

Structural analysis of precomplete classes and closure diagrams in multi-valued logic

A. A. Esin ¹

¹*Incarnet Math Modelling & Complex Networks LTD, Yerevan, Armenia*

¹*Institute for Information Transmission Problems of RAS, Moscow, Russia*

anton.esin@imm.am

Abstract

This paper develops a mathematical framework to ensure the functionality and efficiency of digital electronics built on multi-valued logic (MVL). We investigate the properties of precomplete classes and closure for families of MVL functions, focusing on the existence and structure of \mathcal{R} -closed sets in P_k for $k \geq 4$. Our work provides a systematic analysis of logical bases and their representations through structured diagrams, with a particular focus on Boolean and MVL functions. We rigorously classify \mathcal{R}_2 -complete families in P_3 -spaces and propose extensions for the minimal linear superposition operator. Furthermore, we highlight how MVL significantly enhances information density in memory storage systems, providing pivotal benefits for cutting-edge computational systems. These results bridge theoretical advancements and practical implementations in computer science and logic.

Keywords: Multi-valued logic (MVL), \mathcal{R} -closed sets, precomplete classes, logical bases, structured diagrams, minimal linear operators, computational logic applications.

1 Introduction

Currently, binary computing technology is reaching its peak limits in terms of both computational speed and energy efficiency. There is a rising need for computing power, especially in areas like artificial intelligence, machine learning, and neural networks [1, 36, 37]. This demand extends to managing autonomous robotic systems, verifying Internet of Things infrastructures, and deploying intelligent systems with comprehensive logging protocols across diverse data sources [2]. Additionally, application areas include cyber-physical systems [27], blockchain technologies, quantum cryptography [4], optimising data transmission [17], and innovative data aggregation methods [5, 7, 32], and efficient traffic routing [15, 16].

To overcome the efficiency constraints of binary logic, one promising approach is the use of multi-valued logic (MVL) cells as fundamental computational units. MVL memory blocks are crucial for significantly reducing energy losses and enabling denser information packing [19].

However, binary-to-MVL transition introduces a paradigm shift in computation models. Beyond merely increasing information density, MVL offers the potential to simplify logic circuit design by reducing the number of gates needed for complex operations. This optimisation can lead to more compact, faster, and energy-efficient circuits suitable for integration in next-generation computing architectures. These advantages further align MVL with the demands of emerging applications in neural networks and AI-based systems.

However, the practical implementation and widespread adoption of MVL devices require addressing several significant interdisciplinary challenges. These include:

1. Exploring the potential of contemporary silicon-based materials for constructing MVL devices remains a critical yet insufficiently explored area. Research suggests employing ferroelectrics for designing programmable logic matrices (PLMs), evaluating various PLM implementations and their logical design attributes [19].

2. Recent breakthroughs in materials science have facilitated the synthesis of atomically thin 2D materials with expansive surface areas and high performance. In addition to van der Waals heterojunctions such as molybdenum telluride (MoTe₂)/black phosphorus (BP), other promising candidates include transition metal dichalcogenides (TMDs) and their alloys, which exhibit tunable electronic properties. These materials could enable the fabrication of highly efficient MVL gates while maintaining scalability and robustness under extreme operating conditions [12, 14].

3. Antiambipolar transistors (AATs), particularly those based on organic semiconductors, offer potential in MVL circuits due to their straightforward fabrication and high carrier mobility. Organic semiconductor-based AATs demonstrate significant on/off ratios and operational frequencies suitable for intermediate logic states [21, 26].

4. Interpreting variables in MVL remains a significant challenge. Beyond traditional fuzzy logic mappings, emerging approaches explore hybrid systems combining quantum-inspired states and Galois field arithmetic for optimal representation. These techniques not only streamline operations but also allow integration with quantum computing protocols, enhancing MVL's compatibility with future technologies [35].

These challenges highlight the multifaceted efforts needed to advance MVL technology for broader applications in information technology and interdisciplinary research domains.

1.1 The challenge of ensuring MVL-algorithms correctness

It is essential to highlight the theoretical challenge involved in verifying the correctness of algorithms. Theoretical proof of correctness is crucial for preventing errors during operational phases and for minimizing both development and operational costs.

Verification of algorithm adequacy, particularly those based on Galois fields, is typically performed through experimental means. In the study by Suleimenov and Ibragim [35], this verification was conducted using radio-electronic circuits. While this approach offers empirical validation for MVL, it lacks the rigor necessary to establish the reliability of such circuits.

This study bridges these gaps by proposing a unified framework that integrates algebraic methods and numerical simulations for rigorous theoretical justification. By combining analytical techniques with computational tools, the approach ensures greater reliability and accuracy in validating MVL-algorithms. Such integration allows for handling larger variable spaces and more complex logical operations, especially in high-performance computing systems.

Figure 1 illustrates how algebraic functions operate with arguments taking values in Galois fields when the number of variables in multi-valued logic operations equals a prime number. This visual representation highlights how algebraic functions behave under modular arithmetic in $GF(p)$. In the context of multi-valued logic, these functions provide insights into possible variable mappings and operational properties that extend beyond binary computation, offering new opportunities for optimising logic gate designs.

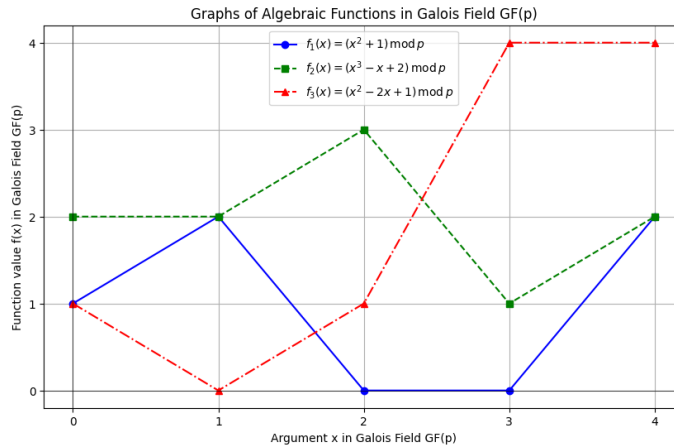


Figure 1: Graph of Algebraic Functions in Galois Fields.

The paper [2] also delves into the significant challenge of modeling and verifying multi-valued logic (MVL) models, which remains a critical issue in computational research. The authors employ 3v-CTL (three-valued conditional commitment logic) to verify their MVL model under conditions of uncertainty, utilizing reduction techniques. Their experimentation focuses on the Smart Contract Mortgage system, leveraging data from multiple sources as outlined in

previous studies. The system’s functionality hinges on protocols involving unconditional commitments amid uncertain conditions.

Such conditional commitment protocols have broader implications for other applications, including autonomous decision-making systems, where uncertainty in state transitions poses significant challenges. Incorporating MVL principles allows for more flexible modeling of these systems, addressing ambiguities through graded logical states. These graded states, unlike binary logic, provide richer semantic interpretations, enabling systems to adapt to incomplete or conflicting information in real-time.

Figure 2 depicts a diagram illustrating the application of three-valued conditional logic (3v-CTL) for modeling and verifying logical systems in scenarios involving uncertainty, exemplified through the “Smart Contract Mortgage” system. The diagram visualizes different states and transitions involved in the system.

The diagram showcases various states and transitions:

- S_0 : Application Submitted
- S_1 : Application Approved
- S_2 : Application Rejected
- S_3 : Contract Signed

Transitions between these states are governed by specific conditions:

- $CH=True$: Transition from S_0 to S_1 (successful credit history check)
- $CH=False$: Transition from S_0 to S_2 (failed credit history check)
- $I=True$: Transition from S_1 to S_3 (sufficient income)
- $I=False$ or $Unknown$: Transition from S_1 to S_2 (insufficient or unknown income)

This diagram visually illustrates how multi-valued logic effectively models and analyzes systems where states and transitions are subject to uncertainty or ambiguity.

The use of three-valued conditional logic (3v-CTL) enables the system to handle ambiguous or incomplete information, which is particularly critical in scenarios such as income verification or credit history checks. Unlike binary logic, MVL allows for intermediate states that enhance the system’s flexibility and reliability. This diagram highlights the adaptability of MVL in real-world applications, such as automated contract verification in financial systems. By visualising the decision paths, it becomes easier to identify potential bottlenecks or failures in the logic design.

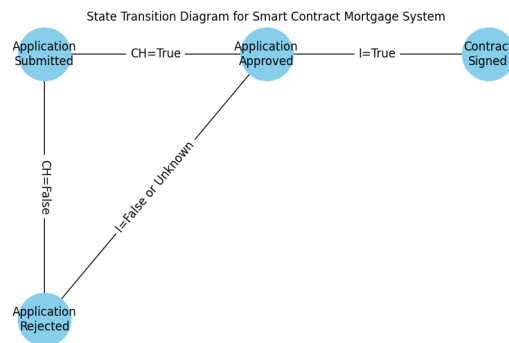


Figure 2: Usage of three-valued conditional logic (3v-CTL) for modeling and verifying logical models.

Experimental validation, while useful, cannot guarantee robustness under diverse operating conditions. This highlights the need for rigorous mathematical frameworks that can universally validate algorithmic correctness across MVL implementations.

1.2 Examining observability in multivalued logical networks (MVLNs)

A critical aspect in the analysis of MVL-networks (MVLNs), especially within AI frameworks such as MVL neural networks [1], is observability. This concept involves subjecting functions to specific perturbations and constructing graph-based structures to assess system behavior. A novel criterion for observability in MVLNs is introduced based on the transition graph depicting indistinguishable state pairs. Additionally, a defined array of candidate pairs encompasses all potentially indistinguishable state pairs. This array of candidate pairs, which satisfies the graph-based condition, provides a robust framework for evaluating observability while taking into account function perturbations, as proposed in [38].

Observability in MVLNs goes beyond simple state transitions. It allows researchers to explore the extent to which system dynamics can be inferred from observed data, making it a cornerstone for developing robust MVL-based AI systems. In particular, observability ensures that every system state corresponds to a unique signature in terms of observable functions, thus avoiding ambiguities that could compromise the network’s reliability.

In MVLNs, the observability graph and the transition configuration among indistinguishable state pairs are vital for understanding and researching system dynamics.

Consider the following example involving an array of candidate pairs of indistinguishable states: suppose an MVLN consists of three states labelled S_1 , S_2 , and S_3 . Transitions within S_1 , S_2 , and S_3 are governed by logical functions yielding values from the tuple $\{0, 1, 2\}$. The observability graph of an MVLN defines which states can be distinguished based on the available observability. For example, if certain state pairs cannot be distinguished by the value of some observable function, these pairs are considered indistinguishable and are grouped into a single vertex in the observability graph. A table above provides a detailed illustration of such state pairs and their transitions.

This merging of indistinguishable states highlights the practical challenges of designing MVLNs with high observability. By understanding which state pairs are indistinguishable, engineers can fine-tune logical functions f or introduce auxiliary observation mechanisms to enhance system clarity.

Take into consideration the truth table for a function f :

S_1	S_2	S_3	$f(S_1, S_2, S_3)$
0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	0
1	0	0	1
1	0	1	2
1	1	0	0
1	1	1	1

Per function f , the states $S_2 = 0$ and $S_3 = 1$ are indistinguishable (i.e., $f(0, 0, 1) = 1$ and $f(1, 0, 1) = 2$). As a result, pairs $(S_2 = 0, S_3 = 1)$ and $(S_1 = 1, S_2 = 0, S_3 = 1)$ merge into a single vertex in the observability graph.

The implications of this analysis are profound. By systematically studying observability graphs, it becomes possible to identify bottlenecks in state transitions and optimize MVLNs for higher performance. Moreover, this approach lays the groundwork for implementing self-correcting mechanisms in MVLNs, where indistinguishable states can be flagged and corrected dynamically.

The requirements for observability in MVLNs assess how easily system states can be differentiated based on observable functions. Variations of f can alter the distinguishability of previously indistinguishable pairs.

1.3 Approaching the MVL completeness issue

Assurance engineering of MVL circuits, devices, and algorithms through rigorous mathematical validation is necessary for their manufacturing and assembly. A critical challenge within this domain is the conceptual difficulty of the completeness issue, which requires strict proof to substantiate the effectiveness of MVL algorithms. A comprehensive and cohesive theoretical background ensures that integrated circuits are stable to risks, mitigate errors, and enhance their functionality and dependability.

The theory of completeness and function class closure is a cornerstone in MVL frameworks [33, 39]. Emil Post extensively studied sets of 2-VL (two-valued logic) functions closed under linear combination transformations [30].

However, transitioning from binary logic to MVL introduces significant complexity. In contrast to 2-VL, MVL encompasses a broader family of function classes, as highlighted in [25, 40]. The enumeration of finitely generated classes

and the characterization of their properties become exponentially more challenging, demanding advanced methods to address these intricacies.

Addressing the completeness issue involves exploring specific operators designed for MVL contexts. A powerful approach is to construct relationships between sublattices of closed classes and the lattice comprising all functions, identifying subsets that preserve closure under linear combination transformations.

The authors of [28, 29] proposed the concept of "0 or 1" linear combination transformation for MVL functions via their binary representations. Additionally, the work in [34] examines implicit completeness requirements in 3-VL via precomplete class applications, providing further insights into the robustness and theoretical foundations of MVL systems. In [22], the authors propose the concepts of fuzzifying closure systems and Birkhoff fuzzifying closure operators, establish their correspondence in fuzzifying mathematics.

Recent progress includes novel methods for identifying the finite generation of closed classes. For example, [9] explores the finite generation of closed classes in two-valued logic, while [17] expands this analysis to superlattices that incorporate precomplete classes of unary functions. These studies lay the groundwork for tackling the broader challenges of completeness in MVL frameworks.

Cheng, in his work [6], delves into the algebraic forms of k -valued logic functions using the semi-tensor product of matrices. These studies introduce conjunctive and disjunctive normal forms and discuss MVL completeness cases via appropriate assemblies of connectives. Malkov's work [24] categorizes subalgebras of finite-valued functions, focusing on their mutual exclusivity and structural properties. Building on these perspectives, another approach to the classification of \mathcal{R} -closed families of multi-valued functions, excluding the use of special filters in residuated lattices, is presented in [13].

This study extends these foundational works by exploring advanced closure operators for MVL functions, particularly focusing on the \mathcal{R}_2 closure operator. This operator distinguishes functions as equivalent if they differ only in fictional variables, or non-equivalent if differences exist in any variable. We aim to generalize these approaches to higher-order MVL systems, establishing structural benchmarks for closed classes in $k \geq 4$.

Additionally, we propose an advanced composition operator to ensure the enumerability of closed classes within Boolean algebra. A fundamental aspect involves defining an operator $\mathcal{I}(\cdot) : P_k \rightarrow P_k$, where for each $f \in P_k$, $\mathcal{I}(f) = P_k$.

1.4 Paper structure

In *Section 2*, we begin by introducing essential notations, defining k -valued logic functions, and outlining preliminary results. We also review seminal contributions by S.V. Yablonski [39] and E.L. Post [30]. Additionally, we examine specific instances of the enhanced closure operator derived in previous research works [9, 17], which serve as foundational elements for our research.

In *Section 3*, we analyze the representation of \mathcal{R}_2 -precomplete classes within P_3 and explore the structure of closed classes in P_k for $k \geq 4$. Our focus is on constructing a non-trivial operator that satisfies the extreme property, emphasizing minimal enhancement of the linear combination operator. Explicit constructions and examples are provided to illustrate our findings, offering practical insights into the theoretical framework.

Finally, the conclusion addresses outstanding challenges and considers potential applications of our theoretical framework, particularly in addressing traffic transmission and optimization issues in MVL-based systems.

2 Preliminary results and definitions on MVL functions

2.1 Defining k -Valued Logic Functions

We begin by examining and defining k -valued logic functions. Let P_k represent the set of all k -VL functions, where $E_k = \{0, 1, \dots, k-1\}$ and $k \geq 2$.

We say that $f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ *essentially* depends on the variable x_i if $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in E_k$ are such that the function $h(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$ is non-constant for all x . In this case, x_i is called an *essential* variable for the function f .

Two functions f_1 and f_2 are equivalent if f_1 can be transformed into f_2 by the addition or removal of essential variables. The notation $|X_f|$ denotes the number of essential parameters in the function f .

Figure 3 presents the plot of the function f in P_3 , illustrating how the inputs \vec{x} are mapped into the values 0, 1, or 2. This graphical representation highlights the bounds of possible outputs for distinct input combinations.

Let \mathcal{A} be a closure operator. A subset $M \subseteq P_k$ is called \mathcal{A} -complete if $\mathcal{A}(M) = P_k$, and \mathcal{A} -precomplete if $\mathcal{A}(M) = M$, and for any $f \in P_k \setminus M$, $\mathcal{A}(M \cup \{f\}) = P_k$.

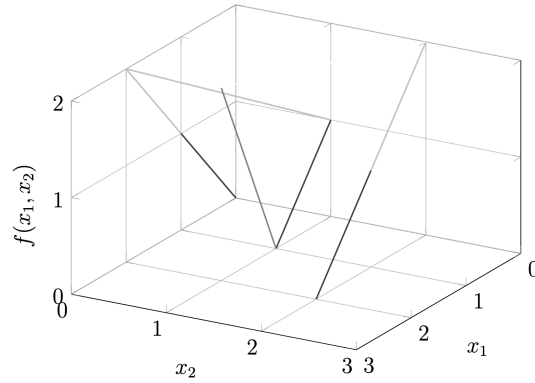


Figure 3: Graph of function $f \in P_3$.

The problem of completeness involves identifying all \mathcal{A} -complete sets for a given closure operator \mathcal{A} [23]. This study specifically focuses on the closure operators \mathcal{R}_1 and \mathcal{R}_2 , which will be defined below.

2.2 Preliminary results

Post’s theorem describes five precomplete classes of binary functions [30]. In 1958, S.V. Yablonski extended this work by defining 18 precomplete classes for 3-VL [39].

Let $\mathcal{R}(M)$ denote the closure of the set M under substitution and equivalence, where equivalence between a function g and an equivalent function $f \sim g$ is defined as:

$$f \sim g \Leftrightarrow \forall \vec{x} [(f(\vec{x}) = g(\vec{x})) \vee (f(\vec{x}), g(\vec{x}) \in \{0, 1\})],$$

with \vec{x} representing a vector and $f(\vec{x}), g(\vec{x})$ denoting arbitrary MVL-functions.

We analyze two closure operators:

1. \mathcal{R}_1 : Functions are equivalent if they differ only by fictitious variables.
2. \mathcal{R}_2 : Functions are non-equivalent if they differ by fictitious variables.

According to [17], \mathcal{R}_1 defines three precomplete classes, two of which form a countable lattice that encompasses all closed subclasses.

Figure 4 illustrates the structural evolution of closed classes in many-valued logic as k increases. As k grows, the number of precomplete classes increases exponentially.

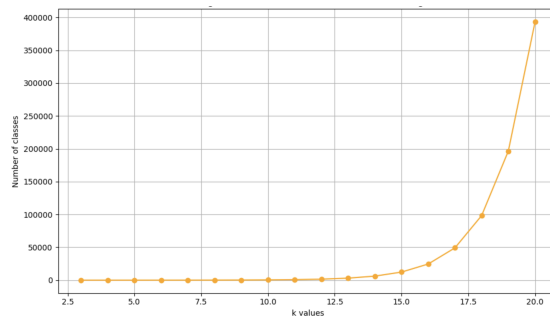


Figure 4: Growth of the number of closed classes with increasing k .

This study explores advanced closure operators for 3-valued logic using \mathcal{R}_2 , confirming the presence of five precomplete classes for $k = 3$. For k -valued logics with $k \geq 4$, a conservative estimate for the number of closed classes is provided.

In his 1942 paper, E. Post identified five precomplete classes in P_2 relative to the substitution operator [30]:

1. $T_0 = \{f \mid f(0, \dots, 0) = 0\}$ — functions preserving zero.

2. $T_1 = \{f \mid f(1, \dots, 1) = 1\}$ — functions preserving one.
3. $M = \{f \mid \vec{\alpha} \leq \vec{\beta} \Rightarrow f(\vec{\alpha}) \leq f(\vec{\beta})\}$ — monotone functions.
4. $S = \{f \mid \sigma(f(x_1, \dots, x_n)) = f(\sigma(x_1), \dots, \sigma(x_n)), \sigma(x) = x + 1 \pmod{2}\}$ — self-dual functions.
5. $L = \{f \mid f(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i + a_{n+1} \pmod{2}\}$ — linear functions.

In 1966, S.V. Yablonski identified 18 precomplete classes in P_3 concerning the substitution operator [39]:

$$1-3 \ T_i = \{f \mid f(i, \dots, i) = i\}, \ i \in \{0, 1, 2\}$$

$$4-6 \ M_i^3 = \{f \mid \vec{\alpha} \leq_i \vec{\beta} \Rightarrow f(\vec{\alpha}) \leq_i f(\vec{\beta})\}, \ i \in \{1, 2, 3\}$$

1. $\leq_0: 0 \leq_1 1 \leq_1 2$
2. $\leq_1: 1 \leq_2 2 \leq_2 0$
3. $\leq_2: 2 \leq_3 0 \leq_3 1$

$$7-9 \ T_A = \{f \mid \vec{\alpha} \in A^n \Rightarrow f(\vec{\alpha}) \in A\}$$

1. $A = \{0, 1\}$
2. $A = \{1, 2\}$
3. $A = \{0, 2\}$

10-12 Let A, B be such that $A \sqcup B = P_3$. Let $a, b \in E_3$, then $aR_A b$ if either $a, b \in A$ or $a, b \in B$. For $\vec{\alpha} = (a_1, \dots, a_n)$, $\vec{\beta} = (b_1, \dots, b_n)$, $\vec{\alpha} R_A \vec{\beta}$ if $a_i R_A b_i$ for $1 \leq i \leq n$.

$$T_{A,B} = \{f \mid \vec{\alpha} R_A \vec{\beta} \Rightarrow f(\vec{\alpha}) R_A f(\vec{\beta})\}$$

1. $A = \{0, 1\}, B = 2$
2. $A = \{1, 2\}, B = 0$
3. $A = \{0, 2\}, B = 1$

13-15 Let A, B be such that $A \sqcup B = P_3$. $H_B = \{f \mid \vec{\alpha} \in A^n, \vec{\beta}(b_i \neq a_i, i = 1, n) \Rightarrow f(\vec{\beta}) \neq c(\vec{\alpha})\}$, where $c : E_3 \in A\}$

1. $A = \{0, 1\}, B = 2$
2. $A = \{1, 2\}, B = 0$
3. $A = \{0, 2\}, B = 1$

16 Sl — the class of functions with at most one core variable, skipping at least one value, i.e., $f(E(f)) \neq E_3$.

$$17 \ S_{x+1} = \{f \mid \sigma(f(x_1, \dots, x_n)) = f(\sigma(x_1), \dots, \sigma(x_n)), \sigma(x) = x + 1 \pmod{3}\}$$

$$18 \ L = \{f \mid f(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i + a_{n+1}\}$$

Suppose M is a set of functions from P_3 . Define $\mathcal{R}_1(M)$ as the closure of M under substitution and the equivalence transition from function g to an equivalent function $f \sim g$, where:

$$f \sim g \Leftrightarrow \forall \vec{x} [(f(\vec{x}) = g(\vec{x})) \vee (f(\vec{x}), g(\vec{x}) \in \{0, 1\})].$$

The following theorem establishes key results on the existence of three precomplete classes within P_3 for the \mathcal{R}_1 operator:

Theorem 2.1 (Completeness). *There exist three \mathcal{R}_1 -precomplete classes in P_3 : T_2 , T_{01} , and T_\sim .*

Here, T_{01} consists of functions preserving the tuple $\{0, 1\}$, T_2 contains functions preserving the value 2, and T_\sim (or $T_{\{01\}, \{2\}}$ ($U(R)$)) consists of functions preserving the equivalence relation \sim .

The lattice configuration of \mathcal{R}_1 -closed subclasses within the class T_\sim was comprehensively investigated in [17], revealing:

Theorem 2.2. For each instance of \mathcal{R}_1 , it has been demonstrated that $\|P_3\|_{\mathcal{R}_1} \geq \aleph_0$ and $\|T_{\sim}\|_{\mathcal{R}_1} = \aleph_0$,

where $\|P_3\|$ denotes the count of closed classes of functions in 3-VL with respect to \mathcal{R}_1 , and $\|T_{\sim}\|$ indicates the cardinality of the closed class set preserving the equivalence relation. \aleph_0 represents the cardinality of the natural numbers \mathbb{N} .

Moreover, the following theorem illustrates completeness within T_{01} :

Theorem 2.3 (Completeness in T_{01}). There exist three \mathcal{R}_1 -precomplete classes in T_{01} : $T_2 \cap T_{01}$, $T_{\sim} \cap T_{01}$, and I^2 .

Here, I^m represents the collection of functions f such that for any $\vec{a}_i = (a_1^i, \dots, a_n^i), 1 \leq i \leq m$, if $(f(\vec{a}_1) \sim \dots \sim f(\vec{a}_m) \sim 2)$, then there exists j such that $a_j^1 \sim \dots \sim a_j^m \sim 2$.

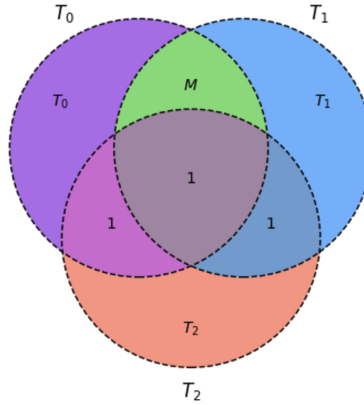


Figure 5: Intersection of Classes T_0 , T_1 , T_2 , and M in Many-Valued Logic.

In Figure 5, a Venn diagram illustrates the relationships between four distinct classes: T_0 , T_1 , T_2 , and M , within the context of MVL. These classes are associated with different properties of functions defined over a three-valued logic system (P_3).

The intersections between the classes show which functions share specific properties. For instance, the intersection between T_0 and T_1 represents functions that preserve both 0 and 1 simultaneously, while the overlap between T_2 and M highlights the functions that are both monotone and preserve 2. Each non-overlapping area in the diagram indicates a set of functions that belong exclusively to one of the classes. For example, the section of the circle labeled T_0 but not intersecting with the other circles represents the functions that preserve 0 but do not preserve 1 or 2. The areas of intersection between multiple circles illustrate shared classes. For instance, the area where T_0 , T_1 , and T_2 overlap represents functions that preserve 0, 1, and 2 in some way, while the intersection of all four circles (if it exists) would denote functions that exhibit properties of all four classes simultaneously.

3 Main results for the closure operator \mathcal{R}_2

3.1 \mathcal{R}_2 -precompleteness in P_3

Consider a set M containing various functions from P_3 . Denote $\mathcal{R}_2(M)$ as the closure of M involving substitution and the equivalence relation between functions f and g : $f \sim g$, where

$$f \sim g \Leftrightarrow \forall \vec{x} [(f(\vec{x}) = g(\vec{x})) \vee (f(\vec{x}), g(\vec{x}) \in \{0, 1, 2\})].$$

This interpretation guarantees that $\mathcal{R}_2(M)$ incorporates all 3-Valued Logic (3-VL) functions generated by substitutions, while maintaining equivalence with functions in M according to the specified criteria.

This study demonstrates the existence of several \mathcal{R}_2 -precomplete classes within P_3 . These classes play a crucial role in understanding the characteristics of \mathcal{R}_2 and its closure properties. For example, T_{01} consists of functions that preserve $\{0, 1\}$, while T_2 includes functions that only preserve the value 2. Furthermore, T_{\sim} (also denoted as $T_{\{01\},\{2\}}$ or as $U(R)$) encompasses functions that retain an equivalence relation \sim among 3-VL functions that accept only the values 0, 1, or 2. The complex structure of \mathcal{R}_2 -closed subclasses within fully structured sets offers a deeper understanding of the closure properties under \mathcal{R}_2 in P_3 .

Figure 6 shows the $\mathcal{R}_2(M)$ -closure diagram, illustrating functions from M along with their \mathcal{R}_2 -closure operation. The arrows represent the transitions between functions enabled by the substitution operator and the equivalence relation denoted by \sim .

Suppose $\alpha^n \in E_3^n$, where $\alpha^n = (a_1, \dots, a_n)$. Let us define $O_2(\alpha^n) = |\{a_i \mid a_i = 2, i \in \{1, n\}\}|$ as the **two-order** of the tuple α^n .

Definition 3.1. *The function $f \in P_3$ with $|X_f| = n$ belongs to ∇_2 if:*

- 1) every $\alpha^n \in E_3^n$ with $O_2(\alpha^n) > 0$, and
- 2) $f(\alpha^n) = 2$. Additionally, ∇_2 includes all constant functions.

This definition characterizes ∇_2 as a class of functions in P_3 distinguished by their behavior on tuples α^n with a non-zero two-order. It includes non-constant functions that return the value 2 for tuples containing at least one 2, as well as constant functions.

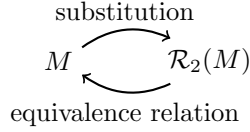


Figure 6: Closure diagram $\mathcal{R}_2(M)$ under substitution and equivalence.

Lemma 3.2. *A class ∇_2 is closed under \mathcal{R}_2 .*

Proof. To prove the closure of ∇_2 under \mathcal{R}_2 , let us refer to the definition: a set is closed under \mathcal{R}_2 if it remains invariant when transitioning to an equivalent function under \sim .

Initially, ∇_2 includes all constant functions, which inherently satisfy the closure condition under \mathcal{R}_2 , as constant functions do not change upon substitution.

Moreover, for non-constant functions in ∇_2 , consider $f \in \nabla_2$. By definition, for each tuple $\alpha^n \in E_3^n$ where $O_2(\alpha^n) > 0$, $f(\alpha^n) = 2$. Upon substituting variables with constants or equivalent functions under \sim , f remains in ∇_2 due to its specific behavior on tuples containing at least one 2.

Now, let us examine compositions within ∇_2 : suppose $h = f(g_1, \dots, g_n)$ where $f \in \nabla_2$ and each $g_i \in \nabla_2$. If h fundamentally depends on all of its variables, any input tuple γ^k containing a 2 will pass through the g_i functions to f , and the property of f ensures that $h(\gamma^k) = 2$, thus validating $h \in \nabla_2$.

Therefore, ∇_2 is closed under substitution and equivalence with respect to \mathcal{R}_2 . \square

Lemma 3.3. *∇_2 is complete under \mathcal{R}_2 .*

Proof. Lemma 3.2 establishes that ∇_2 is \mathcal{R}_2 -closed. Let's examine $\varphi(x, y)$, with $\varphi^9 = (0, 0, 2, 0, 0, 2, 2, 2, 2)^T$. Remarkably, we note that for all g such that $g \sim \varphi$, $|X_g| = 2$ and $g \in \nabla_2$, and φ has two degrees of freedom on y , suggesting the presence of $\varphi_i \sim \varphi$, which ensures that $\varphi_i \notin M_i^3$, and that $f^{M_1^3}, f^{M_2^3}, f^{M_3^3} \in \nabla_2$ exist.

Take into consideration a set denoted as $l_1 \in \nabla_2$, where the vector $l_1^1 = (2, 0, 0)^T$ is its first element, belonging neither to $T_{01}, T_{12}, T_0, T_1, H_1, T_\sim, T_{\{1,2\},0}, L$, nor S_{x+1} . Furthermore, the element 1 is not in either T_{02} or T_2 . Let F be an element from ∇_2 , such that $F^2 = (1, 0, 2, 0, 0, 2, 2, 2, 2)^T$, signifying that F is not in $T_{0[1]}, T_{\{0,2\},1}$, or Sl . This indicates that F does not belong to sets $T_{0[1]}, T_{\{0,2\},1}$, or Sl .

Therefore, ∇_2 does not align with any predefined category through substitution, excluding H_2 . The assertion to be proved is that ∇_2 is contained within H_2 . Assume $f \in \nabla_2$, and $f \notin H_2$, and the cardinality of the subset X_f is $n \geq 1$. This causes the existence of a tuple of indices $\alpha^n = (a_1, \dots, a_n)$ so that $f(\alpha^n) = a \neq 2$. Suppose $\beta^n = (b_1, \dots, b_n)$ – a distinct tuple where each $b_i \in \{a_i, 2\}$ and $\beta^n \neq \alpha^n$. As defined by ∇_2 , if β^n contains 2, then $f(\beta^n) = 2$. This goes against the assertion that $f(\beta^n) \notin \{a, 2\}$.

It is evident that constants are included within the set of H_2 . Let's examine δ featuring $\delta^9 = (0, 0, 2, 0, 0, 0, 2, 2, 2)^T$. Despite δ being a member of H_2 , we observe that δ is not part of ∇_2 since it is not in conformity with the requirement for ∇_2 . Henceforth, due to the presence of functions outside H_2 within ∇_2 , we come to the conclusion that $\nabla_2 \subseteq H_2$.

We shall now establish that $\mathcal{R}_2(\nabla_2 \cup f^{\nabla_2}) = P_3$. If $f^{\nabla_2} \notin H_2$, then the proof process concludes. Let $f = f^{\nabla_2} \in H_2$, leading to $f \notin \nabla_2$ and $f \in H_2$. Assuming $|X_f| = n \geq 2$. Take into account a vector $\alpha^n = (a_1, \dots, a_n)$ with $f(\alpha^n) \neq 2$, and a minimum of one index i so that $a_i = 2$ and $O_2(\alpha^n) = \min_{\beta^n = (b_1, \dots, b_n): f(\beta^n) \neq 2, \exists j \text{ with } b_j = 2} O_2(\beta^n)$. The presence of α^n is guaranteed by $f \notin \nabla_2$.

With $f \in H_2$ it is guaranteed to be \mathcal{R}_2 -closed. We have left to prove that g is within $\mathcal{R}_2(\nabla_2 \cup f^{\nabla_2})$ for any $g \in P_3$.

As indicated in Lemma 3.2, if g is a member of ∇_2 then g is included in $\mathcal{R}_2(\nabla_2) \subseteq \mathcal{R}_2(\nabla_2 \cup f^{\nabla_2})$. In the straightforward case where $g = f$, it is apparent: $g \in \mathcal{R}_2(\nabla_2 \cup f^{\nabla_2})$. When g is neither in ∇_2 nor equal to f , in light of the definition of α^n , there is a β^n such that $g(\beta^n) = 2$. In such a way, it would be deduced that g is within $\mathcal{R}_2(\nabla_2 \cup f^{\nabla_2})$, thereby confirming that $\mathcal{R}_2(\nabla_2 \cup f^{\nabla_2}) = P_3$.

As mentioned earlier, there exists $f^{Sl} \in \nabla_2$, and according to the Slupetski theorem [10], the combination $\mathcal{R}_2(f^{\nabla_2} \cup \nabla_2) = \mathcal{R}_2(P_3(x) \cup f^{Sl}) = P_3$.

Let's assume that $O_2(\alpha^n) = 1$, and for the sake of simplicity, suppose $f(\alpha^n) = 0$. We shall examine the function $h(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$. If $h(2) = 0$ and $2 \in \{h(0), h(1)\}$, then the proof becomes quite simple because h must be in one of the forms $h^1 = (2, 0, 0)^T$, $(0, 2, 0)^T$, or $(2, 2, 0)^T$. Therefore, $\mathcal{R}_2(h \cup g \cup \delta) = P_3$, $\delta \in H_2$ and $\delta \notin \nabla_2$. If $1 \notin \{h(0), h(1)\}$, then $f \in H_2$ since the pair $\bar{2}, (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)$ would hold higher priority in this case.

Let's now explore a scheme where $f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) \equiv 0$. From f , we derive f^{H_2} . We choose a function g such that $f \sim g$ and g matches f everywhere except $\beta^n = (a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n)$, where $g(\beta^n) = 1$. Observe, when such a function g exists, it is a member of $\mathcal{R}_2(\nabla_2 \cup f^{\nabla_2})$. To validate the appropriateness of extending f to g , it is mandatory to affirm an equality $|X_f| = |X_g|$. The extension of f to g does not augment the count of key variables, taking into account that $|X_f| = n$. Through a contradiction, we shall demonstrate no crucial variables are forfeited during this extension.

Assuming for contradiction's sake: a variable x_k loses its indispensability after the extension. For convenience, let's assume $k < n$. Define \mathcal{S} as a group of permutations on E_3 , and let $\sigma \in \mathcal{S}$ such that $\sigma(a_k^i) = i$ for $i \in E_3$.

To further scrutinize the inference from an assumption, let's examine the action of permutation σ on f . By applying σ to the relevant coordinates, we can observe the transformation effect on the essentiality of x_k . If x_k is no longer essential, it would mean there is a permutation σ leaving the value of f unchanged with respect to x_k . But this disproves the initial assumption that x_k is significant in f . Subsequently, the assumption that x_k becomes non-essential after the extension is incorrect, confirming that all variables remain essential for the transitive mapping from f to g . Thus, X_f and X_g have equal cardinality, validating the extension $f \sim g$.

Let's examine a segment of diagrams for f :

$x_1 \dots x_k \dots x_i \dots x_n$	$f(\bar{x})$
...	...
$a_1 \dots a_k^0 \dots 0 \dots a_n$	0
$a_1 \dots a_k^1 \dots 0 \dots a_n$	1
$a_1 \dots a_k^2 \dots 0 \dots a_n$	1
...	...

If $a_k^0 = 2$, then the pair $\bar{2}, (a_1, \dots, a_k^1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n)$ becomes indicative, which means $f \notin H_2$. Similarly, if either $a_k^1 = 2$ or $a_k^2 = 2$, then $f \notin H_2$ as the indicative pair in these cases would be $\bar{2}, (a_1, \dots, a_k^0, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n)$.

Hence, the validity of the extension is verified, demonstrating the existence of a function $g \in \mathcal{R}_2(\nabla_2 \cup f^{\nabla_2})$ such that

$$g(\bar{2}) = 1,$$

and that $\mathcal{R}_2(\nabla_2 \cup g) = P_3$, thus concluding the proof. \square

Lemma 3.4. *The class $P_3(x)$ is \mathcal{R}_2 -predcomplete.*

Let's prove it. The class $P_3(x)$ is \mathcal{R}_2 -closed because $[P_3(x)] = P_3(x)$. By definition, if $f \sim g$ and $|X_f| = |X_g|$, it implies that the extension stays within the boundaries of $P_3(x)$.

To show that $\mathcal{R}_2(P_3(x) \cup f^{P_3(x)}) = P_3$, we examine the functions $g_1(x) = x + 2$ and $g_2(x) = x^2$. It's evident that $g_1 \notin M_1^3 \cup M_2^3 \cup M_3^3 \cup T_{01} \cup T_{02} \cup T_{12} \cup T_0 \cup T_1 \cup T_2 \cup H_0 \cup H_1 \cup H_2 \cup T_{\sim} \cup T_{\{0,2\},1} \cup T_{\{1,2\},0}$, and $g_2 \notin L \cup S_{x+1}$. Based on the characterization of Sl , it follows that $P_3(x) \subseteq Sl$. To demonstrate that $\mathcal{R}(P_3(x) \cup f^{P_3(x)}) = P_3$, we will examine the ensuing cases:

- If $f^{P_3(x)} \notin Sl$, as stated by Slupecki's theorem [10], the proof concludes.
- Assume $f = f^{P_3(x)} \in Sl$. If $|X_f| = n$, then we need to consider a vector $f^n = (f_1, \dots, f_{3^n})^T$. As stipulated $\{0, 1, 2\} \notin \{f_1, \dots, f_{3^n}\}$. Therefore, there is a unique $a \in \{0, 1\}$ such that for all i $f_i \in \{a, 2\}$. For simplicity, let $a = 0$.

Let's consider three cases:

1. If $O_2(f^n) \leq 3^n - 2$, one can identify α_1^n, α_2^n such that $f(\alpha_1^n) = f(\alpha_2^n) = 0$. By introducing g that matches with f except at α_1^n such that $g(\alpha_1^n) = 1$, and since $1 \notin \{f_1, \dots, f_{3^n}\}$, the new extension remains valid and $g \notin Sl$.
2. If $O_2(f^n) = 3^n - 1$, there is a unique α_1^n such that $f(\alpha_1^n) = 0$. Let us select $g' \in P_3(x)$ such that $g'^1 = (2, 2, 0)^T$. Now, consider $h = g'(f)$. It becomes obvious that $|X_f| = |X_h|$ and $h^n = (h_1, \dots, h_{3^n})$ with $h_i \in \{0, 2\}$. Accordingly, case (2) reduces to case (1).
3. If $O_2(f^n) = 0$, a function $f'(\tilde{x}) = f(\tilde{x}) + 1 \pmod k$ ($k \geq 3$) can be constructed. For f' , either case (1) or (2) holds. Consequently, $f^{Sl} \in \mathcal{R}(P_3(x) \cup f^{P_3(x)})$, demonstrating that $P_3(x)$ is \mathcal{R} -precomplete.

Suppose we interchange 0 and 1 in a vector f ($|X_f| \geq 1$), resulting in a new function vector g ($|X_g| < |X_f|$). By altering constants in f , we derive a function h within the domain of $\tilde{P}_2(x)$. Thus, the following assertion is valid:

Lemma 3.5. *h on the domain of $\tilde{P}_2(x)$ is obtained by interchanging $0 \leftrightarrow 1$ or $1 \leftrightarrow 0$ in f ($|X_f| \geq 1$).*

Proof. Let x_i in vector f become irrelevant after substitution. This implies the existence of $\alpha_1^n, \alpha_2^n, \alpha_3^n$ and a permutation σ such that $f(\alpha_1^n), f(\alpha_2^n), f(\alpha_3^n) \in E_2$, and $l, m \in E_3$ exist where $f(\alpha_l^n) \neq f(\alpha_m^n)$. Hence, $h(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$ meets all conditions. \square

3.2 Completeness of \mathcal{R}_2

In this section, we address the completeness problem for the operator \mathcal{R}_2 within the class P_3 . We identify and describe five distinct \mathcal{R}_2 -precomplete categories in P_3 : T_2 , T_{01} , T_{\sim} , ∇_2 , and $P_3(x)$. These categories play a crucial role in demonstrating the comprehensive nature of the elements in P_3 under the action of \mathcal{R}_2 , allowing us to precisely address the completeness criterion. The main result is summarized as follows:

Theorem 3.6 (Completeness). *There exist five \mathcal{R}_2 -precomplete classes within P_3 :*

$$T_2, T_{01}, T_{\sim}, \nabla_2, P_3(x).$$

Proof. Consider the system M composed of functions $f^{T_2}, f^{T_{01}}, f^{T_{\sim}}, f^{\nabla_2}, f^{P_3(x)}$. The goal is to prove that $\mathcal{R}_2(M) = P_3$.

Select the function $f^{T_2} \in M$. By combining the variables of f^{T_2} , we obtain a function $g \notin T_2$ with $|X_g| \leq 1$. In accordance with the definition of \mathcal{R}_2 , the function g is included in $\mathcal{R}_2(M)$.

The set $\overline{T}_2(x)$ can be partitioned into four equivalence classes:

$$\overline{T}_2(x) = \bigsqcup_{i=1}^4 M_i,$$

where the equivalence classes are defined as follows:

$$\begin{aligned} M_1 &= \{f \mid f^1 = (a_1, b, a_2)^T, a_i \in E_2, b = 2\}, \\ M_2 &= \{f \mid f^1 = (b, a_1, a_2)^T, a_i \in E_2, b = 2\}, \\ M_3 &= \{f \mid f^1 = (a_1, a_2, a_3)^T, a_i \in E_2, |X_f| = 1\}, \\ M_4 &= \{f \mid f^1 = (b, b, a)^T, a \in E_2, b = 2\}. \end{aligned}$$

Moreover, there exists a case where combining the variables of g results in a constant value. Let us denote this case as $M_0 = \{0, 1\}$, representing the constant values.

We have demonstrated that: if $g \in M_1$ or $g \in M_2$, then $P_3(x) \subseteq \mathcal{R}_2(M)$. Consequently, $M_3 \subseteq \mathcal{R}_2(M)$ and $M_4 \subseteq \mathcal{R}_2(M)$.

Since $M_0 \subseteq [M_3] \subseteq \mathcal{R}_2(M_3)$, the situation where $g \in M_3$ simplifies to $g \in M_0$.

Thus, we are left with two remaining scenarios to consider:

1. $g \in M_4$,
2. $g \in M_0$.

In the case where $g \in M_4$, it is evident that $g \notin T_{01}$. In this scenario, it suffices to analyze the system $M = \{g, f^{T_{\sim}}, f^{\nabla_2}, f^{P_3(x)}\}$. Given the definition of M_4 , we establish that $M_4 = \mathcal{R}_2(g) \subseteq \mathcal{R}_2(M)$.

We identify a function $h_0 \in M_4$ such that $h_0^1 = (2, 2, 0)^T$. This function satisfies the property $h_0(h_0(x)) = h_1(x)$, where $h_1^1 = (0, 0, 2)^T$. It is important to note that $h_1 \in \mathcal{R}_2(M)$ and that h_1 has two degrees of freedom, which ensures its contribution to the completeness of the system.

Thus, there exists a function $f^{M_i^3} \in \mathcal{R}_2(M)$ for $1 \leq i \leq 3$, which supports the inclusion of all necessary elements under the action of \mathcal{R}_2 .

Using the classification of pre-complete functions for the substitution operator, we observe that

$$h_1 \notin \{T_0, T_1, T_2, T_{01}, T_{12}, H_1, T_{\{1,2\},0}, L, S_{x+1}\}.$$

This observation further confirms that h_1 belongs to a distinct class, aligning with the requirements for $\mathcal{R}_2(M)$.

Furthermore, we identify another function $h_2 \in \mathcal{R}_2(M)$, which is similar to h_1 , with $h_2^1 = (1, 1, 2)^T$. Analogously, $h_2 \notin \{T_{02}, H_0, T_{\{0,2\},1}\}$, which demonstrates its membership in a different category from those listed.

Let $l = f^{P_3(x)}$, and let $n = |X_l|$, denoting the number of essential variables in l . From l , derive the function f^{Sl} . If $l \notin Sl$, the completeness condition is already satisfied.

If $l \in Sl$, we proceed as follows:

a) If $O_2(l^n) < 3^n - 1$, then there exists a function $l_1 \sim l$ obtained by substituting a non-zero element a of l^n with its complement \bar{a} . This transition to l_1 is valid, and it follows that $|X_{l_1}| = |X_l| = n$. Since $l_1 \in \mathcal{R}_2(M)$, it implies that $l_1 \notin Sl$.

b) If $O_2(l^n) = 3^n - 1$, assume the non-zero element in l^n is $a = 0$ (a similar argument holds if $a = 1$). Consider $l_2(\tilde{x}) = h_0(l(\tilde{x}))$, where $h_0 \in M$ and $h_0^1 = (2, 2, 0)^T$. By combination principles, we have $|X_{l_2}| = |X_l| = n$. Given l_2 , it follows that $O_2(l_2^n) = 1$, which is less than $3^n - 1$. Therefore, the reasoning in part (a) can be applied to l_2 .

Define $m = f^{\nabla_2}$ with $k' = |X_m|$ and $k = 3^{k'}$. Assume $k' > 1$; otherwise, the previous results apply. There are two cases to consider:

1) **Case 1:** If $m(\bar{2}) \neq 2$, then there exists an $\alpha^k = (a_1, \dots, a_k)$, where $a_i \in E_3$, such that $m(\alpha^k) \neq 2$ and $\alpha^k \neq \bar{2}$. If $m(\alpha^k) \neq m(\bar{2})$, then α^k and $\bar{2}$ provide evidence for H_2 , implying $m \notin H_2$.

If $m(\alpha^k) = m(\bar{2})$, consider a new function $m_1 \sim m$, differing only in its last element: $m_1(\bar{2}) = \overline{m(\bar{2})}$. If $m_1 \in \mathcal{R}_2(M)$, then α^k and $\bar{2}$ still provide evidence for H_2 , implying $m_1 \notin H_2$.

If $|X_m| > |X_{m_1}|$, suggesting that x_j is irrelevant, consider $\beta^k = (b_1, \dots, b_k)$, where $b_i = 2$ for $i \neq j$ and $b_j \neq 2$. Then, $m(\beta^k) \sim m(\bar{2})$, but $m(\beta^k) \neq m(\bar{2})$, and both β^k and $\bar{2}$ provide evidence for H_2 . Therefore, $m \notin H_2$.

If $O_2(m^{k'}) = k - 1$, assuming $m(\bar{2}) = 0$, consider $m_1(\tilde{x}) = h_0(m(\tilde{x}))$. Construct $m_2(\tilde{x})$ similar to m_1 , except at α^k , where $m_2(\alpha^k) = \overline{m_1(\alpha^k)}$. If β^k is derived from α^k by substituting a single element with 2, then $m_2(\alpha^k) \sim m_2(\beta^k)$, and $m_2(\alpha^k) \neq m_2(\beta^k)$. Thus, α^k and $\bar{2}$ provide evidence for H_2 , implying that $m_2 \notin H_2$.

2) **Case 2:** Suppose $m(\bar{2}) = 2$. Then there exists $\alpha^k = (a_1, \dots, a_k)$ where $a_i \in E_3$ and $a_j = 2$, such that $m(\alpha^k) = a' \neq 2$. Assume $a' = 0$. Let $m_2 \sim m$ be obtained by replacing all elements in m that are not equal to 2 with a' . Now, consider $m_3(\tilde{x}) = h_0(m_2(\tilde{x}))$, where $h_0 \in M$. Then $|X_{m_3}| = |X_{m_2}| = |X_m|$, yet we have $m_3(\bar{2}) \neq 2$. Thus, we can apply the reasoning from case 1.

If the transition from m to m_2 is not exact (i.e., $|X_m| > |X_{m_2}|$), this indicates a situation where a specific variable x_j is incorrectly determined.

As a result, distinct tuples α^k and β^k may be identified, where $a_i = b_i$ for $i \neq j$, $a_j \neq b_j$, $m(\alpha^k) \sim m(\beta^k)$, but $m(\alpha^k) \neq m(\beta^k)$. Suppose $a_j \neq 2$; then α^k and $\bar{2}$ provide evidence for H_2 , proving that $\mathcal{R}_2(M)$ does not belong to H_2 .

Based on this reasoning, we conclude that $f^{T_{\sim}}$ exists in M , implying $\mathcal{R}_2(M) = P_3$.

Next, suppose $f = f^{T_{01}}$ in M , and let $g \in M_3$. Since M_3 is an \mathcal{R} -equivalence class, it follows that $M_3 \subseteq \mathcal{R}_2(g)$. By substituting functions from M_3 into f , we obtain a function f' such that $2 \in \{f'(\bar{0}), f'(\bar{1})\}$. Therefore, f satisfies the condition that 2 is in $\{f(\bar{0}), f(\bar{1})\}$.

By identifying all the variables of f , we find that $\tilde{f} \notin T_{01}$ and that $|X_{\tilde{f}}| \leq 1$.

Consider the set $\overline{T_{01}} = \{f \mid f \notin T_{01}\}$. Clearly, $\overline{T_{01}}(x)$ can be decomposed into five equivalence classes:

$$\overline{T_{01}}(x) = \bigsqcup_{i=1}^5 N_i,$$

where:

1. $N_1 = \{f \mid f^1 = (a_1, b, a_2)^T, a_i \in E_2, b = 2\}$,
2. $N_2 = \{f \mid f^1 = (b, a_1, a_2)^T, a_i \in E_2, b = 2\}$,
3. $N_3 = \{f \mid f^1 = (b, b, a)^T, a \in E_2, b = 2\}$,
4. $N_4 = \{f \mid f^1 = (a, b, b)^T, a \in E_2, b = 2\}$,
5. $N_5 = \{f \mid f^1 = (b, a, b)^T, a \in E_2, b = 2\}$.

This decomposition guarantees that each $f \notin T_{01}$ is uniquely represented by the values in its vector f^1 within one of these classes.

Furthermore, we can identify the argument values of f under which we obtain the constant set $N_0 = \{2\}$.

It should be noted that $N_1 = M_1$, $N_2 = M_2$, and $N_3 = M_4$. Thus, situations where $\tilde{f} \in N_1$ and $\tilde{f} \in N_2$ can be simplified to $\tilde{f} \in N_3$. As a result, $\tilde{f} \in M_4$, and by using the above argumentation, the theorem will be established.

Given that $N_0 \subseteq [N_4] \subseteq \mathcal{R}_2(N_4)$ and $N_0 \subseteq [N_5] \subseteq \mathcal{R}_2(N_5)$, the cases where \tilde{f} is in both N_4 and N_5 can be simplified to $\tilde{f} \in N_0$, signifying that $\tilde{f} \equiv 2$. Hence, by demonstrating the completeness of $M^* = \{g, 2, f^{U_{01}}, f^{\nabla_2}, f^{P_3(x)}\}$, with $g \in M_0$, the proof is completed.

Note that all fixed values lie within M^* . As a result, if there exists $\omega \in \mathcal{R}_2(M^*)\tilde{P}_2(x)$, it implies the existence of h within the closure $\overline{T_{\sim}}(x)$ such that $h \in \mathcal{R}_2(M^*)$. Since $P_3(x)$ is \mathcal{R}_2 -complete, we deduce that $\mathcal{R}_2(M^*) = P_3$. Hence, all transitions from f to g (considering the \mathcal{R}_1 operator) for $f \in M^*$ are correct with respect to the \mathcal{R}_2 operator. If a transition is incorrect, then $\mathcal{R}_2(M^*) = P_3$, thus validating the theorem.

Take $f = f^{\nabla_2}$ with $n = |X_f|$. Define $\nabla_2 = \{f \mid f \notin \nabla_2\}$. We have two scenarios:

1. $|X_f| = 1$: the proof is completed since $\nabla_2(x) = T_2(x)$.
2. $|X_f| \geq 2$: by definition, there exists $\alpha_2^n = (a_1, \dots, a_i, \dots, a_n)$ with $a_i = 2$ and $f(\alpha_2^n) \neq 2$. Consider $\alpha_0^n = (a_1, \dots, a_i^0, \dots, a_n)$ and $\alpha_1^n = (a_1, \dots, a_i^1, \dots, a_n)$, where $\{a_i^0, a_i^1\} = E_2$. Two subcases arise:
 - a) $f(\alpha_0^n)$, $f(\alpha_1^n)$, and $f(\alpha_2^n)$ are not all equal. Let $f'(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$. Then $f'(x) \in \nabla_2(x)$ and $f' \in \mathcal{R}_2(M^*)$, satisfying case 1.
 - b) $f(\alpha_0^n) = f(\alpha_1^n) = f(\alpha_2^n)$. Take $g \sim f$, where g matches f except at α_2^n , and $g(\alpha_2^n) = f(\alpha_2^n)$. Thus, $g \in \mathcal{R}_2(M^*)$, satisfying case (a).

□

3.3 Structure of closed classes in P_k for $k \geq 4$

Consider a collection M comprising functions $f \in P_k$, where $k \geq 4$. Define $\mathcal{R}(M)$ as the closure of M under relation substitution and the equivalence $f \sim g$, given by:

$$f \sim g \iff \forall \vec{x} [(f(\vec{x}) = g(\vec{x})) \vee (f(\vec{x}), g(\vec{x}) \in \{0, 1\})].$$

Any function derived from M through substitution, and functions that agree on all inputs or differ only on inputs mapped to 0 and 1, are contained in $\mathcal{R}(M)$ according to this definition. Thus, $\mathcal{R}(M)$ is the family of functions that are closed with respect to both substitution and the equivalence relation defined above.

Our goal is to determine the limit closure $\mathcal{R}(M)$, ensuring that $\|P_k\|_{\mathcal{R}(M)} < \infty$. This requires the construction of an enhanced linear combination operator that ensures the Boolean algebra of closed classes remains countable.

A trivial solution might use an operator $\mathcal{I}(\cdot) : P_k \rightarrow P_k$ where $\mathcal{I}(f) = P_k$ for all $f \in P_k$. However, our goal is to develop a non-trivial operator that optimally refines the linear combination operator, ideally adding minimal transitive relations beyond those in the original identity matrix.

In the domain of 3-valued logic (3-VL), a minimal enhancement would involve incorporating just one additional transitive relation into the foundational set of relations used by the linear combination operator. This approach aims to strike a balance between complexity and efficacy, ensuring that the resulting algebraic framework remains both tractable and precisely defined.

By carefully constructing this additional transitive relation, we aim to augment the operator's capabilities without unnecessarily complicating the underlying logical structure. This ensures that the theoretical analysis and practical applications of the framework are both robust and manageable. This approach forms the basis for understanding how $\mathcal{R}(M)$ operates within P_k for $k \geq 4$, providing a solid foundation for solving modern electronic challenges.

The following theorem, Theorem 3.7, highlights the relationship between higher-dimensional parameter spaces and the structure of closed classes, emphasizing the balance achieved even as the dimensionality increases.

Theorem 3.7. *For all $k \geq 4$, there exists a \mathcal{R} -closed class in P_k with a countable basis.*

Proof. We aim to prove that for every function f_m , it is not an element of the set formed by excluding itself from M .

Consider the function f_i defined as follows:

$$f_i(x_1, \dots, x_i) = \begin{cases} 2, & \text{if } x_1 = x_2 = \dots = x_{j-1} = x_{j+1} = \dots = x_i = 0 \text{ and } x_j = 1, \\ 3, & \text{otherwise,} \end{cases} \quad (1)$$

where $i = 2, 3, \dots$

These functions f_i are explicitly designed to return 2 only when a specific pattern of inputs is satisfied, namely, all inputs are 0 except for one. Under this condition, f_i returns 2; otherwise, it returns 3.

To demonstrate the validity of the statement, we verify that each f_i belongs to P_k . Each f_i is distinct under the \mathcal{R} -closure due to its unique conditional behavior, which cannot be replicated by compositions of the functions $\{f_2, f_3, \dots\}$ alone.

Thus, the set $\{f_2, f_3, \dots\}$ forms a countable basis for \mathcal{R} -closed classes within P_k for $k \geq 4$, fulfilling the conditions of the theorem. Therefore, the existence of a countable basis for \mathcal{R} -closed classes in P_k for $k \geq 4$ is established, confirming the theorem.

Since the equivalence relation $0 \sim 1$ is not valid for the function values in M , we may redefine M as $M = \{f_2, f_3, \dots\}$. Our goal is to demonstrate:

$$f_m \notin \{f_2, \dots, f_{m-1}, f_{m+1}, \dots\}, \quad \forall m = 2, 3, \dots$$

Therefore, the equivalence relation $0 \sim 1$ does not apply to values within M , and M should be redefined accordingly.

To explore a contradiction, assume that $f_m \in \mathcal{R}(\{f_2, \dots, f_{m-1}, f_{m+1}, \dots\})$. According to the definition of f_m , this assumption implies that f_m can be represented as a composition of functions from $\{f_2, \dots, f_{m-1}, f_{m+1}, \dots\}$. Hence, there must exist some number r and formulas $\mathcal{U}_1, \dots, \mathcal{U}_r$ such that:

$$f_m(x_1, \dots, x_m) = f_r(\mathcal{U}_1[f_2, \dots, f_{m-1}, f_{m+1}, \dots], \dots, \mathcal{U}_r[f_2, \dots, f_{m-1}, f_{m+1}, \dots]), \quad r \neq m.$$

Assume all formulas $\mathcal{U}_1, \dots, \mathcal{U}_r$ are variables. We examine the following cases:

1. If $r < m$, then $f_m(x_1, \dots, x_m) = f_r(x_1, \dots, x_r)$, implying that f_m depends on fewer variables than m , which contradicts its definition as a function of m variables.
2. If $r > m$, then some variable x_p must appear more than once in the expression for f_m . Setting $x_1 = \dots = x_{p-1} = x_{p+1} = \dots = x_m = 0$ and $x_p = 1$, we observe that f_m returns 2, while f_r returns 3. This demonstrates that the repeated variables in f_m lead to different outputs when specific values are substituted, hence $f_m \neq f_r$.
3. If there exists a formula \mathcal{U}_j that is not a variable symbol, then the output of f_r would be fixed at either 2 or 3, which contradicts the definition of f_m . Therefore, this case cannot occur.

Thus, the assumption that f_m can be written as a composition of functions from $\{f_2, \dots, f_{m-1}, f_{m+1}, \dots\}$ leads to a contradiction, proving that the set $\{f_2, f_3, \dots\}$ indeed forms a countable basis for the class of \mathcal{R} -closed functions in P_k .

Hence, the theorem is established. □

The established theorem guarantees that for all $k \geq 4$, there exists an \mathcal{R} -closed class in P_k with a countable basis. These results confirm the existence of explicit \mathcal{R} -closed functions within P_k . The significance of \mathcal{R}_1 -closed and \mathcal{R}_2 -closed classes lies in their hierarchical relationship: every \mathcal{R}_1 -closed class inherently possesses \mathcal{R}_2 -closed properties. Properties preserved under \mathcal{R}_1 are also preserved under \mathcal{R}_2 .

The idea of operators extending to the set P_k for $k \geq 4$ involves establishing a framework in which the interrelations and connections between classes may be understood through a lattice structure. Every two elements possess a distinct supremum and infimum in a lattice, and the continuity of the lattice structure ensures that, as we explore larger sets P_k , the relationships among the closed classes remain well-organized and foreseeable.

Thus, we deduce the following significant consequence of Theorem 3.7:

Theorem 3.8 (Corollary). *An \mathcal{R}_1 -closed class is also \mathcal{R}_2 -closed. The operators \mathcal{R}_1 and \mathcal{R}_2 , extended to the set P_k for $k \geq 4$, form a connected lattice continuum of closed classes.*

This corollary offers a clear understanding of the concept of "hierarchical closure" as applied in program schemes [3], fuzzy logics [31], and quantifier classes in infinitary logic [11]. It underscores the hierarchical relationship between \mathcal{R}_1 -closed and \mathcal{R}_2 -closed classes, confirming that properties preserved under \mathcal{R}_1 are also preserved under \mathcal{R}_2 .

4 Conclusion

The primary outcome of this study lies in the formulation of theoretical models that ensure the assembly of reliable MVL digital circuits. The research establishes the presence of \mathcal{R} -closed sets in P_k with a countable basis for $k \geq 4$. Moreover, a precise method for constructing such a family is provided.

Examples of logical families, presented as diagrams with their bases, were introduced to support these conceptual ideas. In particular, the structural aspects of the logical families were explored in greater detail, illustrating how the algebraic operations interact with the closure properties. Tables 1 and 2 in Appendix A offer tools for organizing and analyzing various types of logical functions, their bases, and properties. These tables serve as a key resource for further classifications and are instrumental for both theoretical exploration and practical application. This classification is valuable not only for practical implementations but also for disciplines such as informatics, computational theory, and algebraic formalism. The tables provide the semantics for logical circuits, computational methods, and complexity theory.

The main theoretical conclusion of this research involves the analysis of the \mathcal{R}_2 operator and \mathcal{R}_2 -precloseness in P_3 . The exploration of \mathcal{R}_2 -precloseness yields new insights into the foundational structure of logical function spaces and sets the stage for a more refined understanding of MVL systems. Findings are supported by lemmas and theorems established in this work. Some of these results extend, complement, and generalize earlier ones derived in publications [9, 17, 18]. Special attention was given to the completeness principle (Theorem 3.6) for P_3 , which plays a central role in establishing the stability and closure properties of these classes.

This research also provides a brief overview of the applied relevance of MVL outcomes. In particular, it highlights the practical advantages of MVL in overcoming the limitations of traditional binary logic, with the potential to transform modern electronics. The opportunities for utilizing MVL as a replacement for, or extension of, binary logic were described, alongside the diverse benefits of MVL in contemporary electronics:

- MVL offers significant reductions in energy consumption and enhanced storage capacity. The application of MVL memory modules can minimize energy loss and improve data storage efficiency.
- The exploration of modern materials, such as silicon, for integrating MVL devices has been advanced. Overcoming interdisciplinary obstacles could propel the MVL-electronics industry towards a prosperous future.
- Establishing a solid mathematical foundation is essential to ensure the reliability, capability, and performance of MVL digital systems and algorithms. This research calls for a deeper, more detailed analysis of function families within MVL to further strengthen these systems' theoretical underpinnings.

While this study focuses on precomplete classes and closure properties within P_k spaces, further research is needed to explore other classes of functions, including those that do not conform to strict closure or monotonicity criteria. A related problem has been partially addressed in the investigation of structurally finite classes of order-preserving three-valued logic maps, which provides valuable insights into monotonicity and order-preservation in smaller logical frameworks [8]. However, many questions remain open, particularly in extending these results to broader classes of functions and higher-dimensional multi-valued logic systems. Understanding these structures could provide deeper insights into the fundamental behaviour of logical operations in multi-valued contexts. In conclusion, we note the following key directions for future research:

- Further investigation is necessary to fully realize the potential of modern materials for MVL mechanisms. New material innovations are crucial for pushing the boundaries of MVL technology.
- Full-scale testing of MVL devices in applications such as AI and integrated physical systems is imperative to evaluate their real-world performance and scalability.
- A comprehensive investigation of the theoretical principles underpinning MVL is needed, with particular attention to enhancing the effectiveness of MVL programs and circuits. Questions related to completeness and closure within functional categories remain open and require further exploration.

MVL represents a promising direction in computer engineering, offering the potential for significant improvements in performance and energy efficiency. Further research and refinement of MVL may lead to revolutionary advancements in high-speed computing and sustainable energy use. In addition, the integration of MVL into quantum computing holds great promise for addressing complex problems in quantum state linear combination superposition [20].

Table 1: Logical Classes, Their Bases, and Order

Class	Example of Basis	Order
T_{\sim}	$\xi(x); x \vee y$	2
A_1	$x \vee y; \xi(\omega_2(x, y))$	2
A_2	$x \wedge y; \xi(\omega_2(x, y))$	2
A_3	$x \vee y; x \wedge \xi(\omega_2(y, z))$	3
A_4	$0; 2; x \vee y; x \wedge y$	2
A_5	$2; x \vee y; x \wedge y$	2
A_6	$x \vee y; x \wedge y$	2
B_1	$h_2(x, \xi(y), \xi(z))$	3
B_2	$h_2(x, y, \xi(z))$	3
B_3	$h_2(x, y, z)$	3
M_1	$2; \omega_2(x, y)$	2
M_2	$\omega_2(x, y)$	2
M_3	$\xi(\omega_2(x, y))$	2
M_4	$\xi(\omega_3(x, y, z))$	3
M_5	$\omega_3(x, y, z)$	3
N_1	$0; 2; x \vee y$	2
N_2	$2; x \vee y$	2
N_3	$0; x \vee y$	2
N_4	$x \vee y$	2
N_5	$0; 2; x \wedge y$	2
N_6	$2; x \wedge y$	2
N_7	$0; x \wedge y$	2
N_8	$x \wedge y$	2
U_1	$0; \xi(x)$	1
U_2	$0; 2; x$	1
U_3	$2; x$	1
U_4	$\xi(x)$	1
U_5	$0; x$	1
U_6	x	1
U_7	$0; 2$	1
U_8	2	1
U_9	0	1

Table 2: Boolean Function Classes and Their k-Dependent Characteristics

Class	Example of Basis	Order
$C_k, k \geq 1$	$x \vee \xi(y); h_{k+1}^*(\tilde{x})$	$k+2$
C_∞	$x \vee \xi(y)$	2
$D_k, k \geq 1$	$x \vee (y \wedge \xi(z)); h_{k+1}^*(\tilde{x})$	$k+2$
D_∞	$x \vee (y \wedge \xi(z))$	3
$E_k, k \geq 1$	$2; h_{k+1}^*(\tilde{x})$	$k+2$
E_∞	$2; x \vee (y \wedge z)$	3
F_1	$x \vee (y \wedge z); h_2^*(x, y, z)$	3
$F_k, k \geq 2$	$h_{k+1}^*(\tilde{x})$	$k+2$
F_∞	$x \vee (y \wedge z)$	3
G_1	$x \wedge (y \vee z); h_2(x, y, z)$	3
$G_k, k \geq 2$	$h_{k+1}(\tilde{x})$	$k+2$
G_∞	$x \wedge (y \vee z)$	3
$H_k, k \geq 1$	$0; h_{k+1}(\tilde{x})$	$k+2$
H_∞	$0; x \wedge (y \vee z)$	3
$J_k, k \geq 1$	$x \wedge \xi(y); h_{k+1}(\tilde{x})$	$k+2$
J_∞	$x \wedge \xi(y)$	2
$K_k, k \geq 1$	$x \wedge (y \vee \xi(z)); h_{k+1}(\tilde{x})$	$k+2$
K_∞	$x \wedge (y \vee \xi(z))$	3

Acknowledgement

The author wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

A Examples of logical classes and their bases

Table 1 presents logical classes, their basis and orders. This table illustrates the theorem concerning the existence of \mathcal{R} -closed classes with a countable basis and the lattice structure of closed classes, which remains continuous for $k \geq 4$. Table 2 presents various classes of Boolean functions and their primary characteristics. These classes are denoted as $C_k, D_k, E_k, F_k, G_k, H_k, J_k$, and K_k .

References

- [1] I. Aizenberg, R. Belardi, M. Bindi, F. Grasso, S. Manetti, A. Luchetta, M. C. Piccirilli, *A neural network classifier with multi-valued neurons for analog circuit fault diagnosis*, Electronics, **10**(3) (2021), 349. <https://doi.org/10.3390/electronics10030349>
- [2] G. Alwhishi, J. Bentahar, A. Elwhishi, W. Pedrycz, N. Drawel, *Multi-valued model checking IoT and intelligent systems with commitment protocols in multi-source data environments*, Information Fusion, **102** (2024), 102048. <https://doi.org/10.1016/j.inffus.2023.102048>
- [3] A. A. Arratia-Quesada, S. R. Chauhan, I. A. Stewart, *Hierarchies in classes of program schemes*, Journal of Logic and Computation, **9**(6) (1999), 915-957. <https://doi.org/10.1093/logcom/9.6.915>
- [4] H. M. H. Babu, *Multiple-valued computing in quantum molecular biology: Sequential circuits, memory devices, programmable logic devices, and nanoprocessors*, CRC Press, 2023. <https://doi.org/10.1201/9781003381921>
- [5] G. Bocewicz, J. Pempera, V. Toporkov, *Performance evaluation models for distributed service networks*, Springer International Publishing, 2021. <https://doi.org/10.1007/978-3-030-67063-4>
- [6] D. Cheng, J. E. Feng, J. Zhao, S. Fu, *A minimum adequate set of multi-valued logic*, Control Theory and Technology, **19** (2021), 425-429. <https://doi.org/10.1007/s11768-021-00064-w>

- [7] A. A. Esin, *Analysis and design principles of modern control systems based on multi-valued logic models*, Upravlenie Bol'shimi Sistemami, (88) (2020), 69-98. <https://doi.org/10.25728/UBS.2020.88.4>
- [8] A. A. Esin, *Characteristics of structurally finite classes of order-preserving three-valued logic maps*, Logic Journal of the IGPL, (2024), jzae128. <https://doi.org/10.1093/jigpal/jzae128>
- [9] A. A. Esin, R. Yavorskiy, N. Zemtsov, *Brief announcement monitoring of linear distributed computations*, Springer Berlin Heidelberg, (2006), 566-568. https://doi.org/10.1007/11864219_47
- [10] F. A. Gadzhiev, *The Kolmogorov-Slupecki theorem for a three-dimensional sphere*, Russian Mathematical Surveys, **39**(5) (1984). <https://doi.org/10.1070/rm1984v039n05abeh004086>
- [11] E. Gradel, G. L. McColm, *Hierarchies in transitive closure logic, stratified Datalog and infinitary logic*, Proceedings., 33rd Annual Symposium on Foundations of Computer Science, (1992), 167-176. <https://doi.org/10.1109/SFCS.1992.267775>
- [12] Y. Hassan, et al., *Phase-engineered molybdenum telluride/black phosphorus van der waals heterojunctions for tunable multivalued logic*, ACS Applied Materials and Interfaces, **12**(12) (2020), 14119-14124. <https://doi.org/10.1021/acsami.9b20041>
- [13] L. C. Holdon, A. Borumand Saeid, *Regularity in residuated lattices*, Iranian Journal of Fuzzy Systems, **16**(6) (2019), 107-126. <https://doi.org/10.22111/IJFS.2019.5023>
- [14] S. B. Jo, J. Kang, J. H. Cho, *Recent advances on multivalued logic gates: A materials perspective*, Advanced Science, **8**(8) (2021), 2004216. <https://doi.org/10.1002/advs.202004216>
- [15] E. Y. Kalimulina, *Analysis of unreliable open queueing network with dynamic routing*, Springer International Publishing, (2017), 355-367. https://doi.org/10.1007/978-3-319-66836-9_30
- [16] E. Y. Kalimulina, *On exponential convergence of dynamic queueing network and its applications*, Springer International Publishing, (2020), 463-474. https://doi.org/10.1007/978-3-030-66471-8_35
- [17] E. Y. Kalimulina, *Lattice structure of some closed classes for three-valued logic and its applications*, Mathematics, **10**(1) (2022), 94. <https://doi.org/10.3390/math10010094>
- [18] E. Y. Kalimulina, *Finiteness of one-valued function classes in many-valued logic*, Fractal and Fractional, **8**(1) (2024), 29. <https://doi.org/10.3390/fractalfract8010029>
- [19] V. Levashenko, I. Lukyanchuk, E. Zaitseva, M. Kvassay, J. Rabcan, P. Rusnak, *Development of programmable logic array for multiple-valued logic functions*, IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, **39**(12) (2020), 4854-4866. <https://doi.org/10.1109/TCAD.2020.2966676>
- [20] B. Li, X. B. Liang, S. M. Fei, *Characterizing the superposition of arbitrary random quantum states and a known quantum state*, Results in Physics, **49** (2023), 106510. <https://doi.org/10.1016/j.rinp.2023.106510>
- [21] D. U. Lim, S. B. Jo, B. Sae J. H. Cho, *Monolithic tandem vertical electrochemical transistors for printed multi-valued logic*, Advanced Materials, **35**(9) (2023), 2208757. <https://doi.org/10.1002/adma.202208757>
- [22] X. Luo, J. Fang, *Fuzzifying closure systems and closure operators*, Iranian Journal of Fuzzy Systems, **8**(1) (2011), 77-94. <https://doi.org/10.22111/ijfs.2011.239>
- [23] S. S. Marchenkov, *Existence of finite bases in closed classes of Boolean functions*, Algebra and Logic, **23**(1) (1984), 66-74. <https://doi.org/10.1007/BF01979700>
- [24] M. A. Malkov, *Algebra of finite-valued functions: Classification of functions and subalgebras, essential and fictitious subalgebras*, Pure and Applied Mathematics Journal, **8**(2) (2019), 30-36. <https://doi.org/10.11648/j.pamj.20190802.11>
- [25] A. Mikhailovich, *Some closed classes of three-valued logic generated by symmetric functions*, arXiv, (2015). <https://doi.org/10.48550/arXiv.1503.05998>
- [26] D. Panigrahi, R. Hayakawa, Y. Wakayama, *High-performance multivalued logic circuits based on optically tunable antiambipolar transistors*, Journal of Materials Chemistry C, **10**(14) (2022), 5559-5566. <https://doi.org/10.1039/D1TC05858D>

- [27] A. Platzer, *Logical foundations of cyber-physical systems*, Springer International Publishing, 2018. <https://doi.org/10.1007/978-3-319-63588-0>
- [28] D. K. Podol'ko, *Classes of functions closed with respect to a special superposition operation*, Moscow University Mathematics Bulletin, **68** (2013), 303-306. <https://doi.org/10.3103/s0027132213060090>
- [29] D. K. Podol'ko, *A family of classes of functions closed with respect to a strengthened superposition operation*, Moscow University Mathematics Bulletin, **70**(2) (2015), 79-83. <https://doi.org/10.3103/S0027132215020059>
- [30] E. L. Post, *The two-valued iterative systems of mathematical logic. (AM-5)*, Princeton University Press, 1941.
- [31] E. Rogova, P. Chountas, Panagiotis, *On imprecision intuitionistic fuzzy sets and OLAP – The case for KNOLAP*, Springer Berlin Heidelberg, (2007), 11-20. https://doi.org/10.1007/978-3-540-72434-6_2
- [32] C. F. Silva, S. Ferlin, O. Alay, A. Brunstrom, B. Y. L. Kimura, *IoT traffic offloading with multiPath TCP*, IEEE Communications Magazine, **59**(4) (2021), 51-57. <https://doi.org/10.1109/MCOM.001.2000915>
- [33] S. M. Srivastava, *A course on mathematical logic*, Universitext, 2013. <https://doi.org/10.1007/978-1-4614-5746-6>
- [34] M. V. Starostin, *Implicitly maximal classes and implicit completeness criterion in the three-valued logic*, Moscow University Mathematics Bulletin, **73**(2) (2018), 82-84. <https://doi.org/10.3103/S0027132218020067>
- [35] I. E. Suleimenov, Y. S. Vitulyova, S. B. Kabdushev, A. S. Bakirov, *Improving the efficiency of using multivalued logic tools*, Scientific Reports, **13**(1) (2023). <https://doi.org/10.1038/s41598-023-28272-1>
- [36] J. Vipra, Jai, S. M. West, *Computational power and AI*, AI Now Institute, 2023.
- [37] Y. S. Vitulyova, D. Matrassulova, I. Suleimenov, A. Bakirov, *Convolutional neural networks from the perspective of the problems of multivalued logic*, Research Square Platform LLC, (2023), 1-19. <https://doi.org/10.21203/rs.3.rs-3224688/v1>
- [38] S. Wang, H. Li, *Graph-based function perturbation analysis for observability of multivalued logical networks*, IEEE Transactions on Neural Networks and Learning Systems, **32**(11) (2021), 4839-4848. <https://doi.org/10.1109/TNNLS.2020.3025912>
- [39] S. V. Yablonskii, *Functional constructions in a k -valued logic*, Trudy Matematicheskogo Instituta Imeni VA Steklova, Russian Academy of Sciences, Steklov Mathematical Institute of Russian, **51** (1958), 5-142.
- [40] Y. I. Yanov, A. A. Muchnik, *The existence of a k -valued closed class that has no finite basis*, Doklady Akad. Nauk SSSR, **127**(1) (1959), 44.