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New results about idempotent semi-t-operators on bounded lattices

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Abstract

More recently, Hua conducted an investigation into the properties and constructions of idempotent semi-t-operators on bounded lattices. In her paper, there were two corollaries (without proof) showing the nonexistence of idempotent semi-t-operators on the bounded lattice which contains one of the eight sublattices mentioned in those corollaries. Unfortunately, it has been discovered that the conclusions drawn from those two corollaries are not entirely correct. This work aims to not only identify the errors in those corollaries through three counterexamples but also supplement all findings pertaining to the existence of idempotent semi-t-operators on any bounded lattice.

Keywords: Semi-t-operators, bounded lattices, idempotent semi-t-operators.

1 Introduction

The concepts of t-operators and nullnorms were initially introduced by Mas et al. [11] and Calvo et al. [2], respectively. Although their concepts different, it has been demonstrated in [11] that they share coincident structural characterizations. Several years ago, semi-t-operators (semi-nullnorms) were proposed by Drygaś [3] as generalizations of t-operators (nullnorms), removing the commutativity aspect. Unlike nullnorms and t-operators, semi-nullnorms and semi-t-operators have different structural characterizations, that is, they have different block structures on the unit square. These generalized operators serve as significant aggregation functions and are worthy of investigation due to their applicability in various fields such as fuzzy quantifiers, decision making, neural networks and expert systems [4, 6, 10].

In 2018, Fang and Hu [5] extended the definition of semi-t-operators from the real unit interval to any bounded lattice. Since then, many scholars have done a lot of research on the properties and structural characterizations of semi-t-operators on bounded lattices [8, 12, 13, 14, 15, 16]. Recently, Hua conducted an investigation into the properties and constructions of idempotent semi-t-operators on bounded lattices. In [7], Hua presented two theorems showing two methods for obtaining idempotent semi-t-operators on two specific cases of bounded lattices, as well as two corollaries indicating the nonexistence of idempotent semi-t-operators on other cases of bounded lattices. However, we found that the conclusions drawn from those two corollaries are not entirely correct. Motivated by this discovery, we will continue to explore the existence of idempotent semi-t-operators on any bounded lattice. On one hand, this study elucidates the errors in those two corollaries through three counterexamples. On the other hand, it supplements all results related to the existence of idempotent semi-t-operators on any bounded lattice.

The content of this work is outlined as follows. In Section 2, we review the definition and properties of idempotent semi-t-operators on bounded lattices. In Section 3, the errors of the corollaries presented in [7] are identified by three counterexamples. In Section 4, we explore the existence of idempotent semi-t-operators on any bounded lattice. This work is concluded in Section 5.

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2 Preliminaries

We assume that the readers are acquainted with the fundamental theory of lattices and aggregation functions. The definitions, notations and properties of lattices and aggregation functions can be found in [1, 9]. In this section, we will solely review the definition and properties of idempotent semi-t-operators on bounded lattices. Prior to commencing, let us first present the notations that will emerge in this paper. Let L be a bounded lattice and $a, b \in L\backslash\{0_L, 1_L\}$, $I_a^b = \{x \in L \mid x \parallel a \text{ and } x \not\parallel b\}$ represents the set whose elements are incomparable with a but comparable with b, $I_b^a = \{x \in L \mid x \parallel b \text{ and } x \not\parallel a\}$ represents the set whose elements are incomparable with b but comparable with a, $I_{a,b} = \{x \in L \mid x \parallel a \text{ and } x \parallel b\}$ represents the set whose elements are incomparable with a and b.

Definition 2.1. [5] *Let $a, b \in L\backslash\{0_L, 1_L\}$, where L is a bounded lattice. A binary operation $F : L^2 \to L$ is called a semi-t-operator if it is associative, non-decreasing in each variable, fulfills $F(0_L, 0_L) = 0_L, F(1_L, 1_L) = 1_L$ and such that $F_{0_L} \in \mathcal{M}(L, [0_L, a]\backslash I_b^a)$, $F^{0_L} \in \mathcal{M}(L, [0_L, b]\backslash I_a^b)$, $F_{1_L} \in \mathcal{M}(L, [b, 1_L]\backslash I_a^b)$, $F^{1_L} \in \mathcal{M}(L, [a, 1_L] \backslash I_b^a)$, where $\mathcal{M}(X, Y)$ denotes the family of all surjective mappings from X to Y and $F_{0_L}(x) = F(0_L, x), F_{1_L}(x) = F(1_L, x), F^{0_L}(x) = F(x, 0_L)$, $F^{1_L}(x) = F(x, 1_L)$, $F(0_L, 1_L) = a$ and $F(1_L, 0_L) = b$.*

Definition 2.2. [7] *Let F be a semi-t-operator on a bounded lattice L. F is called an idempotent semi-t-operator if $F(x, x) = x$ for all $x \in L$.*

Wang et al. showed the relationship between semi-t-operators on L with $a \leq b$ and semi-t-operators on L with $a \geq b$.

Lemma 2.3. [13] *Let $a, b \in L\backslash\{0_L, 1_L\}$, where L is a bounded lattice. The function $F : L^2 \to L$ is a semi-t-operator on L with $a = F(0_L, 1_L)$ and $b = F(1_L, 0_L)$ if and only if the function $G : L^2 \to L$ given by $G(x, y) = F(y, x)$ for all $x, y \in L$ is a semi-t-operator on L with $b = G(0_L, 1_L)$ and $a = G(1_L, 0_L)$.*

In this work, we will focus solely on the case of semi-t-operators on L with $a < b$. This is due to the result in Lemma 2.3 and the fact that semi-t-operators degenerate into semi-nullnorms when $a = b$.

Proposition 2.4. [7] *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice. If F is an idempotent semi-t-operator on L, then the following conclusions hold.*

(i) $F(x, y) = x \vee y$ *when* $x, y \in [0_L, a]$ *and* $F(x, y) = x \wedge y$ *when* $x, y \in [b, 1_L]$;

(ii) $F(x, 1_L) = x \vee a$ *and* $F(0_L, x) = x \wedge a$ *for* $x \in I_a^b$;

(iii) $F(x, 0_L) = x \wedge b$ *and* $F(1_L, x) = x \vee b$ *for* $x \in I_b^a$;

(iv) $F(0_L, x) = x \wedge a$ *and* $F(1_L, x) = x \vee b$ *for* $x \in I_{a,b}$;

(v) $F(x, y) = x \vee (y \wedge a)$ *for* $(x, y) \in [0_L, a] \times \{I_a^b \cup I_{a,b}\}$;

(vi) $F(x, y) = x \wedge (y \vee b)$ *for* $(x, y) \in [b, 1_L] \times \{I_b^a \cup I_{a,b}\}$;

(vii) $F(x, y) \in]a, b]$ *for* $(x, y) \in I_a^b \times [b, 1_L]$;

(viii) $F(x, y) \in]a, b[$ *for* $(x, y) \in I_b^a \times [0_L, a]$;

3 Notes on "Idempotent semi-t-operators on bounded lattices"

In this section, we will point out the errors in the two corollaries given by Hua through three counterexamples.

Corollary 3.1. [7, Corollary 4.7] *Let $(L, \wedge, \vee, 0_L, 1_L)$ be a bounded lattice, $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$. If there are only two elements $u, v \in I_a^b$ in a sublattice which is contained by L and is isomorphic to one of sublattices characterized by Hasse diagram in Fig.1-Fig.4, then there is no idempotent semi-t-operator on L.*

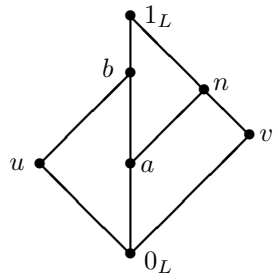


Fig.1.

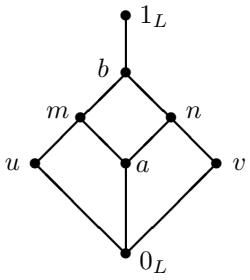


Fig.2.

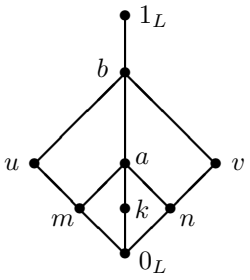


Fig.3.

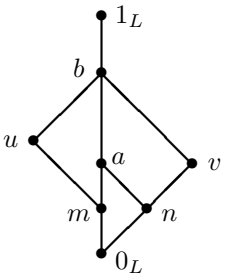


Fig.4.

The following counterexample illustrates that the conclusions of Corollary 3.1 are not entirely correct.

Counterexample 1. Let $(L_1 = \{0_{L_1}, a, u, v, m, n, b, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ be a bounded lattice characterized by Fig.2. We define two functions F_1 and F_2 on L_1 shown in Table 1 and Table 2. It is easy for us to verify that F_1 and F_2 are two idempotent semi-t-operators on L_1.

Table 1 F_1 on L_1

F_1	0_{L_1}	a	m	n	b	1_{L_1}	u	v
0_{L_1}	0_{L_1}	a	m	n	b	b	0_{L_1}	0_{L_1}
a	a	a	m	n	b	b	a	a
m	a	a	m	n	b	b	m	n
n	a	a	m	n	b	b	m	n
b	a	a	m	n	b	b	m	n
1_{L_1}	a	a	m	n	b	1_{L_1}	m	n
u	0_{L_1}	a	m	n	b	b	u	v
v	0_{L_1}	a	m	n	b	b	u	v

Table 2 F_2 on L_1

F_2	0_{L_1}	a	m	n	b	1_{L_1}	u	v
0_{L_1}	0_{L_1}	a	m	n	b	b	0_{L_1}	0_{L_1}
a	a	a	m	n	b	b	a	a
m	a	a	m	n	b	b	m	a
n	a	a	m	n	b	b	a	n
b	a	a	m	n	b	b	m	n
1_{L_1}	a	a	m	n	b	1_{L_1}	m	n
u	0_{L_1}	a	m	n	b	b	u	0_{L_1}
v	0_{L_1}	a	m	n	b	b	0_{L_1}	v

Corollary 3.2. [7, Corollary 4.8] *Let $(L, \wedge, \vee, 0_L, 1_L)$ be a bounded lattice, $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$. If there are only two elements $s, t \in I_b^a$ in a sublattice which is contained in L and is isomorphic to one of sublattices characterized by Hasse diagram in Fig.5-Fig.8, then there is no idempotent semi-t-operator on L.*

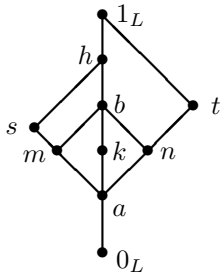


Fig.5.

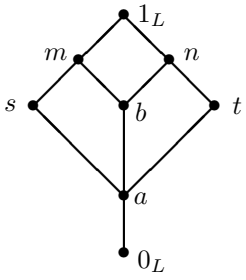


Fig.6.

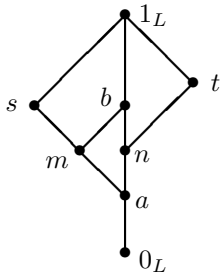


Fig.7.

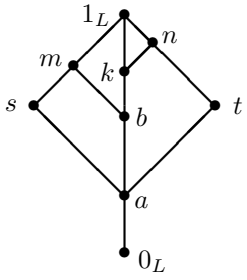


Fig.8.

The following two counterexamples explain that the conclusions of Corollary 3.2 are not entirely correct.

Counterexample 2. Let $(L_2 = \{0_{L_2}, a, m, k, n, b, h, s, t, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ be a bounded lattice given by Fig.5. We define a function F_3 on L_2 shown in Table 3. It is easy to check that F_3 is an idempotent semi-t-operator on L_2.

Table 3 F_3 on L_2

F_3	0_{L_2}	a	m	k	n	b	h	1_{L_2}	s	t
0_{L_2}	0_{L_2}	a	m	k	n	b	b	b	m	n
a	a	a	m	k	n	b	b	b	m	n
m	a	a	m	k	n	b	b	b	m	b
k	a	a	m	k	n	b	b	b	b	b
n	a	a	m	k	n	b	b	b	b	n
b	a	a	m	k	n	b	b	b	b	b
h	a	a	m	k	n	b	h	h	h	h
1_{L_2}	a	a	m	k	n	b	h	1_{L_2}	h	1_{L_2}
s	a	a	m	k	n	b	h	h	s	h
t	a	a	m	k	n	b	h	1_{L_2}	h	t

Counterexample 3. Let $(L_3 = \{0_{L_3}, a, m, n, b, s, t, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ be a bounded lattice characterized by Hasse diagram in Fig.7. We define two functions F_4 and F_5 on L_3 shown in Table 4 and Table 5. We can easily verify that F_4 and F_5 are idempotent semi-t-operators on L_3.

Table 4 F_4 on L_3

F_4	0_{L_3}	a	m	n	b	1_{L_3}	s	t
0_{L_3}	0_{L_3}	a	m	n	b	b	m	n
a	a	a	m	n	b	b	m	n
m	a	a	m	n	b	b	m	b
n	a	a	m	n	b	b	b	n
b	a	a	m	n	b	b	b	b
1_{L_3}	a	a	m	n	b	1_{L_3}	1_{L_3}	1_{L_3}
s	a	a	m	n	b	1_{L_3}	s	1_{L_3}
t	a	a	m	n	b	1_{L_3}	1_{L_3}	t

Table 5 F_5 on L_3

F_5	0_{L_3}	a	m	n	b	1_{L_3}	s	t
0_{L_3}	0_{L_3}	a	m	n	b	b	m	n
a	a	a	m	n	b	b	m	n
m	a	a	m	n	b	b	m	n
n	a	a	m	n	b	b	m	n
b	a	a	m	n	b	b	b	b
1_{L_3}	a	a	m	n	b	1_{L_3}	1_{L_3}	1_{L_3}
s	a	a	m	n	b	1_{L_3}	s	t
t	a	a	m	n	b	1_{L_3}	s	t

4 New results about idempotent semi-t-operators on bounded lattices

Hua pointed out that there exists an idempotent semi-t-operator on a bounded lattice L when L satisfies $I_b^a = \emptyset$, $I_{a,b} = \emptyset$, $I_a^b = \{u, v\}$ and $(u \vee a) \wedge (v \vee a) = (u \wedge a) \vee (v \wedge a) = a$ (or L satisfies $I_a^b = \emptyset$, $I_{a,b} = \emptyset$, $I_b^a = \{s, t\}$ and $(s \vee b) \wedge (t \vee b) = (s \wedge b) \vee (t \wedge b) = b$). In this work, we will discuss in detail whether there is an idempotent semi-t-operator on a bounded lattice L where L has one of the following two situations: (1) $I_b^a = \emptyset$, $I_{a,b} = \emptyset$, $I_a^b = \{u, v\}$, and $(u \vee a) \wedge (v \vee a) \neq a$ or $(u \wedge a) \vee (v \wedge a) \neq a$; (2) $I_a^b = \emptyset$, $I_{a,b} = \emptyset$, $I_b^a = \{s, t\}$, and $(s \vee b) \wedge (t \vee b) \neq b$ or $(s \wedge b) \vee (t \wedge b) \neq b$.

For situation (1), there are three cases that need to be discussed: (i) $(u \wedge a) \vee (v \wedge a) = a$ and $(u \vee a) \wedge (v \vee a) \neq a$; (ii) $(u \vee a) \wedge (v \vee a) = a$ and $(u \wedge a) \vee (v \wedge a) \neq a$; (iii) $(u \vee a) \wedge (v \vee a) \neq a$ and $(u \wedge a) \vee (v \wedge a) \neq a$. Let us first discuss the case (i).

Theorem 4.1. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ and $I_a^b = \{u, v\}$. If $(u \wedge a) \vee (v \wedge a) = a$, $(u \vee a) \wedge (v \vee a) \neq a$ and $(u \vee a) \nparallel (v \vee a)$, then there is no idempotent semi-t-operator on L.*

Proof. Suppose that there is an idempotent semi-t-operator F on L. Before starting, let $u \wedge a = m$, $v \wedge a = n$, $u \vee a = s$ and $v \vee a = t$, then we have $m \parallel n$, $m \vee n = a$, $m, n \in [0_L, a[$, $s \wedge t \neq a$ and $s, t \in]a, b]$. From the monotonicity of F and Proposition 2.4, it follows that $m = F(m, 0_L) \leq F(u, 0_L) \leq F(u, u) = u$ and $n = F(n, 0_L) \leq F(v, 0_L) \leq F(v, v) = v$. Then we can deduce that $F(u, 0_L) \in \{m, u\}$ and $F(v, 0_L) \in \{n, v\}$ since $u \wedge a = m$ and $v \wedge a = n$. Thus, we obtain that $F(u, 0_L) = m$ and $F(v, 0_L) = n$ from the fact $F^{0_L}(x) = F(x, 0_L) \in \mathcal{F}(L, [0_L, b]\backslash I_a^b)$ in Definition 2.1. Therefore, $F(u, n) = F(u, F(0_L, n)) = F(F(u, 0_L), n) = F(m, n) = m \vee n = a$ from the associativity of F.

The condition $(u \vee a) \nparallel (v \vee a)$ means that $u \vee a \leq v \vee a$ or $u \vee a \geq v \vee a$, that is, $s \leq t$ or $s \geq t$. Without losing generality, we assume that $s \leq t$.

When $s < t$. It follows that $F(u, 1_L) = u \vee a = s$ and $F(v, 1_L) = v \vee a = t$ from Proposition 2.4. We can get $F(u, s) = F(u, F(u, 1_L)) = F(F(u, u), 1_L) = F(u, 1_L) = s$ from the associativity and idempotency of F. So we have $F(u, t) = s$ from the fact that $s = F(u, s) \leq F(u, t) \leq F(u, 1_L) = s$. In addition, it follows that $F(u, v) = s$ from an inequality $a = F(u, n) \leq F(u, v) \leq F(u, 1_L) = s \leq b$ and an equation $F(F(u, v), 1_L) = F(u, F(v, 1_L)) = F(u, t) = s$. However, $s = F(s, 0_L) = F(F(u, v), 0_L) = F(u, F(v, 0_L)) = F(u, n) = a$, which is a contradiction.

When $s = t$. It follows that $F(u, 1_L) = u \vee a = s = t = v \vee a = F(v, 1_L)$ from Proposition 2.4. we can obtain $F(u, v) = s$ from an inequality $a = F(u, n) \leq F(u, v) \leq F(u, 1_L) = s \leq b$ and an equation $F(F(u, v), 1_L) = F(u, F(v, 1_L)) = F(u, t) = F(u, s) = F(u, F(u, 1_L)) = F(F(u, u), 1_L) = F(u, 1_L) = s$. However, $s = F(s, 0_L) = F(F(u, v), 0_L) = F(u, F(v, 0_L)) = F(u, n) = a$, which is a contradiction.

So we have the result $s \not\leq t$. Similarly, we can also obtain the result $s \not\geq t$. Consequently, we arrive at the conclusion that $s \parallel t$, which contradicts the condition $(u \vee a) \nparallel (v \vee a)$. In summary, there is no idempotent semi-t-operator on L. □

Theorem 4.2. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ and $I_a^b = \{u, v\}$. If $(u \vee a) \wedge (v \vee a) \neq a$ and $(u \vee a) \parallel (v \vee a)$, then the function $F_1^a : L^2 \to L$ defined as*

$$F_1^a(x, y) = \begin{cases} x \vee y & \text{if } (x, y) \in [0_L, a]^2, \\ a & \text{if } (x, y) \in [0_L, a] \times]a, b[\cup [0_L, a] \times [b, 1_L] \cup \{(u, v), (v, u)\}, \\ x & \text{if } (x, y) \in]a, b[\times L \cup \{(u, u), (v, v)\}, \\ x \wedge y & \text{if } (x, y) \in [b, 1_L]^2, \\ b & \text{if } (x, y) \in [b, 1_L] \times \{L\backslash[b, 1_L]\}, \\ x \vee a & \text{if } (x, y) \in \{u\} \times [u \vee a, 1_L] \cup \{v\} \times [v \vee a, 1_L], \\ (x \wedge a) \vee (y \wedge a) & \text{otherwise,} \end{cases} \tag{1}$$

is an idempotent semi-t-operator on L.

Proof. There are two cases that need to be discussed: (i) $(u \wedge a) \vee (v \wedge a) = a$; (ii) $(u \wedge a) \vee (v \wedge a) \neq a$. Here we only present the proof of the case (i) since the proof of the case (ii) can be similarly obtained.

According to the proof of Theorem 4.1 in [7], we are required to demonstrate the monotonicity and associativity of F_1^a for the following cases.

Monotonicity.

1. For $x \leq y$.

1.1. If $x \in [0_L, u \wedge a]$, $y = u$ and $z = v$, then $F_1^a(x, z) = (x \wedge a) \vee (v \wedge a) \leq a = F_1^a(u, v) = F_1^a(y, z)$.

1.2. If $x \in [0_L, v \wedge a]$, $y = v$ and $z = u$, then $F_1^a(x, z) = (x \wedge a) \vee (u \wedge a) \leq a = F_1^a(v, u) = F_1^a(y, z)$.

1.3. If $x = u$, $y \in [u \vee a, b[$ and $z = v$, then $F_1^a(x, z) = F_1^a(u, v) = a < u \vee a \leq y = F_1^a(y, z)$.

1.4. If $x = u$, $y \in [u \vee a, b[$ and $z \in]a, b[\backslash[u \vee a, b[$, then $F_1^a(x, z) = (u \wedge a) \vee (z \wedge a) = a \leq y = F_1^a(y, z)$.

1.5. If $x = v$, $y \in [v \vee a, b[$ and $z = u$, then $F_1^a(x, z) = F_1^a(v, u) = a < v \vee a \leq y = F_1^a(y, z)$.

1.6. If $x = v$, $y \in [v \vee a, b[$ and $z \in]a, b[\backslash[v \vee a, b[$, then $F_1^a(x, z) = (v \wedge a) \vee (z \wedge a) = a \leq y = F_1^a(y, z)$.

2. For $y \leq z$.

2.1. If $x = u$, $y = v$ and $z \in [v \vee a, b[$, then $F_1^a(x, y) = F_1^a(u, v) = a \leq u \vee a = F_1^a(x, z)$.

2.2. If $x = v$, $y = u$ and $z \in [u \vee a, b[$, then $F_1^a(x, y) = F_1^a(v, u) = a \leq v \vee a = F_1^a(x, z)$.

Associativity.

1. $x \in [0_L, a]$ and $y \in I_a^b$.

1.1. If $z \in]a, b[\backslash[y \vee a, b[$, then $F_1^a(F_1^a(x, y), z) = F_1^a(x \vee (y \wedge a), z) = a = F_1^a(x, a) = F_1^a(x, (y \wedge a) \vee (z \wedge a)) = F_1^a(x, F_1^a(y, z))$.

1.2. If $z \in I_a^b$ and $y \neq z$, then $F_1^a(F_1^a(x,y),z) = F_1^a(x \vee (y \wedge a),z) = x \vee (y \wedge a) \vee (z \wedge a) = x \vee a = a = F_1^a(x,a) = F_1^a(x,F_1^a(y,z))$.

2. $x \in [b,1_L]$ and $y \in I_a^b$.

2.1. If $z \in]a,b[\backslash[y \vee a,b[$, then $F_1^a(F_1^a(x,y),z) = F_1^a(b,z) = b = F_1^a(x,a) = F_1^a(x,(y \wedge a) \vee (z \wedge a)) = F_1^a(x,F_1^a(y,z))$.

2.2. If $z \in I_a^b$ and $y \neq z$, then $F_1^a(F_1^a(x,y),z) = F_1^a(b,z) = b = F_1^a(x,,a) = F_1^a(x,F_1^a(y,z))$.

3. $x,y \in I_a^b$ and $x \neq y$.

3.1. If $z \in]a,b[\backslash[y \vee a,b[$, then $F_1^a(F_1^a(x,y),z) = F_1^a(a,z) = a = (x \wedge a) \vee a = F_1^a(x,a) = F_1^a(x,(y \wedge a) \vee (z \wedge a)) = F_1^a(x,F_1^a(y,z))$.

3.2. If $z \in I_a^b$ and $y = z$, then $F_1^a(F_1^a(x,y),z) = F_1^a(a,z) = a = F_1^a(x,y) = F_1^a(x,F_1^a(y,z))$.

3.3. If $z \in I_a^b$ and $y \neq z$, then $F_1^a(F_1^a(x,y),z) = F_1^a(a,z) = a = F_1^a(x,a) = F_1^a(x,F_1^a(y,z))$.

4. $x,y \in I_a^b$ and $x = y$.

4.1. If $z \in]a,b[\backslash[y \vee a,b[$, then $F_1^a(F_1^a(x,y),z) = F_1^a(x,z) = (x \wedge a) \vee (z \wedge a) = a = (x \wedge a) \vee a = F_1^a(x,a) = F_1^a(x,(y \wedge a) \vee (z \wedge a)) = F_1^a(x,F_1^a(y,z))$.

4.2. If $z \in I_a^b$ and $y \neq z$, then $F_1^a(F_1^a(x,y),z) = F_1^a(x,z) = a = (x \wedge a) \vee a = F_1^a(x,a) = F_1^a(x,F_1^a(y,z))$.

Therefore, F_1^a is an idempotent semi-t-operator on L. □

There is an example to help readers easily get the idea in Theorem 4.2 for obtaining an idempotent semi-t-operator on bounded lattice.

Example 4.3. *Let $(L^\ddagger = \{0_{L^\ddagger}, m, n, a, u, v, x_0, r, s, t, b, 1_{L^\ddagger}\}, \leq, 0_{L^\ddagger}, 1_{L^\ddagger})$ be a bounded lattice, and its lattice diagram is shown in Fig.9. It is evident that the bounded lattice $L^\ddagger$ satisfies the conditions $I_b^a = \emptyset$, $I_{a,b} = \emptyset$, $I_a^b = \{u,v\}$, $(u \wedge a) \vee (v \wedge a) = a$, $(u \vee a) \parallel (v \vee a)$ and $(u \vee a) \wedge (v \vee a) \neq a$.*

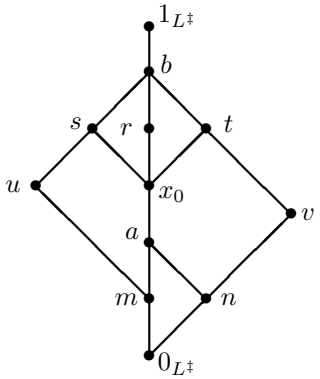


Fig.9 The lattice $L^\ddagger$

Then by the method in Theorem 4.2, we can obtain an idempotent semi-t-operator F_1^a on $L^\ddagger$ shown in Table 6.

Table 6 F_1^a *on* $L^\ddagger$

F_1^a	$0_{L^\ddagger}$	m	n	a	x_0	s	r	t	b	$1_{L^\ddagger}$	u	v
$0_{L^\ddagger}$	$0_{L^\ddagger}$	m	n	a	x_0	s	r	t	b	b	m	n
m	m	m	a	a	x_0	s	r	t	b	b	m	a
n	n	a	n	a	x_0	s	r	t	b	b	a	n
a	a	a	a	a	x_0	s	r	t	b	b	a	a
x_0	a	a	a	a	x_0	s	r	t	b	b	a	a
s	a	a	a	a	x_0	s	r	t	b	b	s	a
r	a	a	a	a	x_0	s	r	t	b	b	a	a
t	a	a	a	a	x_0	s	r	t	b	b	a	t
b	a	a	a	a	x_0	s	r	t	b	b	s	t
$1_{L^\ddagger}$	a	a	a	a	x_0	s	r	t	b	$1_{L^\ddagger}$	s	t
u	m	m	a	a	x_0	s	r	t	b	b	u	a
v	n	a	n	a	x_0	s	r	t	b	b	a	v

Next we will discuss the case (ii) $(u \vee a) \wedge (v \vee a) = a$ and $(u \wedge a) \vee (v \wedge a) \neq a$.

Theorem 4.4. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ and $I_a^b = \{u, v\}$. If $(u \wedge a) \vee (v \wedge a) \neq a$ and $(u \vee a) \wedge (v \vee a) = a$, then the function $F_2^a : L^2 \to L$ defined as*

$$F_2^a(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times]a, b[\cup [0_L, a] \times [b, 1_L], \\ b & \text{if } (x,y) \in [b, 1_L] \times \{L\backslash[b, 1_L]\}, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \vee a & \text{if } (x,y) \in I_a^b \times [b, 1_L], \\ (x \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in [0_L, a] \times I_a^b \cup I_a^b \times [0_L, a] \cup \{(u,v),(v,u)\}, \\ (x \vee a) \wedge y & \text{if } (x,y) \in I_a^b \times]a, b[, \\ x & \text{otherwise,} \end{cases} \tag{2}$$

is an idempotent semi-t-operator on L.

Proof. According to the proof of Theorem 4.2, we are required to verify the monotonicity and associativity of F_2^a for the following cases.

Monotonicity.

1. For $x \leq y$.

1.1. If $x \in [0_L, u \wedge a]$, $y = u$ and $z = v$, then $F_2^a(x,z) = x \vee (v \wedge a) \leq (u \wedge a) \vee (v \wedge a) = F_2^a(u,v) = F_2^a(y,z)$.

1.2. If $x \in [0_L, v \wedge a]$, $y = v$ and $z = u$, then $F_2^a(x,z) = x \vee (u \wedge a) \leq (v \wedge a) \vee (u \wedge a) = F_2^a(v,u) = F_2^a(y,z)$.

1.3. If $x = u$, $y \in [u \vee a, b]$ and $z = v$, then $F_2^a(x,z) = F_2^a(u,v) = (u \wedge a) \vee (v \wedge a) \leq a \leq y = F_2^a(y,z)$.

1.4. If $x = v$, $y \in [v \vee a, b]$ and $z = u$, then $F_2^a(x,z) = F_2^a(v,u) = (v \wedge a) \vee (u \wedge a) \leq a \leq y = F_2^a(y,z)$.

2. For $y \leq z$.

2.1. If $x = u$, $y \in [0_L, v \wedge a]$ and $z = v$, then $F_2^a(x,y) = (u \wedge a) \vee y \leq (u \wedge a) \vee (v \wedge a) = F_2^a(u,v) = F_2^a(x,z)$.

2.2. If $x = v$, $y \in [0_L, u \wedge a]$ and $z = u$, then $F_2^a(x,y) = (v \wedge a) \vee y \leq (v \wedge a) \vee (u \wedge a) = F_2^a(v,u) = F_2^a(x,z)$.

2.3. If $x = u$, $y = v$ and $z \in [v \vee a, b[$, then $F_2^a(x,y) = F_2^a(u,v) = (u \wedge a) \vee (v \wedge a) \leq a = (u \vee a) \wedge (v \wedge a) \leq (u \vee a) \wedge z = F_2^a(x,z)$.

2.4. If $x = v$, $y = u$ and $z \in [u \vee a, b[$, then $F_2^a(x,y) = F_2^a(v,u) = (v \wedge a) \vee (u \wedge a) \leq a = (v \vee a) \wedge (u \vee a) \leq (v \vee a) \wedge z = F_2^a(x,z)$.

Associativity.

1. If $x \in [0_L, a]$, $y, z \in I_a^b$ and $y \neq z$, then $F_2^a(F_2^a(x,y),z) = F_2^a(x \vee (y \wedge a), z) = x \vee (y \wedge a) \vee (z \wedge a) = F_2^a(x, (y \wedge a) \vee (z \wedge a)) = F_2^a(x, F_2^a(y,z))$.

2. If $x \in [b, 1_L]$, $y, z \in I_a^b$ and $y \neq z$, then $F_2^a(F_2^a(x,y),z) = F_2^a(b,z) = b = F_2^a(x, (y \wedge a) \vee (z \wedge a)) = F_2^a(x, F_2^a(y,z))$.

3. If $x, y, z \in I_a^b$ and $x = y \neq z$, then $F_2^a(F_2^a(x,y),z) = F_2^a(F_2^a(x,x),z) = F_2^a(x,z) = (x \wedge a) \vee (z \wedge a) = (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) = F_2^a(x, (y \wedge a) \vee (z \wedge a)) = F_2^a(x, F_2^a(y,z))$.

4. If $x, y, z \in I_a^b$ and $x \neq y = z$, then $F_2^a(F_2^a(x,y),z) = F_2^a((x \wedge a) \vee (y \wedge a), z) = (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) = (x \wedge a) \vee (y \wedge a) = F_2^a(x,y) = F_2^a(x, F_2^a(y,z))$.

5. If $x, y, z \in I_a^b$ and $x \neq y \neq z$, then $F_2^a(F_2^a(x,y),z) = F_2^a((x \wedge a) \vee (y \wedge a), z) = (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) = F_2^a(x, (y \wedge a) \vee (z \wedge a)) = F_2^a(x, F_2^a(y,z))$.

Thus, F_2^a is an idempotent semi-t-operator on L. □

The following example is to help readers better understand the construction method for idempotent semi-t-operators on bounded lattices as presented in Theorem 4.4.

Example 4.5. *Let $(L^* = \{0_{L^*}, s, t, x_0, a, u, v, m, n, b, 1_{L^*}\}, \leq, 0_{L^*}, 1_{L^*})$ and $(L^\star = \{0_{L^\star}, r, x_0, a, u, v, m, n, b, 1_{L^\star}\}, \leq, 0_{L^\star}, 1_{L^\star})$ be two bounded lattices, and their lattice diagrams are shown in Fig.10 and Fig.11.*

It is clear that the bounded lattices L^* and $L^\star$ satisfy the conditions $I_b^a = \emptyset$, $I_{a,b} = \emptyset$, $I_a^b = \{u, v\}$, $(u \wedge a) \vee (v \wedge a) \neq a$ and $(u \vee a) \wedge (v \vee a) = a$. Then by the method in Theorem 4.4, we can obtain the idempotent semi-t-operators F_2^a on L^* and $L^\star$ shown in Table 7 and Table 8.

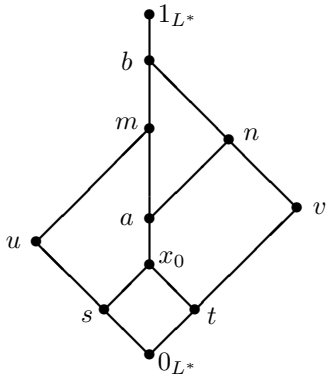


Fig.10 The lattice L^*

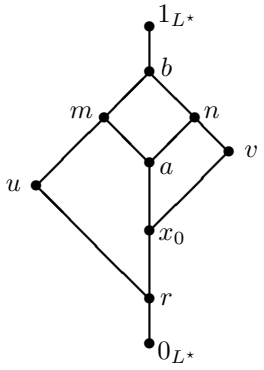


Fig.11 The lattice $L^\star$

Table 7 *F_2^a on L^**

F_2^a	0_{L^*}	s	t	x_0	a	m	n	b	1_{L^*}	u	v
0_{L^*}	0_{L^*}	s	t	x_0	a	m	n	b	b	s	t
s	s	s	x_0	x_0	a	m	n	b	b	s	x_0
t	t	x_0	t	x_0	a	m	n	b	b	x_0	t
x_0	x_0	x_0	x_0	x_0	a	m	n	b	b	x_0	x_0
a	a	a	a	a	a	m	n	b	b	a	a
m	a	a	a	a	a	m	n	b	b	m	a
n	a	a	a	a	a	m	n	b	b	a	n
b	a	a	a	a	a	m	n	b	b	m	n
1_{L^*}	a	a	a	a	a	m	n	b	1_{L^*}	m	n
u	s	s	x_0	x_0	a	m	n	b	b	u	x_0
v	t	x_0	t	x_0	a	m	n	b	b	x_0	v

Table 8 *F_2^a on $L^\star$*

F_2^a	$0_{L^\star}$	r	x_0	a	m	n	b	$1_{L^\star}$	u	v
$0_{L^\star}$	$0_{L^\star}$	r	x_0	a	m	n	b	b	r	x_0
r	r	r	x_0	a	m	n	b	b	r	x_0
x_0	x_0	x_0	x_0	a	m	n	b	b	x_0	x_0
a	a	a	a	a	m	n	b	b	a	a
m	a	a	a	a	m	n	b	b	m	a
n	a	a	a	a	m	n	b	b	a	n
b	a	a	a	a	m	n	b	b	m	n
$1_{L^\star}$	a	a	a	a	m	n	b	$1_{L^\star}$	m	n
u	r	r	x_0	a	m	n	b	b	u	x_0
v	x_0	x_0	x_0	a	m	n	b	b	x_0	v

Finally, we will discuss the case (iii) $(u\wedge a)\vee(v\wedge a)\neq a$ and $(u\vee a)\wedge(v\vee a)\neq a$. For the condition $(u\wedge a)\vee(v\wedge a)\neq a$, there are three possibilities: $u\wedge a=v\wedge a$; $u\wedge a<v\wedge a<a$ (or $v\wedge a<u\wedge a<a$); $(u\wedge a)\parallel(v\wedge a)$ and $(u\wedge a)\vee(v\wedge a)<a$. For the condition $(u\vee a)\wedge(v\vee a)\neq a$, there are three possibilities: $u\vee a=v\vee a$; $a<u\vee a<v\vee a$ (or $a<v\vee a<u\vee a$);

$(u \vee a) \parallel (v \vee a)$ *and* $(u \vee a) \wedge (v \vee a) > a$.

The following five theorems are all conclusions regarding the existence of idempotent semi-t-operators on L with the case (iii).

Theorem 4.6. *Let* $a, b \in L\backslash\{0_L, 1_L\}$ *and* $a < b$, *where* L *is a bounded lattice with* $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ *and* $I_a^b = \{u, v\}$. *If* $(u \wedge a) \vee (v \wedge a) \neq a$, $(u \vee a) \wedge (v \vee a) \neq a$, u *and* v *satisfy one of the following two cases:*

(i) $(u \vee a) \nparallel (v \vee a)$ *and* $(u \wedge a) \parallel (v \wedge a)$;

(ii) $(u \vee a) \nparallel (v \vee a)$ *and* $u \vee a \neq v \vee a$, $(u \wedge a) \nparallel (v \wedge a)$ *and* $u \wedge a \neq v \wedge a$.

Then there is no idempotent semi-t-operator on L.

Proof. The proof follows a similar way to that of Theorem 4.2 in [13]. □

Theorem 4.7. *Let* $a, b \in L\backslash\{0_L, 1_L\}$ *and* $a < b$, *where* L *is a bounded lattice with* $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ *and* $I_a^b = \{u, v\}$. *If* $u \wedge a = v \wedge a$ *and* $(u \vee a) \wedge (v \vee a) \neq a$, *then the function* $F_3^a : L^2 \to L$ *defined as*

$$F_3^a(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times]a, b[\cup [0_L, a] \times [b, 1_L], \\ x & \text{if } (x,y) \in]a, b[\times L \cup I_a^b \times I_a^b, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \vee a & \text{if } (x,y) \in I_a^b \times]a, b[\cup I_a^b \times [b, 1_L], \\ (x \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in [0_L, a] \times I_a^b \cup I_a^b \times [0_L, a], \\ b & \text{otherwise,} \end{cases} \tag{3}$$

is an idempotent semi-t-operator on L.

Proof. The proof follows a similar way to that of Theorem 4.4. □

The following example contains those three possibilities of the condition $(u \vee a) \wedge (v \vee a) \neq a$ in Theorem 4.7.

Example 4.8. *Let* $(L^{\triangleleft} = \{0_{L^{\triangleleft}}, m, a, u, v, s, b, 1_{L^{\triangleleft}}\}, \leq, 0_{L^{\triangleleft}}, 1_{L^{\triangleleft}})$, $(L^{\diamond} = \{0_{L^{\diamond}}, m, a, x_0, u, v, s, t, b, 1_{L^{\diamond}}\}, \leq, 0_{L^{\diamond}}, 1_{L^{\diamond}})$ *and* $(L^{\triangleright} = \{0_{L^{\triangleright}}, m, a, u, v, s, b, 1_{L^{\triangleright}}\}, \leq, 0_{L^{\triangleright}}, 1_{L^{\triangleright}})$ *be three bounded lattices, and their lattice diagrams are shown in Fig.12, Fig.13 and Fig.14.*

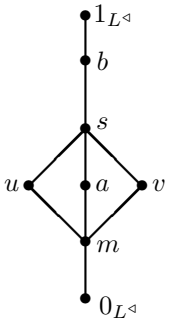


Fig.12 The lattice $L^{\triangleleft}$

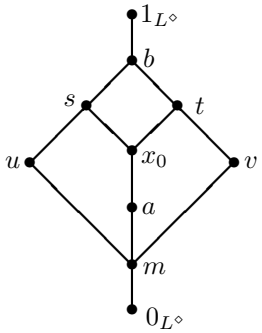


Fig.13 The lattice $L^{\diamond}$

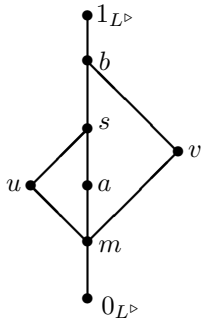


Fig.14 The lattice $L^{\triangleright}$

All three bounded lattices $L^{\triangleleft}$, $L^{\diamond}$ *and* $L^{\triangleright}$ *satisfy* $u \wedge a = v \wedge a$ *and* $(u \vee a) \wedge (v \vee a) \neq a$. *Then by the method in Theorem 4.7, we can obtain three idempotent semi-t-operators on* $L^{\triangleleft}$, $L^{\diamond}$ *and* $L^{\triangleright}$ *shown by Tables 9, 10 and 11.*

Table 9 F_3^a *on* $L^{\triangleleft}$

F_3^a	$0_{L^{\triangleleft}}$	m	a	s	b	$1_{L^{\triangleleft}}$	u	v
$0_{L^{\triangleleft}}$	$0_{L^{\triangleleft}}$	m	a	s	b	b	m	m
m	m	m	a	s	b	b	m	m
a	a	a	a	s	b	b	a	a
s	a	a	a	s	b	b	s	s
b	a	a	a	s	b	b	s	s
$1_{L^{\triangleleft}}$	a	a	a	s	b	$1_{L^{\triangleleft}}$	s	s
u	m	m	a	s	b	b	u	v
v	m	m	a	s	b	b	u	v

Table 10 F_3^a *on* $L^{\diamond}$

F_3^a	$0_{L^{\diamond}}$	m	a	x_0	s	t	b	$1_{L^{\diamond}}$	u	v
$0_{L^{\diamond}}$	$0_{L^{\diamond}}$	m	a	x_0	s	t	b	b	m	m
m	m	m	a	x_0	s	t	b	b	m	m
a	a	a	a	x_0	s	t	b	b	a	a
x_0	a	a	a	x_0	s	t	b	b	s	t
s	a	a	a	x_0	s	t	b	b	s	t
t	a	a	a	x_0	s	t	b	b	s	t
b	a	a	a	x_0	s	t	b	b	s	t
$1_{L^{\diamond}}$	a	a	a	x_0	s	t	b	$1_{L^{\diamond}}$	s	t
u	m	m	a	x_0	s	t	b	b	u	v
v	m	m	a	x_0	s	t	b	b	u	v

Table 11 F_3^a *on* $L^{\triangleright}$

F_3^a	$0_{L^{\triangleright}}$	m	a	s	b	$1_{L^{\triangleright}}$	u	v
$0_{L^{\triangleright}}$	$0_{L^{\triangleright}}$	m	a	s	b	b	m	m
m	m	m	a	s	b	b	m	m
a	a	a	a	s	b	b	a	a
s	a	a	a	s	b	b	s	b
b	a	a	a	s	b	b	s	b
$1_{L^{\triangleright}}$	a	a	a	s	b	$1_{L^{\triangleright}}$	s	b
u	m	m	a	s	b	b	u	v
v	m	m	a	s	b	b	u	v

Theorem 4.9. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ and $I_a^b = \{u, v\}$. If $u \vee a = v \vee a$ and $v \wedge a \leq u \wedge a < a$, then the function $F_4^a : L^2 \to L$ defined as*

$$F_4^a(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times]a, b[\cup [0_L, a] \times [b, 1_L], \\ b & \text{if } (x,y) \in [b, 1_L] \times \{L\backslash[b, 1_L]\}, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \vee a & \text{if } (x,y) \in I_a^b \times [b, 1_L] \cup I_a^b \times]a, b[, \\ (x \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in [0_L, a] \times I_a^b \cup I_a^b \times [0_L, a], \\ y & \text{if } (x,y) \in \{v\} \times \{u\}, \\ x & \text{otherwise,} \end{cases} \tag{4}$$

is an idempotent semi-t-operator on L.

Proof. According to the proof of Theorem 4.2, we are required to verify the monotonicity and the associativity of F_5^a for the following cases.

Monotonicity.

1. For $x \leq y$.

1.1. If $x \in [0_L, v \wedge a]$, $y = v$ and $z = u$, then $F_4^a(x,z) = x \vee (u \wedge a) \leq (v \wedge a) \vee (u \wedge a) \leq (u \wedge a) \vee (u \wedge a) = u \wedge a < u = F_4^a(v,u) = F_4^a(y,z)$.

1.2. If $x = v$, $y \in [v \vee a, b[$ and $z = u$, then $F_4^a(x,z) = F_4^a(v,u) = u \leq u \vee a = v \vee a \leq y = F_4^a(y,z)$.

2. For $y \leq z$.

2.1. If $x = v$, $y \in [0_L, u \wedge a]$ and $z = u$, then $F_4^a(x,y) = (v \wedge a) \vee y \leq (v \wedge a) \vee (u \wedge a) = u \wedge a < u = F_4^a(v,u) = F_4^a(x,z)$.

2.2. If $x = v$, $y = u$ and $z \in [u \vee a, b[$, then $F_4^a(x,y) = F_4^a(v,u) = u < u \vee a = v \vee a = F_4^a(x,z)$.

Associativity.

1. If $x \in [0_L, a]$, $y = v$ and $z = u$, then $F_4^a(F_4^a(x,y),z) = F_4^a(x \vee (v \wedge a), z) = x \vee (v \wedge a) \vee (u \wedge a) = x \vee (u \wedge a) = F_4^a(x,u) = F_4^a(x, F_4^a(y,z))$.

2. If $x \in [b, 1_L]$, $y = v$ and $z = u$, then $F_4^a(F_4^a(x,y),z) = F_4^a(b,z) = b = F_4^a(x,u) = F_4^a(x, F_4^a(y,z))$.

3. If $x = v$, $y = u$ and $z \in I_a^b$, then $F_4^a(F_4^a(x,y),z) = F_4^a(u,z) = u = F_4^a(v,u) = F_4^a(x, F_4^a(y,z))$.

4. If $x \in I_a^b$, $y = v$ and $z = u$, then $F_4^a(F_4^a(x,y),z) = F_4^a(F_4^a(x,v),u) = u = F_4^a(x,u) = F_4^a(x, F_4^a(y,z))$.

Thus, F_4^a is an idempotent semi-t-operator on L. □

Theorem 4.10. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ and $I_a^b = \{u, v\}$. If $u \vee a = v \vee a$ and $u \wedge a \leq v \wedge a < a$, then the function $F_5^a : L^2 \to L$ defined as*

$$F_5^a(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times]a, b[\cup [0_L, a] \times [b, 1_L], \\ b & \text{if } (x,y) \in [b, 1_L] \times \{L\backslash[b, 1_L]\}, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \vee a & \text{if } (x,y) \in I_a^b \times [b, 1_L] \cup I_a^b \times]a, b[, \\ (x \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in [0_L, a] \times I_a^b \cup I_a^b \times [0_L, a], \\ y & \text{if } (x,y) \in \{u\} \times \{v\}, \\ x & \text{otherwise,} \end{cases} \tag{5}$$

is an idempotent semi-t-operator on L.

Proof. The proof follows a similar way to that of Theorem 4.9. □

So far, we have analyzed all cases of the existence of idempotent semi-t-operators on bounded lattices for situation (1) $I_b^a = \emptyset$, $I_{a,b} = \emptyset$, $I_a^b = \{u, v\}$, and $(u \vee a) \wedge (v \vee a) \neq a$ or $(u \wedge a) \vee (v \wedge a) \neq a$. At the second part of this section, we will discuss the existence of idempotent semi-t-operators for situation (2) $I_a^b = \emptyset$, $I_{a,b} = \emptyset$, $I_b^a = \{s, t\}$, and $(s \vee b) \wedge (t \vee b) \neq b$ or $(s \wedge b) \vee (t \wedge b) \neq b$.

For situation (2), there are also three cases that need to be discussed: (i) $(s \wedge b) \vee (t \wedge b) = b$ and $(s \vee b) \wedge (t \vee b) \neq b$; (ii) $(s \vee b) \wedge (t \vee b) = b$ and $(s \wedge b) \vee (t \wedge b) \neq b$; (iii) $(s \vee b) \wedge (t \vee b) \neq b$ and $(s \wedge b) \vee (t \wedge b) \neq b$. Since the idea of analyzing situation (1) and situation (2) is the same, we will only present the results of the existence of idempotent semi-t-operators on L for situation (2) and omit the proofs. All conclusions regarding the existence of idempotent semi-t-operators on bounded lattices for situation (2) are presented below.

Theorem 4.11. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and $I_b^a = \{s, t\}$. If $(s \wedge b) \vee (t \wedge b) = b$ and $(s \vee b) \wedge (t \vee b) \neq b$, then the function $F_1^b : L^2 \to L$ defined as*

$$F_1^b(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\}, \\ b & \text{if } (x,y) \in [b, 1_L] \times [0_L, a] \cup [b, 1_L] \times]a, b[, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \wedge b & \text{if } (x,y) \in I_b^a \times [0_L, a], \\ (x \vee b) \wedge (y \vee b) & \text{if } (x,y) \in [b, 1_L] \times I_b^a \cup I_b^a \times [b, 1_L] \cup \{(s,t),(t,s)\}, \\ (x \wedge b) \vee y & \text{if } (x,y) \in I_b^a \times]a, b[, \\ x & \text{otherwise,} \end{cases} \tag{6}$$

is an idempotent semi-t-operator on L.

Theorem 4.12. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and $I_b^a = \{s, t\}$. If $(s \wedge b) \vee (t \wedge b) \neq b$ and $(s \wedge b) \parallel (t \wedge b)$, then the function $F_2^b : L^2 \to L$ defined as*

$$F_2^b(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\}, \\ x & \text{if } (x,y) \in]a, b[\times L \cup \{(s,s),(t,t)\}, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ b & \text{if } (x,y) \in [b, 1_L] \times [0_L, a] \cup [b, 1_L]\times]a, b[, \\ x \wedge b & \text{if } (x,y) \in \{s\} \times [0_L, s \wedge b] \cup \{t\} \times [0_L, t \wedge b], \\ (x \vee b) \wedge (y \vee b) & \text{otherwise,} \end{cases} \tag{7}$$

is an idempotent semi-t-operator on L.

Theorem 4.13. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and $I_b^a = \{s, t\}$. If $(s \wedge b) \vee (t \wedge b) \neq b$, s and t satisfy one of the following three cases:*

(i) *$(s \vee b) \wedge (t \vee b) = b$ and $(s \wedge b) \nparallel (t \wedge b)$;*

(ii) *$(s \vee b) \wedge (t \vee b) \neq b$, $(s \wedge b) \nparallel (t \wedge b)$ and $(s \vee b) \parallel (t \vee b)$;*

(iii) *$(s \vee b) \wedge (t \vee b) \neq b$, $(s \vee b) \nparallel (t \vee b)$ and $s \vee b \neq t \vee b$, $(s \wedge b) \nparallel (t \wedge b)$ and $s \wedge b \neq t \wedge b$.*

Then there is no idempotent semi-t-operator on L.

Theorem 4.14. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and $I_b^a = \{s, t\}$. If $s \vee b = t \vee b$ and $(s \wedge b) \vee (t \wedge b) \neq b$, then the function $F_3^b : L^2 \to L$ defined as*

$$F_3^b(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\}, \\ b & \text{if } (x,y) \in [b, 1_L] \times [0_L, a] \cup [b, 1_L]\times]a, b[, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \wedge b & \text{if } (x,y) \in I_b^a \times [0_L, a] \cup I_b^a\times]a, b[, \\ (x \vee b) \wedge (y \vee b) & \text{if } (x,y) \in [b, 1_L] \times I_b^a \cup I_b^a \times [b, 1_L], \\ x & \text{otherwise,} \end{cases} \tag{8}$$

is an idempotent semi-t-operator on L.

Theorem 4.15. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and $I_b^a = \{s, t\}$. If $s \wedge b = t \wedge b$ and $b < s \vee b \leq t \vee b$, then the function $F_4^b : L^2 \to L$ defined as*

$$F_4^b(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\}, \\ b & \text{if } (x,y) \in [b, 1_L] \times [0_L, a] \cup [b, 1_L]\times]a, b[, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \wedge b & \text{if } (x,y) \in I_b^a \times [0_L, a] \cup I_b^a\times]a, b[, \\ (x \vee b) \wedge (y \vee b) & \text{if } (x,y) \in [b, 1_L] \times I_b^a \cup I_b^a \times [b, 1_L], \\ y & \text{if } (x,y) \in \{t\} \times \{s\}, \\ x & \text{otherwise,} \end{cases} \tag{9}$$

is an idempotent semi-t-operator on L.

Theorem 4.16. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a < b$, where L is a bounded lattice with $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and $I_b^a = \{s, t\}$. If $s \wedge b = t \wedge b$ and $b < t \vee b \leq s \vee b$, then the function $F_5^b : L^2 \to L$ defined as*

$$F_5^b(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\}, \\ b & \text{if } (x,y) \in [b, 1_L] \times [0_L, a] \cup [b, 1_L] \times]a, b[, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ x \wedge b & \text{if } (x,y) \in I_b^a \times [0_L, a] \cup I_b^a \times]a, b[, \\ (x \vee b) \wedge (y \vee b) & \text{if } (x,y) \in [b, 1_L] \times I_b^a \cup I_b^a \times [b, 1_L], \\ y & \text{if } (x,y) \in \{s\} \times \{t\}, \\ x & \text{otherwise,} \end{cases} \tag{10}$$

is an idempotent semi-t-operator on L.

5 Conclusions

In this study, we have examined the presence of idempotent semi-t-operators on a bounded lattice L under two scenarios: (1) $I_b^a = \emptyset$, $I_{a,b} = \emptyset$, $I_a^b = \{u, v\}$, and $(u \vee a) \wedge (v \vee a) \neq a$ or $(u \wedge a) \vee (v \wedge a) \neq a$; (2) $I_a^b = \emptyset$, $I_{a,b} = \emptyset$, $I_b^a = \{s, t\}$, and $(s \vee b) \wedge (t \vee b) \neq b$ or $(s \wedge b) \vee (t \wedge b) \neq b$. As a result, this study not only identified the errors in those corollaries proposed by Hua through three counterexamples but also supplemented all findings related to the existence of idempotent semi-t-operators on any bounded lattice. In our future research, we will continue to investigate the properties and conduct structural characterization of semi-t-operators on any bounded lattice.

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