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# The characterization for the sobriety of $L$-convex spaces

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**Abstract**

With a commutative integral quantale $L$ as the truth value table, this study focuses on the characterizations of the sobriety of stratified $L$-convex spaces introduced by Liu and Yue in 2024. It is shown that a stratified sober $L$-convex space $Y$ is a sobrification of a stratified $L$-convex space $X$ if and only if there exists a quasihomeomorphism from $X$ to $Y$; a stratified $L$-convex space is sober if and only if it is a strictly injective object in the category of stratified $S_0$ $L$-convex spaces.

*Keywords:* Quantale, $L$-convex space, sobriety, quasihomeomorphism, strictly injective object.

## 1 Introduction

A convex structure, also known as an algebraic closure system, on a set is a family of subsets closed under arbitrary intersections and directed unions, including the empty set. A set equipped with a convex structure is called a convex space, or an algebraic closure space. Convex structures can be viewed as abstractions of traditional convex sets in Euclidean spaces. Monograph [27] provides a comprehensive overview of the theory of convex structures. Convex structures have close connections with various mathematical structures, including ordered structures [2, 4, 25, 28], algebraic structures [10, 11, 14, 16], and topological structures [8, 26, 37].

Sobriety is an interesting property in the realm of non-Hausdorff topology [6] and domain theory [5]. It fosters the interplay between ordered and topological structures [31]. The theory of convex spaces also addresses the so-called sobriety as an important concept. The notion of sobriety in convex structures can be traced back to the pioneering work of Erné [2], where he established a Stone-type duality between sober convex spaces and algebraic lattices. Shen et al. [22] explored properties of sober convex spaces similar to those of sober topological spaces and shed light on more connections between convex spaces and domain theory. In [36], Yao and Zhou established a categorical isomorphism between join-semilattices and sober convex spaces. Sun and Pang [24] proposed an alternative form of sober convex spaces and demonstrated its equivalence to Erné's sober convex spaces [2]. Xia in [30] investigated further properties of sober convex spaces, contributing to the development of pointfree convex geometry research methods [13].

With the development of fuzzy set theory and its applications, fuzzy convex spaces have been extensively studied (see [12, 17, 18, 19, 21, 23]). Recently, Liu and Yue [10] extended the concept of algebraic irreducible convex set to the fuzzy setting, leading to a fuzzy type of sobriety of $L$-convex spaces. In [29], we extended the notion of polytopes to the fuzzy setting and introduced an alternative type of fuzzy sober convex spaces. In [10], Liu and Yue mainly studied the reflectivity and their sobriety in the category of $L$-convex spaces. Different from Liu and Yue's work, we primarily investigated $L$-join-semilattice completions of $L$-ordered sets via our notion of sobriety in [29], establishing more connections between fuzzy ordered structures and fuzzy convex structures. In the classical case, since the notions of polytopes and algebraic irreducible convex sets are equivalent, one can conclude that these two types of sobriety in classical convex spaces are also equivalent. However, in the fuzzy setting, while Liu and Yue's notion of sobriety clearly implies ours, the reverse implication was left as an unresolved problem in [29]. Fortunately, in this paper, we provide a counterexample showing that the reverse implication does not hold.

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In topology, it is well known that sober topological spaces are precisely the strictly injective objects in the category of $T_0$ topological spaces [3, 5]. Xia [30] provided a corresponding result for sober convex spaces. In the theory of fuzzy convex structures, it is natural to ask whether sober $L$-convex spaces are exactly the strictly injective objects in the category of $S_0$ $L$-convex spaces. This paper will give a positive answer. Moreover, Xia in [30] presented a characterization for the sobrification of convex spaces with help of the notion of quasihomeomorphism. As an application of fuzzy orders, we also give a characterization for the sobrifications of $L$-convex spaces and provide an example to demonstrate that this result is helpful in identifying the sobrifications of $L$-convex spaces.

In this paper, we select a commutative integral quantale $L$ as the truth value table. The paper is structured as follows: In Section 2, we review fundamental concepts and results related to $L$-orders and $L$-convex spaces. In Section 3, we revisit essential findings on sober $L$-convex spaces and provide additional propositions concerning sobriety. In Section 4, we introduce the concepts of quasihomeomorphism and strict embedding within the framework of $L$-convex spaces, offering characterizations for the sobrification and sobriety of an $L$-convex space, respectively.

## 2 Preliminaries

We refer to [20] for contents of quantales, to [7] for notions of fuzzy sets, and to [32, 35, 39] for contents of fuzzy posets.

Let $L$ be a complete lattice with a bottom element 0 and a top element 1 and let $\otimes$ be a binary operation on $L$ such that $(L, \otimes, 1)$ forms a commutative monoid. The pair $(L, \otimes)$ is called a *commutative integral quantale*, or a *complete residuated lattice*, if the operation $\otimes$ is distributive over joins, i.e.,

$$a \otimes (\bigvee S) = \bigvee_{s \in S} (a \otimes s).$$

For a commutative integral quantale $(L, \otimes)$, the operation $\otimes$ has a corresponding operation $\rightarrow: L \times L \longrightarrow L$ defined via the adjoint property:

$$a \otimes b \leq c \iff a \leq b \rightarrow c \ (\forall a, b, c \in L).$$

In this paper, unless otherwise specified, the truth value table $L$ is always assumed to be a commutative integral quantale.

Every mapping $A : X \longrightarrow L$ is called an $L$-*subset* of $X$, denoted by $A \in L^X$. Let $Y \subseteq X$ and $A \in L^X$. Define $A|_Y \in L^Y$ by $A|_Y(y) = A(y)$ $(\forall y \in Y)$. For an element $a \in L$, the notation $a_X$ denotes the constant $L$-subset of $X$ with the value $a$, i.e., $a_X(x) = a$ $(\forall x \in X)$. For all $a \in L$ and $A \in L^X$, we write $a \otimes A$ for the $L$-subset given by $(a \otimes A)(x) = a \otimes A(x)$.

For each $a \in X$, define the characteristic function $1_{a,X} : X \longrightarrow L$ by

$$1_{a,X}(x) = \begin{cases} 1, & x = a; \\ 0, & x \neq a. \end{cases}$$

We always write $1_a$ instead of $1_{a,X}$ when there is no confusion. Readers can identify the domain of a given characteristic function based on the context.

**Definition 2.1.** [32, 39] A mapping $e : P \times P \longrightarrow L$ is called an $L$-*order* if

(E1) $\forall x \in P$, $e(x, x) = 1$;

(E2) $\forall x, y, z \in P$, $e(x, y) \otimes e(y, z) \leq e(x, z)$;

(E3) $\forall x, y \in P$, if $e(x, y) \wedge e(y, x) = 1$, then $x = y$.

The pair $(P, e)$ is called an $L$-*ordered set*. It is customary to write $P$ for the pair $(P, e)$.

To avoid confusion, we sometimes use $e_P$ to emphasize that the background set is $P$. A mapping $f : P \longrightarrow Q$ between two $L$-ordered sets is said to be $L$-*order-preserving* if for all $x, y \in P$, $e_P(x, y) \leq e_Q(f(x), f(y))$; $f$ is said to be an $L$-*order-isomorphism* if $f$ is a bijection and $e_P(x, y) = e_Q(f(x), f(y))$ for all $x, y \in P$, denoted by $P \cong Q$.

**Example 2.2.** [32, 39] Define $\mathrm{sub}_X : L^X \times L^X \longrightarrow L$ by

$$\mathrm{sub}_X(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x) \ (\forall A, B \in L^X).$$

Then $\mathrm{sub}_X$ is an $L$-order on $L^X$, which is called the *inclusion* $L$-*order* on $L^X$. If the background set $X$ is clear, we can delete the subscript $X$ in $\mathrm{sub}_X$.

A mapping $f : X \longrightarrow Y$ can induce two mappings naturally; those are, $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$, which are respectively defined by

$$f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)\ (\forall A \in L^X), \qquad f^{\leftarrow}(B) = B \circ f\ (\forall B \in L^Y).$$

These two mappings are called the *Zadeh extensions* of $f$.

**Lemma 2.3.** [35] *For each mapping $f : X \longrightarrow Y$, the Zadeh extensions $f^{\rightarrow}$ and $f^{\leftarrow}$are $L$-order-preserving with respect the inclusion $L$-order, and $f^{\rightarrow}$ is left adjoint to $f^{\leftarrow}$, denoted $f^{\rightarrow} \dashv f^{\leftarrow}$; this means that*

$$\mathrm{sub}_Y(f^{\rightarrow}(A), B) = \mathrm{sub}_X(A, f^{\leftarrow}(B))\ (\forall A \in L^X, B \in L^Y).$$

Define $\uparrow x$ and $\downarrow x$ respectively by $\uparrow x(y) = e(x, y)$, $\downarrow x(y) = e(y, x)$ $(\forall x, y \in P)$.

**Definition 2.4.** [32, 39] Let $P$ be an $L$-ordered set. An element $x_0$ is called a *supremum* of an $L$-subset $A$ of $P$, in symbols $x_0 = \sqcup A$, if

(1) $\forall x \in P,\ A(x) \le e(x, x_0)$;

(2) $\forall y \in P,\ \bigwedge_{x\in P} A(x) \to e(x, y) \le e(x_0, y)$.

An element $x_0$ is called an *infimum* of an $L$-subset $A$ of $P$, in symbols $x_0 = \sqcap A$, if

(3) $\forall x \in P,\ A(x) \le e(x_0, x)$;

(4) $\forall y \in P,\ \bigwedge_{x\in P} A(x) \to e(y, x) \le e(y, x_0)$.

It is easy to verify that if the supremum (resp., infimum) of an $L$-subset in an $L$-ordered set exists, then it must be unique. The following provides an equivalent definition of supremum and infimum, respectively.

**Lemma 2.5.** [32, 39] *Let $P$ be an $L$-ordered set. An element $x_0 \in P$ is the supremum (resp., infimum) of $A \in L^P$, i.e., $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), iff*

$$e(x_0, y) = \mathrm{sub}(A, \downarrow y)\ (\forall y \in P)\ (resp., e(y, x_0) = \mathrm{sub}(A, \uparrow y)\ (\forall y \in P).$$

An $L$-ordered set $P$ is said to be *complete* if its every $L$-subset has a supremum, or equivalently, has an infimum.

**Example 2.6.** [32] For every $L$-subset $\mathcal{A}$ in the $L$-ordered set $(L^X, \mathrm{sub})$, the supremum (resp., infimum) of $\mathcal{A}$ exists; that is

$$\sqcup \mathcal{A} = \bigvee_{B\in L^X} \mathcal{A}(B) \otimes B \text{ (resp., } \sqcap \mathcal{A} = \bigwedge_{B\in L^X} \mathcal{A}(B) \to B).$$

Hence, the $L$-ordered set $(L^X, \mathrm{sub})$ is complete.

**Definition 2.7.** Let $P$ and $Q$ be two complete $L$-ordered sets. The mapping $f : P \longrightarrow Q$ is said to be *supremum-preserving* (resp., *infimum-preserving*) if for every $A \in L^P$, $f(\sqcup A) = \sqcup f^{\rightarrow}(A)$ (resp., $f(\sqcap A) = \sqcap f^{\rightarrow}(A)$).

**Example 2.8.** [34, Proposition 5.1],[35, Theorem 3.5] Let $f : X \longrightarrow Y$ be a mapping between two sets. Then $f^{\rightarrow} : (L^X, \mathrm{sub}_X) \longrightarrow (L^Y, \mathrm{sub}_Y)$ is supremum-preserving, and $f^{\leftarrow} : (L^Y, \mathrm{sub}_Y) \longrightarrow (L^X, \mathrm{sub}_X)$ is both supremum-preserving and infimum-preserving.

In the following, we recall the contents of $L$-convex spaces/algebraic $L$-closure spaces.

**Definition 2.9.** [9, Definitions 2.4, 2.5] Let $X$ be a set and $\mathcal{C} \subseteq L^X$. The family $\mathcal{C}$ is called an *$L$-convex structure*, or *algebraic $L$-closure structure*, on $X$ if it satisfies the following conditions:

(C1) $0_X, 1_X \in \mathcal{C}$;

(C2) $\bigvee^{\uparrow}_{i\in I} C_i \in \mathcal{C}$ for every directed subset $\{C_i \mid i \in I\}$ of $\mathcal{C}$;

(C3) $\bigwedge_{j\in J} C_j \in \mathcal{C}$ for every subset $\{C_j \mid j \in J\}$ of $\mathcal{C}$;

The pair $(X, \mathcal{C})$ is called an *$L$-convex space*, or *algebraic $L$-closure space*; each element of $\mathcal{C}$ is called a *convex set* of $(X, \mathcal{C})$.

The $L$-convex space $(X, \mathcal{C})$ is said to be *stratified* if it furthermore satisfies:

(S) $p \to C \in \mathcal{C}$ for all $p \in L$ and $C \in \mathcal{C}$.

In this paper, every $L$-convex space is assumed to be stratified, so we omit the the word "stratified". As usual, we often write $X$ instead of $(X, \mathcal{C})$ for an $L$-convex space and write $\mathcal{C}(X)$ for the $L$-convex structure on $X$.

**Definition 2.10.** [9] Let $X$ be an $L$-convex space. Define a mapping $co_X : L^X \longrightarrow L^X$ by

$$co_X(A) = \bigwedge\{C \in \mathcal{C}(X) \mid A \leq C\} \ (\forall A \in L^X),$$

called the *hull operator* of $X$ and $co_X(A)$ is called the *hull of* $A$. We write $co$ instead of $co_X$ when no ambiguity can arise.

**Definition 2.11.** [17] Let $f : X \longrightarrow Y$ be a mapping between two $L$-convex spaces. Then $f$ is said to be

- *convexity-preserving* if for every $D \in \mathcal{C}(Y)$, $f^{\leftarrow}(D) \in \mathcal{C}(X)$;
- *convex-to-convex* if for every $C \in \mathcal{C}(X)$, $f^{\rightarrow}(C) \in \mathcal{C}(Y)$;
- *homeomorphic* if it is bijective, convexity-preserving and convex-to-convex.

One easily verifies that a bijection $f : X \longrightarrow Y$ is a homeomorphism if and only if $f : X \longrightarrow Y$ and its inverse mapping $f^{-1} : Y \longrightarrow X$ are both convexity-preserving.

Given a convexity-preserving mapping $f : X \longrightarrow Y$, we obtain an assignment from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$, which sends $D \in \mathcal{C}(Y)$ to $f^{\leftarrow}(D) \in \mathcal{C}(X)$. When no confusion can arise, we also use

$$f^{\leftarrow} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X),$$

to denote this mapping. For each $L$-convex space $X$, $(\mathcal{C}(X), \mathrm{sub}_X)$ is a complete $L$-ordered set. Specifically, for every $L$-subset $\mathcal{A}$ of $(\mathcal{C}(X), \mathrm{sub}_X)$, the infimum of $\mathcal{A}$ exists; that is $\sqcap\mathcal{A} = \bigwedge_{C\in\mathcal{C}(X)} \mathcal{A}(C) \rightarrow C$. If $f : X \longrightarrow Y$ is convexity-preserving, then

$$f^{\leftarrow} : (\mathcal{C}(Y), \mathrm{sub}_Y) \longrightarrow (\mathcal{C}(X), \mathrm{sub}_X),$$

is infimum-preserving.

**Lemma 2.12.** [38, Proposition 2.8] *Let $f : X \longrightarrow Y$ be a mapping between two $L$-convex spaces. Then $f$ is convexity-preserving iff $f^{\rightarrow}(co_X(A)) \subseteq co_Y(f^{\rightarrow}(A))$ iff $co_Y(f^{\rightarrow}(co_X(A))) = co_Y(f^{\rightarrow}(A))$ $(\forall A \in L^X)$.*

**Lemma 2.13.** [9, Proposition 2.10] *Let $X$ be an $L$-convex space. Then $\forall A \in L^X$ and $B \in \mathcal{C}(X)$, we have* $\mathrm{sub}(A, B) = \mathrm{sub}(co(A), B)$.

# 3 Sober $L$-convex spaces

In [9], Liu and Yue introduced a notion of sober $L$-convex spaces, extending the study of sober convex spaces to the fuzzy setting. Making use of the inclusion $L$-order between convex sets in $L$-convex spaces, they defined algebraic irreducible convex sets in $L$-convex spaces and subsequently introduced a notion of sober $L$-convex spaces. In this section, we first recall the concept of sober $L$-convex spaces defined by Liu and Yue and explore additional properties concerning this type of sobriety of $L$-convex spaces.

**Definition 3.1.** [9, Definition 2.12] Let $X$ be an $L$-convex space. A convex set $F \in \mathcal{C}(X)$ is said to be *algebraic irreducible* if

(1) $\bigvee_{x\in X} F(x) = 1$;

(2) $\mathrm{sub}(F, \bigvee^{\uparrow}_{i\in I} C_i) = \bigvee^{\uparrow}_{i\in I} \mathrm{sub}(F, C_i)$ for every directed family $\{C_i \mid i \in I\} \subseteq \mathcal{C}(X)$.

Let $\mathrm{irr}(X)$ denote the collection of all algebraic irreducible convex sets in $X$. An algebraic irreducible convex set is called a *compact convex set* in [29].

**Definition 3.2.** [9, Definition 2.13] An $L$-convex space $X$ is said to be *sober* if for each algebraic irreducible convex set $F \in \mathrm{irr}(X)$, $F$ is the hull of $1_a$ for a unique $a \in X$, i.e., $F = co(1_a)$.

Since we introduced another type of sobriety for $L$-convex spaces in [29], we provide the following standing assumption to prevent confusion.

**Standing Assumption**. In this paper, the terminology "sober" mentioned in this paper always refers to the definition above which was given by Liu and Yue, unless otherwise specified.

Recall that an $L$-convex space $X$ is said to be $S_0$ if for every $x, y \in X$, $C(x) = C(y)$ ($\forall C \in \mathcal{C}$) implies $x = y$. Using hull operator, Liu and Yue ([9, Proposition 2.8]) showed that an $L$-convex space $X$ is $S_0$ if and only if for all $x, y \in X$, $co(1_x) = co(1_y)$ implies $x = y$. Clearly, for every $x \in X$, $co(1_x)$ is an algebraic irreducible convex set. Thus, every sober $L$-convex space is $S_0$.

The following example is the same as Example 3.4 in [29]. We now provied a detailed proof that the $L$-convex structure defined below is indeed a sober $L$-convex spaces in the sense of Definition 3.1.

**Example 3.3.** Let $L = ([0,1], \otimes)$ be a commutative integral quantale with $\otimes$ being the meet operation $\wedge$. That is to say, $L$ is a frame. Define a stratified $L$-convex structure

$$\mathcal{C} = \{a \wedge \phi \mid a \in [0,1], \phi : [0,1] \longrightarrow [0,1] \text{ is increasing}, \phi \geq id\},$$

on $[0,1]$, where $id$ sends each $x \in [0,1]$ to $x \in [0,1]$. Specifically, a function $\mu : [0,1] \to [0,1]$ is a member of $\mathcal{C}$ if and only if $\mu$ is an increasing function; and there exists some $a \in [0,1]$ such that $\mu(x) \geq x$ if $x \in [0,a)$ and $\mu(x) = a$ if $x \in [a,1]$.

We next show that in $([0,1], \mathcal{C})$, every algebraic irreducible convex set is the hull of $1_x$ for a unique $x \in [0,1]$. Let $F \in \mathrm{irr}([0,1])$. Since $\bigvee_{x \in [0,1]} F(x) = 1$, we have $F(1) = 1$. Write $b = \inf\{x \in [0,1] \mid F(x) = 1\}$ and $A = \{x \in [0,1] \mid F(x) < 1\}$. Clearly, $A = [0,b)$ or $A = [0,b]$. We claim that for every $x \in A$, $F(x) = x$. Suppose, for the sake of contradiction, that there exists $x_0 \in A$ such that $F(x_0) \in (x_0, 1)$. For every $n \in \mathbb{N}$, define $F_n : [0,1] \longrightarrow [0,1]$ by

$$F_n(x) = \begin{cases} \frac{(n-1)F(x)+x}{n}, & x \in A; \\ 1, & otherwise. \end{cases}$$

It is clear that $F_n \in \mathcal{C}$ and $\bigvee_{n \in \mathbb{N}} F_n = F$. Hence, $\mathrm{sub}(F, \bigvee_{n \in \mathbb{N}} F_n) = 1$. For every $n$, we have $\mathrm{sub}(F, F_n) \leq F(x_0) \to F_n(x_0) = F_n(x_0)$. Thus,

$$\bigvee_{n \in \mathbb{N}} \mathrm{sub}(F, F_n) \leq \bigvee_{n \in \mathbb{N}} F_n(x_0) = F(x_0) < 1;$$

this is a contradiction. Thus, for every $x \in A$, $F(x) = x$.

We now claim that $b \notin A$, i.e., $F(b) = 1$. Assume, for the sake of contradiction, that $b \in A$, i.e., $F(b) = b$. For every $n \in \mathbb{N}$, define $G_n : [0,1] \longrightarrow [0,1]$ by

$$G_n(x) = \begin{cases} x, & x \in [0,b]; \\ \frac{n}{2}(x-b) + \frac{b+1}{2}, & (b, \frac{1+(n-1)b}{n}]; \\ 1, & otherwise. \end{cases}$$

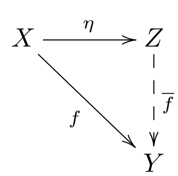
It is clear that $G_n \in \mathcal{C}$ and $\bigvee_{n \in \mathbb{N}} G_n = F$. For every $n$, $\mathrm{sub}(F, G_n) = \frac{b+1}{2}$. Therefore,

$$\bigvee_{n \in \mathbb{N}} \mathrm{sub}(F, G_n) = \frac{b+1}{2}.$$

This contradicts the fact that $\mathrm{sub}(F, \bigvee_{n \in \mathbb{N}} G_n) = 1$. Thus, $b \notin A$. This shows that $F(x) = x$ for all $x \in [0,b)$ and $F(x) = 1$ otherwise. It is clear that $F = co(1_b)$. The uniqueness of $b$ is clear. Thus, $([0,1], \mathcal{C})$ is a sober $L$-convex space. □

Given an $L$-convex space $X$, for every $C \in \mathcal{C}(X)$, define an $L$-subset $\phi(C)$ of $\mathrm{irr}(X)$ by $\phi(C)(F) = \mathrm{sub}(F, C)$ ($\forall F \in \mathrm{irr}(X)$). Liu and Yue ([9, Proposition 4.1]) showed that $\{\phi(C) \mid C \in \mathcal{C}(X)\}$ forms an $L$-convex structure on $\mathrm{irr}(X)$. In this paper, we denote the resulting convex structure by $\mathcal{C}(\mathrm{irr}(X))$ and the resulting $L$-convex space $(\mathrm{irr}(X), \mathcal{C}(\mathrm{irr}(X)))$ by $\mathbb{S}(X)$. [9, Proposition 4.2] states that the $L$-convex space $\mathbb{S}(X)$ is sober. Liu and Yue referred to $\mathbb{S}(X)$ as the *sobrification* of $X$. Here we provide the standard definition of sobrifications corresponding to $L$-convex spaces.

**Definition 3.4.** Let $X$ be an $L$-convex space, let $Z$ be a sober $L$-convex space and let $\eta : X \longrightarrow Z$ be a convexity-preserving mapping. Then $Z$ with the convexity-preserving mapping $\eta$, or $Z$ for short, is called a *sobrification* of $X$ if for every sober $L$-convex space $Y$ and every convexity-preserving mapping $f : X \longrightarrow Y$, there exists a unique convexity-preserving mapping $\overline{f} : Z \longrightarrow Y$ such that $f = \overline{f} \circ \eta$ (see the following figure).



By the universal property of sobrification, it is easy to see that the sobrification of an $L$-convex space is unique up to homeomorphism.

Let $X$ be an $L$-convex space. Define $\eta_X : X \longrightarrow \mathbb{S}(X)$ by $\eta_X(x) = co(1_x)$. By [9, Lemma 4.3], $X$ is sober if and only if the corresponding mapping $\eta_X$ is a homeomorphism. In this paper, we use $L$-**CS** (resp., $L$-$\mathbf{CS}_0$, $L$-**SobCS**) to denote the category of $L$-convex spaces (resp., $S_0$ $L$-convex spaces, sober $L$-convex spaces) with convexity-preserving mappings as morphisms.

Liu and Yue have shown that $L$-**SobCS** is reflective in $L$-**CS** (see Theorem 5.1 in [9]). It follows that $\mathbb{S}(X)$, equipped with the mapping $\eta_X$, is a sobrificaion of $X$ in the sense of Definition 3.4. Therefore, it is reasonable for Liu and Yue to directly call $\mathbb{S}(X)$ the sobrificaion of $X$.

For an $L$-convex space $X$ and $Y \subseteq X$, define $\mathcal{C}(X)|_Y := \{C|_Y \mid C \in \mathcal{C}(X)\}$. The family $\mathcal{C}(X)|_Y$ forms an $L$-convex structure on the background set $Y$. The convex space $(Y, \mathcal{C}(X)|_Y)$ is referred to as a *subspace* of $(X, \mathcal{C}(X))$.

For $f : X \longrightarrow Y$, denote $f^\circ : X \longrightarrow f(X)$ as the mapping defined by $f^\circ(x) = f(x)$ for all $x \in X$. The mapping $f$ is called a *subspace embedding* if $f^\circ$ is a homeomorphism from $X$ to $f(X)$, where $f(X)$ is considered as a subspace of $Y$.

We present some properties of sober $L$-convex spaces.

**Proposition 3.5.** *Let $f$ and $g$ be two convexity-preserving mappings from a sober $L$-convex space $X$ to an $S_0$ $L$-convex space $Y$. When $Z = \{x \in X \mid f(x) = g(x)\}$ is considered as a subspace of $X$, it is a sober $L$-convex space.*

*Proof.* Let $F$ be an algebraic irreducible convex set of $Z$. Define $F^* \in L^X$ by $F^*(x) = F(x)$ for $x \in Z$ and $F^*(x) = 0$ for $x \in X - Z$. We claim that $co_X(F^*) \in \mathrm{irr}(X)$. Indeed, for every directed family $\{S_i \mid i \in I\} \subseteq \mathcal{C}(X)$, by $F \in \mathrm{irr}(Z)$ and Lemma 2.13,

$$\begin{aligned}
\mathrm{sub}_X(co_X(F^*), \textstyle\bigvee^{\uparrow}_{i\in I} S_i) &= \mathrm{sub}_X(F^*, \textstyle\bigvee^{\uparrow}_{i\in I} S_i) \\
&= \mathrm{sub}_Z(F, \textstyle\bigvee^{\uparrow}_{i\in I} S_i|_Z) \\
&= \textstyle\bigvee^{\uparrow}_{i\in I} \mathrm{sub}_Z(F, S_i|_Z) \\
&= \textstyle\bigvee^{\uparrow}_{i\in I} \mathrm{sub}_X(F^*, S_i) \\
&= \textstyle\bigvee^{\uparrow}_{i\in I} \mathrm{sub}_X(co_X(F^*), S_i).
\end{aligned}$$

Since $X$ is sober, we can let $x_0 \in X$ be the unique element satisfying $co_X(F^*) = co_X(1_{x_0,X})$. We claim that $x_0 \in Z$; that is, $f(x_0) = g(x_0)$. Since $Y$ is $S_0$, we only need to show that $co_Y(1_{f(x_0)}) = co_Y(1_{g(x_0)})$. For every $E \in \mathcal{C}(Y)$, by $f^{\rightarrow} \dashv f^{\leftarrow}$,

$$\begin{aligned}
1_{f(x_0)} \leq E &\Leftrightarrow f^{\rightarrow}(1_{x_0,X}) \leq E \\
&\Leftrightarrow 1_{x_0,X} \leq f^{\leftarrow}(E) \\
&\Leftrightarrow co_X(F^*) = co_X(1_{x_0,X}) \leq f^{\leftarrow}(E) \\
&\Leftrightarrow F^* \leq f^{\leftarrow}(E) \\
&\Leftrightarrow f^{\rightarrow}(F^*) \leq E.
\end{aligned}$$

Similarly, $1_{g(x_0)} \leq E$ if and only if $g^{\rightarrow}(F^*) \leq E$ for every $E \in \mathcal{C}(Y)$. Notice that due to the definition of $Z$ and the fact that $F^*$ only takes non-zero values on $Z$, we have $f^{\rightarrow}(F^*) = g^{\rightarrow}(F^*)$. Altogether, we have $1_{f(x_0)} \leq E$ if and only if $1_{g(x_0)} \leq E$, which shows that $co_Y(1_{f(x_0)}) = co_Y(1_{g(x_0)})$. Thus $f(x_0) = g(x_0)$, i.e., $x_0 \in Z$. We claim that $co_Z(F) = co_Z(1_{x_0,Z})$. In fact, for all $C \in \mathcal{C}(X)$,

$$\begin{aligned}
F \leq C|_Z &\Leftrightarrow F^* \leq C \\
&\Leftrightarrow co_X(F^*) \leq C \\
&\Leftrightarrow co_X(1_{x_0,X}) \leq C \\
&\Leftrightarrow 1_{x_0,X} \leq C \\
&\Leftrightarrow 1_{x_0,Z} \leq C|_Z.
\end{aligned}$$

This shows that $co_Z(F) = co_Z(1_{x_0,Z})$. Thus $Z$ is a sober $L$-convex space. □

The preceding discussion shows that the equalizer of two convexity-preserving mappings from a sober $L$-convex space to an $S_0$ $L$-convex spaces is sober in categorical terms. For the notion of equalizer, please refer to [1]. Proposition 3.5 is a counterpart of $L$-topological setting (see [33, Proposition 5.9]).

Let $X$ and $Y$ be two $L$-convex spaces. The space $Y$ is called a *retraction kernel* of $X$ in $L$-**CS** if there exist two convexity-preserving mappings $r : X \longrightarrow Y$ and $d : Y \longrightarrow X$ such that $r \circ d = id_Y$. It is clear that $d$ is an injection and $r$ is a surjection.

**Proposition 3.6.** *Each retraction kernel of a sober $L$-convex space is a sober $L$-convex space.*

*Proof.* Let $Y$ be a retraction kernel of $X$ in $L$-**CS**. Then there exist two convexity-preserving mappings $r : X \longrightarrow Y$ and $d : Y \longrightarrow X$ such that $r \circ d = id_Y$. We prove that $Y$ is a sober $L$-convex space. Let $G \in \mathrm{irr}(Y)$. Then by [9, Lemma 4.4], $co_X(d^{\rightarrow}(G)) \in \mathrm{irr}(X)$. By the sobriety of $X$, there exists a unique $x \in X$ such that $co_X(d^{\rightarrow}(G)) = co_X(1_x)$. We claim that $G = co_Y(1_{r(x)})$. On one hand, by Lemma 2.12,

$$\begin{aligned} G &= (r \circ d)^{\rightarrow}(G) \\ &\le r^{\rightarrow}(co_X(d^{\rightarrow}(G))) \\ &= r^{\rightarrow}(co_X(1_x)) \\ &\le co_Y(1_{r(x)}). \end{aligned}$$

On the other hand, since $co_X(d^{\rightarrow}(G)) = co_X(1_x)$, we have

$$r^{\rightarrow}(co_X(d^{\rightarrow}(G)))(r(x)) \ge co_X(d^{\rightarrow}(G))(x) = 1.$$

Thus, $r^{\rightarrow}(co_X(d^{\rightarrow}(G)))(r(x)) = 1$. Since

$$\begin{aligned} r^{\rightarrow}(co_X(d^{\rightarrow}(G)))(r(x) &\le co_Y((r \circ d)^{\rightarrow}(G))(r(x)) \\ &= co_Y(G)(r(x)) \\ &= G(r(x)), \end{aligned}$$

we have $G(r(x)) = 1$, which shows that $co_Y(1_{r(x)}) \le G$. Thus $G = co_Y(1_{r(x)})$. To verify the uniqueness of $r(x)$, we need to show that $Y$ is $S_0$. Let $y_1, y_2 \in Y$ and $y_1 \ne y_2$. Since $d$ is an injection, we have $d(y_1) \ne d(y_2)$. Since $X$ is $S_0$, there exists $C \in \mathcal{C}(X)$ such that $C(d(y_1)) \ne C(d(y_2))$. Therefore, $d^{\leftarrow}(C)(y_1) \ne d^{\leftarrow}(C)(y_2)$. Since $d$ is convexity-preserving, we have $d^{\leftarrow}(C) \in \mathcal{C}(Y)$. This shows that $Y$ is $S_0$. ☐

# 4 The characterization of sobriety

Xia in [30] introduced the notions of quasihomeomorphisms and strict embeddings in the framework of convex spaces. He showed that sober convex spaces are precisely injective $S_0$ convex spaces related to strict embeddings. Following Xia's step, we aim to extend the notions of quasihomeomorphisms and strict embeddings to those in the category of $L$-convex spaces. Making use of these notions, we shall provide the characterizations for sobrification and sobriety, respectively.

**Definition 4.1.** Let $f : X \longrightarrow Y$ be a mapping between two $L$-convex spaces.

- The map $f$ is called a *quasihomeomorphism* if $f^{\leftarrow} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ is a bijection.
- The map $f$ is called a *strict embedding* if it is both a quasihomeomorphism and a subspace embedding.

**Example 4.2.** Given an $L$-convex space $X$, $\eta_X : X \longrightarrow \mathbb{S}(X)$ is a quasihomeomorphism. It is routine to check that for every $C \in \mathcal{C}(X)$, $\eta_X^{\leftarrow}(\phi(C)) = C$ and $\phi \circ \eta_X^{\leftarrow}(\phi(C)) = \phi(C)$. Thus, $\eta_X^{\leftarrow} \circ \phi = id_{\mathcal{C}(X)}$ and $\phi \circ \eta_X^{\leftarrow} = id_{\mathcal{C}(\mathrm{irr}(X))}$.

The following shows that in the category of $S_0$ $L$-convex spaces, quasihomeomorphisms coincide with strict embeddings.

**Proposition 4.3.** *Let $f : X \longrightarrow Y$ be a mapping between two $S_0$ $L$-convex spaces. Then $f$ is a subspace embedding if and only if $f^{\leftarrow} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ is a surjection.*

*Proof. Necessity.* For every $C \in \mathcal{C}(X)$, since $f$ is a subspace embedding, we have $(f^{\circ})^{\rightarrow}(C) \in \mathcal{C}(Y)|_{f(X)}(:= \{D|_{f(X)} \mid D \in \mathcal{C}(Y)\})$. So, there exists $D \in \mathcal{C}(Y)$ such that $(f^{\circ})^{\rightarrow}(C) = D|_{f(X)}$. We prove that $f^{\leftarrow}(D) = C$. For every $x \in X$, since $f$ is an injection, we have

$$\begin{aligned} f^{\leftarrow}(D)(x) &= D(f(x)) \\ &= D|_{f(X)}(f(x)) \\ &= (f^{\circ})^{\rightarrow}(C)(f(x)) \\ &= C(x). \end{aligned}$$

Thus $f^{\leftarrow}(D) = C$. By the arbitrariness of $C \in \mathcal{C}(X)$, we have that $f^{\leftarrow} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ is a surjection.

*Sufficiency.* First, we show that $f$ is an injection. Let $x, y \in X$ and $x \neq y$. Since $X$ is $S_0$, there exists $C \in \mathcal{C}(X)$ such that $C(x) \neq C(y)$. As $f^{\leftarrow}$ is a surjection, there exists $E \in \mathcal{C}(Y)$ such that $f^{\leftarrow}(E) = C$. Therefore

$$E(f(x)) = f^{\leftarrow}(E)(x) \neq f^{\leftarrow}(E)(y) = E(f(y)).$$

Thus $f(x) \neq f(y)$, showing that $f$ is an injection.

Next, we show that $f^{\circ} : (X, \mathcal{C}(X)) \longrightarrow (f(X), \mathcal{C}(Y)|_{f(X)})$ is a convex-to-convex mapping. Let $C \in \mathcal{C}(X)$. Since $f^{\leftarrow}$ is a surjection, there exists $E \in \mathcal{C}(Y)$ such that $f^{\leftarrow}(E) = C$. We claim that $(f^{\circ})^{\rightarrow}(C) = E|_{f(X)}$. In fact, for every $y \in f(X)$, assume $y = f(x)$ for some $x \in X$,

$$\begin{aligned}(f^{\circ})^{\rightarrow}(C)(y) &= (f^{\circ})^{\rightarrow}(f^{\leftarrow}(E))(y)\\ &= (f^{\circ})^{\rightarrow}(f^{\leftarrow}(E))(f(x)) = f^{\leftarrow}(E)(x)\\ &= E(f(x)) = E(y).\end{aligned}$$

Thus, $(f^{\circ})^{\rightarrow}(C) = E|_{f(X)}$, showing that $f^{\circ} : (X, \mathcal{C}(X)) \longrightarrow \mathcal{C}(Y)|_{f(X)})$ is convex-to-convex. In conclusion, $f$ is a subspace embedding. □

Now we obtain the characterization for sobrification. The $L$-ordered structure on the family of convex sets plays a crucial role in this characterization. We first see a lemma.

**Lemma 4.4.** *Let $f : P \longrightarrow Q$ be a bijection between two complete $L$-ordered sets. Then $f$ is supremum-preserving (or, infimum-preserving) if and only if $f$ is an $L$-order-isomorphism.*

*Proof.* We assume that $f$ is supremum-preserving. The proof of the case that $f$ is infimum-preserving is similar.

*Necessity.* For every $x, y \in P$, we have $f(y) = f(\sqcup{\downarrow}y) = \sqcup f^{\rightarrow}({\downarrow}y)$. By Condition (1) in Definition 2.4,

$$e_P(x, y) = f^{\rightarrow}({\downarrow}y)(f(x)) \leq e_Q(f(x), f(y)).$$

On the other hand,

$$f(\sqcup f^{\leftarrow}({\downarrow}f(y))) = \sqcup f^{\rightarrow}(f^{\leftarrow}({\downarrow}f(y))) = \sqcup{\downarrow}f(y) = f(y).$$

Since $f$ is a bijection, $\sqcup f^{\leftarrow}({\downarrow}f(y)) = y$. By Condition (1) in Definition 2.4,

$$f^{\leftarrow}({\downarrow}f(y))(x) = e_Q(f(x), f(y)) \leq e_P(x, y).$$

Thus $e_P(x, y) = e_Q(f(x), f(y))$, which shows that $f$ is an $L$-order-isomorphism.

*Sufficiency.* Let $g$ be the inverse mapping of $f$. For every $A \in L^P$ and $x \in Q$,

$$\begin{aligned}\mathrm{sub}(f^{\rightarrow}(A), {\downarrow}x) &= \bigwedge_{y \in Q} f^{\rightarrow}(A)(y) \rightarrow e(y, x)\\ &= \bigwedge_{t \in P} A(t) \rightarrow e(f(t), x)\\ &= \bigwedge_{t \in P} A(t) \rightarrow e(t, g(x))\\ &= \mathrm{sub}(A, {\downarrow}g(x))\\ &= e_P(\sqcup A, g(x))\\ &= e_Q(f(\sqcup A), x).\end{aligned}$$

Thus $f(\sqcup A) = \sqcup f^{\rightarrow}(A)$. This shows that $f$ is a supremum-preserving mapping. □

**Theorem 4.5.** (Characterization Theorem I: for sobrification) *Let $X$ and $Y$ be two $L$-convex spaces. If $Y$ is sober, then the following are equivalent:*

(1) *$Y$ is a sobrification of $X$;*

(2) *there exists a quasihomeomorphism from $X$ to $Y$;*

(3) *$(\mathcal{C}(X), \mathrm{sub}_X)$ and $(\mathcal{C}(Y), \mathrm{sub}_Y)$ are $L$-order-isomorphic.*

*Proof.* (1) ⇒ (2): Since $Y$ and $\mathbb{S}(X)$ are sobrifications of $X$, there exists a homeomorphism $f : \mathbb{S}(X) \longrightarrow Y$. By Example 4.2, $\eta_X : X \longrightarrow \mathbb{S}(X)$ is a quasihomeomorphism. Since a composition of quasihomeomorphism and homeomorphism is a quasihomeomorphism, it follows that $f \circ \eta_X$ is a quasihomeomorphism from $X$ to $Y$.

(2) ⇒ (3): It is clear that $j^{\leftarrow} : (\mathcal{C}(Y), \mathrm{sub}_Y) \to (\mathcal{C}(X), \mathrm{sub}_X)$ is an infimum-preserving bijection. By Lemma 4.4, we have $j^{\leftarrow}$ is an $L$-order-isomorphism.

(3) ⇒ (1): For an $L$-convex space $X$, it is easy to see that $\mathrm{irr}(X)$ and $\mathcal{C}(\mathrm{irr}(X))$ are defined by the corresponding $L$-ordered structure on $(\mathcal{C}(X), \mathrm{sub}_X)$. Therefore,

$$(\mathrm{irr}(Y), \mathcal{C}(\mathrm{irr}(Y))) \cong (\mathrm{irr}(X), \mathcal{C}(\mathrm{irr}(X))).$$

Since $Y$ is sober, it follows that $(Y, \mathcal{C}(Y)) \cong (\mathrm{irr}(Y), \mathcal{C}(\mathrm{irr}(Y)))$. Thus

$$(Y, \mathcal{C}(Y)) \cong (\mathrm{irr}(X), \mathcal{C}(\mathrm{irr}(X))).$$

Note that $\mathrm{irr}(X)$ is a sobrification of $X$. Hence, $Y$ is also a sobrification of $X$. □

The following example shows that Theorem 4.5(3) can be effectively used to identify the sobrification of an $L$-convex space.

**Example 4.6.** Let $L = ([0,1], \otimes)$ be a commutative integral quantale with $\otimes$ being $\wedge$. Define an $L$-convex structure on $[0,1)$ as follows:

$$\mathcal{C}' = \{a \wedge \psi \mid a \in [0,1], \psi : [0,1) \longrightarrow [0,1] \text{ is increasing}, \psi \geq id\},$$

where $id$ sends each $x \in [0,1)$ to $x \in [0,1]$. Clearly, $([0,1), \mathcal{C}')$ is a subspace of $([0,1], \mathcal{C})$ defined in Example 3.3. We now show that $id : [0,1) \longrightarrow [0,1]$ is an algebraic irreducible convex set. Obviously, we have $id \in \mathcal{C}$ and $\bigvee_{x \in [0,1)} id(x) = 1$. For every $\mu \in \mathcal{C}$, there exists some $a \in [0,1]$ such that $\mu(x) \geq x$ if $x \in [0,a)$ and $\mu(x) = a$ if $x \in [a,1)$. Thus, we have

$$\mathrm{sub}(id, \mu) = \bigvee_{x \in [0,1)} \mu(x) = a.$$

For a directed subset $\{\mu_i \mid i \in I\} \subseteq \mathcal{C}$, we have

$$\mathrm{sub}(id, {\bigvee}^{\uparrow}_{i \in I} \mu_i) = \bigvee_{x \in [0,1)} {\bigvee}^{\uparrow}_{i \in I} \mu_i(x) = {\bigvee}^{\uparrow}_{i \in I} \bigvee_{x \in [0,1)} \mu_i(x) = {\bigvee}^{\uparrow}_{i \in I} \mathrm{sub}(id, \mu_i).$$

Thus, $id$ is an algebraic irreducible convex set. However, $id$ is not the hull of $1_x$ for a unique $x \in [0,1)$. Therefore, $([0,1), \mathcal{C}')$ is not sober. It is straightforward to check that $(\mathcal{C}, \mathrm{sub}_{[0,1]}) \cong (\mathcal{C}', \mathrm{sub}_{[0,1)})$. By Theorem 4.5 (3), the sober $L$-convex space $([0,1], \mathcal{C})$ is a sobrification of $([0,1), \mathcal{C}')$. □

**Remark 4.7.** In Example 4.6, similar to Example 3.4 in [29], we can deduce that $A$ is nonempty finite $L$-subset of $[0,1)$ if and only if there exists a nonempty finite subset $K \subseteq_{fin} [0,1)$ such that $A = \chi_K$. For the definition of finite $L$-subset, please refer to [29, Definition 3.1]. For $K \subseteq_{fin} [0,1)$, write $\min(K) = b_0$. It is clear that $co(\chi_K) = co(1_{b_0})$. Thus, $\mathcal{C}$ is sober in the sense of Wu and Yao (see Definition 3.1 in [29]). However, Example 4.6 shows that $\mathcal{C}$ is not sober in the sense of Liu and Yue (see Definition 3.2). Thus, we here have provided an answer to the unresolved problem propopsed in [29]. To prevent confusion, we can rename the sobriety defined in [29] as *"weak sobriety"*.

For every convexity-preserving mapping $f : X \longrightarrow Y$, by [9, Lemma 4.4], we can define a mapping

$$\mathbb{S}(f) : \mathbb{S}(X) \longrightarrow \mathbb{S}(Y)$$

by $\mathbb{S}(f)(F) = co_Y(f^{\rightarrow}(F))$ for every $F \in \mathrm{irr}(X)$.

**Lemma 4.8.** (1) *The mapping* $\mathbb{S}(f) : \mathbb{S}(X) \longrightarrow \mathbb{S}(Y)$ *is convexity-preserving;*

(2) *The equality* $\mathbb{S}(f) \circ \eta_X = \eta_Y \circ f$ *holds.*

*Proof.* (1) For every $E$ in $\mathcal{C}(Y)$ and $G \in \mathrm{irr}(Y)$, it holds that

$$\begin{aligned}\mathbb{S}(f)^{\leftarrow}(\phi(E))(G) &= \phi(E)(co_Y(f^{\rightarrow}(G)) \\ &= \mathrm{sub}_Y(co_Y(f^{\rightarrow}(G)), E) \\ &= \mathrm{sub}_Y(f^{\rightarrow}(G), E) \\ &= \mathrm{sub}_X(G, f^{\leftarrow}(E)) \\ &= \phi(f^{\leftarrow}(E))(G).\end{aligned}$$

Thus, $\mathbb{S}(f)^{\leftarrow}(\phi(E)) = \phi(f^{\leftarrow}(E)) \in \mathcal{C}(\mathrm{irr}(X))$, which shows that $\mathbb{S}(f)$ is convexity-preserving.

(2) The equality $\mathbb{S}(f) \circ \eta_X = \eta_Y \circ f$ holds by Lemma 2.12. □

Let $f : X \longrightarrow Y$ be a mapping, with $X' \subseteq X$ and $Y' \subseteq Y$. If $f(X') \subseteq Y'$, we denote $f|_{X'}^{Y'} : X' \longrightarrow Y'$ as the mapping defined by $f|_{X'}^{Y'}(x) = f(x)$ for all $x \in X'$.

**Proposition 4.9.** *Let $X$ and $Y$ be two $L$-convex spaces, and let $f : X \longrightarrow Y$ be a convexity-preserving mapping. Then following are equivalent:*
(1) *$f$ is a quasihomeomorphism;*
(2) *$\mathbb{S}(f) : \mathbb{S}(X) \longrightarrow \mathbb{S}(Y)$ is a homeomorphism.*

*Proof.* (1) ⇒ (2): Let $f_* : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ be the inverse mapping of $f^{\leftarrow} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$. Clearly, $f_*$ is an $L$-order-isomorphism. Notice that the notion of algebraic irreducible convex sets is given by means of the inclusion $L$-order between convex sets. So, $f_*|_{\mathrm{irr}(X)}^{\mathrm{irr}(Y)} : \mathrm{irr}(X) \longrightarrow \mathrm{irr}(Y)$ is an $L$-order-isomorphism, whose inverse mapping is $f^{\leftarrow}|_{\mathrm{irr}(Y)}^{\mathrm{irr}(X)} : \mathrm{irr}(Y) \longrightarrow \mathrm{irr}(X)$. We claim that $f_*|_{\mathrm{irr}(X)}^{\mathrm{irr}(Y)} = \mathbb{S}(f)$; that is, for all $F \in \mathrm{irr}(X)$, $f_*(F) = \mathbb{S}(f)(F)(= co_Y(f^{\rightarrow}(F)))$. In fact, for every $D \in \mathcal{C}(Y)$, we have

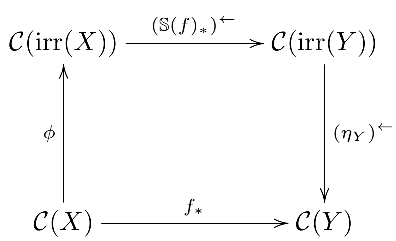
$$f^{\rightarrow}(F) \leq D \Longleftrightarrow F \leq f^{\leftarrow}(D) \Longleftrightarrow f_*(F) \leq D.$$

Notice that $f_*(F) \in \mathcal{C}(Y)$. We have $co_Y(f^{\rightarrow}(F)) = f_*(F)$. By Lemma 4.8, $\mathbb{S}(f) = f_*|_{\mathrm{irr}(X)}^{\mathrm{irr}(Y)} : \mathrm{irr}(X) \longrightarrow \mathrm{irr}(Y)$ is convexity-preserving. We now only need to show that $\mathbb{S}(f)$ is convex to convex. For every $C \in \mathcal{C}(X)$ and $G \in \mathrm{irr}(Y)$,

$$\begin{aligned}(\mathbb{S}(f))^{\rightarrow}(\phi(C))(G) &= \phi(C)((\mathbb{S}(f))^{-1}(G))\\ &= \phi(C)(f^{\leftarrow}(G))\\ &= \mathrm{sub}_X(f^{\leftarrow}(G), C)\\ &= \mathrm{sub}_Y(G, f_*(C))\\ &= \phi(f_*(C))(G).\end{aligned}$$

Thus, $(\mathbb{S}(f))^{\rightarrow}(\phi(C)) = \phi(f_*(C))$. This shows that $\mathbb{S}(f) : \mathrm{irr}(X) \longrightarrow \mathrm{irr}(Y)$ is convex to convex, hence a homeomorphism.

(2) ⇒ (1): Let $\mathbb{S}(f)_*$ denote the inverse mapping of $\mathbb{S}(f)$. Define $f_* : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ as the composition $(\eta_Y)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow} \circ \phi$ (see figure below).




For every $C \in \mathcal{C}(X)$, by Lemma 4.8(2), we have

$$\begin{aligned}f^{\leftarrow} \circ f_*(C) &= f^{\leftarrow} \circ (\eta_Y)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow} \circ \phi(C)\\ &= (\eta_Y \circ f)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow}(\phi(C))\\ &= (\mathbb{S}(f) \circ \eta_X)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow}(\phi(C))\\ &= \eta_X^{\leftarrow} \circ \mathbb{S}(f)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow}(\phi(C))\\ &= \eta_X^{\leftarrow} \circ ((\mathbb{S}(f)_* \circ \mathbb{S}(f))^{\leftarrow}(\phi(C))\\ &= \eta_X^{\leftarrow}(\phi(C))\\ &= C.\end{aligned}$$

Therefore, $f^{\leftarrow} \circ f_* = id_{\mathcal{C}(X)}$.

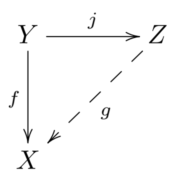
For every $D \in \mathcal{C}(Y)$, by Lemma 4.8(1), we have

$$\begin{aligned}f_* \circ f^{\leftarrow}(D) &= (\eta_Y)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow} \circ \phi(f^{\leftarrow}(D))\\ &= (\eta_Y)^{\leftarrow} \circ ((\mathbb{S}(f)_*)^{\leftarrow}(\mathbb{S}(f))^{\leftarrow}(\phi(D))\\ &= (\eta_Y)^{\leftarrow}(\mathbb{S}(f) \circ \mathbb{S}(f)_*)^{\leftarrow}(\phi(D))\\ &= (\eta_Y)^{\leftarrow}(\phi(D))\\ &= D.\end{aligned}$$

Therefore, $f_* \circ f^{\leftarrow} = id_{\mathcal{C}(Y)}$. Thus, $f^{\leftarrow} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ is a bijection; hence $f$ is a quasihomeomorphism. □

For a category $\mathcal{C}$, we use the symbol $Mor(\mathcal{C})$ to denote the class of morphisms of $\mathcal{C}$. Readers can refer to [5, Section II-3] for the notion of a $J$-injective object in the terminology of category. For convenience, we list it below.

**Definition 4.10.** [5] Let $\mathcal{M} \subseteq Mor(\mathcal{C})$ be a class of monomorphisms which is closed under multiplication with isomorphisms, i.e., for every $j \in \mathcal{M}$ and every isomorphism $i$, ones has $j \circ i \in \mathcal{M}$ and $i \circ j \in \mathcal{M}$. Then an object $X$ is said to be $\mathcal{M}$-*injective* if for every mapping $j : Y \longrightarrow Z$ in $\mathcal{M}$ and every morphism $f : Y \longrightarrow X$, there exists a morphism $g : Z \longrightarrow X$ with $f = g \circ j$; that is the following diagram commutes.



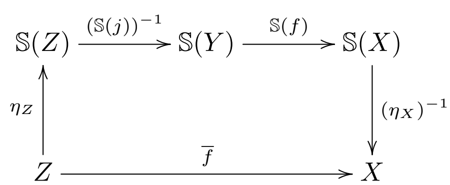

In $L$-$\mathbf{CS}_0$, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the classes of all subspace embeddings and all strict embeddings, respectively. An $S_0$ $L$-topological space $X$ is said to be *injective* (resp., *strictly injective*) if $X$ is $\mathcal{M}_1$-injective (resp., $\mathcal{M}_2$-injective) in the category $L$-$\mathbf{CS}_0$.

We now provide the characterization of sober $L$-convex space in $L$-$\mathbf{CS}_0$.

**Theorem 4.11.** (Characterization Theorem II: for sober $L$-convex space) *The $L$-convex space $X$ is a strictly injective $S_0$ $L$-convex space if and only if $X$ is a sober $L$-convex space.*

*Proof. Necessity.* It follows from Example 4.2 that $\eta_X$ is a strict embedding. As $X$ is strictly injective, we obtain a convexity-preserving mapping $r_X : \mathbb{S}(X) \longrightarrow X$ such that $r_X \circ \eta_X = id_X$. It follows from Proposition 3.6 that $X$ is also a sober $L$-convex space.

*Sufficiency.* Let $X$ be a sober $L$-convex space, and let $j : Y \longrightarrow Z$ be a strict embedding between two $S_0$ $L$-convex spaces. Then $\eta_X : X \longrightarrow \mathbb{S}(X)$ is a bijection. By Proposition 4.9, $\mathbb{S}(j) : \mathbb{S}(Y) \longrightarrow \mathbb{S}(Z)$ is a homeomorphism. For every convexity-preserving mapping $f : Y \longrightarrow X$, define $\overline{f} : Z \longrightarrow X$ as the composition $(\eta_X)^{-1} \circ \mathbb{S}(f) \circ (\mathbb{S}(j))^{-1} \circ \eta_Z$ (see figure below).




By Lemma 4.8(2), we have

$$\begin{aligned} \overline{f} \circ j &= (\eta_X)^{-1} \circ \mathbb{S}(f) \circ (\mathbb{S}(j))^{-1} \circ \eta_Z \circ j \\ &= (\eta_X)^{-1} \circ \mathbb{S}(f) \circ (\mathbb{S}(j))^{-1} \circ \mathbb{S}(j) \circ \eta_Y \\ &= (\eta_X)^{-1} \circ \mathbb{S}(f) \circ \eta_Y \\ &= (\eta_X)^{-1} \circ \eta_X \circ f = f. \end{aligned}$$

Thus, $\overline{f} \circ j = f$. Therefore, $X$ is a strictly injective $S_0$ $L$-convex space. □

**Remark 4.12.** It is straightforward to verify that the mapping $\overline{f}$ constructed in the above proof is unique. In fact, if $g : Z \longrightarrow X$ satisfies $f = g \circ j$, then by Lemma 4.8(2), we have

$$\begin{aligned} \overline{f} &= (\eta_X)^{-1} \circ \mathbb{S}(f) \circ (\mathbb{S}(j))^{-1} \circ \eta_Z \\ &= (\eta_X)^{-1} \circ \mathbb{S}(g \circ j) \circ (\mathbb{S}(j))^{-1} \circ \eta_Z \\ &= (\eta_X)^{-1} \circ \mathbb{S}(g) \circ \mathbb{S}(j) \circ (\mathbb{S}(j))^{-1} \circ \eta_Z \\ &= (\eta_X)^{-1} \circ \mathbb{S}(g) \circ \eta_Z \\ &= (\eta_X)^{-1} \circ \eta_X \circ g \\ &= id_X \circ g = g. \end{aligned}$$

# 5 Conclusion remarks

The primary focus of this paper is to characterize sober $L$-convex spaces with a commutative integral quantale $L$ as the truth value table. By means of the fuzzy ordered methods, we have derived a characterization theorem for sobrification (see Theorem 4.5). From a categorical perspective, we have proved that an $L$-convex space $X$ is sober if and only if it is a strictly injective $S_0$ $L$-convex space. Moreover, we have solved a problem left in [29] by presenting a counterexample; specifically, the sobriety defined by Wu and Yao [29] does not imply the sobriety defined by Liu and Yue [9] (see Remark 4.7). Consequently, the fuzzy ordered methods and categorical methods can be effectively combined in the study of fuzzy convex theory.

The methods of studying the sobriety of (fuzzy) topological spaces in [5, 15, 40] can also be applied in studying the sobriety of (fuzzy) convex spaces. We have found that many results in [5, 15, 40] also hold in the framework of (fuzzy) convex spaces. This raises a natural question: why does this phenomenon occur? In the future, we will investigate the fundamental principles behind this phenomenon, abstract out a unified method, and further explore the categorical properties of sobriety in more general closure spaces as presented in [2] and [38].

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# References

[1] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and concrete categories*, Wiley, New York, 1990.

[2] M. Erné, *General stone duality*, Topology and Its Applications, **137** (2004), 125-158. https://doi.org/10.1016/S0166-8641(03)00204-9

[3] M. H. Escardó, R. C. Flagg, *Semantic domains, injective spaces and monad: Extended abstract*, Electronic Notes in Theoretical Computer Science, **20** (1999), 229-244. https://doi.org/10.1016/S1571-0661(04)80077-X

[4] S. P. Franklin, *Some results on order convexity*, The American Mathematical Monthly, **69** (1962), 357-359. https://doi.org/10.1080/00029890.1962.11989897

[5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *Continuous lattices and domains*, Cambridge University Press, Cambridge, 2003.

[6] J. Goubault-Larrecq, *Non-Hausdorff topology and domain theory*, Cambridge University Press, Cambridge, 2013.

[7] U. Höhle, *Many-valued topology and its applications*, Kluwer Academic Publishers, Boston, Dordrecht, London, 2001.

[8] H. Komiya, *Convexity on a topological space*, Fundamenta Mathematicae, **111** (1981), 107-113. https://doi.org/10.4064/fm-111-2-107-113

[9] M. Y. Liu, Y. L. Yue, *The reflectivity of the category of stratified L-algebraic closure spaces*, Iranian Journal of Fuzzy Systems, **21** (2024), 117-127. https://doi.org/10.22111/IJFS.2024.47190.8314

[10] M. Y. Liu, Y. L. Yue, X. W. Wei, *Frame-valued Scott open set monad and its algebras*, Fuzzy Sets and Systems, **460** (2023), 52-71. https://doi.org/10.1016/j.fss.2022.11.002

[11] E. Marczewski, *Independence in abstract algebras results and problems*, Colloquium Mathematicum, **14** (1966), 169-188. https://doi.org/10.4064/cm-14-1-169-188

[12] Y. Maruyama, *Lattice-valued fuzzy convex geometry*, Computational Geometry and Discrete Mathematics, **164** (2009), 22-37.

[13] Y. Maruyama, *Fundamental results for pointfree convex geometry*, Annals of Pure and Applied Logic, **161** (2010), 1486-1501. https://doi.org/10.1016/j.apal.2010.05.002

[14] J. Nieminen, *The ideal structure of simple ternary algebras*, Colloquium Mathematicum, **40** (1978), 23-29. https://doi.org/10.4064/cm-40-1-23-29

[15] R. Noor, A. K. Srivastava, *The categories $L$-$\mathbf{Top}_0$ and $L$-$\mathbf{Sob}$ as epireflective hulls*, Soft Computing, **18** (2014), 1865-1871. https://doi.org/10.1007/s00500-014-1315-8

[16] B. Pang, *Quantale-valued convex structures as lax algebras*, Fuzzy Sets and Systems, **473** (2023), 108737. https://doi.org/10.1016/j.fss.2023.108737

[17] B. Pang, F. G. Shi, *Subcategories of the category of L-convex spaces*, Fuzzy Sets and Systems, **313** (2017), 61-74. https://doi.org/10.1016/j.fss.2016.02.014

[18] B. Pang, F. G. Shi, *Fuzzy counterparts of hull operators and interval operators in the framework of L-convex spaces*, Fuzzy Sets and Systems, **369** (2019), 20-39. https://doi.org/10.1016/j.fss.2018.05.012

[19] M. V. Rosa, *A study of fuzzy convexity with special reference to separation properties*, Cochin University of Science and Technology, Cochin, India, 1994.

[20] K. I. Rosenthal, *Quantales and their applications*, Longman House, Burnt Mill, Harlow, 1990.

[21] C. Shen, F. G. Shi, *Characterizations of L-convex spaces via domain theory*, Fuzzy Sets and Systems, **380** (2020), 44-63. https://doi.org/10.1016/j.fss.2019.02.009

[22] C. Shen, S. J. Yang, D. S. Zhao, F.G. Shi, *Lattice-equivalence of convex spaces*, Algebra University, **80** (2019), 26. https://doi.org/10.1007/s00012-019-0600-x

[23] S. H. Su, Q. G. Li, *The category of algebraic L-closure systems*, Journal of Intelligent and Fuzzy Systems, **33** (2017), 2199-2210. https://doi.org/10.3233/JIFS-16452

[24] L. C. Sun, B. Pang, *Completely Scott closed set and its applications*, Topology and Its Applications, Preprint, 2024.

[25] M. Van De Vel, *Binary convexities and distributive lattices*, Proceedings of the London Mathematical Society, **48** (1984), 1-33. https://doi.org/10.1112/plms/s3-48.1.1

[26] M. Van De Vel, *On the rank of a topological convexity*, Fundamenta Mathematicae, **119** (1984), 17-48. https://doi.org/10.4064/fm-119-2-101-132

[27] M. Van De Vel, *Theory of convex spaces*, North-Holland, Amsterdam, 1993.

[28] M. Y. Wu, L. K. Guo, Q. G. Li, *New representatons of algebraic domains and algebraic L-domains via closure systems*, Semigroup Forum, **103** (2021), 700-712. https://doi.org/10.1007/s00233-021-10209-7

[29] G. J. Wu, W. Yao, *Sober L-convex spaces and L-join-semilattices*, Iranian Journal of Fuzzy Systems, **21** (2024), 163-177. https://doi.org/10.22111/ijfs.2024.48576.8572

[30] C. C. Xia, *Some further results on pointfree convex geometry*, Algebra University, **85** (2024), 20. https://doi.org/10.1007/s00012-024-00847-7

[31] L. S. Xu, X. X. Mao, *Formal topological characterizations of various continuous domains*, Computational and Applied Mathematics, **56** (2008), 444-452. https://doi.org/10.1016/j.camwa.2007.12.010

[32] W. Yao, *Quantitative domains via fuzzy sets: Part I: Continuity of fuzzy directed-complete poset*, Fuzzy Sets and Systems, **161** (2010), 983-987. https://doi.org/10.1016/j.fss.2009.06.018

[33] W. Yao, *A survey of fuzzifications of frames, the Papert-Papert-Isbell adjunction and sobriety*, Fuzzy Sets and Systems, **190** (2012), 63-81. https://doi.org/10.1016/j.fss.2011.07.013

[34] W. Yao, S. E. Han, *A stone-type duality for $sT_0$ stratified Alexandrov L-topological spaces*, Fuzzy Sets and Systems, **282** (2016), 1-20. https://doi.org/10.1016/j.fss.2014.12.012

[35] W. Yao, L. X. Lu, *Fuzzy Galois connections on fuzzy posets*, Mathematical Logic Quarterly, **55** (2009), 84-91. https://doi.org/10.1002/malq.200710079

[36] W. Yao, C. J. Zhou, *Representation of sober convex spaces by join-semilattices*, Journal of Nonlinear and Convex Analysis, **21** (2020), 2715-2724.

[37] Y. L. Yue, W. Yao, W. K. Ho, *Applications of Scott-closed sets in convex structures*, Topology and Its Applications, **314** (2022), 108093. https://doi.org/10.1016/j.topol.2022.108093

[38] Z. X. Zhang, *A general categorical reflection for various completions of closure spaces and $\mathcal{Q}$-ordered sets*, Fuzzy Sets and Systems, **473** (2023), 1-18. https://doi.org/10.1016/j.fss.2023.108736

[39] Q. Y. Zhang, L. Fan, *Continuity in quantitative domains*, Fuzzy Sets and Systems, **154** (2005), 118-131. https://doi.org/10.1016/j.fss.2005.01.007

[40] B. Zhao, Y. J. Zhang, *Epireflections in the category of $T_0$ stratified Q-cotopological spaces*, Soft Computing, **27** (2023), 2443-2451. https://doi.org/10.1007/s00500-022-07809-y