq_{L^I} \mathbf{z}$,i.e. $x_k \leq z_k$. It follows from $J_1\big(z_2, J_2(z_k, y_t)\big) \leq J_1\big(x_2, J_2(z_k, y_t)\big) \leq J_1\big(x_2, J_2(x_k, y_t)\big)$ and $J_1\big(z_1, J_2(z_k, y_2)\big) \leq J_1\big(x_1, J_2(z_k, y_2)\big) \leq J_1\big(x_1, J_2(x_k, y_2)\big)$ that

$$\mathcal{I}^{kt}_{J_1J_2}(\mathbf{z},\mathbf{y}) = \big[J_1\big(z_2, J_2(z_k, y_t)\big), J_1\big(z_1, J_2(z_k, y_2)\big)\big] \\ \leq_{L^I} \big[J_1\big(x_2, J_2(x_k, y_t)\big), J_1\big(x_1, J_2(x_k, y_2)\big)\big] = \mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y}),$$

for $k,t \in \{1,2\}$.

(I2) Let us show that $\mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y}) \leq_{L^I} \mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{z})$ for every elements $\mathbf{x},\mathbf{y},\mathbf{z} \in L^I$ with $\mathbf{y} \leq_{L^I} \mathbf{z}$, i.e, $y_k \leq z_k$. It follows from $J_1\big(x_2, J_2(x_k, y_t)\big) \leq J_1\big(x_2, J_2(x_k, z_t)\big)$ and $J_1\big(x_1, J_2(x_k, y_2)\big) \leq J_1\big(x_1, J_2(x_k, z_2)\big)$ that

$$\mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y}) = \big[J_1\big(x_2, J_2(x_k, y_t)\big), J_1\big(x_1, J_2(x_k, y_2)\big)\big] \\ \leq_{L^I} \big[J_1\big(x_2, J_2(x_k, z_t)\big), J_1\big(x_1, J_2(x_k, z_2)\big)\big] = \mathcal{I}^{kt}_{J_1J_2}(\mathbf{x},\mathbf{y}),$$

for $k,t \in \{1,2\}$.

(I3)

$$\mathcal{I}^{kt}_{J_1J_2}(\mathbf{0},\mathbf{0}) = \big[J_1\big(0, J_2(0,0)\big), J_1\big(0, J_2(0,0)\big)\big] \\ = [J_1(0,1), J_1(0,1)] \\ = [1,1].$$

(I4)

$$\mathcal{I}^{kt}_{J_1J_2}(\mathbf{1},\mathbf{1}) = \big[J_1\big(1, J_2(1,1)\big), J_1\big(1, J_2(1,1)\big)\big] \\ = [J_1(1,1), J_1(1,1)] \\ = [1,1].$$

(I5)

$$\mathcal{I}^{kt}_{J_1J_2}(\mathbf{1},\mathbf{0}) = \big[J_1\big(1, J_2(1,0)\big), J_1\big(1, J_2(1,0)\big)\big] \\ = [J_1(1,0), J_1(1,0)] \\ = [0,0].$$

□

Example 3.8. *Let us give an example for Theorem 3.7 and obtain the implication $\mathcal{I}^{kt}_{J_1J_2}$, where the L^I is as given in Figure 2, J_1, J_2 are as in the Tables 7 and 6, respectively.*

I_1	0	a	b	1
0	1	1	1	1
a	0	1	0	1
b	0	0	1	1
1	0	0	0	1

Table 6: The implication I_1 on L,

and

I_2	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Table 7: The implication I_2 on L.

By applying the formula (8) in Theorem 3.7 with $k = 1$ and $t = 2$, the implication $\mathcal{I}^{12}_{I_2 I_1}$ can be obtained as in Table 8.

$\mathcal{I}^{12}_{I_2 I_1}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,b]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[b,b]$	$[1,1]$	$[b,b]$	$[1,1]$	$[b,b]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[b,b]$	$[a,a]$	$[a,a]$	$[1,1]$	$[a,a]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,1]$	$[0,b]$	$[1,1]$	$[0,b]$	$[1,1]$	$[0,b]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[b,1]$	$[0,a]$	$[0,a]$	$[1,1]$	$[0,a]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$

Table 8: The implication $\mathcal{I}^{12}_{I_2 I_1}$.

Remark 3.9. *In Example 3.8 we have $\mathcal{I}^{12}_{I_2 I_1}([0,1],[0,1]) = [1,1]$ from the Table 8, which shows that the implication $\mathcal{I}^{kt}_{J_1 J_2}$ given by Theorem 3.7 does not have to be i-representable on L^I.*

Example 3.10. *Let $J_1 = I_{GG}$ and $J_2 = I_{GD}$,*

$$I_{GG}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \frac{y}{x}, & \text{if } x > y, \end{cases}$$

$$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ y, & \text{if } x > y, \end{cases}$$

are two fuzzy implications on the unit interval $[0,1]$ (see [3]).
The implication $\mathcal{I}^{11}_{I_{GG} I_{GD}}$ on $[0,1]$ with $k = t = 1$ is obtained as follows:

$$\mathcal{I}^{11}_{I_{GG} I_{GD}} = \begin{cases} [1,1], & \text{if } x_1 \le y_1, \\ [\frac{y_1}{x_2}, 1], & \text{if } x_1 > y_1 \text{ and } x_1 \le y_2, \\ [\frac{y_1}{x_2}, \frac{y_2}{x_1}], & \text{if } x_1 > y_1 \text{ and } x_1 > y_2. \end{cases}$$

In the following theorem, we offer a construction method for implication operators on L^I by using an order-reversing function $\theta : L \to L$ and an arbitrary fix element of L.

Theorem 3.11. *Let $(L, \le, 0, 1)$ be a bounded lattice, $\theta : L \to L$ be an order-reversing function, $c \in \overline{\{\theta(x) : x \in L\}}$ and $k \in \{1,2\}$. Then, the mapping $\mathcal{I}^k_{\theta c} : (L^I)^2 \to L^I$ defined by, for all $\mathbf{x}, \mathbf{y} \in L^I$,*

$$\mathcal{I}^k_{\theta c}(\mathbf{x}, \mathbf{y}) = \begin{cases} [1,1], & \text{if } [x_1,x_2] = [0,0] \text{ or } [y_1,y_2] = [1,1], \\ [0,0], & \text{if } [x_1,x_2] = [1,1] \text{ and } [y_1,y_2] = [0,0], \\ [\theta(x_k), c], & \text{otherwise}, \end{cases} \tag{9}$$

is an implication on L^I.

Proof. I3, I4 and I5 are obtained directly from the definition of $\mathcal{I}^k_{\theta c}$.

(I1) Let us show that $\mathcal{I}^k_{\theta c}(\mathbf{z},\mathbf{y})\leq_{L^I}\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{y})$ for every elements $\mathbf{x},\mathbf{y},\mathbf{z}\in L^I$ with $\mathbf{x}\leq_{L^I}\mathbf{z}$. The inequality $\mathcal{I}^k_{\theta c}(\mathbf{z},\mathbf{y})\leq_{L^I}\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{y})$ is directly obtained by the definiton of $\mathcal{I}^k_{\theta c}$ if $[x_1,x_2]=[0,0]$ or $[y_1,y_2]=[1,1]$. Similarly, if $[z_1,z_2]=[1,1]$ and $[y_1,y_2]=[0,0]$, the proof is obivous as well. Otherwise, it follows that

$$\mathcal{I}^k_{\theta c}(\mathbf{z},\mathbf{y})=[\theta(z_k),c]\leq_{L^I}[\theta(x_k),c]=\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{y}).$$

(I2) Let us show that $\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{y})\leq_{L^I}\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{z})$ for every elements $\mathbf{x},\mathbf{y},\mathbf{z}\in L^I$ with $\mathbf{y}\leq_{L^I}\mathbf{z}$. The inequality $\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{y})\leq_{L^I}\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{z})$ is directly obtained by the definiton of $\mathcal{I}^k_{\theta c}$ if $[x_1,x_2]=[0,0]$ or $[z_1,z_2]=[1,1]$. Similarly, if $[x_1,x_2]=[1,1]$ and $[y_1,y_2]=[0,0]$, the proof is obivous as well. Otherwise, it follows that

$$\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{y})=[\theta(x_k),c]=\mathcal{I}^k_{\theta c}(\mathbf{x},\mathbf{z}).$$

□

Example 3.12. *Consider the lattice L^I as given in Fig. 2 and the order-reversing function θ as follows:*

$$\theta(x)=\begin{cases}1, & \text{if } x=0,\\ a, & \text{if } x=a,\\ b, & \text{if } x=b,\\ 0, & \text{if } x=1.\end{cases} \tag{10}$$

Then, c must be 1*. By applying the formula (9) in Theorem 3.11 with $k=1$, the implication $\mathcal{I}^1_{\theta 1}$ can be obtained as in Table 9,*

$\mathcal{I}^1_{\theta 1}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,b]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[1,1]$
$[b,b]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[1,1]$
$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[1,1]$
$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[b,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$

Table 9: The implication $\mathcal{I}^1_{\theta 1}$ on L^I.

Remark 3.13. *In Example 3.12 we have $\mathcal{I}^1_{\theta 1}([0,1],[0,1])=[1,1]$ from the Table 9, which shows that the implication $\mathcal{I}^k_{\theta c}$ given by the formula (9) in Theorem 3.11 does not have to be i-representable on L^I.*

Corollary 3.14. *In Theorem 3.11, if the order-reversing function $\theta:L\to L$ satisfies $\theta(0)=1$, the implication $\mathcal{I}^k_{\theta 1}:L^I\to L^I$ is obtained on L^I as follows.*

$$\mathcal{I}^k_{\theta 1}(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & \text{if } [y_1,y_2]=[1,1],\\ [0,0], & \text{if } [x_1,x_2]=[1,1] \text{ and } [y_1,y_2]=[0,0],\\ [\theta(x_k),1], & \text{otherwise,}\end{cases}$$

is an implication on L^I.

In the following theorem, we offer a construction method for implication operators on L^I by using an order-preserving function $\beta:L\to L$ and an arbitrary fix element of L.

Theorem 3.15. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $\beta : L \to L$ *be an order-preserving function,* $c \in \underline{\{\beta(x) \ : \ x \in L\}}$ *and* $k \in \{1, 2\}$. *Then, the mapping* $\mathcal{I}^{*}_{\beta c} : (L^I)^2 \to L^I$ *defined by, for all* $\S, \dagger \in L^I$,

$$\mathcal{I}^{k*}_{\beta c}(\mathbf{x}, \mathbf{y}) = \begin{cases} [1,1], & \text{if } [x_1, x_2] = [0,0] \text{ or } [y_1, y_2] = [1,1], \\ [0,0], & \text{if } [x_1, x_2] = [1,1] \text{ and } [y_1, y_2] = [0,0], \\ [c, \beta(y_k)], & \text{otherwise,} \end{cases} \tag{11}$$

is an implication on L^I.

Proof. The proof can be done in an analogous way to the proof of Theorem 3.11. Therefore, we omit it. □

Example 3.16. *Consider the lattice* L^I *as in Example 3.3 and the order-preserving function* β *as* $\beta(x) = x$ *for all* $x \in L$. *Then, it is clear that* $c = 0$. *By applying the formula (11) in Theorem 3.15 with* $k = 1$, *the implication* $\mathcal{I}^{1*}_{\beta 0}$ *can be obtained as in Table 10.*

$\mathcal{I}^{1*}_{\beta 0}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[0,b]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[a,a]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[b,b]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[0,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[a,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[b,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[1,1]$

Table 10: The implication $\mathcal{I}^{1*}_{\beta 0}$ on L^I.

Remark 3.17. *Consider the implication* $\mathcal{I}^{1*}_{\beta 0}$ *given by Table 10 in Example 3.16. It fulfills that* $\mathcal{I}^{1*}_{\beta 0}([0,1], [0,0]) = [0,0]$. *The implication* $\mathcal{I}^{k*}_{\beta c}$ *given by the formula (11) in Theorem 3.15 does not have to be i-representable on* L^I.

Corollary 3.18. *In Theorem 3.15, if the order-preserving function* $\beta : L \to L$ *satisfies* $\beta(0) = 0$, *the implication* $\mathcal{I}^{k*}_{\beta 0} : L^I \to L^I$ *is obtained on* L^I *as follows.*

$$\mathcal{I}^{k*}_{\beta 0}(\mathbf{x}, \mathbf{y}) = \begin{cases} [1,1], & \text{if } [x_1, x_2] = [0,0] \text{ or } [y_1, y_2] = [1,1], \\ [0, \beta(y_k)], & \text{otherwise.} \end{cases}$$

In Theorems 3.19,3.23, 3.27 and 3.29 we add various new construction methods of implication operators on L^I by means of an arbitrary fix element of L and an order-preserving or an order-reversing functions on L.

Theorem 3.19. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $\beta : L \to L$ *be an order-preserving function,* $c \in \overline{\{\beta(x) \ : \ x \in L\}}$ *and* $k \in \{1, 2\}$. *Then, the mapping* $\mathcal{I}^{k}_{\beta c} : (L^I)^2 \to L^I$ *defined by, for all* $\mathbf{x}, \mathbf{y} \in L^I$,

$$\mathcal{I}^{k}_{\beta c}(\mathbf{x}, \mathbf{y}) = \begin{cases} [1,1], & \text{if } [x_1, x_2] = [0,0] \text{ or } [y_1, y_2] = [1,1], \\ [0,0], & \text{if } [x_1, x_2] = [1,1] \text{ and } [y_1, y_2] = [0,0], \\ [\beta(y_k), c], & \text{otherwise,} \end{cases} \tag{12}$$

is an implication on L^I.

Proof. The proof can be done in an analogous way to the proof of Theorem 3.11. Therefore, we omit it. □

Example 3.20. *Consider the lattice* L^I *as given in Figure 2 and the order-preserving function* β *as* $\beta(x) = x$ *for all* $x \in L$. *Then, it follows that* $c = 1$. *By applying the formula (12) in Theorem 3.19 with* $k = 1$, *the implication* $\mathcal{I}^{1}_{\beta 1}$ *can be obtained as in Table 11.*

$\mathcal{I}^1_{\beta 1}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[a,a]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[b,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[a,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[b,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$

Table 11: The implication $\mathcal{I}^1_{\beta 1}$ on L^I.

Remark 3.21. *Consider L^I as given in Figure 2 and the order-preserving function as follows:*

$$\beta(x) = \begin{cases} 0, & \textit{if } \ x \in \{0, b\}, \\ a, & \textit{otherwise.} \end{cases}$$

It must be $c = a$. By applying the formula (12) in Theorem 3.19 with $k = 1$, we have that:

$$\mathcal{I}^1_{\beta a}([0,1],[0,1]) = [\beta(0), a] = [0,a] \neq [0,1].$$

Then, $\mathcal{I}^k_{\beta a}$ does not need to be i-representable on L^I.

Corollary 3.22. *In Theorem 3.19, if the order-preserving function $\beta : L \to L$ satisfies $\beta(1) = 1$, the implication $\mathcal{I}^k_{\beta 1} : L^I \to L^I$ is obtained on L^I as follows:*

$$\mathcal{I}^k_{\beta 1}(\mathbf{x},\mathbf{y}) = \begin{cases} [1,1], & \textit{if } [x_1,x_2] = [0,0], \\ [0,0], & \textit{if } [x_1,x_2] = [1,1] \textit{ and } [y_1,y_2] = [0,0], \\ [\beta(y_k), 1], & \textit{otherwise,} \end{cases}$$

is an implication on L^I.

Theorem 3.23. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $\theta : L \to L$ be an order-reversing function, $c \in \underline{\{\theta(x) \ : \ x \in L\}}$ and $k \in \{1,2\}$. Then, the mapping $\mathcal{I}^{k*}_{\theta c} : (L^I)^2 \to L^I$ defined by, for all $\mathbf{x},\mathbf{y} \in L^I$,*

$$\mathcal{I}^{k*}_{\theta c}(\mathbf{x},\mathbf{y}) = \begin{cases} [1,1], & \textit{if } [x_1,x_2] = [0,0] \textit{ or } [y_1,y_2] = [1,1], \\ [0,0], & \textit{if } [x_1,x_2] = [1,1] \textit{ and } [y_1,y_2] = [0,0], \\ [c, \theta(x_k)], & \textit{otherwise,} \end{cases} \tag{13}$$

Proof. The proof can be done in an analogous way to the proof of Theorem 3.11. Therefore, we omit it. □

Example 3.24. *Consider the lattice L^I as given in Figure 2 and the order-reversing function θ as given in the formula (10). Therefore $c = 0$. By applying the formula (13) in Theorem 3.23 with $k = 1$, the implication $\mathcal{I}^{1*}_{\theta 0}$ can be obtained as in Table 12.*

$\mathcal{I}^{1*}_{\theta 0}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[0,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[a,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[1,1]$
$[b,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[1,1]$
$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[a,1]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[0,a]$	$[1,1]$
$[b,1]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[0,b]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[1,1]$

Table 12: The implication $\mathcal{I}^{1*}_{\theta 0}$ on L^I.

Remark 3.25. *From the Table (12) in the Example 3.24, it is seen that* $\mathcal{I}_{\theta 0}^{1*}([1,1],[0,1]) = [0,0]$. *Then,* $\mathcal{I}_{\theta 0}^{1*}$ *is an implication operator which does not have to be i-representable on* L^I.

Corollary 3.26. *In Theorem 3.23, if the order-reversing function* $\theta : L \to L$ *satisfies* $\theta(1) = 0$, *the implication* $\mathcal{I}_{\theta 0}^{k*} : (L^I)^2 \to L^I$ *is obtained on* L^I *as follows*

$$\mathcal{I}_{\theta 0}^{k*}(\mathbf{x},\mathbf{y}) = \begin{cases} [1,1], & \text{if } [x_1,x_2]=[0,0] \text{ or } [y_1,y_2]=[1,1], \\ [0,\theta(x_k)], & \text{otherwise,} \end{cases}$$

is an implication on L^I.

Theorem 3.27. *Let* $(L,\leq,0,1)$ *be a bounded lattice,* β *be an order-preserving function and* θ *be an order-reversing function on* L. *Then, the mapping* $\mathcal{I}_{\beta\theta} : (L^I)^2 \to L^I$ *defined by, for all* $\mathbf{x},\mathbf{y} \in L^I$,

$$\mathcal{I}_{\beta\theta}(\mathbf{x},\mathbf{y}) = \begin{cases} [0,0], & \text{if } [x_1,x_2]=[1,1] \text{ and } [y_1,y_2]=[0,0], \\ [1,1], & \text{if } \beta(x_1)\leq\beta(y_1) \text{ and } \beta(x_2)\leq\beta(y_2), \\ [\theta(x_2),\theta(x_1)], & \text{otherwise,} \end{cases} \tag{14}$$

is an implication on L^I.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.11. Therefore, we omit it. □

Remark 3.28. *Consider the implication* $\mathcal{I}_{\beta\theta}$ *given by the formula (14) in Theorem 3.27. It holds that* $\mathcal{I}_{\beta\theta}([0,1],[0,1]) = [1,1]$. *Then,* $\mathcal{I}_{\beta\theta}$ *is an implication operator which is not i-representable on* L^I.

Theorem 3.29. *Let* $(L,\leq,0,1)$ *be a bounded lattice and* β *be an order-preserving function on* L. *Then, the mapping* $\mathcal{I}_{\beta} : (L^I)^2 \to L^I$ *defined by, for all* $\mathbf{x},\mathbf{y} \in L^I$,

$$\mathcal{I}_{\beta}(\mathbf{x},\mathbf{y}) = \begin{cases} [0,0], & \text{if } [x_1,x_2]=[1,1] \text{ and } [y_1,y_2]=[0,0], \\ [1,1], & \text{if } \beta(x_1)\leq\beta(y_1) \text{ and } \beta(x_2)\leq\beta(y_2), \\ [\beta(y_1),\beta(y_2)], & \text{otherwise,} \end{cases} \tag{15}$$

is an implication on L^I.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.11. Therefore, we omit it. □

Remark 3.30. *Consider the implication* $\mathcal{I}_{\beta}$ *given by the formula (14) in Theorem 3.29. Since* $\mathcal{I}_{\beta}([0,1],[0,1]) = [1,1] \neq [0,1]$, $\mathcal{I}_{\beta}$ *is a non-i-representable implication* L^I.

The following theorem reveal a construction method for implication operators on L^I under an additional condition by using a t-norm $T : L^2 \to L$, an order-reversing function $\theta : L \to L$ and an order-preserving function $\beta : L \to L$.

Theorem 3.31. *Let* $(L,\leq,0,1)$ *be a bounded lattice,* T *be a t-norm,* θ *be an order-reversing function and* β *be an order-preserving function on* L. *Then, the mapping* $\mathcal{I}_{T\theta\beta} : (L^I)^2 \to L^I$ *defined by, for all* $\mathbf{x},\mathbf{y} \in L^I$,

$$\mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{y}) = \begin{cases} [1,1], & \text{if } [x_1,x_2]=[0,0] \text{ or } [y_1,y_2]=[1,1], \\ \left[T\big(\theta(x_2),\beta(y_1)\big),T\big(\theta(x_1),\beta(y_2)\big)\right], & \text{otherwise,} \end{cases} \tag{16}$$

is an implication on L^I *if only if* $T\big(\theta(1),\beta(0)\big) = 0$.

Proof. Let $T\big(\theta(1),\beta(0)\big) = 0$.

I3 and I4 are obtained directly from the definition of $\mathcal{I}_{T\theta\beta}$.

(I1) Let us show that $\mathcal{I}_{T\theta\beta}(\mathbf{z},\mathbf{y}) \leq_{L^I} \mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{y})$ for every elements $\mathbf{x},\mathbf{y},\mathbf{z} \in L^I$ with $\mathbf{x} \leq_{L^I} \mathbf{z}$. The definiton of $\mathcal{I}_{T\theta\beta}$ directly follows that $\mathcal{I}_{T\theta\beta}$ is a decreasing function in the first place if $[x_1,x_2]=[0,0]$ or $[y_1,y_2]=[1,1]$. Otherwise, it follows that

$$\mathcal{I}_{T\theta\beta}(\mathbf{z},\mathbf{y}) = \left[T\big(\theta(z_2),\beta(y_1)\big),T\big(\theta(z_1),\beta(y_2)\big)\right] \leq_{L^I} \left[T\big(\theta(x_2),\beta(y_1)\big),T\big(\theta(x_1),\beta(y_2)\big)\right] = \mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{y}).$$

(I2) Let us show that $\mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{y})\leq_{L^I}\mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{z})$ for every elements $\mathbf{x},\mathbf{y},\mathbf{z}\in L^I$ with $\mathbf{y}\leq_{L^I}\mathbf{z}$. From definiton of $\mathcal{I}_{T\theta\beta}$ it follows that $\mathcal{I}_{T\theta\beta}$ is an increasing function in the second place if $[x_1,x_2]=[0,0]$ or $[z_1,z_2]=[1,1]$. Otherwise, it follows that

$$\mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{y})=\big[T\big(\theta(x_2),\beta(y_1)\big),T\big(\theta(x_1),\beta(y_2)\big)\big]\leq_{L^I}\big[T\big(\theta(x_2),\beta(z_1)\big),T\big(\theta(x_1),\beta(z_2)\big)\big]=\mathcal{I}_{T\theta\beta}(\mathbf{x},\mathbf{z}).$$

(I5) $\mathcal{I}_{T\theta\beta}(\mathbf{1},\mathbf{0})=\big[T\big(\theta(1),\beta(0)\big),T\big(\theta(1),\beta(0)\big)\big]=[0,0]$.

The converse of the proof is immediate. □

Example 3.32. *Consider the lattice L^I as given in Figure 2, the order-reversing function θ as given in the formula (10), the t-norm T as given in Table 3 and the order-preserving function β as $\beta(x)=x$ for all $x\in L$. By applying formula (16) in Theorem 3.31, the implication $\mathcal{I}_{T\theta\beta}$ can be obtained as in Table 13.*

$\mathcal{I}_{T\theta\beta}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[0,b]$	$[0,1]$	$[a,1]$	$[0,1]$	$[1,1]$
$[0,b]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,a]$	$[0,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[a,a]$	$[0,0]$	$[0,a]$	$[0,0]$	$[a,a]$	$[0,0]$	$[0,a]$	$[a,a]$	$[0,a]$	$[1,1]$
$[b,b]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,b]$	$[0,b]$	$[0,b]$	$[1,1]$
$[0,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[0,a]$	$[0,b]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$
$[a,1]$	$[0,0]$	$[0,a]$	$[0,0]$	$[0,a]$	$[0,0]$	$[0,a]$	$[0,a]$	$[0,a]$	$[1,1]$
$[b,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,b]$	$[0,b]$	$[0,b]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[1,1]$

Table 13: The implication $\mathcal{I}_{T\theta\beta}$ on L^I.

Remark 3.33. *If we consider Example 3.32, then we know that $\mathcal{I}_{T\theta\beta}([0,1],[0,0])=[0,0]\neq[0,1]$. This shows that $\mathcal{I}_{T\theta\beta}$ is an implication operator on L^I which is not i-representable.*

Example 3.34. *Let $T=T_p(x,y)=xy$, is a t-norm on the unit interval $[0,1]$ (see [17]), the order-reversing function $\theta(x)=1-x$ and the order-preserving function $\beta(x)=x$. The implication $\mathcal{I}_{T_p\theta\beta}$ on $[0,1]$ is obtained as follows:*

$$\mathcal{I}_{T_p\theta\beta}(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & \text{if } [x_1,x_2]=[0,0] \text{ or } [y_1,y_2]=[1,1],\\ [y_1-x_2y_1,y_2-x_1y_2], & \text{otherwise.}\end{cases}$$

We present construction method for implication operators on L^I under some additional conditions by using a t-norm $T:L^2\to L$, an order-reversing function $\theta:L\to L$ and an order-preserving function $\beta:L\to L$ in Theorem 3.35.

Theorem 3.35. *Let $(L,\leq,0,1)$ be a bounded lattice, S be a t-conorm, θ be an order-reversing function and β be an order-preserving function on L. Then, the mapping $\mathcal{I}_{S\theta\beta}:(L^I)^2\to L^I$ defined by, for all $\mathbf{x},\mathbf{y}\in L^I$,*

$$\mathcal{I}_{S\theta\beta}(\mathbf{x},\mathbf{y})=\begin{cases}[0,0], & \text{if } [x_1,x_2]=[1,1] \text{ and } [y_1,y_2]=[0,0],\\ \big[S\big(\theta(x_2),\beta(y_1)\big),S\big(\theta(x_1),\beta(y_2)\big)\big], & \text{otherwise,}\end{cases} \tag{17}$$

is an implication on L^I if and only $S\big(\theta(1),\beta(1)\big)=1$ and $S\big(\theta(0),\beta(0)\big)=1$.

Proof. The proof can be done in an analogous way to the proof of Theorem 3.11. Therefore, we omit it. □

Example 3.36. *Consider the lattice L^I as given in Figure 2, the order-reversing function θ as given in the formula (10), the t-conorm S as in Table 3 and the order-preserving function function β as in Example 3.32. By applying the formula (17) in Theorem 3.35, the implication $\mathcal{I}_{S\theta\beta}$ can be obtained as in Table 14.*

$\mathcal{I}_{S\theta\beta}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[a,1]$	$[a,1]$	$[a,1]$	$[a,1]$	$[1,1]$	$[a,1]$	$[a,1]$	$[1,1]$	$[1,1]$
$[0,b]$	$[b,1]$	$[b,1]$	$[b,1]$	$[1,1]$	$[1,1]$	$[b,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[a,a]$	$[a,a]$	$[a,1]$	$[a,a]$	$[1,1]$	$[a,1]$	$[a,1]$	$[1,1]$	$[1,1]$
$[b,b]$	$[b,b]$	$[b,1]$	$[b,1]$	$[1,1]$	$[1,1]$	$[b,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[a,1]$	$[0,a]$	$[0,a]$	$[0,1]$	$[a,a]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[b,1]$	$[0,b]$	$[0,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$

Table 14: The implication $\mathcal{I}_{S\theta\beta}$ on L^I.

Remark 3.37. *Consider L^I as given in Figure 2 and the order reserving function θ as given in the formula (10) and the order-preserving β function as follows:*

$$\beta(x) = \begin{cases} a, & \text{if } x \in \{0,a\}, \\ 1, & \text{otherwise,} \end{cases}$$

By applying the formula (17) in Theorem 3.35, it can be obtained:

$$\mathcal{I}_{S\theta\beta}([0,1],[0,1]) = \big[S\big(\theta(1),\beta(0)\big), S\big(\theta(1),\beta(0)\big)\big] = [S(0,a),S(0,a)] = [a,a] \neq [0,1].$$

Then, $\mathcal{I}_{S\theta\beta}$ does not need to be i-representable on L^I.

In the following example, we compare the construction methods that we present and the internal implication (see [20] for more detail information about the Brouwer lattice and the internal implication).

Example 3.38. *Consider the lattice L^I as given in Figure 2. One can easily check that the lattice L^I is a Brouwer lattice and the internal implication $\mathcal{I}_{int}$ is obtained as in Table 15.*

$\mathcal{I}_{\text{int}}$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,a]$	$[b,b]$	$[1,1]$	$[b,b]$	$[1,1]$	$[b,b]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,b]$	$[a,a]$	$[a,a]$	$[1,1]$	$[a,a]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[b,b]$	$[b,1]$	$[b,b]$	$[1,1]$	$[b,b]$	$[b,1]$	$[1,1]$	$[b,1]$	$[1,1]$
$[b,b]$	$[a,a]$	$[a,a]$	$[a,1]$	$[a,a]$	$[1,1]$	$[a,1]$	$[a,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,0]$	$[a,a]$	$[b,b]$	$[a,a]$	$[b,b]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,1]$	$[0,0]$	$[0,a]$	$[b,b]$	$[a,a]$	$[b,b]$	$[b,1]$	$[1,1]$	$[b,1]$	$[1,1]$
$[b,1]$	$[0,0]$	$[a,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[a,1]$	$[a,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,a]$	$[0,b]$	$[a,a]$	$[b,b]$	$[0,1]$	$[a,1]$	$[b,1]$	$[1,1]$

Table 15: The implication $\mathcal{I}_{\text{int}}$ on L^I.

It can be easily seen that the implication $\mathcal{I}_{\text{int}}$ is different from the construction methods that we present by means of above examples.

The following theorem provides construction method to generate implication operators on L^I by means of the negation $N : L \to L$, the order-reversing function $\theta : L \to L$, the order-preserving function $\beta : L \to L$, and arbitrary fix element $[c,d]$ of L^I.

Theorem 3.39. *Let $(L, \leq, 0, 1)$ be a bounded chain, N be a negation on L, β be an order-preserving function with $\beta(1) \neq \beta(0)$, θ be an order-reversing function with $\theta(1) \neq \theta(0)$ on L and $[c,d] \in L^I$. Then, the mapping $\mathcal{I}_{N\theta\beta cd}$:*

$(L^I)^2 \to L^I$ *defined by, for all* $\mathbf{x}, \mathbf{y} \in L^I$,

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = \begin{cases} [1,1], & \text{if } [y_1,y_2]=[1,1], \\ [N(x_2)\vee y_1 \vee c, N(x_1)\vee y_2 \vee d], & \text{if } \beta(x_1)\le\beta(y_1) \text{ and } \theta(x_2)\ge\theta(y_2), \\ [c,d], & \text{if } \beta(x_1)\le\beta(y_1) \text{ and } \theta(x_2)<\theta(y_2), \\ [0,d], & \text{if } \beta(x_1)>\beta(y_1) \text{ and } \theta(x_2)\ge\theta(y_2), \\ [0,0], & \text{otherwise,} \end{cases} \tag{18}$$

is an implication on L^I.

Proof. Observe that the function $\mathcal{I}_{N\theta\beta cd}$ satisfies I_3, I_4 and I_5.

(I1) We need to show that $\mathcal{I}_{N\theta\beta cd}$ is a decreasing function on the first variable. Then it should be $\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) \le_{L^I} \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z})$ for $\mathbf{x},\mathbf{y},\mathbf{z} \in L^I$ and $\mathbf{x} \le_{L^I} \mathbf{y}$. If $\mathbf{z}=\mathbf{1}$, the proof is clear.

Then, we assume $\mathbf{z} \neq \mathbf{1}$ and review the following cases:

1. Let $\beta(y_1) \le \beta(z_1)$ and $\theta(y_2) \ge \theta(z_2)$.

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) = [N(y_2)\vee z_1\vee c, N(y_1)\vee z_2\vee d] \le_{L^I} [N(x_2)\vee z_1\vee c, N(x_1)\vee z_2\vee d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

2. Let $\beta(y_1) \le \beta(z_1)$ and $\theta(y_2) < \theta(z_2)$.

2.1. If $\theta(x_2) \ge \theta(z_2)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) = [c,d] \le_{L^I} [N(x_2)\vee z_1\vee c, N(x_1)\vee z_2\vee d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

2.2. If $\theta(x_2) < \theta(z_2)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) = [c,d] \le_{L^I} [c,d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

3. Let $\beta(y_1) > \beta(z_1)$ and $\theta(y_2) \ge \theta(z_2)$.

3.1. If $\beta(x_1) \le \beta(z_1)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) = [0,d] \le_{L^I} [N(x_2)\vee z_1\vee c, N(x_1)\vee z_2\vee d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

3.2. If $\beta(x_1) > \beta(z_1)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) = [0,d] \le_{L^I} [0,d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

4. Let $\beta(y_1) > \beta(z_1)$ and $\theta(y_2) < \theta(z_2)$.

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{y},\mathbf{z}) = [0,0] \le_{L^I} \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

(I2) We need to show that $\mathcal{I}_{N\theta\beta cd}$ is an increasing function on the second variable. Then it should be $\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) \le_{L^I} \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z})$ for $\mathbf{x},\mathbf{y},\mathbf{z} \in L^I$ and $\mathbf{y} \le_{L^I} \mathbf{z}$. If $\mathbf{z} \in \{\mathbf{1}\}$, the proof is immediate. Then, we assume $\mathbf{z} \notin \{\mathbf{1}\}$ and check the following cases.

1. Let $\beta(x_1) \le \beta(y_1)$ and $\theta(x_2) \ge \theta(y_2)$.

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = [N(x_2)\vee y_1\vee c, N(x_1)\vee y_2\vee d] \le_{L^I} [N(x_2)\vee z_1\vee c, N(x_1)\vee z_2\vee d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

2. Let $\beta(x_1) \le \beta(y_1)$ and $\theta(x_2) < \theta(y_2)$.

2.1. If $\theta(x_2) \ge \theta(z_2)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = [c,d] \le_{L^I} [N(x_2)\vee z_1\vee c, N(x_1)\vee z_2\vee d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

2.2. If $\theta(x_2) < \theta(z_2)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = [c,d] \le_{L^I} [c,d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

3. Let $\beta(x_1) > \beta(y_1)$ and $\theta(x_2) \ge \theta(y_2)$.

3.1. If $\beta(x_1) \le \beta(z_1)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = [0,d] \le_{L^I} [N(x_2)\vee z_1\vee c, N(x_1)\vee z_2\vee d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

3.2. If $\beta(x_1) > \beta(z_1)$, then

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = [0,d] \leq_{L^I} [0,d] = \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

4. Let $\beta(x_1) > \beta(y_1)$ and $\theta(x_2) < \theta(y_2)$.

$$\mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{y}) = [0,0] \leq_{L^I} \mathcal{I}_{N\theta\beta cd}(\mathbf{x},\mathbf{z}).$$

□

Example 3.40. *Consider the bounded chain ($L = \{0, c, d, 1\}, \leq, 0, 1$) characterized by the Hasse diagram in Figure 3.*

Figure 3: The lattice L.

It is easy to obtain the lattice L^I as in Figure 4.

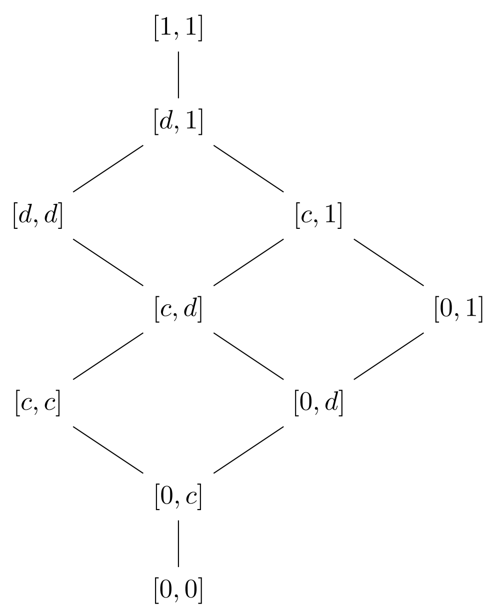

Figure 4: The lattice L^I.

Take the order-preserving function, the order-reversing function θ and the negation N as follows respectively,

$$\beta(x) = x$$

$$\theta(x) = \begin{cases} 1, & \text{if } x = 0, \\ d, & \text{if } x \in \{c, d\}, \\ 0, & \text{if } x = 1, \end{cases}$$

and

$$N(x) = \begin{cases} 1, & \textit{if } x = 0, \\ d, & \textit{if } x = c, \\ c, & \textit{if } x = d, \\ 0, & \textit{if } x = 1. \end{cases}$$

By applying the formula (18) in Theorem 3.39, the implication $\mathcal{I}_{N\theta\beta cd}$ is obtained as in Table 16.

$\mathcal{I}_{N\theta\beta cd}$	$[0,0]$	$[0,c]$	$[0,d]$	$[0,1]$	$[c,c]$	$[c,d]$	$[c,1]$	$[d,d]$	$[d,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,c]$	$[c,d]$	$[d,1]$	$[d,1]$	$[d,1]$	$[d,1]$	$[d,1]$	$[d,1]$	$[d,1]$	$[d,1]$	$[1,1]$
$[0,d]$	$[c,d]$	$[c,1]$	$[c,1]$	$[c,1]$	$[c,1]$	$[c,1]$	$[c,1]$	$[d,1]$	$[d,1]$	$[1,1]$
$[0,1]$	$[c,d]$	$[c,d]$	$[c,d]$	$[c,1]$	$[c,d]$	$[c,d]$	$[c,1]$	$[c,d]$	$[d,1]$	$[1,1]$
$[c,c]$	$[0,0]$	$[0,d]$	$[0,d]$	$[0,d]$	$[d,d]$	$[d,d]$	$[d,1]$	$[d,d]$	$[d,1]$	$[1,1]$
$[c,d]$	$[0,0]$	$[0,d]$	$[0,d]$	$[0,d]$	$[c,d]$	$[c,d]$	$[c,1]$	$[d,d]$	$[d,1]$	$[1,1]$
$[c,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,d]$	$[c,d]$	$[c,d]$	$[c,1]$	$[c,d]$	$[d,1]$	$[1,1]$
$[d,d]$	$[0,0]$	$[0,d]$	$[0,d]$	$[0,d]$	$[0,d]$	$[0,d]$	$[0,d]$	$[d,d]$	$[d,1]$	$[1,1]$
$[d,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,d]$	$[0,0]$	$[0,0]$	$[0,d]$	$[c,d]$	$[d,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,d]$	$[0,0]$	$[0,0]$	$[0,d]$	$[0,0]$	$[0,d]$	$[1,1]$

Table 16: The implication $\mathcal{I}_{N\theta\beta cd}$ on L^I.

Remark 3.41. *In Example 3.40 we have $\mathcal{I}_{N\theta\beta cd}([0,1],[0,1]) = [c,1] \neq [0,1]$, then the implication $\mathcal{I}_{N\theta\beta cd}$ given by the formula (18) in Theorem 3.39 does not have to be i-representable on L^I.*

Example 3.42. *Consider the lattice L^I as given in Figure 4. One can easily check that the lattice L^I is a Brouwer lattice and the internal implication $\mathcal{I}_{int}$ is obtained as in Table 17.*

$\mathcal{I}_{\text{int}}$	$[0,0]$	$[0,c]$	$[0,d]$	$[0,1]$	$[c,c]$	$[c,d]$	$[c,1]$	$[d,d]$	$[d,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,c]$	$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,d]$	$[0,0]$	$[c,c]$	$[1,1]$	$[1,1]$	$[c,c]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,0]$	$[c,c]$	$[d,d]$	$[1,1]$	$[c,c]$	$[d,d]$	$[1,1]$	$[d,d]$	$[1,1]$	$[1,1]$
$[c,c]$	$[0,0]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[c,d]$	$[0,0]$	$[0,c]$	$[0,1]$	$[0,1]$	$[c,c]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[c,1]$	$[0,0]$	$[0,c]$	$[0,d]$	$[0,1]$	$[c,c]$	$[d,d]$	$[1,1]$	$[d,d]$	$[1,1]$	$[1,1]$
$[d,d]$	$[0,0]$	$[0,c]$	$[0,1]$	$[0,1]$	$[c,c]$	$[c,1]$	$[c,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[d,1]$	$[0,0]$	$[0,c]$	$[0,d]$	$[0,1]$	$[c,c]$	$[c,d]$	$[c,1]$	$[d,d]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,c]$	$[0,d]$	$[0,1]$	$[c,c]$	$[c,d]$	$[c,1]$	$[d,d]$	$[d,1]$	$[1,1]$

Table 17: The implication $\mathcal{I}_{\text{int}}$ on L^I.

Remark 3.43. *It is worth nothing that the construction method $\mathcal{I}_{N\theta\beta cd}$ given in Theorem (3.39) and the internal implication $\mathcal{I}_{int}$ are different from each other by means of Tables 16 and 17.*

Example 3.44. *Consider the bounded lattice $(L = \{0, a, c, d, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 5.*

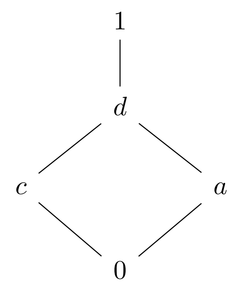

Figure 5: The lattice L.

It is easy to obtain the lattice bounded lattice $(L^I = \{[0,0],[0,a],[0,c],[0,d],[a,a],[c,c],[d,d],[c,d],[a,d],$ $[0,1],[a,1],[c,1],[d,1],[1,1]\}, \leq_{L^I}, [0,0],[1,1])$ *as in Fig. 6.*

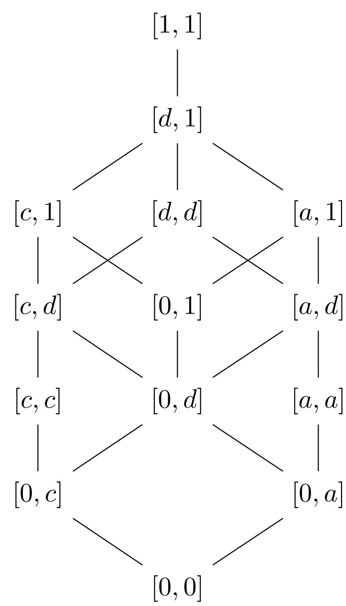

Figure 6: The lattice L^I.

Consider the order-reversing function θ*, the order-preserving function* β *and the negation* N *as follows respectively,*

$$\theta(x) = \begin{cases} 1, & \text{if } x = 0, \\ d, & \text{if } x \in \{a, c, d\}, \\ 0, & \text{if } x = 1, \end{cases}$$

$$\beta(x) = x$$

and

$$N(x) = \begin{cases} 1, & \text{if } x = 0, \\ a, & \text{if } x = a \\ c, & \text{if } x = c, \\ 0, & \text{if } x \in \{d, 1\}. \end{cases}$$

In this example, one can easily show that the $\mathcal{I}_{N\theta\beta cd}$ *given in Theorem 3.39 is not an implication operator.*

4 Conclusions

In this study, construction methods for implications on the interval valued lattices L^I have been investigated via implication operators, t-norms, t-conorms, negations, order-preserving and order-reversing functions, fix arbitrary elements of L, where also some constraints on the bounded lattice L have been put if they are nececessary. Also, the given construction methods are clarified by examples. For the future works, we aim to introduce novel construction methods for interval-valued t-norms, t-conorms, uninorms, etc.

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