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Fuzzy portfolio selection with different risk attitudes based on machine learning

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Abstract

This paper studies a fuzzy portfolio selection program with different risk attitudes based on Machine Learning to help investors make reasonable investment decisions. Firstly, the asset's return is assumed as a fuzzy variable, which is estimated by the Long-Short Term Memory, Convolutional Neural Networks, Support Vector Regression and Random Forest, respectively. Secondly, the possibilistic mean and possibilistic variance with different risk attitudes are calculated within the framework of possibilistic theory. Thirdly, considering the borrowing constraints, transaction costs, and threshold constraints, a fuzzy portfolio selection model with different risk attitudes based on Machine Learning is proposed. According to the possibilistic measure, the proposed model is transformed into a quadratic programming problem, and the pivoting algorithm is designed to obtain the optimal solution. Finally, the possibility equal-weight portfolio model and possibility mean-variance portfolio model based on history quartile are selected as benchmarks, and a comparing numerical example is provided to demonstrate the effectiveness of the proposed model.

Keywords: The fuzzy portfolio selection, possibilistic mean, possibilistic variance, different risk attitudes, machine learning.

1 Introduction

To avoid damage from risk factors such as inflation, individuals and institutions often need asset preservation and appreciation by trade of return and risk. The mean-variance model proposed by Markowitz (1952)[28] first used mathematical methods to balance returns and risks, and laid the foundation for modern portfolio theory.

Many researchers studied portfolio optimization problems in the stochastic environment. However, there are many factors, such as economic, social, policy, and people's psychological, that are related to asset return, and not suited for description by random variables. Thus, some researchers introduced the fuzzy set theory into the framework of asset management and studied fuzzy portfolio optimization. For example, Zhang et al. (2009)[44] proposed a possibilistic mean-variance portfolio selection model and obtained the optimal solution by a sequential minimal optimization algorithm. Qin (2017)[33] formulated a random fuzzy mean absolute deviation portfolio selection model, the optimal solution can be calculated by random fuzzy simulation and simulation-based genetic algorithm. Ghahtarani (2021)[12] presented a new fuzzy portfolio selection model by application of Z-number theory and fuzzy neural network. Tsaur et al. (2021)[36] studied a possibilistic portfolio selection problem with a guaranteed rate of return. Kumar (2023)[23] studied the three-dimensional mixed intuitionistic fuzzy assignment problems. There are also many extended researches based on downside risk. For example, Zhou et al. (2018)[45] presented a mean lower absolute deviation fuzzy portfolio selection model. Zhang et al. (2022)[42] proposed a fuzzy mean semi-variance portfolio optimization model with sell orders. Chen (2015)[4] used semi-absolute-deviation to measure risk and presented a fuzzy mean-semi-absolute-deviation portfolio selection model. Pahade and Jha (2022)[31] proposed a possibilistic mean semi-absolute deviation portfolio model with V-shaped transaction cost. Deng and Chen (2024)[6] studied a fuzzy mean-semi-absolute deviation portfolio selection problem with a power membership function. Considering some realistic constraints, such as transaction

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costs and cardinality constraints, Zhang and Zhang (2014)[43] proposed a multiperiod mean absolute deviation fuzzy portfolio selection model. Mehlawat et al. (2018)[30] and Mansour et al. (2019)[27] studied fuzzy multi-objective portfolio optimization problems. Zhang and Dang (2021) [41] present a new weighted lower and upper admissible mean downside semi-variance portfolio selection model. Gong et al. (2022)[13] proposed two multiperiod coherent fuzzy portfolio selection models, which include investor's attitude. Deng et al. (2022)[7] presented a fuzzy mean semi-absolute deviation-Yager entropy portfolio selection model with dynamic risk tolerance and used an NSGA-II to solve the model. Kharrim (2023)[20] studied a multi-period possibilistic mean-semivariance portfolio problem and used the dynamic programming method to solve the problem. In addition, the fuzzy set theory is also applied in some realistic scenarios. For instance, Zabaleta et al. (2022)[39] present a Fuzzy DEA model to evaluate water security under the Pressure-State-Response approach. Deveci et al. (2023)[9] applied a type-2 neutrosophic number method to handle uncertainty in dealing with intelligent transportation systems. Deveci et al. (2023)[8] used a multi-criteria decision-making method to evaluate the alternative implementation options for autonomous vehicles in the metaverse.Rasoulzadeh et al. (2024)[34] incorporate returns represented as trapezoidal intuitionistic fuzzy numbers, and introduce an innovative four-objective model that intricately blends risk, return, and efficiency considerations.

Predicted asset features, such as price, return, and risk, are essential for portfolio management. With the development of computer technology, Machine Learning has more advantages in dealing with nonlinear, high-dimensional, large-scale datasets, and is widely applied in forecasting fields. For example, Fischer and Krauss (2018)[11] used LSTM to predict the out-of-sample direction movements of stock. Beniwal et al. (2023)[1] predicted a long-term stock price by a forward-validating Genetic Algorithm Optimization for SVR. Henriques and Sadorsky (2023)[16] applied RF, RNN, and SVR to forecast direction of daily rare earth stock price. Koo and Kim (2024)[22] used LSTM, SVR, and Multi-layer perceptron to predict the Bitcoin price. Jha et al. (2024)[19] forecasted crude oil spot prices by SVR. Some researchers try to improve the prediction accuracy of LSTM by optimal hyperparameters. For instance, Gülmez (2023)[14] presented an optimized deep LSTM network to predict stock prices, in which the hyperparameters were optimized by an artificial rabbits optimization algorithm. Wang et al. (2024)[38] dealt with visual sequence data by the LSTM, in which hyperparameters were optimized by the sparrow search algorithm. Other researchers improve prediction accuracy based on input features. For example, Khodaee et al. (2022)[21] converted input features into 2D images and used the hybrid CNN and LSTM to predict the turning points of stock prices. Jeribi et al. (2024)[18] predicted the stock market by Deep Learning, in which the input features were extracted and selected by multiple technologies.

The application of machine learning in portfolio management is another field, which is concerned by researchers. For example, Lee and Yoo (2020)[25] used a Neural Network to optimal the threshold-based portfolio. Paiva et al. (2019)[32] used SVM to forecast the asset return trends and proposed a mean-variance portfolio model based on the predictive returns. Mazraeh et al. (2022)[29] used Machine Learning, technical analysis, and Markov Chains to filter stock exchanges and predicted buy signals, the MOGWO and NSGA-II were used for optimal portfolios. Du (2022)[10] proposed a predicted mean-variance portfolio selection model based on Machine Learning. Sharma and Shekhawat (2022)[35] used a deep Belief-Recurrent Neural Network to predict return, and Harris Hawks-Deer Hunting Optimization was used to select assets and calculate the optimal weight. Ma et al. (2023)[26] studied a worst-case omega portfolio selection problem based on LSTM with an autoencoder for feature extraction. Habbab and Kampouridis (2024)[15] applied LR, SVR, KNN, and LSTM to predict future prices of stocks and bonds and optimized portfolios with a Genetic Algorithm.

It is not difficult to find that Machine Learning can improve prediction accuracy and has been applied in asset management. However, it can be found from **Table 1.**: portfolio optimization researches based on Machine Learning are focused on random environments, and few studies apply Machine Learning to fuzzy portfolio optimization. Based on this, we proposed a predicted fuzzy portfolio selection model based on Machine Learning, which considers risk attitudes, transaction costs, borrowing constraints, and threshold constraints. A pivoting algorithm is designed to solve the optimal solution. The main novel contributions of this paper can be summarized as follows:

(1) We first define the possibilistic mean, variance, and covariance with different risk attitudes and analyze their mathematical property. Considering the borrowing constraints, transaction costs, and threshold constraints, a predicted fuzzy portfolio selection model with different risk attitudes is proposed. To our knowledge, it has not yet been studied in previous research.

(2) We use hybrid Machine Learning to predict asset fuzzy numbers. First, hybrid Machine Learning methods, including the LSTM, CNN, SVR, and RF, are used to predict the opening, closing, highest, and lowest prices, respectively. Then, based on the prediction prices, the predicted fuzzy numbers of asset prices can be obtained by the "Price Rank" method.

(3) We designed a pivoting algorithm of quadratic programming to solve the optimal solution. This algorithm is highly efficient because it does not require adding variables or changing the form, allowing for quick identification and removal of redundant constraints.

The rest of this paper is organized as follows: In Section 2, four Machine Learning models, i.e., LSTM, CNN, SVR, and RF, are introduced. In Section 3, the possibilistic mean, variance, and covariance with different risk attitudes are defined, several properties associated with these definitions are presented. In Section 4, considering the borrowing constraints, transaction costs, and threshold constraints, a new predicted fuzzy portfolio selection model with different risk attitudes is proposed, and the pivoting algorithm of the quadratic programming is used to solve the model. In Section 5, the in-sample and out-of-sample analysis are given to illustrate the effectiveness of the proposed model. At last, some conclusions are given in Section 6.

Table 1: The relevant researches in portfolio selection and their details

Reference	Environment	Machine Learning	Transaction Cost	Threshold Constraint	No Short Selling	Borrow and Lending	Behavioral Factors
Chen (2015)	fuzzy	×	✓	✓	✓	×	×
Qin (2017)	fuzzy	×	×	✓	✓	×	×
Mehlawat et al. (2018)	fuzzy	×	×	✓	✓	×	×
Zhou et al. (2018)	fuzzy	×	✓	×	×	×	✓
Mansour et al. (2019)	fuzzy	×	×	×	✓	×	×
Tsaur et al. (2021)	fuzzy	×	×	✓	×	×	×
Gong et al. (2022)	fuzzy	×	✓	✓	✓	×	×
Deng et al. (2022)	fuzzy	×	✓	✓	✓	×	×
Pahade and Jha (2022)	fuzzy	×	✓	✓	✓	×	×
Kharrim (2023)	fuzzy	×	✓	✓	×	✓	×
Deng and Chen (2024)	fuzzy	×	×	×	✓	×	✓
Paiva et al. (2019)	stochastic	✓	×	✓	✓	×	×
Du (2022)	stochastic	✓	×	×	✓	×	×
Mazraeh et al. (2022)	stochastic	✓	×	×	✓	×	×
Ma et al. (2023)	stochastic	✓	×	×	✓	×	×
Habbab and Kampouridis (2024)	stochastic	×	×	×	×	×	×
Wang et al. (2024)	stochastic	✓	×	×	✓	×	×
This Paper	fuzzy	✓	✓	✓	✓	✓	✓

2 Machine learning

2.1 Long-Short Term Memory networks

Long Short-Term Memory (LSTM) was proposed by Hochreiter and Schmidhuber (1997)[17]. It autonomously controls information flows by introducing input, forget, and output gates. It also has advantages in capturing long-term dependency information by dealing with gradient vanishing or exploding problems.

The operating principle of LSTM can be summarized as follows: Firstly, the information flows x_{t-1} and hidden state outputs h_{t-1} are put into the forget, input, candidate memory, and output gates. Then, the sigmoid and tanh active functions are used to calculate each gate's outputs. Secondly, the degree of information preservation can be adjusted through the forget gates and input gates. For example, the long-term information is preserved when F_t=1 and I_t=0, and it is dropped when F_t=1 and I_t=0. Finally, the outputs of the memory and hidden state of period t can be calculated according to formula (1) and formula (2).

$$C_t = F_t \odot C_{t-1} + I_t \odot \widetilde{C}_t. \tag{1}$$

$$H_t = O_t \odot \tanh\left(C_t\right). \tag{2}$$

where X_t is information flows, W_{xf}, W_{hf}, W_{xi}, W_{hi}, W_{xo}, W_{ho}, W_{xc} and W_{hc} are the weight matrix of the forget, input, candidate memory, and output gates, b_f, b_i, b_o, b_c are the bias vector of each gate.

$$\mathrm{F}_t = \sigma\left(\mathrm{X}_t\mathrm{W}_{xf} + \mathrm{H}_{t-1}\mathrm{W}_{hf} + b_f\right), \mathrm{I}_t = \sigma\left(\mathrm{X}_t\mathrm{W}_{xi} + \mathrm{H}_{t-1}\mathrm{W}_{hi} + b_i\right),$$

$$\mathrm{O}_t = \sigma\left(\mathrm{X}_t\mathrm{W}_{xo} + \mathrm{H}_{t-1}\mathrm{W}_{ho} + b_o\right), \widetilde{C}_t = \tanh\left(\mathrm{X}_t\mathrm{W}_{xc} + \mathrm{H}_{t-1}\mathrm{W}_{hc} + b_c\right).$$

2.2 Convolutional neural networks

Convolutional Neural Networks (CNN) was proposed by LeCun et al. (1989)[24]. It can improve efficiency through local connections and weight sharing. It also can reduce computational complexity and overfitting risk by introducing the pooling layer.

CNN consists of input, convolutional, pooling, fully connected, and output layers. And the operating principle of CNN can be summarized as follows:

Firstly, the information flows are input to the convolutional layers, and then the basic features of the flows can be extracted according to formula (3).

$$x_i^l = f(\sum_{i=1}^{j^{l-1}} x_j^{l-1} w_{ij}^l + b_j^l), \tag{3}$$

where ${x_i}^l$ and ${x_i}^l - 1$ is the j_{th} input of the convolution layer l and convolutional layer l - 1, ${w_{ij}}^l$ is the convolutional numerical weight matrix of the convolution layer l, ${b_i}^l$ is the bias vector, $f(\cdot)$ is the non-linear aggressive activation function.

Secondly, the outputs of convolutional layers as new flows are inputted to the pooling layers, which can reduce the number of parameters and avoid the overfitting problem. When using the maximum pooling, the output can be calculated by formula (4).

$$x_i^{l+1} = \max x_i^l(s), \tag{4}$$

where ${x_i}^{l+1}$ is the output of the largest pooling layer. ${x_i}^l(s)$ is the s-value in the pooling area.

Thirdly, the fully connected layers are introduced to aggregate the information from convolutional and pooling layers, which can be expressed by formula (5). Finally, the predicted output can be obtained through the output layers.

$$y_i^l = f(w^l x^{l-1} + b^l), \tag{5}$$

where y^l is the output of the full connection layer, and x^{l-1} is an expanded one-dimensional vector.

2.3 Support vector regression

Support Vector Regression (SVR) was proposed by Vapnik (1998)[37]. This algorithm is based on statistical learning theory and attention to the most effective decision boundary. It is robust and suitable for predicting small sample data. And the nonlinear relationship data can be processed by setting kernel functions. The operating principle of SVR can be summarized as follows:

This algorithm minimizes the sum of squared errors adds a penalty function as an objective, and seeks a function that is farthest from all training samples. By solving the quadratic programming problem, the optimal hyperplane can be found and the prediction can be calculated.

2.4 Random forest

Random forest (RF) was proposed by Breiman (2001)[2]. It combines multiple decision tree results and has the advantages of high accuracy, strong interpretability, and parallel computing. Therefore, it is suitable for handling high-dimensional, imbalanced, and missing datasets. The operating principle of RF can be summarized as follows:

(1) Construct some sub-training sets by bootstrap randomly from the training set.

(2) Construct a decision tree. Firstly, K feature are randomly selected as candidate features for each sub-training set. Then, a feature and a threshold are selected to maximize the node purity, the best splitting point can be found. Finally, repeat feature selection and optimal splitting process until max-depth or min-sample-nodes.

(3) Decision integrations. The predicted result is the average of each decision tree's prediction.

3 Fuzzy mean and variance with different risk attitudes

This section introduces some related definitions in this paper. A fuzzy number A is a fuzzy set of the real line R with a normal, fuzzy convex, and continuous membership function of bounded support. The family of fuzzy numbers is denoted by $\mathcal{F}$.

Definition 3.1. *Carlsson and Fullér (2002)*[3] *defined the lower and upper possibilistic mean values of fuzzy number* P_i *with* γ*-level set* $[P_i]^\gamma = [a_{i1}(\gamma), a_{i2}(\gamma)]$, $(\gamma \in [0,1])$ *as*

$$M_*(P_i) = \frac{\int_0^1 a_{i1}(\gamma) Pos(P_i \le a_{i1}(\gamma))d\gamma}{\int_0^1 Pos(P_i \le a_{i1}(\gamma))d\gamma} = 2\int_0^1 \gamma a_{i1}(\gamma)d\gamma, \tag{6}$$

and

$$M^*(P_i) = \frac{\int_0^1 a_{i2}(\gamma) Pos(P_i \ge a_{i2}(\gamma))d\gamma}{\int_0^1 Pos(P_i \ge a_{i2}(\gamma))d\gamma} = 2\int_0^1 \gamma a_{i2}(\gamma)d\gamma, \tag{7}$$

where Pos denotes possibility, i.e.,

$$Pos(P_i \le a_{i1}(\gamma)) = \sup_{u \le a_{i1}(\gamma)} P_i(u) = \gamma, Pos(P_i \ge a_{i2}(\gamma)) = \sup_{u \ge a_{i2}(\gamma)} P_i(u) = \gamma. \tag{8}$$

Definition 3.2. *Let P_i be a fuzzy number, in which the lower and upper possibilistic mean values are $M_*(P_i)$ and $M^*(P_i)$; θ be different risk attitudes associated with $M^*(P_i)$. Considering different risk attitudes, the possibilistic mean with different risk attitudes of P_i can be defined as:*

$$M(P_i(\theta)) = (1-\theta)M_*(P_i) + \theta M^*(P_i), \tag{9}$$

where $0 \le \theta \le 1$, the bigger θ means that investors are more optimistic due to the higher return, and θ = 0, θ = 1/2, and θ =1 represents investors' pessimistic, neutral, and optimistic, respectively. Between the two extreme cases, for $0 < \theta < 1$, the investor's attitude towards return is neither too pessimistic nor too optimistic.

*According to **Definition 3.1** and **Definition 3.2**, the **Theorem 3.3** can be obtained.*

Theorem 3.3. *Let P_i and P_j be fuzzy numbers, λ and μ are nonnegative real numbers, θ be the different risk attitudes associated with $M^*(P_i)$ and $M^*(P_j)$, $\theta \le \theta \le 1$, $M(P_i(\theta)) = (1-\theta)M_*(P_i) + \theta M^*(P_i)$ and $M(P_j(\theta)) = (1-\theta)M_*(P_j) + \theta M^*(P_j)$, then*

$$M(\lambda P_i(\theta)) = \lambda M(P_i(\theta)), \tag{10}$$

$$M(\lambda P_i(\theta) + \mu) = \lambda M(P_i(\theta)) + \mu, \tag{11}$$

$$M(\lambda P_i(\theta) + \mu P_j(\theta)) = \lambda M(P_i(\theta)) + \mu M(P_j(\theta)). \tag{12}$$

Definition 3.4. [44] *Let P_i is a fuzzy number with γ-level set $[P_i]^\gamma = [a_{i1}(\gamma), a_{i2}(\gamma)]$, $\gamma \in [0,1]$, the lower and upper possibilistic variances of fuzzy number P_i can be defined as:*

$$Var_*(P_i) = 2\int_0^1 \gamma(M_*(P_i) - a_{i1}(\gamma))^2 d\gamma, \tag{13}$$

$$Var^*(P_i) = 2\int_0^1 \gamma(M^*(P_i) - a_{i2}(\gamma))^2 d\gamma. \tag{14}$$

Definition 3.5. *Let P_i be a fuzzy number that the lower and upper possibilistic variance values are $Var_*(P_i)$ and $Var^*(P_i)$, θ be different risk attitudes associated with $Var_*(P_i)$. The possibilistic variance with different risk attitudes of P_i can be defined as:*

$$Var(P_i(\theta)) = \theta Var_*(P_i) + (1-\theta)Var^*(P_i), \tag{15}$$

where $0 \le \theta \le 1$, the bigger θ means that the investors are more optimistic due to the lower variance. And θ = 0, θ = 1/2, and θ =1 represents investors' pessimistic, neutral, and optimistic, respectively. Between the two extreme cases, for $0 < \theta < 1$, the investor's attitude towards variance is neither too pessimistic nor too optimistic.

Definition 3.6. [44] *Let P_i and P_j be fuzzy numbers with γ-level set $[P_i]^\gamma = [a_{i1}(\gamma), a_{i2}(\gamma)]$ and $[P_j]^\gamma = [a_{j1}(\gamma), a_{j2}(\gamma)]$, $\gamma \in [0,1]$. Then, the lower and upper possibilistic covariance of fuzzy numbers P_i and P_j can be defined as:*

$$Cov_*(P_i, P_j) = 2\int_0^1 \gamma(M_*(P_i) - a_{i1}(\gamma))(M_*(P_j) - a_{j1}(\gamma))d\gamma, \tag{16}$$

$$Cov^*(P_i, P_j) = 2\int_0^1 \gamma(M^*(P_i) - a_{i2}(\gamma))(M^*(P_j) - a_{j2}(\gamma))d\gamma. \tag{17}$$

Definition 3.7. *Let P_i and P_j be fuzzy numbers, in which the lower and upper possibilistic covariance values are $Cov_*(P_i, P_j)$ and $Cov^*(P_i, P_j)$, θ be the different risk attitudes associated with $Cov_*(P_i, P_j)$, $0 \leq \theta \leq 1$. Then, the possibilistic covariance with different risk attitudes of R_i and R_j can be defined as:*

$$Cov(P_i(\theta), P_j(\theta)) = \theta Cov_*(P_i, P_j) + (1-\theta)Cov^*(P_i, P_j). \tag{18}$$

According to **Definition 3.5** and **Definition 3.7**, the **Theorem 3.8** can be obtained.

Theorem 3.8. *Let P_i and P_j be fuzzy numbers with γ-level set, $[P_i]^\gamma = [a_{i1}(\gamma), a_{i2}(\gamma)]$ and $[P_j]^\gamma = [a_{j1}(\gamma), a_{j2}(\gamma)]$, $\gamma \in [0,1]$, θ be different risk attitudes, λ and μ are nonnegative real numbers. Then,*

$$Var(\lambda P_i(\theta)) = \lambda^2 Var(P_i(\theta)), \tag{19}$$

$$Var(\lambda P_i(\theta) + \mu) = \lambda^2 Var(P_i(\theta)), \tag{20}$$

$$Cov(\lambda P_i(\theta), \mu P_j(\theta)) = \lambda\mu Cov(P_i(\theta), P_j(\theta)). \tag{21}$$

Let $P_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$ be a trapezoidal fuzzy number, which membership function can be expressed as:

$$\mu_{P_i}(x) = \begin{cases} \frac{x-a_{i1}}{a_{i2}-a_{i1}}, & \text{if } x \in [a_{i1}, a_{i2}] \\ 1, & \text{if } x \in [a_{i2}, a_{i3}] \\ \frac{a_{i4}-x}{a_{i4}-a_{i3}}, & \text{if } x \in [a_{i3}, a_{i4}] \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

where $a_{i2} - a_{i1}$ and $a_{i4} - a_{i3}$ are positive numbers i.e., $a_{i2} - a_{i1}$, $a_{i4} - a_{i3} > 0$. Thus, the γ-level set of trapezoidal fuzzy number P_i can be expressed as $[P_i]^\gamma = [a_{i2} - (1-\gamma)(a_{i2} - a_{i1}), a_{i3} + (1-\gamma)(a_{i4} - a_{i3})]$, for all $\gamma \in [0,1]$.

The **Theorem 3.9** can be obtained based on the **Theorem 3.3**.

Theorem 3.9. *Let $P_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$ be a trapezoidal fuzzy number, $[P_i]^\gamma = [a_{i2} - (1-\gamma)(a_{i2} - a_{i1}), a_{i3} + (1-\gamma)(a_{i4} - a_{i3})]$, and $\gamma \in [0,1]$; θ be different risk attitudes associated with $M^*(P_i)$, $0 \leq \theta \leq 1$. The possibilistic mean with different risk attitudes can be expressed as:*

$$M(P_i(\theta)) = (1-\theta)(a_{i2} - (a_{i2} - a_{i1})/3) + \theta(a_{i3} + (a_{i4} - a_{i3})/3). \tag{23}$$

The following **Theorem 3.10** can be obtained based on the **Theorem 3.8**.

Theorem 3.10. *Let $P_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$ and $P_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4})$ be trapezoidal fuzzy numbers, $[P_i]^\gamma = [a_{i2} - (1-\gamma)(a_{i2} - a_{i1}), a_{i3} + (1-\gamma)(a_{i4} - a_{i3})]$, $[P_j]^\gamma = [a_{j2} - (1-\gamma)(a_{j2} - a_{j1}), a_{j3} + (1-\gamma)(a_{j4} - a_{j3})]$, and $\gamma \in [0,1]$; θ be different risk attitudes associated with $Var^*(P_i)$ and $Cov^*(P_i, P_j)$, $0 \leq \theta \leq 1$. The possibilistic variance and covariance with different risk attitudes can be expressed as:*

$$Var(P_i(\theta)) = \frac{\theta(a_{i2} - a_{i1})^2 + (1-\theta)(a_{i4} - a_{i3})^2}{18}, \tag{24}$$

$$Cov(P_i(\theta), P_j(\theta)) = \frac{\theta(a_{i2} - a_{i1})(a_{j_2} - a_{j_1}) + (1-\theta)(a_{i4} - a_{i3})(a_{j_4} - a_{j_3})}{18}. \tag{25}$$

4 The predicted fuzzy portfolio selection with different risk attitudes

This section discusses the portfolio selection problem with fuzzy returns. The problem description and notations are introduced first. Then, the possibilistic mean, covariance, and variance with different risk attitudes of the portfolio are formulated. Finally, considering the borrowing constraints, threshold constraints, and transaction costs, a predicted fuzzy portfolio selection model with different risk attitudes is proposed.

4.1 Problem description and notations

Assuming investors allocate wealth into n risky assets and one risk-free asset. Let W_0 be the initial wealth; p_{0i} be the initial price of risky asset i; r_f be the price of risk-free asset; ω_{0i} be the initial investment number of risky asset i; ω_i be the investment number of risky asset i, where $\omega = (\omega_1, \omega_2, ..., \omega_n)'$, the prime ($'$) denotes matrix transposition; ω_f be the investment amount of risk-free asset, where $\omega_f = W_0 - [(\omega_1 - \omega_{01}) * p_{01} + (\omega_2 - \omega_{02}) * p_{02} + \ldots + (\omega_n - \omega_{0n}) * p_{0n}]$; P_i be the price of risky asset i,$i = 1, ..., n$, $\overset{\wedge}{P_i} = (\overset{\wedge}{a_{i1}}, \overset{\wedge}{a_{i2}}, \overset{\wedge}{a_{i3}}, \overset{\wedge}{a_{i4}})$ is the estimated value of P_i, which is predicted by LSTM, CNN, SVR, and RF, respectively.

4.2 Possibilistic mean, variance and real constraints of portfolio selection

Considering different risk attitudes, the possibilistic mean value of the portfolio $\omega = (\omega_1, \omega_2, ..., \omega_n)'$ can be expressed as:

$$\begin{aligned} W_p = \sum_{i=1}^{n} M(P_i(\theta))\omega_i + r_f\omega_f = \sum_{i=1}^{n} ((1-\theta)(\overset{\wedge}{a_{i2}} - (\overset{\wedge}{a_{i2}} - \overset{\wedge}{a_{i1}})/3) \\ + \theta(\overset{\wedge}{a_{i3}} + (\overset{\wedge}{a_{i4}} - \overset{\wedge}{a_{i3}})/3))\omega_i + r_f(W_0 - \sum_{i=1}^{n} p_{0i}(\omega_i - \omega_{0i})). \end{aligned} \tag{26}$$

Let c_i be the unit transaction cost of risky asset i with V-shaped function. Then, the transaction cost for asset i is $c_i p_{0i}|\omega_i - \omega_{0i}|$, and the total transaction costs can be expressed as:

$$C(x) = \sum_{i=1}^{n} c_i p_{0i} \, |\omega_i - \omega_{0i}|. \tag{27}$$

Thus, the net wealth the portfolio ω can be denoted as

$$W_N = \sum_{i=1}^{n} ((1-\theta)(\overset{\wedge}{a_{i2}} - (\overset{\wedge}{a_{i2}} - \overset{\wedge}{a_{i1}})/3) + \theta(\overset{\wedge}{a_{i3}} + (\overset{\wedge}{a_{i4}} - \overset{\wedge}{a_{i3}})/3))\omega_i + r_f(W_0 - \sum_{i=1}^{n} p_{0i}(\omega_i - \omega_{0i})) - \sum_{i=1}^{n} c_i p_{0i} \, |\omega_i - \omega_{0i}|. \tag{28}$$

Considering different risk attitudes, the possibilistic variance of the portfolio ω can be expressed as:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} (\frac{\theta(\overset{\wedge}{a_{i2}} - \overset{\wedge}{a_{i1}})(\overset{\wedge}{a_{j2}} - \overset{\wedge}{a_{j1}}) + (1-\theta)(\overset{\wedge}{a_{i4}} - \overset{\wedge}{a_{i3}})(\overset{\wedge}{a_{j4}} - \overset{\wedge}{a_{j3}})}{18})\omega_i\omega_j. \tag{29}$$

Let ${\omega^f}_b$ be the minimum value of ω^f, where ${\omega^f}_b \leq 0$, the borrowing constraint of the risk-free asset can be expressed as:

$$\omega_f = W_0 - \sum_{i=1}^{n} p_{0i}(\omega_i - \omega_{0i}) \geq \omega_f^b. \tag{30}$$

Let u_i be the upper bound of disposable wealth, where $u_i \geq 0$, the threshold constraints of risky asset i can be expressed as:

$$0 \leq p_{0i}\omega_i \leq u_i(W_0 - \omega_f^b). \tag{31}$$

4.3 The predicted fuzzy mean variance portfolio selection model

Considering the borrowing constraints, transaction costs and threshold constraints, a predicted fuzzy mean variance portfolio selection model with different risk attitudes can be expressed as:

$$\min \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\theta(\hat{a_{i2}}-\hat{a_{i1}})(\hat{a_{j2}}-\hat{a_{j1}})+(1-\theta)(\hat{a_{i4}}-\hat{a_{i3}})(\hat{a_{j4}}-\hat{a_{j3}})}{18}\right)\omega_i\omega_j,$$

$$\text{s.t}\begin{cases}\sum_{i=1}^{n}((1-\theta)(\hat{a_{i2}}-(\hat{a_{i2}}-\hat{a_{i1}})/3)+\theta(\hat{a_{i3}}+(\hat{a_{i4}}-\hat{a_{i3}})/3))\omega_i \\ +r_f(W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i}))-\sum_{i=1}^{n}c_ip_{0i}\left|\omega_i-\omega_{0i}\right|\geq W_{0p} & \text{(a)}\\ W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i})\geq\omega_f^b & \text{(b)}\\ 0\leq p_{0i}\omega_i\leq u_i(W_0-\omega_f^b), i=1,\cdots,n & \text{(c)}\end{cases} \tag{32}$$

where constraint (a) denotes that the net wealth of the portfolio is not less than the present value W_0p; constraint (b) indicates the borrowing constraints; constraint (c) states the threshold constraints of risky asset i. When $\theta=0$, $\theta=\frac{1}{2}$, and $\theta=1$, the Model (32) indicates that the investor is pessimistic, neutral, and optimistic, respectively.

Let $\pi_i=|\omega_i-\omega_{0i}|$. Then the Model (32) can be rewritten as follows:

$$\min \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\theta(\hat{a_{i2}}-\hat{a_{i1}})(\hat{a_{j2}}-\hat{a_{j1}})+(1-\theta)(\hat{a_{i4}}-\hat{a_{i3}})(\hat{a_{j4}}-\hat{a_{j3}})}{18}\right)\omega_i\omega_j,$$

$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}\left((1-\theta)\left(\hat{a_{i2}}-\left(\hat{a_{i2}}-\hat{a_{i1}}\right)/3\right)+\theta\left(\hat{a_{i3}}+\left(\hat{a_{i4}}-\hat{a_{i3}}\right)/3\right)\right)\omega_i \\ +r_f(W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i}))-\sum_{i=1}^{n}c_ip_{0i}\pi_i\geq W_{0p}\\ W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i})\geq\omega_f^b\\ 0\leq p_{0i}\omega_i\leq u_iW_0-\omega_f^b,\quad i=1,\ldots,n\\ \pi_i\geq\omega_i-\omega_{0i},\quad \pi_i\geq-(\omega_i-\omega_{0i})\\ \pi_i\geq 0\end{cases} \tag{33}$$

When the covariance matrix is semi-definite, the objective of portfolio model is quadratic function. The constraints are linear constraints, so the Model (33) is a convex quadratic programming problem, and can be solved by the pivoting algorithm.

In Model (33), the constraint $0\leq p_{0i}\omega_i\leq u_i(W_0-\omega_f^b)$,and the conditions $\pi_i\geq\omega_i-\omega_{0i},\quad \pi_i\geq-(\omega_i-\omega_{0i})$, can be turned into the following constraints:

$$\omega_i\geq 0,\quad -\omega_i\geq -u_i(W_0-\omega_f^b)/p_{0i}, -\omega_i+\pi_i\geq-\omega_{0i},\quad \omega_i+\pi_i\geq\omega_{0i}. \tag{34}$$

According to formula (34), the Model (33) can be turned into the following model:

$$\min \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\theta(\hat{a_{i2}}-\hat{a_{i1}})(\hat{a_{j2}}-\hat{a_{j1}})+(1-\theta)(\hat{a_{i4}}-\hat{a_{i3}})(\hat{a_{j4}}-\hat{a_{j3}})}{18}\right)\omega_i\omega_j,$$

$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}\left((1-\theta)\left(\hat{a_{i2}}-\left(\hat{a_{i2}}-\hat{a_{i1}}\right)/3\right)+\theta\left(\hat{a_{i3}}+\left(\hat{a_{i4}}-\hat{a_{i3}}\right)/3\right)\right)\omega_i \\ +r_f(W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i}))-\sum_{i=1}^{n}c_ip_{0i}\pi_i\geq W_{0p} \\ W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i})\geq\omega_f^b \\ \omega_i\geq 0,\quad -\omega_i\geq-\left(u_i(W_0-\omega_f^b)/p_{0i}\right. \\ -\omega_i+\pi_i\geq-x_{0i},\quad \omega_i+\pi_i\geq-\omega_{0i} \\ \pi_i\geq 0,\quad i=1,\ldots,n\end{cases}\tag{35}$$

Let ρ_1 and ρ_2 be the Lagrange multipliers associated with constraint (1) and constraint (2) in Model (35), ϕ_i and $\overline{\phi_i}$, $i=1,\ldots,n$, be the Lagrange multipliers associated with $\omega_i\geq 0$, $-\omega_i\geq -u_i(W_0-\omega_f^b)/p_{0i}$, τ_i and $\overline{\tau_i}$ be the Lagrange multipliers associated with $-\omega_i+\pi_i\geq-\omega_{0i}$, $\omega_i+\pi_i\geq\omega_{0i}$, and o_i, $i=1,\ldots,n$, be the Lagrange multipliers associated with $\pi_i\geq 0$. Let $\sigma_{ij}=(\theta(\hat{a_{i2}}-\hat{a_{i1}})(\hat{a_{j2}}-\hat{a_{j1}})+(1-\theta)(\hat{a_{i4}}-\hat{a_{i3}})(\hat{a_{j4}}-\hat{a_{j3}}))/18$, $M(P_i(\theta))=((1-\theta)(\hat{a_{i2}}-(\hat{a_{i2}}-\hat{a_{i1}})/3)+\theta(\hat{a_{i3}}+(\hat{a_{i4}}-\hat{a_{i3}})/3)$. The KKT conditions for the Model (35) are as follows:

$$\begin{cases}\sigma_{i1}\omega_1+\sigma_{i2}\omega_2+\cdots+\sigma_{in}\omega_n-\rho_1(M(P_i(\theta))-r_fp_{0i})+\rho_2p_{0i}-\phi_i+\overline{\phi_i}+\tau_i-\overline{\tau_i}=0,\quad i=1,2,\ldots,n, \\ -\tau_i-\overline{\tau_i}-o_i=0,\quad i=1,2,\ldots,n \\ \rho_1\geq 0,\quad \rho_1\left(\sum_{i=1}^{n}M(P_i(\theta))\omega_i+r_f(W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i})-\sum_{i=1}^{n}c_ip_{0i}\pi_i-W_{0p}\right)=0 \\ \rho_2\geq 0,\quad \rho_2\left(\sum_{i=1}^{n}M(P_i(\theta))\omega_i-\omega_{0i}-\omega_f^b\right)=0 \\ \phi_i\geq 0,\quad \phi_i\omega_i=0,\quad i=1,2,\ldots,n \\ \overline{\phi_i}\geq 0,\quad \overline{\phi_i}(-\omega_i+u_i(W_{0\,f}^b)/p_{0i})=0,\quad i=1,2,\ldots,n \\ \tau_i\geq 0,\quad \tau_i(-\omega_i+\pi_i+\omega_{0i}),\quad i=1,2,\ldots,n \\ \overline{\tau_i}\geq 0,\quad \overline{\tau_i}(\omega_i+\pi_i-\omega_{0i}),\quad i=1,2,\ldots,n \\ o_i\geq 0,\quad o_i\pi_i\geq 0,\quad i=1,2,\ldots,n \\ \sum_{i=1}^{n}M(P_i(\theta))\omega_i+r_f(W_0-\sum_{i=1}^{n}p_{0i}(\omega_i-\omega_{0i})-\sum_{i=1}^{n}c_ip_{0i}\pi_i\geq W_{0p} \\ \omega_i\geq 0,\quad -\omega_i\geq -u_i(W_0-\omega_f^b)/p_{0i},\quad i=1,2,\ldots,n \\ -\omega_i+\pi_i\geq-\omega_{0i},\quad \omega_i+\pi_i\geq\omega_{0i},\quad \pi_i\geq 0,\quad i=1,2,\ldots,n\end{cases}\tag{36}$$

where there are $7n+2$ variables, and $12n+4$ linear equalities or inequalities in Model (36). The pivoting algorithm of the quadratic programming is used to obtain the optimal solution(Zhang, 2019)[40] .

5 Numerical example

In this section, we assume that risky asset price is a trapezoidal fuzzy number, which can be predicted by Machine Learning. The sample assets are chosen from the component stocks of the SSE 50 Index. To ensure the completeness of the numerical examples, the 34 stocks are finally selected while the sample with missing data is dropped. The assets codes are as follows: SH_1 (600000), SH_2 (600028), SH_3 (600030), SH_4 (600031), SH_5 (600036),SH_6 (600048), SH_7 (600050), SH_8 (600104), SH_9 (600196), SH_{10} (600276), SH_{11} (600309), SH_{12} (600436), SH_{13} (600438), SH_{14} (600519), SH_{15} (600547), SH_{16} (600570), SH_{17} (600585), SH_{18} (600588), SH_{19} (600690), SH_{20} (600745), SH_{21} (600809), SH_{22}

(600837), SH_{23} (600887), SH_{24} (600893), SH_{25} (600900), SH_{26} (601088), SH_{27} (601166), SH_{28} (601318), SH_{29} (601398), SH_{30} (601601), SH_{31} (601628), SH_{32} (601857), SH_{33} (601899), SH_{34} (601919). When $W_0 = 1000000$, $c_i = 0.003$ $(i = 1, \ldots, 34)$, $\omega_f^b = -0.1$, $r_f = 0.00125$, $l_i = 0$, $u_i = 0.5$ $(i = 1, \ldots, 34)$, $\omega_0 = (0, 0, \ldots, 0)$, the numerical example includes estimating prices based on machine learning, in-sample analysis and the out-of-sample analysis. The framework of numerical examples is shown in **Figure 1**. The main contents are as follows:

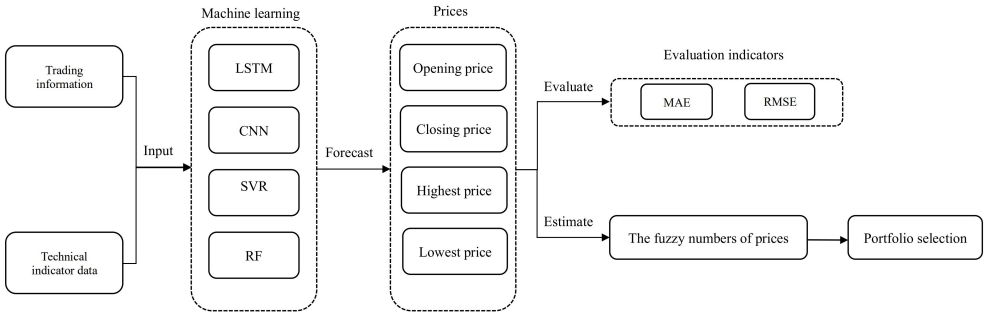


Figure 1: The framework of numerical examples

5.1 The estimation of the fuzzy prices

5.1.1 The input indicators and hyperparameters

Machine Learning is trained based on input indicators. In this paper, the 18 input indicators,which include price and technique data, are selected to predict stock price. The input indicators are shown in **Table 2**. Each indicator is standardized by z-score for improving prediction effectiveness.

$$\tilde{x} = \frac{x_i - x_{mean}}{\sigma_x}, \tag{37}$$

where x_i is the feature of stock i, x_{mean} and σ_x denote the mean and standard deviation, respectively.

Hyperparameters are essential to Machine Learning performance. This paper uses GridSearchCV to optimize the hyperparameter of four Machine Learning, i.e., LSTM, CNN, SVR, and RF. And 5-fold cross-validation is used in Machine Learning to avoid overfitting problems. The hyperparameters are shown in **Table 3**.

5.1.2 Evaluation of machine learning

In this paper, the mean absolute error (MAE) and root mean squared error ($RMSE$) are used to evaluate the prediction effectiveness of Machine Learning. The formulas are as follows:

$$MAE = \frac{1}{m}\sum_{i=1}^{m} |p_i - \hat{p}_i\,|. \tag{38}$$

$$RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m} (p_i - \hat{p}_i)^2}. \tag{39}$$

where m is the number of validation set, p_i and $\hat{p}_i$ denote the actual price and the predicted price, respectively. The prediction errors of LSTM, CNN, SVR, and RF are shown in **Table 4**.

The statistics of average predicted errors of Machine Learning are displayed in Table 4, which include mean, minimum, quartile 25%, median, quartile 75%, maximum, standard deviation, kurtosis, and skewness. From Table 4, it can be found that: when predicting the opening price, the MAE and $RMSE$ of CNN, SVR, and RF are similar and smaller than the error of LSTM. Similarly, it also can be seen that: the CNN and SVR performed better than RF and LSTM when predicting the closing prices; the SVR and RF performed better than CNN and LSTM when predicting the highest prices and lowest prices, respectively. Obviously, there is no one model that performs better for predicting all price. Thus, this paper takes the average value of predicted prices as the predicted value of prices. Let

Table 2: Indicators and Descriptions

Indicator	Description	Indicator	Description
P_t^o	The opening price at t	P_t^l	The lowest price at t
P_{t-1}^o	The opening price at $t-1$	P_{t-1}^l	The lowest price at $t-1$
P_{t-2}^o	The opening price at $t-2$	P_{t-2}^l	The lowest price at $t-2$
P_t^c	The closing price at t	TR	Turnover Rate
P_{t-1}^c	The closing price at $t-1$	MA	Moving Average
P_{t-2}^c	The closing price at $t-2$	$MACD$	Moving Average Convergence Divergence
P_t^h	The highest price at t	RSI	Relative Strength Index
P_{t-1}^h	The highest price at $t-1$	MI	Momentum Indicator
P_{t-2}^h	The highest price at $t-2$	STD	Standard Deviation

Table 3: The hyperparameters for Machine Learning

Method	Parameter	Value
LSTM	*Learning rate*	{0.01, 0.001, 0.0001}
	Dropout rate	{0.2, 0.3, 0.4}
CNN	*filters*	{32, 64, 128}
	Kernel size	$\{2, 3, 5\}$
SVR	*kernel*	{rbf, linear}
	C	$\{0.1, 1, 10\}$
	epsilon	{0.01, 0.1, 0.2}
RF	*n_estimators*	{100, 200, 300}
	max_depth	{None, 10, 20}
	min_samples_split	{2, 5, 10}

$\hat{P}_i = (\hat{a}_{i1}, \hat{a}_{i2}, \hat{a}_{i3}, \hat{a}_{i4})$ be a trapezoidal fuzzy number, where $\hat{a}_{i1} = \left(\hat{P}_{io}, \hat{P}_{ic}, \hat{P}_{ih}, \hat{P}_{il}\right)$, $\hat{a}_{i2} = \text{secmin}\left(\hat{P}_{io}, \hat{P}_{ic}, \hat{P}_{ih}, \hat{P}_{il}\right)$, $\hat{a}_{i3} = thi\text{min}\left(\hat{P}_{io}, \hat{P}_{ic}, \hat{P}_{ih}, \hat{P}_{il}\right)$, $\hat{a}_{i4} = \max\left(\hat{P}_{io}, \hat{P}_{ic}, \hat{P}_{ih}, \hat{P}_{il}\right)$, the possibilistic numbers of prices are shown in **Table 5**.

5.2 In-sample numerical examples

In this section, we perform the in-sample analysis to explore the impacts of parameters on the portfolio efficient frontier, including different risk attitudes, borrowing constraints, and the upper bound. The monthly data of the above 34 assets are collected as the sample from June 1, 2010, to May 31, 2022. The sample data are randomly split into a training set with 80% and a validation set with 20%, and the LSTM, CNN, SVR, and RF are used to predict prices. The results of the in-sample analysis are shown in **Figure 2** to **Figure 4**.

5.2.1 The impact of different risk attitudes

When $\theta = 0,\ 0.1,\ 0.2, \ldots, 1$, $u_i = 0.5$, $\omega_f^b = 0$, $c_i = 0.003$, $\omega_0 = (0, 0, \ldots, 0)$, the effective frontiers are shown in **Figure 2**.

Table 4: The prediction errors of LSTM, CNN, SVR and RF

Model	Statistic	Opening Price		Closing Price		Highest Price		Lowest Price	
		MAE	RMSE	MAE	RMSE	MAE	RMSE	MAE	RMSE
LSTM	Mean value	17.744	23.090	15.888	20.589	19.686	26.640	18.953	24.739
	Minimum	0.449	0.613	0.424	0.604	0.497	0.708	0.425	0.539
	Quartile 1	2.480	3.009	2.404	2.910	2.692	3.391	2.600	3.248
	Median	5.507	6.821	5.372	6.970	6.136	7.585	5.100	6.340
	Quartile 3	10.832	13.876	10.032	13.490	11.496	15.693	10.459	13.550
	Maximum	266.467	339.226	236.251	296.905	292.224	392.358	312.234	391.856
	Standard deviation	46.559	59.722	41.017	51.949	51.201	69.124	53.934	68.254
	Kurtosis	26.405	25.383	27.060	25.913	25.992	25.326	28.533	27.204
	Skewness	4.957	4.847	5.025	4.900	4.911	4.840	5.195	5.048
CNN	Mean value	5.749	8.422	8.958	12.226	8.125	12.211	7.934	11.977
	Minimum	0.228	0.296	0.315	0.425	0.322	0.429	0.326	0.444
	Quartile 1	0.852	1.161	1.548	1.997	1.242	1.625	1.123	1.455
	Median	1.876	2.593	3.192	3.916	2.367	3.554	2.796	3.630
	Quartile 3	3.751	5.344	5.991	8.071	4.937	7.484	5.336	7.146
	Maximum	88.495	123.168	139.434	190.181	128.049	187.509	118.780	170.494
	Standard deviation	15.282	21.629	23.889	32.727	22.038	32.671	20.616	30.364
	Kurtosis	27.997	25.629	29.085	28.529	28.617	26.912	27.182	23.987
	Skewness	5.131	4.873	5.251	5.192	5.202	5.014	5.049	4.711
SVR	Mean value	5.737	8.185	8.556	11.496	5.930	8.781	4.735	7.374
	Minimum	0.213	0.308	0.281	0.394	0.223	0.335	0.204	0.286
	Quartile 1	1.038	1.323	1.393	1.804	0.934	1.231	0.897	1.210
	Median	1.983	2.693	2.775	3.750	1.923	2.664	1.874	2.514
	Quartile 3	3.910	5.713	5.554	7.669	4.013	6.159	3.587	5.170
	Maximum	86.916	114.304	136.902	170.033	85.900	126.989	66.434	101.284
	Standard deviation	14.919	20.081	23.540	29.716	14.924	22.197	11.426	17.730
	Kurtosis	28.606	25.199	28.866	26.260	26.753	26.041	27.657	25.435
	Skewness	5.196	4.823	5.231	4.943	4.992	4.914	5.083	4.844
RF	Mean value	5.970	8.467	9.260	12.379	6.421	9.302	4.618	7.016
	Minimum	0.208	0.299	0.277	0.384	0.228	0.328	0.199	0.265
	Quartile 1	1.004	1.276	1.410	1.833	0.942	1.238	0.865	1.139
	Median	1.951	2.660	2.885	3.802	1.981	2.851	1.859	2.522
	Quartile 3	3.767	5.461	5.535	7.629	3.939	6.144	3.602	5.081
	Maximum	93.190	122.915	152.615	188.080	97.100	136.935	64.400	98.526
	Standard deviation	16.034	21.540	26.236	32.821	16.835	23.937	11.075	17.100
	Kurtosis	28.568	25.795	29.143	26.653	27.370	26.187	27.620	26.594
	Skewness	5.195	4.890	5.264	4.988	5.063	4.932	5.079	4.969

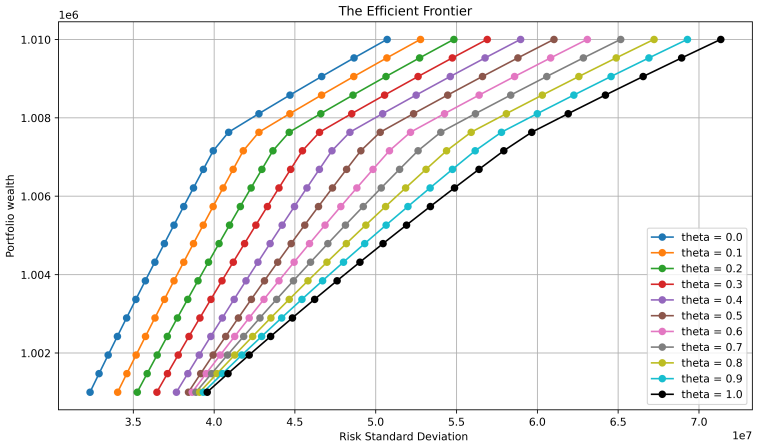


Figure 2: The efficient frontier with different risk attitudes

Table 5: The possibilistic numbers of prices ($P_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$)

Asset	a1	a2	a3	a4	Asset	a1	a2	a3	a4
Asset1	7.711	7.921	8.026	8.327	Asset18	16.645	19.035	21.883	22.942
Asset2	4.150	4.256	4.317	4.442	Asset19	22.921	23.872	25.453	26.355
Asset3	17.585	19.027	20.340	21.055	Asset20	64.481	69.682	81.757	85.954
Asset4	14.427	16.018	17.012	18.504	Asset21	237.799	257.368	265.586	279.112
Asset5	38.869	41.326	45.082	47.518	Asset22	8.496	8.972	9.687	10.115
Asset6	16.030	17.244	17.819	19.141	Asset23	35.679	37.090	38.492	39.561
Asset7	3.372	3.452	3.502	3.638	Asset24	33.529	39.644	43.215	44.834
Asset8	14.742	15.781	16.581	16.955	Asset25	22.131	22.664	23.096	24.317
Asset9	40.163	45.061	50.289	53.825	Asset26	27.014	29.059	30.003	32.142
Asset10	26.412	29.107	35.805	38.867	Asset27	19.814	20.591	21.199	22.846
Asset11	70.888	74.591	81.766	89.079	Asset28	41.922	43.867	46.445	49.369
Asset12	277.343	303.772	316.249	329.030	Asset29	4.681	4.744	4.838	4.898
Asset13	37.210	41.482	44.636	45.951	Asset30	19.299	20.236	21.850	23.241
Asset14	1676.385	1703.927	1781.434	1843.433	Asset31	23.668	25.749	26.294	28.044
Asset15	18.514	19.761	21.537	23.459	Asset32	4.910	5.270	5.377	5.573
Asset16	30.941	37.663	42.394	43.577	Asset33	10.130	10.959	11.327	12.608
Asset17	37.533	39.549	40.210	42.826	Asset34	13.171	14.174	15.093	16.723

It can be found from **Figure 2** that the effective frontier changes with risk attitude θ in the opposite direction. This is because investors are optimistic and eager to acquire risky assets when θ decreases. In contrast, pessimistic investors require more return for compensation when risk increases. Thus, the effective frontier for larger θ is lower.

5.2.2 The impact of the borrowing constraints

When $\omega_f^b = 0,\ -0.2,\ -0.4,\ -0.6,\ -0.8,\ -1.0,\ \theta = 0.5,\ u_i = 0.5,\ c_i = 0.003,\ \omega_0 = (0, 0, \ldots, 0)$, the effective frontiers are shown in **Figure 3**.

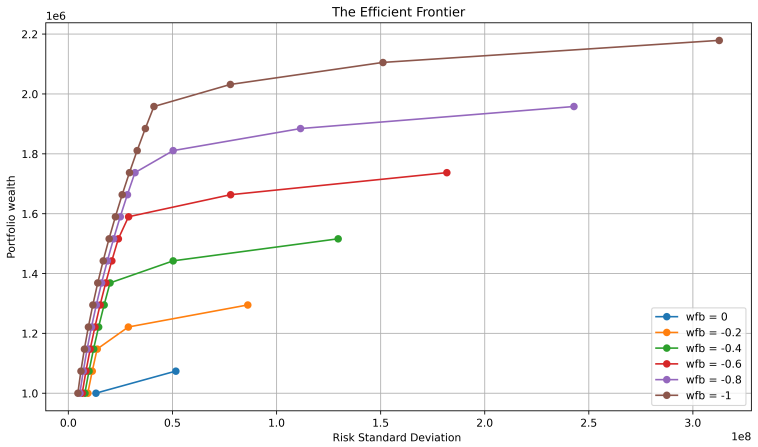


Figure 3: The efficient frontiers with different borrowing constraints

From **Figure 3**, it can be seen that the effective frontier changes with borrowing constraints in the opposite direction. However, there is no significant difference between the effective frontier of different borrowing constraints. The reason is lower ω_f^b means investors can borrow more wealth from others. When investors expect higher portfolio wealth, they need to borrow wealth to invest in risky assets. In other words, the borrowing constraint becomes a tight constraint while investors require higher portfolio wealth. Thus, the effective frontier is higher with borrowing constraints decreasing. However, while the expected portfolio wealth is low, such as lower than 1.2*10e6, investors achieve this goal by allocating their wealth. In other words, the borrowing constraints become loose constraints, so the effective frontiers of different ω_f^b are similar.

5.2.3 The impact of the upper bounds

When $u_i = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ \omega_f^b = -0.2,\ \theta = 0.5,\ c_i = 0.003,\ \omega_0 = (0, 0, \ldots, 0)$, the effective frontiers are shown in **Figure 4**.

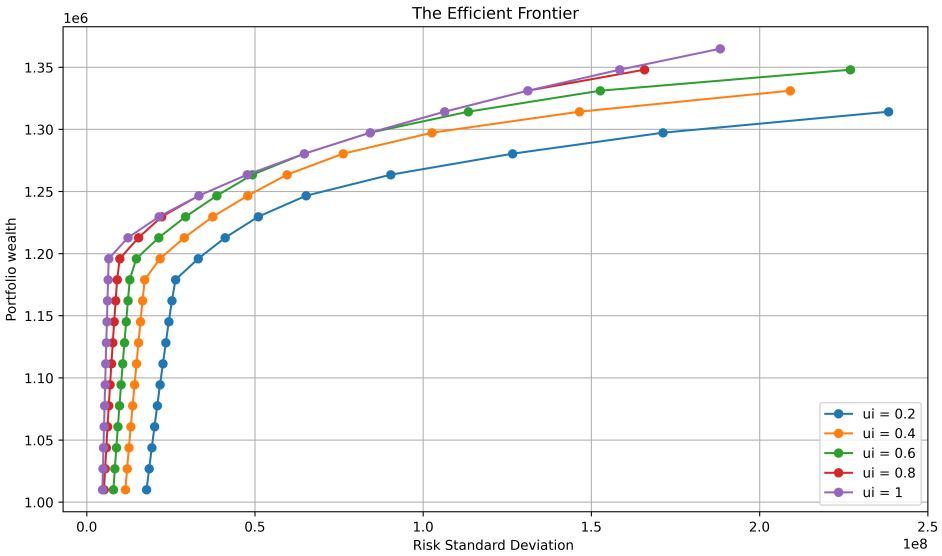


Figure 4: The efficient frontiers with different the upper bounds

From **Figure 4**, it can be concluded that the efficient frontier is higher with the upper bounds greater. The possible explanation is the investor's demand for risky assets increases with the higher excepted portfolio wealth. For the same expected portfolio wealth, the smaller upper bound means investors need to hold an inferior portfolio with greater risk. Therefore, the effective frontier of smaller u_i is lower.

5.3 Out-of-sample numerical examples

In this section, the out-of-sample analysis is conducted based on the 34 stocks selected above by using the "rolling window"(DeMiguel et al., 2009)[5]. Firstly, we analyze the impact of risk attitudes on optimal portfolio performance. Secondly, The possibility equal-weight portfolio model and the possibility mean-variance portfolio model based on the historical quartile are selected as basic models, and then the effectiveness of the proposed model can be evaluated by comparing it with basic models on return. Thirdly, a robust test is conducted by using higher frequency data to verify the reality of the conclusion. Finally, another robust test is performed by extending data to test the conclusion's reality.

5.3.1 The compare analysis of different risk attitudes

In this section, the monthly data of the above 34 stocks are collected as the sample from Jun 1, 2012, to May 31, 2022, and a total of 119 observations are obtained after eliminating the missing data. A "rolling window" method is applied with a window size of 59. After each rolling, the first data is dropped and a new one data is added, resulting in a total of 60 rolling data. At each rolling, the samples are split into a training set with 80% and a test set with 20%. The LSTM, CNN, SVR, and RF are used to predict prices, and the price of the asset can be calculated according to the predicted price of machine learning. We analyze the impact of risk attitudes on the proposed model's performance, which includes the wealth, return, excess net wealth, risk, W_ratio, and information ratio. The evaluation index can be calculated as follows:

$$W_p = \sum_{i=1}^{n} p_i\omega_i + r_f(W_0 - \sum_{i=1}^{n} p_{0i}(\omega_i - \omega_{0i})) - \sum_{i=1}^{n} c_i p_{0i}|\omega_i - \omega_{0i}|, \tag{40}$$

$$Return = (W_p - W_0)/W_0, \tag{41}$$

$$Excess\ net\ wealth = W_p - W_0(1 + r_f) \tag{42}$$

$$Risk = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\theta(\hat{a_{i2}} - \hat{a_{i1}})(\hat{a_{j2}} - \hat{a_{j1}}) + (1-\theta)(\hat{a_{i4}} - \hat{a_{i3}})(\hat{a_{j4}} - \hat{a_{j3}})}{18}\right)\omega_i\omega_j. \tag{43}$$

$$W_{ratio} = (W_p - W_0(1 + r_f))/Risk. \tag{44}$$

$$Inforamtion\ ratio = \frac{Excess \quad net \quad wealth}{standard \quad deviation}. \tag{45}$$

where the *standard deviation* is calculated by the volatility of monthly excess return.

When θ = 0.2, 0.4, 0.6, 0.8, ω_f^b = -0.1, u_i = 0.2, c_i = 0.003, and $\omega_0 = (0, 0, \ldots, 0)$, the impact of risk attitudes on wealth and return are shown in **Figure 5**.

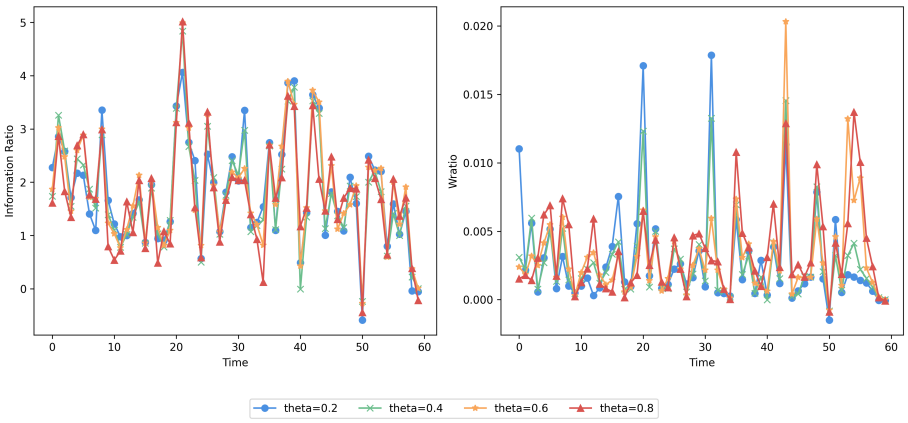


Figure 5: The impact of risk attitudes

From **Figure 5**, it can be seen that wealth and return fluctuate, which reflects the uncertainty of the financial market. In addition, the volatility of wealth and return increase with greater risk attitudes. It means that the portfolio is more affected by financial market uncertainty for optimistic investors.

When θ = 0.2, 0.4, 0.6, 0.8, ω_f^b = -0.1, u_i = 0.2, c_i = 0.003, and $\omega_0 = (0, 0, \ldots, 0)$, the impact of different risk attitudes on *excess return*, *risk* and W_{ratio} are shown in **Figure 6**.

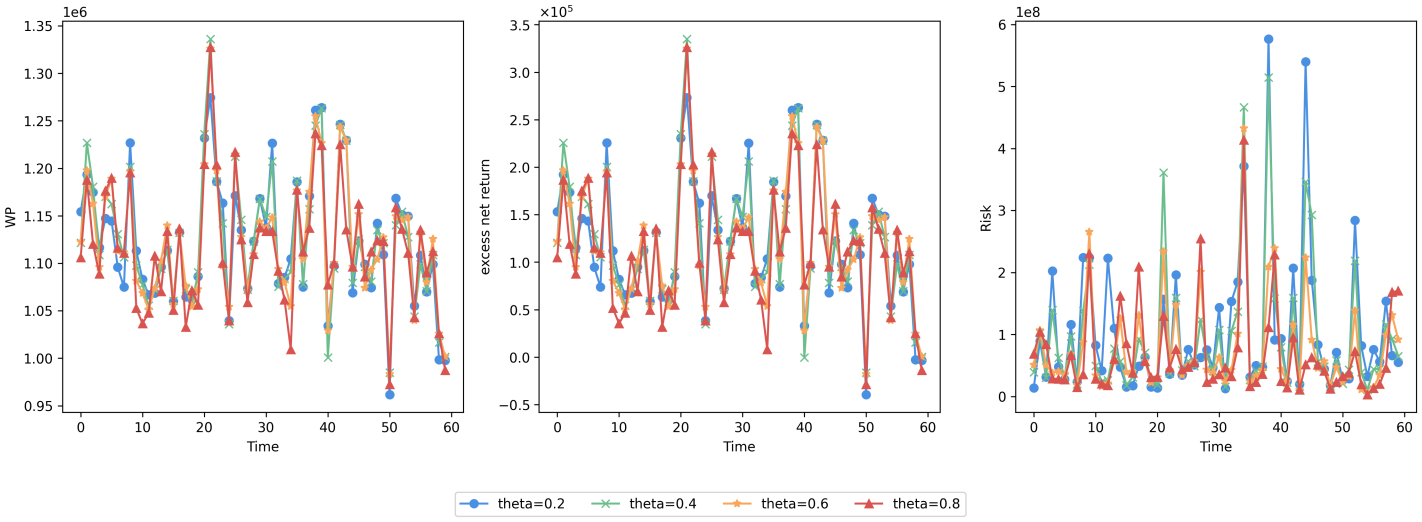


Figure 6: The impact of risk attitudes

From **Figure 6**, it can be seen that the excess return, risk, and W_{ratio} are volatile, which reflects the uncertainty of the financial market. Additionally, the volatility of excess return increases with risk attitudes, while the volatility of risk and W_{ratio} have no obvious relation with risk attitude.

When θ = 0.2, 0.4, 0.6, 0.8, ω_f^b = -0.1, u_i = 0.2, c_i = 0.003, and $\omega_0 = (0, 0, \ldots, 0)$, the impact of different risk attitudes on information ratio is shown in **Figure 7**.

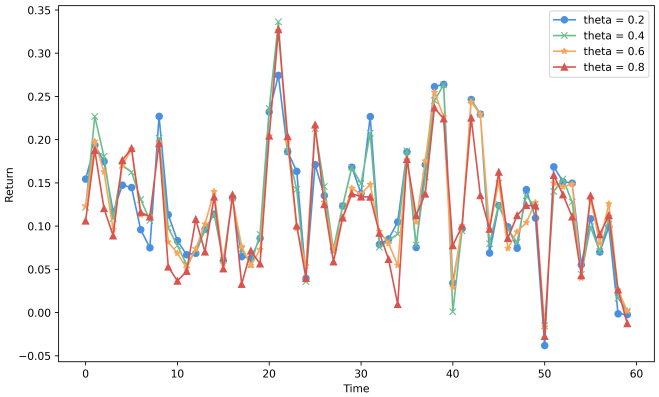


Figure 7: The impact of risk attitudes

From **Figure 7**, it can be seen that the information ratio is volatile, and the volatility of the information ratio increases with risk attitudes.

5.3.2 The compare analysis of different models

This section tests the effectiveness of the proposed model by "rolling windows". The monthly data are collected as the sample from Jun 1, 2012, to May 31, 2022, and a total of 119 observations are obtained after eliminating the missing data. The window size is 59, and the rolling times are 60. In each rolling, the sample data is used to train the machine learning models and to predict asset prices. We select the possibility equal-weight portfolio model (AEW), and the possibility mean-variance portfolio model based on the historical quartile (HQMV) as the benchmark, and the return is used to evaluate the performance of different models. When θ = 0, 0.5, 1, u_i = 0.1, 0.2, 0.3, ω_f^b = -0.1, c_i = 0.003, and $\omega_0 = (0, 0, \ldots, 0)$, the out-of-sample return of PPMV, HQMV and AEW are shown in **Figure 8**.

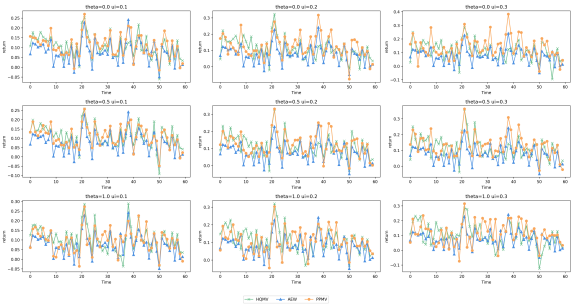


Figure 8: The out-of-smaple of different models

It can be seen from **Figure 8** that the PPMV performs well most time. Specifically, when θ=0,u_i=0.1, the return of PPMV is higher than HQMV and AEW on the whole. The PPMV performs better than HQMV and EMW significantly while θ=0.5,u_i=0.1,Further, some parameters are changed to test the reliability of the above conclusion. The results show that the conclusion is robust, and the advantage of PPMV is more clear with u_i increasing.

5.3.3 The compare analysis based on higher frequency data

In order to test the robustness of the conclusion that "the PPMV performs better than HQMV and EMW", this section uses higher frequency data for out-of-sample analysis. The weekly data with 511 observations from June 1, 2012, to May 31, 2022, are collected as the sample. The window size is 100, and the rolling times are 9. The first 80 sample data are used to train the machine learning models, and the last 20 sample data are used to predict asset prices. the first 20 sample data are dropped and the new 20 sample data are added for the next rolling. When θ = 0, 0.5, 1, u_i = 0.1, 0.2, 0.3, ω_f^b = -0.1, c_i = 0.003, and $\omega_0 = (0, 0, \ldots, 0)$, the out-of-sample returns of PPMV model, HQMV model and AEW model based on higher frequency data are shown in **Figure 9**.

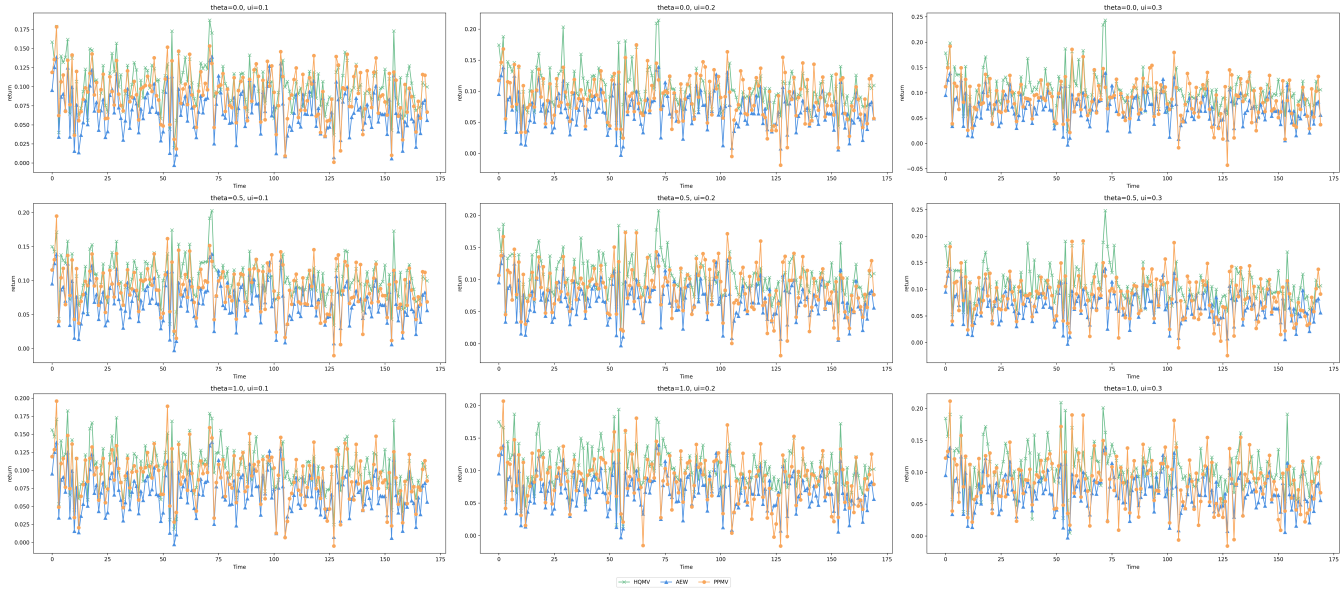


Figure 9: The out-of-smaple of different models

From **Figure 9,** it can be seen that the returns of the PPMV and HQMV are greater than the AEW most time. When the risk attitude becomes small, the return volatility of the PPMV becomes great compared with HQMV and AEW.

5.3.4 The compare analysis of different sample sizes

For more reliable validation, this section extends the length of the sample data for out-of-sample analysis. The monthly data with 139 observations from June 1, 2010 to May 31, 2022, are collected as the sample. The window size is 69, and the rolling times are 72. The first 68 sample data are used to train the machine learning models, and the last sample data are used to predict asset prices. the first sample data are dropped and one new sample data are added for the next rolling. When $\theta = 0, 0.5, 1$, $u_i = 0.1, 0.2, 0.3$, ω_f^b = -0.1, $c_i = 0.003$, and $\omega_0 = (0, 0, \ldots, 0)$, the out-of-sample returns of PPMV, HQMV and AEW based on extend data are shown in **Figure 10**.

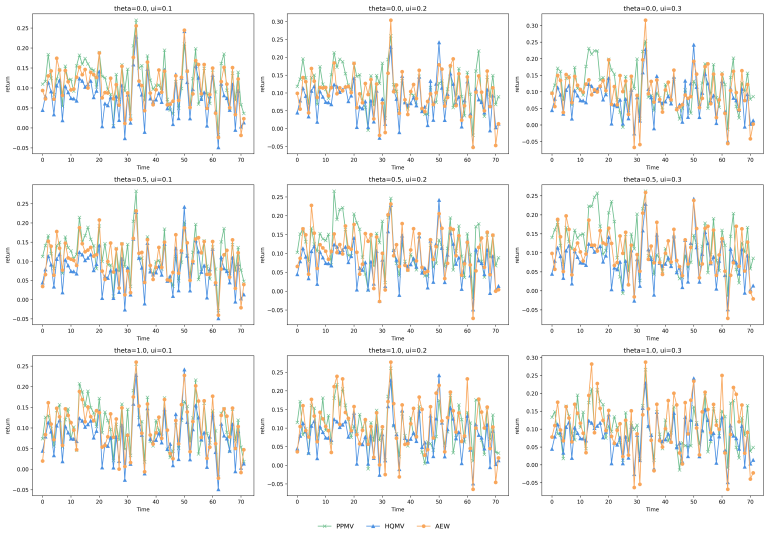


Figure 10: The out-of-smaple of different models

From **Figure 10**, it can be seen that the returns of the PPMV and HQMV model are larger than AEW on the whole, and the PPMV performs well with θ increasing.

6 Conclusions

In this paper, the prices of risky assets are modeled as trapezoidal fuzzy numbers, which are predicted using LSTM, CNN, SVR, and RF models. The possibilistic mean and variances with different risk attitudes are calculated based on possibilistic theory. Several constraints, such as borrowing constraints, transaction costs, and threshold constraints, are considered. A new predicted fuzzy portfolio selection model with different risk attitudes is proposed, and a pivoting algorithm for quadratic programming is designed to solve it. Finally, both in-sample and out-of-sample analysis are conducted based on the Shanghai Stock Exchange, China. The empirical results are as follows:

(1) From the in-sample numerical examples, it can be concluded that the efficient frontier is lower with increased risk attitude and borrowing constraints, while the efficient frontier is higher with the upper bound of risky assets.

(2) From the out-of-sample numerical examples, it can be concluded that the volatility of portfolio wealth, return, excess net wealth, and the information ratio are greater with risk attitude, while the volatility of portfolio risk and W_ratio is not clearly related to risk attitude. Additionally, the fuzzy portfolio selection model with different risk attitudes, based on machine learning, performs better than the possibility mean-variance portfolio model based on the historical quartile and the possibility equal-weight portfolio model in terms of portfolio return.

There are several directions for future studies. As is well known, increasing the accuracy of predictions can improve investment performance. One direction would be to incorporate abundant textual information into input features and use ensemble learning to enhance the effectiveness of predictions. Additionally, considering different financial markets or types of assets, such as bonds and options, may provide another avenue for future research.

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Compliance with Ethical Standards

Conflict of Interest: Peng Zhang, Shulin Cui and Beibei Du declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants performed by any of the authors.

Informed consent: Informed consent was obtained from all individual participants included in the study.

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