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Some inequalities for generalized Choquet integrals of triangular fuzzy number-valued functions and its application

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Abstract

Recently, D. Zhang et al. introduced the generalized Choquet integral, extending pseudo-integrals and Choquet-like integrals while exploring their foundational properties. Building on this framework, we introduce the concept of generalized Choquet integrals for triangular fuzzy number (TFN)-valued functions, referred to as TGC-integrals. This work investigates the key properties of TGC-integrals, including monotone non-decreasing convergence theorems and inequalities such as the Fatou type, Jensen type, Minkowski type, and Hölder type inequalities, specifically tailored for TFN-valued functions. Furthermore, we provide illustrative examples that demonstrate practical applications of TGC-integrals, such as TFN-valued Choquet expected utility and pseudo-functional analysis. These results establish a robust theoretical foundation for analyzing TFN-valued functions and highlight their potential for addressing uncertainty and ambiguity in real-world problems.

Keywords: Generalized Choquet integral, Jensen type inequality, triangular fuzzy number, Minkowski type inequality, Hölder type inequality

1 Introduction

The Choquet integral, originally proposed by Choquet [4], has emerged as a significant instrument for modeling uncertain scenarios, notably within the domain of fuzzy measure theory and fuzzy integrals. Its applications have been extensively studied across various domains including decision-making, optimization, and preference modeling [2, 20, 22, 30, 31]. By extending the Choquet integral to encompass fuzzy measures, it has proven useful in multi-criteria decision theory, facilitating the analysis of preference relationships, risk assessment, and group decision-making processes [5, 6, 11, 18, 26, 29].

Of particular note is pseudo-analysis [15, 16, 17], which offers a broader perspective on classical analysis by defining a semiring $[a, b] \subseteq [-\infty, \infty]$ with pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ operations [22, 31, 32]. This approach draws upon mathematical tools from diverse fields including functional equations, variational calculus, measure theory, functional analysis, optimization theory, and semiring theory.

In recent years, D. Zhang et al. [32] introduced generalized Choquet integrals within the framework of pseudo-analysis, extending the scope of pseudo-integrals and Choquet-like integrals. Their work explored fundamental concepts such as monotone convergence and associated inequalities, building upon earlier advancements in pseudo-integrals and related integral frameworks [13, 23]. Recognizing the potential of generalized Choquet integrals to address uncertainty in mathematical models, this research seeks to integrate these integrals with triangular fuzzy numbers (TFNs), which are particularly effective for representing ambiguity and vagueness in practical applications across various domain. TFNs have demonstrated their utility across diverse applications, including decision-making, optimization, and information theory, making them an ideal candidate for extending the reach of generalized Choquet integrals [7, 8, 10, 12, 19, 20,

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21, 28]. This investigation endeavors to explore the interaction between non-additive measures and TFNs through the generalized Choquet integrals, thereby advancing both theoretical and practical understanding in this domain.

The integration of generalized Choquet integrals with TFNs addresses the need for robust and efficient tools capable of modeling uncertainty while ensuring mathematical consistency. By extending the properties and inequalities of Choquet integrals, such as monotone convergence non-decreasing convergence theorem and Fatou type, Jensen type, Minkowski type, Hölder type inequalities, to TFN-valued functions, this framework bridges a critical gap in fuzzy measure theory. Moreover, the introduction of TFNs into generalized Choquet integrals provides a robust approach for managing imprecise data in systems where traditional numeric measures fall short, particularly in contexts such as decision-making under uncertainty or risk.

Beyond their existing scope, Choquet integrals share a profound theoretical relationship with copulas, which are indispensable for modeling dependence structures in multivariate systems. Copulas, widely employed in probability, statistics, and finance, facilitate the analysis of complex relationships between random variables. The flexibility of Choquet integrals as aggregation operators aligns them with copulas in representing joint distributions and dependence measures, enriching their applicability. For instance, the authors [1] demonstrate how the convergence of linear approximations of Archimedean generators, analyzed through Williamson's transform, underscores the analytical power of Choquet integrals in understanding such dependencies. By incorporating TFNs, this framework has the potential to extend these connections, enabling more nuanced analyses of uncertainty in multivariate systems.

Our paper focuses on establishing the framework of triangular fuzzy number-valued generalized Choquet integrals (TGC-integrals), drawing on foundational concepts from generalized Choquet integrals and TFN-valued functions with non-idempotent fuzzy measures. We aim to elucidate their properties, explore monotone non-decreasing convergence theorems, and derive associated inequalities such as Jensen type, Minkowski type, and Hölder type inequalities for TGC-integrals. Furthermore, we provide illustrative examples highlighting the TFN-valued Choquet expected utility, and discuss their practical applications to enhance understanding and relevance including pseudo-functional analysis for the generalized Choquet integral of TFN-valued functions.

2 Preliminary and generalized choquet integrals

In this section, we recall basic contents related to a fuzzy measure, pseudo-arithmetic operators, a semiring, the generalized Choquet integral, the g-Choquet integral, and convergence theorems with inequalities.

Definition 2.1. [31] *Let $[a,b]$ be a closed or semiclosed subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$ and $[a,b]_+ = \{x \in [a,b] : \mathbf{0} \preceq x\}$, where $\mathbf{0}$ is a zero (neutral) element in $[a,b]$. Let the unit element $\mathbf{1} \in [a,b]$, i.e., $\mathbf{1} \otimes x = x$ for each $x \in [a,b]$.*

(i) *A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing with respect to $\preceq$ associative and with $\mathbf{0} \in [a,b]$.*

(ii) *A binary operation $\otimes$ on $[a,b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \otimes z \preceq y \otimes z$ for all $z \in [a,b]_+$, associative and with $\mathbf{1} \in [a,b]$.*

Note that we also suppose $\mathbf{0} \otimes x = \mathbf{0}$ for each $x \in [a,b]$ and that $\otimes$ is distributive over $\oplus$, i.e., $x \oplus (y \otimes z) = (x \otimes y) \oplus (x \otimes z)$ for all $x, y, z \in [a,b]$, which provides that the structure $([a,b], \oplus, \otimes)$ is a semiring.

Definition 2.2. [14, 24, 25] *Let X be a non-empty set and (X,Ω) be a measurable space. Then, a fuzzy measure μ on the set X is a set function $\mu : \Omega \to [a,b]_+$ satisfying the following axioms:*

(i) $\mu(\emptyset) = 0$.

(ii) *For every $A, B \in \Omega$, $A \subseteq B$ implies $\mu(A) \preceq \mu(B)$.*

If $\mu(X) = 1$, then μ is called normalized and we call the triplet (X,Ω,μ) a fuzzy measure space if μ is a fuzzy measure.

Definition 2.3. [15, 16, 31] *For non-idempotent $\otimes$ and $x \in [a,b]_+$, its pseudo-power $x_{\otimes}^{(n)}$ is defined*

$$x_{\otimes}^{(n)} = x \otimes x \otimes \cdots \otimes x, \text{ and } x_{\otimes}^{(\frac{1}{n})} = \sup\{y : y_{\otimes}^{(n)} \preceq x\}, \quad n \in \mathbb{N}.$$

Then, by continuity and monotonicity of $\otimes$, we have $x_{\otimes}^{(\frac{m}{n})} = x_{\otimes}^{(r)}$ is well defined for any rational number $r \in (0,\infty) \cap \mathbb{Q}$. Otherwise, $x_{\otimes}^{(p)} = \sup\{x_{\otimes}^{(r)} : r \in (0,p) \cap \mathbb{Q}\}$ for $p \notin (0,\infty) \cap \mathbb{Q}$. If x is idempotent, then $x_{\otimes}^{(p)} = x$ for all $x \in [a,b]$ and $p \in (0,\infty)$.

Definition 2.4. [31] *For a given semiring* $([a,b],\oplus,\otimes)$, *we consider a set function* $m: Borel([a,b]) \to [a,b]_+$ *satisfying the following properties:*

(i) $m(\emptyset) = \mathbf{0}$ *for non-idempotent* $\oplus$.

(ii) *(monotonicity)* $m(A) \preceq m(B)$, *whenever* $A \subseteq B$ *and* $A, B \in Borel([a,b])$.

(iii) *($\oplus$-additivity)* $m(A \cup B) = m(A) \oplus m(B)$, *whenever* $A \cap B = \emptyset$ *for* $A, B \in \Omega$.

(iv) *($\oplus$-submodularity)* $m(A \cap B) \oplus m(A \cup B) \preceq m(A) \oplus m(B)$ *for* $A, B \in Borel([a,b])$.

(v) *(lower semicontinuity)* $m(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$ *for any increasing sequence* $\{A_n\}_{n=1}^{\infty}$ *as* $A_j \subseteq A_k$, *whenever* $j \le k$.

(vi) *(upper semicontinuity)* $m(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$ *for any decreasing sequence* $\{A_n\}_{n=1}^{\infty}$ *as* $A_j \supseteq A_k$, *whenever* $j \ge k$ *such that* $m(A_{n_0})$ *is pseudo-finite for some* $n_0 \ge 1$.

(vii) *(σ-$\oplus$-additivity)* $m(\cup_{n=1}^{\infty} A_n) = \oplus_{n=1}^{\infty} m(A_n) = \lim_{k\to\infty} \oplus_{n=1}^{k} m(A_n)$ *for any sequence* $\{A_n\}_{n=1}^{\infty}$ *of pairwise disjoint sets of* $Borel([a,b])$.

Using the above pseudo-arithmetic operators and a semiring, we consider the generalized Choquet integral and its properties as follows:

Definition 2.5. [31] *Let* $([a,b],\oplus,\otimes)$ *be a semiring,* $\mu : \Omega \to [a,b]_+$ *be a fuzzy measure, and* $m : Borel([a,b]) \to [a,b]_+$ *be a* σ-$\oplus$-*measure.*

(i) *The generalized Choquet integral of a nonnegative measurable function* $f : X \to [a,b]_+$ *with respect to* μ, m *over* $A \in \Omega$ *is defined by*

$$(GC)\int_A^{\oplus} f \otimes d(\mu, m) = \int_{[a,b]_+}^{\oplus} \mu\left(A \cap (t \prec f)\right) \otimes dm(t),$$

where $(t \prec f) = \{x \in X : t \prec f(x)\}$.

(ii) *If it is pseudo-finite, then we say that* f *is GC-integrable over* A *with respect to* μ, m.

For simplicity, we use the notation $(GC)\int^{\oplus} f \otimes d(\mu,m)$ instead of $(GC)\int_X^{\oplus} f \otimes d(\mu,m)$ when $X = A$. From Definitions 2.4 and 2.5, we recall the basic properties [31] of the GC-integral and fuzzy measures.

Theorem 2.6. *The GC-integrals satisfy the following properties:*

(i) *If either* $f = \mathbf{0}$ *or* $\mu(A) = \mathbf{0}$, *then* $(GC)\int_A^{\oplus} f \otimes d(\mu,m) = \mathbf{0}$.

(ii) *If* $f \preceq g$, *then* $(GC)\int_A^{\oplus} f \otimes d(\mu,m) \preceq (GC)\int_A^{\oplus} g \otimes d(\mu,m)$.

(iii) *If* $A \subseteq B$, *then* $(GC)\int_A^{\oplus} f \otimes d(\mu,m) \preceq (GC)\int_B^{\oplus} f \otimes d(\mu,m)$.

(iv) $(GC)\int_A^{\oplus} f \otimes d(\mu,m) = (GC)\int^{\oplus} (\chi_A \otimes f) \otimes d(\mu,m)$, *where* $\chi_A(x) = \begin{cases} \mathbf{1} & \text{if } x \in A, \\ \mathbf{0} & \text{if } x \notin A. \end{cases}$

(v) $(GC)\int_A^{\oplus} c \otimes d(\mu,m) = m([\mathbf{0},\ c)) \otimes \mu(A)$ *for* $\mathbf{0} \preceq c$.

(vi) *If* $\mu_1 \preceq \mu_2$, *then* $(GC)\int_A^{\oplus} f \otimes d(\mu_1,m) \preceq (GC)\int_A^{\oplus} f \otimes d(\mu_2,m)$.

(vii) *If* $\mu = \mu_1 \oplus \mu_2$, *then* $(GC)\int_A^{\oplus} f \otimes d(\mu,m) = (GC)\int_A^{\oplus} f \otimes d(\mu_1,m) \oplus (GC)\int_A^{\oplus} f \otimes d(\mu_2,m)$.

Lemma 2.7. *Let* $([a,b],\oplus,\otimes)$ *be a semiring and let* $\{\mu_1, \mu_2, \ldots, \mu_n, \ldots, \mu\}$ *be a sequence of set functions from* Ω *to* $[a,b]_+$ *and* $\mu_n \to \mu$, *i.e.,* $\mu_n(A) \to \mu(A)$ *for all* $A \in \Omega$. *Then we have the following properties:*

(i) *If* $\mu_n \to \mu$ *and* μ_n *are fuzzy measures for* $n = 1, 2, \ldots,$ *then* μ *is a fuzzy measure.*

(ii) *If* $\mu_n \to \mu$ *and* μ_n *are* $\oplus$-*additive for* $n = 1, 2, \ldots,$ *then* μ *is* $\oplus$-*additive.*

(iii) *If* $\mu_n \uparrow \mu$, *i.e.*, $\mu_1(A) \preceq \mu_2(A) \preceq \cdots$, *and* $\mu_n(A) \to \mu(A)$ *for all* $A \in \Omega$, *and* μ_n *are lower semicontinuous for* $n = 1, 2, \ldots$, *then* μ *is lower semicontinuous.*

(iv) *If* $\mu_n \uparrow \mu$ *and* μ_n *are* σ-$\oplus$-*additive for* $n = 1, 2, \ldots$, *then* μ *is* σ-$\oplus$-*additive.*

(vi) *If* $\mu_n \downarrow \mu$, *i.e.*, $\mu_1(A) \succeq \mu_2(A) \succeq \cdots$, *and* $\mu_n(A) \to \mu(A)$ *for all* $A \in \Omega$, *and* μ_n *are upper semicontinuous for* $n = 1, 2, \ldots$, *then* μ *is upper semicontinuous.*

Recall that a g-semiring $([a,b], \oplus_g, \otimes_g)$ with the pseudo-operators $\oplus_g$ and $\otimes_g$ on $[a,b]$, for a continuous strictly monotone generator $g : [a,b] \to [0,\infty]$, which are given by

$$x \oplus_g y = g^{-1}(g(x) + g(y)), \quad x \otimes_g y = g^{-1}(g(x) \cdot g(y)) \quad \forall x, y \in [a,b].$$

In the g-semiring $([a,b], \oplus_g, \otimes_g)$ with $m([a,t)) = t$, the GC-integral of $f : X \to [a,b]$ is a measurable function over $A \in \Omega$ in Definition 2.5 can be expressed as

$$(GC)\int_A^{\oplus} f \otimes d(\mu, m) = g^{-1}\left((C)\int_A (g \circ f) d(g \circ \mu)\right) = (C_g)\int_A^{\oplus_g} f \otimes_g d\mu, \tag{1}$$

where the second integral is the Choquet integral and the third integral $(C_g)\int_A^{\oplus_g} f \otimes_g d\mu$ is called the g-Choquet integral of f (For details, see [13, 31]). We note that the convex definition below is used to introduce the Jensen type inequality.

Definition 2.8. [31] *Let* $([a,b], \oplus, \otimes)$ *be a semiring. Then a function* $\varphi : [a,b] \to [a,b]$ *is said to be* $(\oplus, \otimes)$-*convex if and only if*

$$\varphi(\lambda_1 \otimes x_1 \oplus \lambda_2 \otimes x_2) \preceq \lambda_1 \otimes \varphi(x_1) \oplus \lambda_2 \otimes \varphi(x_2),$$

for all $x_1, x_2, \lambda_1, \lambda_2 \in [a,b]$ *with* $\lambda_1 \oplus \lambda_2 = 1$.

We review the properties of the Choquet integral, as discussed in [31], to serve as a foundation for studying the properties of the TGC-integral, as outlined below:

Theorem 2.9. *(Monotone Non-decreasing Convergence Theorem) Let* $([a,b], \oplus, \otimes)$ *be a semiring and let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative measurable functions and* f *be a nonnegative function. If* μ *is lower semicontinuous, then* $f_n \uparrow f$ *implies*

$$(GC)\int^{\oplus} f_n \otimes d(\mu, m) \uparrow (GC)\int^{\oplus} f \otimes d(\mu, m).$$

Theorem 2.10. *(Fatou's Lemma) Let* $([a,b], \oplus, \otimes)$ *be a semiring, and let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative measurable functions and* f *be a nonnegative function. If* μ *is lower semicontinuous, then we have*

$$(GC)\int^{\oplus} \liminf_n f_n \otimes d(\mu, m) \preceq \liminf_n (GC)\int^{\oplus} f_n \otimes d(\mu, m).$$

Theorem 2.11. *(Monotone Non-decreasing Convergence Theorem for fuzzy measure) If* $([a,b], \oplus, \otimes)$ *is a semiring and* $\{\mu_n\}_{n=1}^{\infty}$ *is a sequence of fuzzy measures, with* μ *being a set function, and* $f : X \to [a,b]+$ *is GC-integrable, then* $\mu_n \uparrow \mu$ *implies*

$$(GC)\int^{\oplus} f \otimes d(\mu_n, m) \uparrow (GC)\int^{\oplus} f \otimes d(\mu, m).$$

Theorem 2.12. *(Jensen type inequality) Let* $([a,b], \oplus, \otimes)$ *be a semiring. If* $\varphi : [a,b]_+ \to [a,b]_+$ *is a* $(\oplus, \otimes)$-*convex function, and* $A \in \Omega$ *with* $\mu(A) = 1$, *and both* f *and* $\varphi \circ f$ *are GC-integrable functions, then*

$$\varphi\left((C_g)\int_A^{\oplus} f \otimes d\mu\right) \preceq (C_g)\int_A^{\oplus} (\varphi \circ f) \otimes d\mu.$$

Theorem 2.13. *(Minkowski type inequality) Let* $([a,b], \oplus, \otimes)$ *be a semiring and let* f, h *be measurable functions. If* μ *is a* $\oplus$-*submodular fuzzy measure, then*

$$\left((C_g)\int^{\oplus} (f \otimes h)_{\otimes}^{(p)} \otimes d\mu\right)_{\otimes}^{(\frac{1}{p})} \preceq \left((C_g)\int^{\oplus} f_{\otimes}^{(p)} \otimes d\mu\right)_{\otimes}^{(\frac{1}{p})} \oplus \left((C_g)\int^{\oplus} h_{\otimes}^{(p)} \otimes d\mu\right)_{\otimes}^{(\frac{1}{p})},$$

where $1 \leq p < \infty$.

Theorem 2.14. *(Hölder type inequality) Let* $([a,b],\oplus,\otimes)$ *be a semiring and let* $f,h : X \to [a,b]$ *be measurable functions. If* μ *is a* $\oplus$*-submodular fuzzy measure, then*

$$(C_g)\int^{\oplus}(f\otimes h)d\mu \preceq \left((C_g)\int^{\oplus} f_{\otimes}^{(p)}\otimes d\mu\right)_{\otimes}^{(\frac{1}{p})} \oplus \left((C_g)\int^{\oplus} h_{\otimes}^{(q)}\otimes d\mu\right)_{\otimes}^{(\frac{1}{q})},$$

where $1 < p, q < \infty$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

3 Generalized choquet integral of TFN-valued functions and its properties

In this section, we consider a triangular fuzzy number (TFN), which were introduced in [3, 9], as described below: A triple $\tilde{a} = (a_l, a_m, a_r)$ is defined by a fuzzy set with a triangular membership function $\mathrm{m}_{\tilde{a}}$ given by

$$\mathrm{m}_{\tilde{a}}(x) = \begin{cases} \frac{x-a_l}{a_m-a_l}, & a_l \le x \le a_m, \\ \frac{x-a_r}{a_m-a_r}, & a_m \le x \le a_r, \\ 0, & \text{otherwise.} \end{cases}$$

Let $TFN([a,b])$ be the set of all TFNs on $[a,b]$. The T-full order on $TFN([a,b])$ will be denoted by $\preceq_{\mathrm{T}}$, i.e., $\tilde{a} \preceq_{\mathrm{T}} \tilde{b}$ if and only if $a_l \preceq b_l, a_m \preceq b_m, a_r \preceq b_r$, where $\tilde{a} = (a_l, a_m, a_r)$ and $\tilde{b} = (b_l, b_m, b_r)$ in $TFN([a,b])$ and $\preceq$ is a full order on $([a,b],\oplus,\otimes)$. By using Definition 2.1, we define a pseudo-addition $\bigoplus_T$ and a pseudo-multiplication $\bigotimes_T$ on $TFN([a,b])$.

Definition 3.1. *Let* $([a,b],\oplus,\otimes)$ *be a semiring. Then the pseudo-operations on* $TFN([a,b])$ *are defined as follows: The pseudo-addition* $\bigoplus_T$ *and the pseudo-multiplication* $\bigotimes_T$ *on* $TFN([a,b])$ *are respectively defined by*

$$\tilde{a}\bigoplus_T\tilde{b} = (a_l,a_m,a_r)\bigoplus_T(b_l,b_m,b_r) = (a_l\oplus b_l, a_m\oplus b_m, a_r\oplus b_r),$$

and

$$\tilde{a}\bigotimes_T\tilde{b} = (a_l,a_m,a_r)\bigotimes_T(b_l,b_m,b_r) = (a_l\otimes b_l, a_m\otimes b_m, a_r\otimes b_r),$$

where $\tilde{a} = (a_l, a_m, a_r)$ *and* $\tilde{b} = (b_l, b_m, b_r)$ *in* $TFN([a,b])$.

Note that if $\lambda \in [a,b]$, we define

$$\lambda\bigotimes_T\tilde{a} = (\lambda,\lambda,\lambda)\bigotimes_T(a_l,a_m,a_r) = (\lambda\otimes a_l, \lambda\otimes a_m, \lambda\otimes a_r).$$

Furthermore, using Definition 2.3, the T-pseudo-power of $\tilde{a}$ is defined as

$$\tilde{a}_{\otimes_T}^{(p)} = (a_{l\otimes}^{(p)}, a_{m\otimes}^{(p)}, a_{r\otimes}^{(p)}), \text{ for } p\in(0,\infty).$$

Then we can easily obtain the following result.

Theorem 3.2. $([a,b],\oplus,\otimes)$ *is a semiring if and only if* $(TFN([a,b]),\oplus_T,\otimes_T)$ *is a semiring.*

Let X be a non-empty set, (X,Ω) be a measurable space and $\tilde{f} = (f_l, f_m, f_r) : X \to TFN([a,b]_+)$ be a TFN-valued function. Then, we proceed to define the TGC-integral of a TFN-valued function from X to $TFN([a,b]_+)$.

Definition 3.3. (i) *A TFN-valued function* $\tilde{f} = (f_l, f_m, f_r)$ *is said to be a* T*-measurable function if* $f_l, f_m,$ *and* f_r *are measurable functions.*

(ii) *Let* $\tilde{f} = (f_l, f_m, f_r)$ *be a* T*-measurable function on* X*, then the TFN-valued generalized Choquet integral of* $\tilde{f}$*, simply TGC-integral, is defined by*

$$(TGC)\int_X^{\oplus_T}\tilde{f}\bigotimes_T d(\mu,m) = \left((GC)\int_X^{\oplus} f_l\otimes d(\mu,m), (GC)\int_X^{\oplus} f_m\otimes d(\mu,m), (GC)\int_X^{\oplus} f_r\otimes d(\mu,m)\right).$$

(iii) *We say* $\tilde{f}$ *is TGC-integrable if* $(TGC)\int_X^{\oplus_T}\tilde{f}\otimes_T d(\mu,m)$ *exists.*

It is easily seen that if f_l, f_m, and f_r are GC-integrable, then $\tilde{f} = (f_l, f_m, f_r)$ is TGC-integrable. We denote $TGC(X; [a,b]_+) = \{\tilde{f} : X \to TFN([a,b]_+) \mid \tilde{f}$ is TGC-integrable$\}$ and introduce some properties of TGC-integrals of a TFN-valued function in $TGC(X; [a,b]_+)$.

Theorem 3.4. *The TGC-integrals of $\tilde{f}, \tilde{g}$ in $TGC(X; [a,b]_+)$ satisfy the following properties:*

(i) *If $\tilde{f} = \mathbf{0}$ or $\mu(A) = \mathbf{0}$, then $(TGC)\int_A^{\oplus_T} \tilde{f} \otimes_T d(\mu, m) = \mathbf{0}$.*

(ii) *If $\tilde{f} \preceq_T \tilde{g}$, i.e., $f_l \preceq g_l, f_m \preceq g_m, f_r \preceq g_r$ for $\tilde{f} = (f_l, f_m, f_r), \tilde{g} = (g_l, g_m, g_r)$, then*

$$(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m) \preceq_T (TGC)\int_A^{\oplus_T} \tilde{g} \bigotimes_T d(\mu, m).$$

(iii) *If $A \subseteq B$, then*

$$(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m) \preceq_T (TGC)\int_B^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m).$$

(iv) *If $\tilde{c} = (c_l, c_m, c_r) \in TFN(\mathbb{R}^+)$, then*

$$(TGC)\int_A^{\oplus_T} \tilde{c} \bigotimes_T d(\mu, m) = \big(m([0, c_l) \otimes \mu(A),\ m([0, c_m) \otimes \mu(A),\ m([0, c_r) \otimes \mu(A)\big).$$

(v) *If $\mu_1 \preceq \mu_2$ and $\tilde{f} = (f_l, f_m, f_r)$, then*

$$(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_1, m) \preceq_T (TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_2, m).$$

(vi) *If $\mu = \mu_1 \oplus \mu_2$, then*

$$(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m) = \left((TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_1, m)\right) \bigoplus_T \left((TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_2, m)\right).$$

Proof. For (i), we note that if $\tilde{f} = (f_l, f_m, f_r) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ or $\mu(A) = \mathbf{0}$, then either $f_l = f_m = f_r = \mathbf{0}$ or $\mu(A) = \mathbf{0}$. Theorem 2.6 (i) provides

$$\begin{aligned}(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m) &= \left((GC)\int_A^{\oplus} f_l \otimes d(\mu, m), (GC)\int_A^{\oplus} f_m \otimes d(\mu, m), (GC)\int_A^{\oplus} f_r \otimes d(\mu, m)\right) \\ &= (\mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}.\end{aligned}$$

For (ii), from Theorem 2.6 (ii), we see that

$$\begin{aligned}(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m) &= \left((GC)\int_A^{\oplus} f_l \otimes d(\mu, m), (GC)\int_A^{\oplus} f_m \otimes d(\mu, m), (GC)\int_A^{\oplus} f_r \otimes d(\mu, m)\right) \\ &\preceq_T \left((GC)\int_A^{\oplus} g_l \otimes d(\mu, m), (GC)\int_A^{\oplus} g_m \otimes d(\mu, m), (GC)\int_A^{\oplus} g_r \otimes d(\mu, m)\right) \\ &= (TGC)\int_A^{\oplus_T} \tilde{g} \bigotimes_T d(\mu, m).\end{aligned}$$

The proofs of (iii) and (iv) are similarly obtained from (ii). For (v), Theorem 2.6 (vi) gives that

$$\begin{aligned}(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_1, m) &= \left((GC)\int_A^{\oplus} f_l \otimes d(\mu_1, m), (GC)\int_A^{\oplus} f_m \otimes d(\mu_1, m), (GC)\int_A^{\oplus} f_r \otimes d(\mu_1, m)\right) \\ &\preceq_T \left((GC)\int_A^{\oplus} f_l \otimes d(\mu_2, m), (GC)\int_A^{\oplus} f_m \otimes d(\mu_2, m), (GC)\int_A^{\oplus} f_r \otimes d(\mu_2, m)\right) \\ &= (TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_2, m).\end{aligned}$$

For (vi), Theorem 2.6 (vii) shows that

$$
\begin{aligned}
(TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m) &= \left((GC)\int_A^{\oplus} f_l \otimes d(\mu, m), (GC)\int_A^{\oplus} f_m \otimes d(\mu, m), (GC)\int_A^{\oplus} f_r \otimes d(\mu, m)\right) \\
&= \Big((GC)\int_A^{\oplus} f_l \otimes d(\mu_1, m) \oplus (GC)\int_A^{\oplus} f_l \otimes d(\mu_2, m), (GC)\int_A^{\oplus} f_m \otimes d(\mu_1, m) \\
&\quad \oplus (GC)\int_A^{\oplus} f_m \otimes d(\mu_2, m), (GC)\int_A^{\oplus} f_r \otimes d(\mu_1, m) \oplus (GC)\int_A^{\oplus} f_r \otimes d(\mu_2, m)\Big) \\
&= \left((TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_1, m)\right) \bigoplus_T \left((TGC)\int_A^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_2, m)\right).
\end{aligned}
$$

□

4 Monotone non-decreasing convergence theorem for TGC-integrals

In this section, we define the limits of sequences of TFNs and TFN-valued functions, which are essential for establishing convergence theorems for TGC-integrals. Let (X, Ω, μ) be a fuzzy measure space. Specifically, we define the limit of a sequence $\{\tilde{a}_n\}$ in $TFN([a, b]_+)$ and the limit of a sequence $\{\tilde{f}_n\}$ in $TGC(X; [a, b]_+)$.

Definition 4.1. *We define the following limits for sequences of $\{\tilde{a}_n\}_{n=1}^{\infty}$ and $\{\tilde{f}_n\}_{n=1}^{\infty}$, where $\tilde{a}_n = (a_{l_n}, a_{m_n}, a_{r_n}), \tilde{a} = (a_l, a_m, a_r) \in TFN([a, b]_+)$ and $\tilde{f}_n = (f_{l_n}, f_{m_n}, f_{r_n}), \tilde{f} = (f_l, f_m, f_r) \in TGC(X; [a, b]_+)$.*

(i) *The limit inferior of $\{\tilde{f}_n\}$ is defined by $(\liminf_n)_T \tilde{f}_n = \left(\liminf_n f_{l_n}, \liminf_n f_{m_n}, \liminf_n f_{r_n}\right)$.*

(ii) *The upper limit $\tilde{f}_n \uparrow_T \tilde{f}$ of $\{\tilde{f}_n\}$ is defined by $f_{l_n} \uparrow f_l$, $f_{m_n} \uparrow f_m$, and $f_{r_n} \uparrow f_r$.*

(iii) *The upper limit $\tilde{a}_n \uparrow_T \tilde{a}$ of $\{\tilde{a}_n\}$ is defined by $a_{l_n} \uparrow a_l$, $a_{m_n} \uparrow a_m$, and $a_{r_n} \uparrow a_r$.*

By applying Definitions 4.1(ii) and 3.3(i) along with Theorem 2.9, we derive the monotone non-decreasing convergence theorem for the TGC-integral, stated as follows.

Theorem 4.2. *(Monotone Non-decreasing Convergence Theorem) Let $(TFN([a, b]), \bigoplus_T, \bigotimes_T)$ be a semiring and let $\{\tilde{f}_n\}$ be a sequence of nonnegative measurable functions in $TGC(X; [a, b]_+)$ and $\tilde{f}$ be a nonnegative measurable function in $TGC(X; [a, b]_+)$. If μ is lower semicontinuous and m is a σ-$\oplus$-measure, then $\tilde{f}_n \uparrow_T \tilde{f}$ implies*

$$
(TGC)\int^{\oplus_T} \tilde{f}_n \bigotimes_T d(\mu, m) \uparrow_T (TGC)\int^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m).
$$

Proof. Since $\tilde{f}_n \uparrow_T \tilde{f}$, we have that $f_{l_n} \uparrow f_l, f_{m_n} \uparrow f_m$, and $f_{r_n} \uparrow f_r$. By using Theorem 2.10, we have

$$
(GC)\int^{\oplus} f_{l_n} \otimes d(\mu, m) \uparrow (GC)\int^{\oplus} f_l \otimes d(\mu, m), (GC)\int^{\oplus} f_{m_n} \otimes d(\mu, m) \uparrow (GC)\int^{\oplus} f_m \otimes d(\mu, m),
$$

and

$$
(GC)\int^{\oplus} f_{r_n} \otimes d(\mu, m) \uparrow (GC)\int^{\oplus} f_r \otimes d(\mu, m).
$$

By Definition 4.1 (iii), we have

$$
\begin{aligned}
(TGC)\int^{\oplus_T} \tilde{f}_n \bigotimes_T d(\mu, m) &= \left((GC)\int^{\oplus_T} f_{l_n} \otimes d(\mu, m), (GC)\int^{\oplus_T} f_{m_n} \otimes d(\mu, m), (GC)\int^{\oplus_T} f_{r_n} \otimes d(\mu, m)\right) \\
&\uparrow_T \left((GC)\int^{\oplus_T} f_l \otimes d(\mu, m), (GC)\int^{\oplus_T} f_m \otimes d(\mu, m), (GC)\int^{\oplus_T} f_r \otimes d(\mu, m)\right) \\
&= (TGC)\int^{\oplus_T} \tilde{f} \bigotimes_T d(\mu, m).
\end{aligned}
$$

□

Using Definition 3.3 (i) and 4.1 (i), and Theorem 2.10, we derive the Fatou-type lemma for the TGC-integral, presented as follows.

Theorem 4.3. *(Fatou type Lemma) Let $(TFN([a,b]), \oplus_T, \otimes_T)$ be a semiring, let $\{\tilde{f}_n\}$ be a sequence of nonnegative measurable functions in $TGC(X;[a,b]_+)$ and $\tilde{f}$ be a nonnegative measurable function in $TGC(X;[a,b]_+)$. If μ is lower semicontinuous and m is a σ-$\oplus$-measure, then we have*

$$(TGC)\int^{\oplus_T} \liminf_n \tilde{f}_n \bigotimes_T d(\mu,m) \preceq_T \liminf_n (TGC)\int^{\oplus_T} \tilde{f}_n \bigotimes_T d(\mu,m).$$

Proof. By using Theorem 2.10, we have

$$\begin{aligned}
&(TGC)\int^{\oplus_T} \liminf_n \tilde{f}_n \bigotimes_T d(\mu,m)\\
&= \Big((GC)\int^{\oplus_T} \liminf_n f_{l_n} \otimes d(\mu,m), (GC)\int^{\oplus_T} \liminf_n f_{m_n} \otimes d(\mu,m), (GC)\int^{\oplus_T} \liminf_n f_{r_n} \otimes d(\mu,m)\Big)\\
&\preceq_T \Big(\liminf_n (GC)\int^{\oplus_T} f_{l_n} \otimes d(\mu,m), \liminf_n (GC)\int^{\oplus_T} f_{m_n} \otimes d(\mu,m), \liminf_n (GC)\int^{\oplus_T} f_{r_n} \otimes d(\mu,m)\Big)\\
&= \liminf_n (TGC)\int^{\oplus_T} \tilde{f}_n \bigotimes_T d(\mu,m).
\end{aligned}$$

□

By utilizing Definitions 3.3 (i) and 4.1 (iii) along with Theorem 2.11, we establish the monotone non-decreasing convergence theorem for the TGC-integral, as stated below.

Theorem 4.4. *(Monotone Non-decreasing Convergence Theorem for fuzzy measure) Let $(TFN([a,b]), \oplus_T, \otimes_T)$ be a semiring and let $\{\mu_n\}$ be a sequence of fuzzy measures. Furthermore, if $\tilde{f}$ is nonnegative measurable function in $TGC(X;[a,b]_+)$, then $\mu_n \uparrow \mu$ for a set function μ implies*

$$(TGC)\int^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_n,m) \uparrow_T (TGC)\int^{\oplus_T} \tilde{f} \bigotimes_T d(\mu,m).$$

Proof. By using Theorem 2.11 with $\tilde{f} = (f_l, f_m, f_r)$, we have

$$(GC)\int^{\oplus} f_l \otimes d(\mu_n,m) \uparrow (GC)\int^{\oplus} f_l \otimes d(\mu,m),\ (GC)\int^{\oplus} f_m \otimes d(\mu_n,m) \uparrow (GC)\int^{\oplus} f_m \otimes d(\mu,m),$$

and

$$(GC)\int^{\oplus} f_r \otimes d(\mu_n,m) \uparrow (GC)\int^{\oplus} f_r \otimes d(\mu,m).$$

Also, Definition 4.1 (iii) provides that

$$\begin{aligned}
(TGC)\int^{\oplus_T} \tilde{f} \bigotimes_T d(\mu_n,m) &= \Big((GC)\int^{\oplus} f_l \otimes d(\mu_n,m), (GC)\int^{\oplus} f_m \otimes d(\mu_n,m), (GC)\int^{\oplus} f_r \otimes d(\mu_n,m)\Big)\\
&\uparrow_T \Big((GC)\int^{\oplus} f_l \otimes d(\mu,m), (GC)\int^{\oplus} f_m \otimes d(\mu,m), (GC)\int^{\oplus} f_r \otimes d(\mu,m)\Big)\\
&= (TGC)\int^{\oplus_T} \tilde{f} \bigotimes_T d(\mu,m).
\end{aligned}$$

□

5 Inequalities for TGC-integrals and application

In this section, we aim to define the TFN-valued convex function, denoted as the $(\oplus_T, \otimes_T)$-convex function, using Definitions 2.1 and 2.8, along with Theorem 3.2, as follows.

Definition 5.1. *Let ($TFN([a,b]),\bigoplus_T,\bigotimes_T$) be a semiring. We define a function $\tilde{\varphi}=(\varphi_l,\varphi_m,\varphi_r):TGC(X;[a,b]_+)\to TGC(X;[a,b]_+)$ is ($\bigoplus_T,\bigotimes_T$)-convex function if $\tilde{\varphi}(\tilde{x})=(\varphi_l(x_l),\varphi_m(x_m),\varphi_r(x_r))$ for all $(x_l,x_m,x_r)\in TFN([a,b]_+)$, where φ_l,φ_m, and φ_r are $(\oplus,\otimes)$-convex functions.*

Using the g-semiring $([a,b],\oplus_g,\otimes_g)$ with $m([a,t))=t$ and Theorem 3.2, we define a semiring, referred to as the T_g-semiring, denoted by $(TGC(X;[a,b]_+),\bigoplus_{T_g},\bigotimes_{T_g})$. Within this framework, the TGC-integral of a TFN-valued function $\tilde{f}=(f_l,f_m,f_r)$ over $A\in\Omega$, as given in Definition 3.3, is expressed as

$$\begin{aligned}(TGC)\int_A^{\bigoplus_T}\tilde{f}\bigotimes_T d(\mu,m)=&\Big(g^{-1}\Big((C)\int_A(g\circ f_l)d(g\circ\mu)\Big),g^{-1}\Big((C)\int_A(g\circ f_m)d(g\circ\mu)\Big),g^{-1}\Big((C)\int_A(g\circ f_r)d(g\circ\mu)\Big)\Big)\\ =&\Big((C_g)\int_A^{\oplus}f_l\otimes d\mu,(C_g)\int_A^{\oplus}f_m\otimes d\mu,(C_g)\int_A^{\oplus}f_r\otimes d\mu\Big)\\ =&(TC_g)\int_A^{\bigoplus_{T_g}}\tilde{f}\bigotimes_{T_g}d\mu,\end{aligned}\tag{2}$$

where $(TC_g)\int_A^{\bigoplus_{T_g}}\tilde{f}\bigotimes_{T_g}d\mu$ is referred to as the $\mathrm{TC_g}$-integral of $\tilde{f}$. It is noted that the TGC-integral of $\tilde{f}$ in Eq. (2) can be seen as a $\mathrm{TC_g}$-integral of $\tilde{f}$.

Example 5.2. *Let us consider a pseudo-addition $\oplus$ and a pseudo-multiplication $\otimes$ on $[a,b]$ by*

$$x\oplus y=g^{-1}(g(x)+g(y))\text{ and }x\otimes y=g^{-1}(g(x)\cdot g(y)),$$

where $[a,b]=[0,\infty]$ and a function $g:[0,\infty]\to[0,\infty]$ is given by $g(x)=x^2$. Let $X=[0,1],\Omega=Borel(X)$, and $\mu=\sqrt{\lambda}$, where λ is the Lebesgue measure on Ω and $m([0,t))=t$ for all $t\in X$. Considering $\tilde{1}=(1_l,1_m,1_r)=(0.9,1,1.1)$, a TFN-valued function $\tilde{f}\in TGC(X;[0,\infty])$ is given by

$$\tilde{f}(x)=\tilde{1}\bigotimes_T\sqrt{x+1}=(0.9\sqrt{x+1},\sqrt{x+1},1.1\sqrt{x+1})\quad\text{for all }x\in X.$$

Then, we have

$$(TC_g)\int_X^{\bigoplus_{T_g}}\tilde{f}\bigotimes_{T_g}d\mu=\Big(0.9(C_g)\int_X^{\oplus}\sqrt{x+1}\otimes d\mu,(C_g)\int_X^{\oplus}\sqrt{x+1}\otimes d\mu,\ 1.1(C_g)\int_X^{\oplus}\sqrt{x+1}\otimes d\mu\Big).$$

By Eq. (1), one can obtain that

$$\begin{aligned}(C_g)\int_X^{\oplus}\sqrt{x+1}\otimes d(\mu,m)=&g^{-1}\Big(\int_0^\infty g\circ\mu\big([0,1]\cap(t<g\circ\sqrt{x+1})\big)dt\Big)\\ =&g^{-1}\Big(\int_0^\infty\lambda\big([0,1]\cap(t<x+1)\big)dt\Big)\\ =&g^{-1}\Big(\int_0^1\lambda([0,1])dt+\int_1^2\lambda([t-1,1])dt\Big)\\ =&g^{-1}\Big(1+\int_1^2(2-t)dt\Big)\\ =&g^{-1}(1.5)=\sqrt{1.5}.\end{aligned}$$

Hence, $(TGC)\int_X^{\bigoplus_T}\tilde{f}\bigotimes_T d(\mu,m)=(0.9\sqrt{1.5},\sqrt{1.5},1.1\sqrt{1.5})$.

Using Eq. (2), we establish the following Jensen type inequality, which relates the ($\bigoplus_T,\bigotimes_T$)-convex function of the $\mathrm{TC_g}$-integral to the $\mathrm{TC_g}$-integral of the composition of a ($\bigoplus_T,\bigotimes_T$)-convex function and a TGC-integrable function, as detailed below.

Theorem 5.3. *(Jensen type inequality) Let ($TFN([a,b]),\bigoplus_{T_g},\bigotimes_{T_g}$) be a g-semiring and let a function $\tilde{\varphi}=(\varphi_l,\varphi_m,\varphi_r):TGC(X;[a,b]_+)\to TGC(X;[a,b]_+)$ be ($\bigoplus_{T_g},\bigotimes_{T_g}$)-convex. Suppose $A\in\Omega$ with $\mu(A)=1$. If $\tilde{f}=(f_l,f_m,f_r)$ and $\tilde{\varphi}\circ\tilde{f}$ are TGC-integrable functions, then*

$$\tilde{\varphi}\left((TC_g)\int_A^{\bigoplus_{T_g}}\tilde{f}\bigotimes_{T_g}d\mu\right)\preceq_T(TC_g)\int_A^{\bigoplus_{T_g}}(\tilde{\varphi}\circ\tilde{f})\bigotimes_{T_g}d\mu,$$

where $(\tilde{\varphi}\circ\tilde{f})(x)=\tilde{\varphi}(\tilde{f}(x))=\big(\varphi_l(f_l(x)),\varphi_m(f_m(x)),\varphi_r(f_r(x))\big)$ is the composition of $\tilde{\varphi}$ and $\tilde{f}$.

Proof. Definitions of $\tilde{\varphi}$ and $\mathrm{TC_g}$-integral, and Theorem 2.12 provide that

$$\begin{aligned}\tilde{\varphi}\Big((TC_g)\int_A^{\oplus_{\mathrm{T}_g}} \tilde{f} \bigotimes_{\mathrm{T}_g} d\mu\Big) &= \Big(\varphi_l\Big((C_g)\int_A^{\oplus_g} f_l \otimes_g d\mu\Big), \varphi_m\Big((C_g)\int_A^{\oplus_g} f_m \otimes_g d\mu\Big), \varphi_r\Big((C_g)\int_A^{\oplus_g} f_r \otimes_g d\mu\Big)\Big)\\ &\preceq_{\mathrm{T}} \Big((C_g)\int_A^{\oplus_g} (\varphi_l \circ f_l) \otimes_g d\mu, (C_g)\int_A^{\oplus_g} (\varphi_m \circ f_m) \otimes_g d\mu, (C_g)\int_A^{\oplus_g} (\varphi_r \circ f_r) \otimes_g d\mu\Big)\\ &= (TC_g)\int_A^{\oplus_{\mathrm{T}_g}} (\tilde{\varphi} \circ \tilde{f}) \bigotimes_{\mathrm{T}_g} d\mu.\end{aligned}$$

□

Example 5.4. *Let us consider $X = [0, \infty]$, where $[a, b] = [0, \infty]$, and $\mu = \sqrt{\lambda}$ is a fuzzy measure, with λ being the Lebesgue measure. Let $\{A_k\}_{k=1}^n \subset A = X$ be the collection of disjoint subsets satisfying $\mu(X) = 1$ and $\mu(A_k) = r_k < \infty$ for all $k = 1, 2, \ldots, n$. If $g(x) = x^2$, then it is noted that the pseudo-operations satisfy*

$$x \oplus_g y = \sqrt{x^2 + y^2} \text{ and } x \otimes_g y = xy. \tag{3}$$

Let $\tilde{\varphi} = (\varphi_l, \varphi_m, \varphi_r) : TGC(X; [0, \infty]) \to TGC(X; [0, \infty])$ be a $(\bigoplus_{\mathrm{T}_g}, \otimes_{\mathrm{T}_g})$-convex function with $\tilde{\varphi}(0) = (0, 0, 0)$, and let $\tilde{f} = (f_l, f_m, f_r)$ be a function in $TGC(X; [0, \infty]$ given by

$$\tilde{f}(x) = \bigoplus_{k=1}^{n}{}_T \frac{1}{r_k} \bigotimes_T \tilde{a}_k \chi_{A_k}(x),$$

where $\tilde{a}_k = (a_{k_l}, a_{k_m}, a_{k_r}) \in TFN([0, \infty])$, and the characteristic function χ_{A_k} is given by

$$\chi_{A_k}(x) = \begin{cases} 1 \text{ if } x \in A_k, \\ 0 \text{ if } x \notin A_k, \end{cases} \quad k = 1, 2, \ldots, n.$$

Definition 3.3 with Eqs. (2) *and* (3) *provides the following results:*

$$\begin{aligned}(TC_g)\int^{\oplus_{\mathrm{T}_g}} \tilde{f} \bigotimes_{\mathrm{T}_g} d\mu &= \Big((C_g)\int^{\oplus_g} f_l \otimes_g d\mu, (C_g)\int_A^{\oplus_g} f_m \otimes_g d\mu, (C_g)\int^{\oplus_g} f_r \otimes_g d\mu\Big)\\ &= \Big(\bigoplus_{k=1}^{n}{}_g \int_{A_k}^{\oplus_g} \frac{a_{k_l}}{r_k} \chi_{A_k} \otimes_g d\mu, \bigoplus_{k=1}^{n}{}_g \int_{A_k}^{\oplus_g} \frac{a_{k_m}}{r_k} \chi_{A_k} \otimes_g d\mu, \bigoplus_{k=1}^{n}{}_g \int_{A_k}^{\oplus_g} \frac{a_{k_r}}{r_k} \chi_{A_k} \otimes_g d\mu\Big)\\ &= \Big(\bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\int_{A_k} g\left(\frac{a_{k_l}}{r_k}\right) d(g \circ \mu)\Big), \bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\int_{A_k} g\left(\frac{a_{k_m}}{r_k}\right) d(g \circ \mu)\Big),\\ &\qquad \bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\int_{A_k} g\left(\frac{a_{k_r}}{r_k}\right) d\,(g \circ \mu)\Big)\Big)\\ &= \Big(\bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\frac{a_{k_l}^2}{r_k^2} \lambda(A_k)\Big), \bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\frac{a_{k_m}^2}{r_k^2} \lambda(A_k)\Big), \bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\frac{a_{k_r}^2}{r_k^2} \lambda(A_k)\Big)\Big)\\ &= \Big(\bigoplus_{k=1}^{n}{}_g a_{k_l}, \bigoplus_{k=1}^{n}{}_g a_{k_m}, \bigoplus_{k=1}^{n}{}_g a_{k_r}\Big) = \bigoplus_{k=1}^{n}{}_{\mathrm{T}_g} \tilde{a}_k,\end{aligned} \tag{4}$$

and

$$
\begin{aligned}
&(TC_g)\int^{\oplus_{T_g}} (\tilde{\varphi}\circ \tilde{f}) \bigotimes_{T_g} d\mu \\
&= \Big((C_g)\int^{\oplus_g} \varphi_l \circ f_l \otimes_g d\mu, (C_g)\int^{\oplus_g} \varphi_m \circ f_m \otimes_g d\mu, (C_g)\int^{\oplus_g} \varphi_r \circ f_r \otimes_g d\mu\Big) \\
&= \Big(\bigoplus_{k=1}^{n}{}_g \int^{\oplus_g} \varphi_l\Big(\frac{a_{k_l}}{\sqrt{r_k}}\chi_{A_k}\Big)\otimes_g d\mu, \bigoplus_{k=1}^{n}{}_g \int^{\oplus_g} \varphi_m\Big(\frac{a_{k_m}}{\sqrt{r_k}}\chi_{A_k}\Big)\otimes_g d\mu, \bigoplus_{k=1}^{n}{}_g \int^{\oplus_g} \varphi_r\Big(\frac{a_{k_r}}{\sqrt{r_k}}\chi_{A_k}\Big)\otimes_g d\mu\Big) \\
&= \Big(\bigoplus_{k=1}^{n}{}_g\, g^{-1}\Big(\int_{A_k} g\circ\varphi_l\Big(\frac{a_{k_l}}{r_k}\Big)d\lambda\Big), \bigoplus_{k=1}^{n}{}_g\, g^{-1}\Big(\int_{A_k} g\circ\varphi_m\Big(\frac{a_{k_m}}{r_k}\Big)d\lambda\Big), \bigoplus_{k=1}^{n}{}_g\, g^{-1}\Big(\int_{A_k} g\circ\varphi_r\Big(\frac{a_{k_r}}{r_k}\Big)d\lambda\Big)\Big) \\
&= \Big(\bigoplus_{k=1}^{n}{}_g\, \varphi_l\Big(\frac{a_{k_l}}{r_k}\Big)r_k, \bigoplus_{k=1}^{n}{}_g\, \varphi_m\Big(\frac{a_{k_m}}{r_k}\Big)r_k, \bigoplus_{k=1}^{n}{}_g\, \varphi_r\Big(\frac{a_{k_r}}{r_k}\Big)r_k\Big) = \bigoplus_{k=1}^{n}{}_{T_g}\, \tilde{\varphi}\Big(\frac{\tilde{a}_k}{r_k}\Big)\bigotimes_T r_k.
\end{aligned} \tag{5}
$$

By Theorem 5.3 with the above two results 4 and 5, we have the following Jensen type inequality to the finite series of triangular fuzzy numbers

$$
\tilde{\varphi}\Big(\bigoplus_{k=1}^{n}{}_{T_g} \tilde{a}_k\Big) \preceq_T \bigoplus_{k=1}^{n}{}_{T_g} \tilde{\varphi}\Big(\frac{\tilde{a}_k}{r_k}\Big)\bigotimes_T r_k. \tag{6}
$$

From Eq. (3)*, we obtain the following Jensen type inequality for another expression of Eq.* (6)*:*

$$
\tilde{\varphi}\Big(\bigoplus_{k=1}^{n}{}_g a_{k_l}, \bigoplus_{k=1}^{n}{}_g a_{k_m}, \bigoplus_{k=1}^{n}{}_g a_{k_r}\Big) \preceq_T \Big(\bigoplus_{k=1}^{n}{}_g \varphi_l(\frac{a_{k_l}}{r_k})r_k, \bigoplus_{k=1}^{n}{}_g \varphi_m(\frac{a_{k_m}}{r_k})r_k, \bigoplus_{k=1}^{n}{}_g \varphi_r(\frac{a_{k_r}}{r_k})r_k\Big).
$$

In particular, if $r_k = \frac{1}{n}$ *for* $k = 1, 2, \ldots, n$*, then we have*

$$
\tilde{\varphi}\Big(\bigoplus_{k=1}^{n}{}_g a_{k_l}, \bigoplus_{k=1}^{n}{}_g a_{k_m}, \bigoplus_{k=1}^{n}{}_g a_{k_r}\Big) \preceq_T \Big(\bigoplus_{k=1}^{n}{}_g \frac{1}{n}\varphi_l(na_{k_l}), \bigoplus_{k=1}^{n}{}_g \frac{1}{n}\varphi_m(na_{k_m}), \bigoplus_{k=1}^{n}{}_g \frac{1}{n}\varphi_r(na_{k_r})\Big).
$$

Remark 5.5. *In Theorem 5.3, let $\tilde{\varphi}$ represent a TFN-valued utility function, specifically a convex and continuous function, and let the TC_g-integral denote the TFN-valued Choquet expected utility. If $\tilde{f}$ is a TFN-valued trade volume function in the global semiconductor market (as discussed in* [27]*), then*

- *The expression*

$$
(TC_g)\int_A^{\oplus_{T_g}} (\tilde{\varphi}\circ \tilde{f})\bigotimes_{T_g} d\mu, \tag{7}
$$

  represents the TFN-valued Choquet expected utility of $\tilde{f}$.

- *The expression*

$$
\tilde{\varphi}\Big((TC_g)\int_A^{\oplus_{T_g}} \tilde{f}\bigotimes_{T_g} d\mu\Big), \tag{8}
$$

  represents the utility of the TFN-valued Choquet expected value of $\tilde{f}$.

The difference between (7) *and* (8) *quantifies the impact of preprocessing the TFN-valued trade volume function $\tilde{f}$, providing a numerical value for this contribution.*

We now derive the following Minkowski-type inequality, which establishes that $TGC(X;[a,b]_+)$ forms a pseudo-normed vector space. Consequently, $TGC(X;[a,b]_+)$ is referred to as a pseudo-L^p space for $1 \le p < \infty$.

Theorem 5.6. *(Minkowski type inequality) Let $(TFN([a,b]), \oplus_{T_g}, \otimes_{T_g})$ be a T_g-semiring and let $\tilde{f}, \tilde{h}$ be measurable functions in $TGC(X;[a,b]_+)$. If μ is a $\oplus_g$-submodular fuzzy measure, then*

$$
\Big((TC_g)\int^{\oplus_{T_g}} (\tilde{f} \oplus_{T_g} \tilde{h})^{(p)}_{\otimes_T} \bigotimes_{T_g} d\mu\Big)^{(\frac{1}{p})}_{\otimes_T} \preceq_T \Big((TC_g)\int^{\oplus_{T_g}} \tilde{f}^{(p)}_{\otimes_T} \bigotimes_{T_g} d\mu\Big)^{(\frac{1}{p})}_{\otimes_T} \bigoplus_{T_g} \Big((TC_g)\int^{\oplus_{T_g}} \tilde{h}^{(p)}_{\otimes_T} \bigotimes_{T_g} d\mu\Big)^{(\frac{1}{p})}_{\otimes_T},
$$

where $1 \le p < \infty$.

Proof. Definitions of $\tilde{f}$, $\mathrm{TC_g}$-integral, and T-psuedo-power with Theorem 2.13 provide that

$$\begin{aligned}
&\Big((TC_g)\int^{\bigoplus_{T_g}}(\tilde{f}\bigoplus_{T_g}\tilde{h})^{(p)}_{\otimes_T}\bigotimes_{T_g}d\mu\Big)^{(\frac{1}{p})}_{\otimes_T}\\
&=\Big(\Big((C_g)\int^{\oplus_g}(f_l\oplus_g h_l)^{(p)}_{\otimes}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes},\Big((C_g)\int^{\oplus_g}(f_m\oplus_g h_m)^{(p)}_{\otimes}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes},\Big((C_g)\int^{\oplus_g}(f_r\oplus_g h_r)^{(p)}_{\otimes}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes}\Big)\\
&\preceq_{\mathrm{T}}\Big(\Big((C_g)\int^{\oplus_g}f_{l\otimes}^{(p)}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes}\oplus_g\Big((C_g)\int^{\oplus_g}h_{l\otimes}^{(p)}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes},\Big((C_g)\int^{\oplus_g}f_{m\otimes}^{(p)}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes}\oplus_g\Big((C_g)\int^{\oplus_g}h_{m\otimes}^{(p)}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes},\\
&\qquad\Big((C_g)\int^{\oplus_g}f_{r\otimes}^{(p)}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes}\oplus_g\Big((C_g)\int^{\oplus_g}h_{r\otimes}^{(p)}\otimes_g d\mu\Big)^{(\frac{1}{p})}_{\otimes}\Big)\\
&=\Big((TC_g)\int^{\bigoplus_{T_g}}\tilde{f}^{(p)}_{\otimes_T}\bigotimes_{T_g}d\mu\Big)^{(\frac{1}{p})}_{\otimes_T}\bigoplus_{T_g}\Big((TC_g)\int^{\bigoplus_{T_g}}\tilde{h}^{(p)}_{\otimes_T}\bigotimes_{T_g}d\mu\Big)^{(\frac{1}{p})}_{\otimes_T}.
\end{aligned}$$

□

Theorem 5.6 establishes that the Minkowski-type inequality holds for a finite sequence of triangular fuzzy numbers, as stated below.

Example 5.7. *Consider the set $X=[0,\infty)$, and let $[a,b]\subseteq X$. Define the fuzzy measure $\mu=\sqrt{\lambda}$, where λ is the Lebesgue measure. Let $A=\bigcup_{k=1}^{n_1}A_k$ and $B=\bigcup_{k=1}^{n_2}B_k$, where $\{A_k\}_{k=1}^{n_1}\subset X$ and $\{B_k\}_{k=1}^{n_2}\subset X$ are collections of disjoint subsets satisfying $\mu(A_k)=r_k$ and $\mu(B_k)=s_k$, respectively. Let $g(x)=x^2$ as defined in Ex. 5.4, and let $\tilde{f}=(f_l,f_m,f_r)$ and $\tilde{h}=(h_l,h_m,h_r)$ be functions in $TGC(X;[0,\infty])$, given by*

$$\tilde{f}(x)=\bigoplus_{k=1}^{n_1}{}_{T_g}\tilde{a}_k\chi_{A_k}(x)\quad\text{and}\quad\tilde{h}(x)=\bigoplus_{k=1}^{n_2}{}_{T_g}\tilde{b}_k\chi_{B_k}(x),$$

where $\tilde{a}_k=(a_{k_l},a_{k_m},a_{k_r})$ and $\tilde{b}_k=(b_{k_l},b_{k_m},b_{k_r})\in TFN([0,\infty])$, with $n_1,n_2<\infty$. Then $\tilde{f}$ and $\tilde{h}$ can be expressed as follows without loss of generality:

$$\tilde{f}(x)=\bigoplus_{k=1}^{n}{}_{T_g}\tilde{a}_k\chi_{C_k}(x),\quad\tilde{h}(x)=\bigoplus_{k=1}^{n}{}_{T_g}\tilde{b}_k\chi_{C_k}(x),\quad\mu(C_k)=t_k.$$

Note that

$$(\tilde{f}\bigoplus_{T_g}\tilde{h})^{(p)}_{\otimes_T}=\bigoplus_{k=1}^{n}{}_{T_g}(\tilde{a}_k\bigoplus_{T_g}\tilde{b}_k)^{(p)}_{\otimes_T}\chi_{C_k}.\tag{9}$$

For $1\le p<\infty$, we have

$$\begin{aligned}
\Big((TC_g)\int^{\bigoplus_{T_g}}(\tilde{f}\bigoplus_{T_g}\tilde{h})^{(p)}_{\otimes_T}\bigotimes_{T_g}d\mu\Big)^{(\frac{1}{p})}_{\otimes_T}&=\Big((TC_g)\int^{\bigoplus_{T_g}}\bigoplus_{k=1}^{n}{}_{T_g}(\tilde{a}_k\bigoplus_{T_g}\tilde{b}_k)^{(p)}_{\otimes_T}\chi_{C_k}\bigotimes_{T_g}d\mu\Big)^{(\frac{1}{p})}_{\otimes_T}\\
&=\Big(\bigoplus_{k=1}^{n}{}_{T_g}(TC_g)\int_{C_k}^{\bigoplus_{T_g}}(\tilde{a}_k\bigoplus_{T_g}\tilde{b}_k)^{(p)}_{\otimes_T}\bigotimes_{T_g}d\mu\Big)^{(\frac{1}{p})}_{\otimes_T}\\
&=\Big(\bigoplus_{k=1}^{n}{}_{T_g}\big((\tilde{a}_k\bigoplus_{T_g}\tilde{b}_k)^{(p)}_{\otimes_T}\bigotimes_{T}t_k\big)\Big)^{(\frac{1}{p})}_{\otimes_T},
\end{aligned}$$

and

$$\left((TC_g)\int^{\oplus_{T_g}} \tilde{f}^{(p)}_{\otimes_T} \bigotimes_{T_g} d\mu\right)^{(\frac{1}{p})}_{\otimes_T} \bigoplus_{T_g} \left((TC_g)\int^{\oplus_{T_g}} \tilde{h}^{(q)}_{\otimes_T} \bigotimes_{T_g} d\mu\right)^{(\frac{1}{q})}_{\otimes_T}$$
$$= \left((TC_g)\int^{\oplus_{T_g}} \left(\bigoplus_{k=1}^{n}{}_{T_g} \tilde{a}_k\chi_{C_k}(x)\right) \bigotimes_{T_g} d\mu\right)^{(\frac{1}{p})}_{\otimes_T} \bigoplus_{T_g} \left((TC_g)\int^{\oplus_{T_g}} \left(\bigoplus_{k=1}^{n}{}_{T_g} \tilde{b}_k\chi_{C_k}(x)\right) \bigotimes_{T_g} d\mu\right)^{(\frac{1}{q})}_{\otimes_T}$$
$$= \left(\bigoplus_{k=1}^{n}{}_{T_g} (TC_g)\int_{C_k}^{\oplus_{T_g}} \tilde{a}_k \bigotimes_{T_g} d\mu\right)^{(\frac{1}{p})}_{\otimes_T} \bigoplus_{T_g} \left(\bigoplus_{k=1}^{n}{}_{T_g} (TC_g)\int_{C_k}^{\oplus_{T_g}} \tilde{b}_k \bigotimes_{T_g} d\mu\right)^{(\frac{1}{q})}_{\otimes_T}$$
$$= \left(\bigoplus_{k=1}^{n}{}_{T_g} ((\tilde{a}_k^{(p)})_{\otimes_T} \bigotimes_T t_k)\right)^{(\frac{1}{p})}_{\otimes_T} \bigoplus_{T_g} \left(\bigoplus_{k=1}^{n}{}_{T_g} ((\tilde{b}_k^{(q)})_{\otimes_T} \bigotimes_T t_k)\right)^{(\frac{1}{q})}_{\otimes_T}.$$

Thus, by Theorem 5.6, we have the following inequality:

$$\left(\bigoplus_{k=1}^{n}{}_{T_g} ((\tilde{a}_k + \tilde{b}_k)^{(p)}_{\otimes_T} \bigotimes_T t_k)\right)^{(\frac{1}{p})}_{\otimes_T} \preceq_T \left(\bigoplus_{k=1}^{n}{}_{T_g} ((\tilde{a}_k^{(p)})_{\otimes_T} \bigotimes_T t_k)\right)^{(\frac{1}{p})}_{\otimes_T} \bigoplus_{T_g} \left(\bigoplus_{k=1}^{n}{}_{T_g} ((\tilde{b}_k^{(q)})_{\otimes_T} \bigotimes_T t_k)\right)^{(\frac{1}{q})}_{\otimes_T}.$$

Remark 5.8. *Let $\otimes_{T_g}$ denote the pseudo-scalar multiplication defined as $c \otimes_{T_g} \tilde{f} = (cf_l, cf_m, cf_r)$ for $c \in [a,b]_+$ and $\tilde{f} \in TGC(X;[a,b]_+)$. By Theorem 5.6, it follows that $\big(TGC(X;[a,b]_+), \oplus_{T_g}, \otimes_{T_g}\big)$ forms a pseudo-normed vector space. That is, for $\tilde{f}, \tilde{g} \in TGC(X;[a,b]_+)$, both $c \otimes_{T_g} \tilde{f} \in TGC(X;[a,b]_+)$ and $\tilde{f} \oplus_{T_g} \tilde{g} \in TGC(X;[a,b]_+)$.*

We finally obtain the following Hölder-type inequality, which serves as a fundamental relationship for the $\mathrm{TC_g}$-integral in $TGC(X;[a,b]_+)$. This inequality is indispensable for the study of pseudo-normed vector spaces and their dual spaces within $TGC(X;[a,b]_+)$.

Theorem 5.9. *(Hölder type inequality) Let $(TFN([a,b]), \oplus_{T_g}, \otimes_{T_g})$ be a g-semiring and let $\tilde{f}, \tilde{h}$ be measurable functions in $TGC(X;[a,b]_+)$. If μ is a $\oplus_g$-submodular fuzzy measure, then*

$$(TC_g)\int^{\oplus_{T_g}} (\tilde{f} \bigotimes_{T_g} \tilde{h}) d\mu \preceq_T \left((TC_g)\int^{\oplus_{T_g}} \tilde{f}^{(p)}_{\otimes_T} \bigotimes_{T_g} d\mu\right)^{(\frac{1}{p})}_{\otimes_T} \bigotimes_{T_g} \left((TC_g)\int^{\oplus_{T_g}} \tilde{h}^{(q)}_{\otimes_T} \bigotimes_{T_g} d\mu\right)^{(\frac{1}{q})}_{\otimes_T},$$

where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Definitions of $\tilde{f}$, $\mathrm{TC_g}$-integral, and T-pseudo-power with Theorem 2.14 provide that

$$\begin{aligned}
(TC_g)\int^{\oplus_{T_g}} (\tilde{f} \bigotimes_{T_g} \tilde{h}) d\mu &= \left((C_g)\int^{\oplus_g} (f_l \otimes_g h_l) d\mu, (C_g)\int^{\oplus_g} (f_m \otimes_g h_m) d\mu, (C_g)\int^{\oplus_g} (f_r \otimes_g h_r) d\mu\right) \\
&\preceq_T \Bigg(\left((C_g)\int^{\oplus_g} f_{l\otimes}^{(p)} \otimes_g d\mu\right)^{(\frac{1}{p})}_{\otimes} \otimes_g \left((C_g)\int^{\oplus_g} h_{l\otimes}^{(q)} \otimes_g d\mu\right)^{(\frac{1}{q})}_{\otimes}, \\
&\quad \left((C_g)\int^{\oplus_g} f_{m\otimes}^{(p)} \otimes_g d\mu\right)^{(\frac{1}{p})}_{\otimes} \otimes_g \left((C_g)\int^{\oplus_g} h_{m\otimes}^{(q)} \otimes_g d\mu\right)^{(\frac{1}{q})}_{\otimes}, \\
&\quad \left((C_g)\int^{\oplus_g} f_{r\otimes}^{(p)} \otimes_g d\mu\right)^{(\frac{1}{p})}_{\otimes} \otimes_g \left((C_g)\int^{\oplus_g} h_{r\otimes}^{(q)} \otimes_g d\mu\right)^{(\frac{1}{q})}_{\otimes}\Bigg) \\
&= \left((TC_g)\int^{\oplus_{T_g}} \tilde{f}^{(p)}_{\otimes_T} \bigotimes_{T_g} d\mu\right)^{(\frac{1}{p})}_{\otimes_T} \bigotimes_{T_g} \left((TC_g)\int^{\oplus_{T_g}} \tilde{h}^{(q)}_{\otimes_T} \bigotimes_{T_g} d\mu\right)^{(\frac{1}{q})}_{\otimes_T}.
\end{aligned}$$

□

Theorem 5.9 provides the Hölder-type inequality for a finite sequence of triangular fuzzy numbers, as stated below.

Example 5.10. *Consider $X = [0,\infty]$, $[a,b] = [0,\infty]$, and the fuzzy measure $\mu = \sqrt{\lambda}$, where λ is the Lebesgue measure. Let g be the function given in Ex. 5.4, and $A = \bigcup_{k=1}^{n} A_k$, the union of disjoint subsets $\{A_k\}_{k=1}^{n} \subset X$, satisfying $\mu(A_k) = r_k < \infty$. Define $\tilde{f} = (f_l, f_m, f_r)$ and $\tilde{h} = (h_l, h_m, h_r)$ as functions in $TGC(X;[a,b]_+)$ by*

$$\tilde{f}(x) = \bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{\sqrt{r_k}} \bigotimes_T \tilde{a}_k\chi_{A_k}(x) \quad \textit{and} \quad \tilde{h}(x) = \bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{\sqrt{r_k}} \bigotimes_T \tilde{b}_k\chi_{A_k}(x),$$

where $\tilde{a}_k = (a_{k_l}, a_{k_m}, a_{k_r})$ *and* $\tilde{b}_k = (b_{k_l}, b_{k_m}, b_{k_r}) \in TFN([a,b])$*, with* $n < \infty$*. Then, using the same calculation as in Eq.* (4)*, with* $(\tilde{f} \otimes_{T_g} \tilde{h})(x) = \bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k} \otimes_T (\tilde{a}_k \otimes_{T_g} \tilde{b}_k)\chi_{A_k}(x)$*, we obtain*

$$(TC_g)\int^{\oplus_{T_g}} (\tilde{f} \otimes_{T_g} \tilde{h}) d\mu = \bigoplus_{k=1}^{n}{}_{T_g} (\tilde{a}_k \otimes_{T_g} \tilde{b}_k). \tag{10}$$

Note that if $x \in A_k$ $(k = 1, 2, \ldots, n)$ *and* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$\tilde{f}^{(p)}_{\otimes_T} = \Big(\frac{1}{\sqrt{r_k}} \otimes_T \tilde{a}_k\Big)^{(p)}_{\otimes_T} = \frac{1}{\sqrt{r_k}} \otimes_T (\tilde{a}_k)^{(p)}_{\otimes_T} \quad \textit{and} \quad \tilde{h}^{(q)}_{\otimes_T} = \Big(\frac{1}{\sqrt{r_k}} \otimes_T \tilde{b}_k\Big)^{(q)}_{\otimes_T} = \frac{1}{\sqrt{r_k}} \otimes_T (\tilde{b}_k)^{(q)}_{\otimes_T}. \tag{11}$$

From Eq. (11)*, we have*

$$\tilde{f}^{(p)}_{\otimes_T}(x) = \bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k^{\frac{p}{2}}} \otimes_T (\tilde{a}_k)^{(p)}_{\otimes_T} \chi_{A_k}(x) \quad \textit{and} \quad \tilde{h}^{(q)}_{\otimes_T}(x) = \bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k^{\frac{q}{2}}} \otimes_T (\tilde{b}_k)^{(q)}_{\otimes_T} \chi_{A_k}(x).$$

Using the same calculation as in Eq. (4)*, we have*

$$\begin{aligned}(TC_g)\int^{\oplus_{T_g}} \tilde{f}^{(p)}_{\otimes_T}(x) \otimes_{T_g} d\mu &= \Big(\bigoplus_{k=1}^{n}{}_g \int_{A_k}^{\oplus_g} \frac{(a_{k_l})^{(p)}_{\otimes}}{r_k^{\frac{p}{2}}} \otimes_g d\mu, \bigoplus_{k=1}^{n}{}_g \int_{A_k}^{\oplus_g} \frac{(a_{k_m})^{(p)}_{\otimes}}{r_k^{\frac{p}{2}}} \otimes_g d\mu, \bigoplus_{k=1}^{n}{}_g \int_{A_k}^{\oplus_g} \frac{(a_{k_r})^{(p)}_{\otimes}}{r_k^{\frac{p}{2}}} \otimes_g d\mu\Big) \\ &= \Big(\bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\frac{(a_{k_l})^{(2p)}_{\otimes}}{r_k^{p-2}}\Big), \bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\frac{(a_{k_m})^{(2p)}_{\otimes}}{r_k^{p-2}}\Big), \bigoplus_{k=1}^{n}{}_g g^{-1}\Big(\frac{(a_{k_r})^{(2p)}_{\otimes}}{r_k^{p-2}}\Big)\Big) \\ &= \Big(\bigoplus_{k=1}^{n}{}_g \frac{(a_{k_l})^{(p)}_{\otimes}}{r_k^{\frac{p-2}{2}}}, \bigoplus_{k=1}^{n}{}_g \frac{(a_{k_m})^{(p)}_{\otimes}}{r_k^{\frac{p-2}{2}}}, \bigoplus_{k=1}^{n}{}_g \frac{(a_{k_r})^{(p)}_{\otimes}}{r_k^{\frac{p-2}{2}}}\Big) = \bigoplus_{k=1}^{n}{}_{T_g} \frac{(\tilde{a}_k)^{(p)}_{\otimes_T}}{r_k^{\frac{p-2}{2}}},\end{aligned}$$

and hence,

$$\Big((TC_g)\int^{\oplus_{T_g}} \tilde{f}^{(p)}_{\otimes_T} \otimes_{T_g} d\mu\Big)^{(\frac{1}{p})}_{\otimes_T} = \Big(\bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k^{\frac{p-2}{2}}} \otimes_T (\tilde{a}_k)^{(p)}_{\otimes_T}\Big)^{(\frac{1}{p})}_{\otimes_T}. \tag{12}$$

Similarly, using the same calculation as in Eq. (12)*, we obtain*

$$\Big((TC_g)\int^{\oplus_{T_g}} \tilde{h}^{(q)}_{\otimes_T} \otimes_{T_g} d\mu\Big)^{(\frac{1}{q})}_{\otimes_T} = \Big(\bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k^{\frac{q-2}{2}}} \otimes_T (\tilde{b}_k)^{(q)}_{\otimes_T}\Big)^{(\frac{1}{q})}_{\otimes_T}. \tag{13}$$

By Eqs. (10)*,* (12)*,* (13)*, and Theorem 5.9, we have the following Hölder type inequality related to the* TC_g*-integral for a space of sequences in* $TFN([a,b])$*:*

$$\bigoplus_{k=1}^{n}{}_{T_g} (\tilde{a}_k \otimes_{T_g} \tilde{b}_k) \preceq_T \Big(\bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k^{\frac{p-2}{2}}} \otimes_T (\tilde{a}_k)^{(p)}_{\otimes_T}\Big)^{(\frac{1}{p})}_{\otimes_T} \otimes_{T_g} \Big(\bigoplus_{k=1}^{n}{}_{T_g} \frac{1}{r_k^{\frac{q-2}{2}}} \otimes_T (\tilde{b}_k)^{(q)}_{\otimes_T}\Big)^{(\frac{1}{q})}_{\otimes_T}. \tag{14}$$

In particular, if we take $p = q = 2$ *in Eq.* (14)*, then we have the following Hölder type inequality*

$$\bigoplus_{k=1}^{n}{}_{T_g} (\tilde{a}_k \otimes_{T_g} \tilde{b}_k) \preceq_T \Big(\bigoplus_{k=1}^{n}{}_{T_g} (\tilde{a}_k)^{(2)}_{\otimes_T}\Big)^{(\frac{1}{2})}_{\otimes_T} \otimes_{T_g} \Big(\bigoplus_{k=1}^{n}{}_{T_g} (\tilde{b}_k)^{(2)}_{\otimes_T}\Big)^{(\frac{1}{2})}_{\otimes_T}.$$

Remark 5.11. *Theorems 5.6 and 5.9 allow us to deal with the properties of the pseudo-functional analysis, for examples, a pseudo-algebra, a pseudo-*L^p*-space, a pseudo-dual space, or etc. under* $\big(TGC(X;[a,b]_+), \oplus_{T_g}, \otimes_{T_g}, \otimes_T\big)$*.*

6 Conclusion

This study introduces the TGC-integral applying the concept of the generalized Choquet integrals and triangular fuzzy numbers in Definition 3.3. The TGC-integral is a powerful tool to help model non-deterministic problems using the properties of the triangular fuzz number and the Choquet integrals in Theorem 3.4 to problems in which uncertainty and inaccuracy are embedded.

By thoroughly analyzing the convergence theorems of TGC-integrals in Theorems 4.2, 4.3, and 4.4, we have laid the fundamental concept of analysis for the TGC-integral of TFN-valued functions. These inequalities are not merely mathematical generalization, but provide key insights into the generalized inequality and pseudo-operations between TGC-integrals, for examples, Jensen type inequality is established between the $(\bigoplus_r, \bigotimes_r)$-convex functions of the TC_g-integral and the TC_g-Choquet integral of the pseudo-composition of a $(\bigoplus_r, \bigotimes_r)$-convex function and a TGC-integrable function, Minkowski type inequality allows $TGC(X;[a,b]_+)$ to be a pseudo-normed vector space, and Hölder type inequality is indispensable for the study of the pseudo-normed vector space and pseudo-dual space in $TGC(X;[a,b]_+)$.

A major highlight of our work is the detailed examination of three crucial inequalities: the Jensen type inequality (Theorem 5.3), the Minkowski type inequality (Theorem 5.6), and the Hölder type inequality (Theorem 5.9), specifically tailored for TGC-integrals. These inequalities provide a robust mathematical foundation for analyzing TFN-valued functions within the framework of generalized Choquet integrals. Ex. 5.2 illustrates Theorem 5.3 through an application to TFN-valued Choquet expected utility, using a TFN-valued trade volume function in the global semiconductor market, as elaborated in Remark 5.5. Ex. 5.4 demonstrates the triangular inequality associated with finite series in the space of TFNs, referred to as pseudo-L^p spaces, as an example of Theorem 5.6. Similarly, Ex. 5.7 applies Theorem 5.9 to investigate properties of pseudo-functional analysis, as highlighted in Remark 5.11.

Beyond these theoretical contributions, the framework of TGC-integrals holds potential for concrete applications across various domains. For instance, in hydrology, TGC-integrals could model the uncertainty in water resource management using TFN-valued inputs. In copula modeling, they could enhance the analysis of dependence structures in multivariate data. Furthermore, their use in classification problems could support decision-making processes in machine learning, particularly under uncertainty. Other potential applications include optimization problems in engineering and risk analysis in finance.

Looking forward, several intriguing questions remain open for further exploration. How can TGC-integrals be effectively incorporated into real-world models for hydrological and environmental systems? What novel insights can they offer in dependence modeling when combined with copulas? Additionally, can the properties of pseudo-functional analysis extend to more generalized pseudo-algebraic frameworks, including advanced pseudo-L^p and pseudo-dual spaces? Investigating these and other applications will help to deepen the theoretical and practical impact of TGC-integrals, paving the way for future research.

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