Iranian Journal of Fuzzy Systems

Volume 22, Number 1, (2025), pp. 35-47

Original paper

University of Sistan and Baluchestan

# New characterizations of migrative 2-uninorms

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**Abstract**

This article pays attention to the $\alpha$-migrativity of 2-uninorms with $\alpha \in [0,1]$ deeply. It describes the $\alpha$-migrativity of 2-uninorms completely, which generalizes and unifies some current existing characterizations for the $\alpha$-migrativity of triangular norms, triangular conorms, uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively.

*Keywords:* Uninorm, nullnorm, uni-nullnorm, 2-uninorm, migrativity.

## 1 Introduction

There are many special classes of aggregation operators, such as triangular norms (briefly t-norms), t-conorms, t-operators, nullnorms, uninorms, uni-nullnorms, 2-uninorms, copulas and so on. When using an aggregation technology to solve practical problems, the first problem we face is how to select the most appropriate aggregation function. The selection of aggregation function is determined by its application circumstances, and there are no unified criterions. For example, nullnorms can be applied in the areas of neural networks and decision making [9, 15], and uninorms can be used in various fields like neural networks, decision making, fuzzy logics and fuzzy system modeling [29, 30]. On the other hand, 2-uninorms specifically unify uninorms and nullnorms. They were first investigated by Akella [1], since then, researchers on 2-uninorms have mainly concentrated on the structure of 2-uninorms [37], distributivity of 2-uninorms [21, 26, 33], migrativity of 2-uninorms [10, 14, 27, 32] and $(U^2, N)$-implications of 2-uninorms [35], etc. 2-uninorms are crucial for its practical applications in fuzzy quantifiers or networks as well as theoretical interest [10, 26, 35] because it is closely related to other aggregation operators.

The study of $\alpha$-migrativity of aggregation function has great significance and value in theory and applications. In applications, the migrativity means that when the input is scaled down during the aggregation process, the inherent properties of the aggregation function itself will not be changed [17]. So the migrativity of aggregation function is of great importance in image processing and decision-making [15]. The concept of migrativity comes from an open question, which was proposed by Endre Pap [18]: does there exist a strictly increasing t-norm, that is different from the t-norm given by

$$T(x,y) = \begin{cases} \alpha xy, & (x,y) \in (0,1)^2, \\ \min(x,y), & \text{otherwise}, \end{cases}$$

for the following equation

$$T(\alpha x, y) = T(x, \alpha y) \text{ for any } x, y \in [0,1], \tag{1}$$

for an $\alpha \in (0,1)$? This problem was positively solved by Budinčević and Kurilić [3]. In 2008, the property of $T$ satisfying the equation (1) was called an $\alpha$-migrativity of $T$ by Durante and Sarkoci [6]. By replacing the product $\alpha x$

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Received: July 2024; Revised: November 2024; Accepted: January 2025.

https://doi.org/10.22111/ijfs.2025.49281.8694

in the equation (1) by $T_1(\alpha, x)$ with $T_1$ a t-norm, Fodor and Rudas [7] generalized the $\alpha$-migrativity to the $(\alpha, T_1)$-migrativity of t-norm $T$, which is expressed as follows

$$T(T_1(\alpha, x), y) = T(x, T_1(\alpha, y)) \text{ for each } x, y \in [0, 1]. \tag{2}$$

Indeed, many scholars all over the world explore the migrative property of several binary aggregation operations and their works can be mainly divided into two categories: one is to change the form of Formula (1) or (2); the other is to find more general aggregation functions to replace the t-norms $T$ or $T_1$ in Formula (2). For example, the migrativity of semicopulas, quasicoupulas, copulas [5, 17], fuzzy implications [2, 8], semi-t-operators [25, 34], t-subnorms, t-norms, t-conorms [7, 28], uninorms, nullnorms [11, 16, 20, 36], overlap functions [13], uni-nullnorms [22, 23] and 2-uninorms [10, 14, 27] have been widely studied, respectively. Because t-norm, t-conorm, uninorm, nullnorm, uni-nullnorm and null-uninorm are just subclasses of 2-uninorms and the $\alpha$-migrativity of 2-uninorms is very valuable at least in image processing and decision making [15], Liang and Wang [14] recently characterized $(\alpha, \mathbb{U}_2)$-migrative 2-uninorms $\mathbb{U}_1$ when $e_1 = e_2$ or $a_1 = a_2$ or $f_1 = f_2$ where $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$ are 2-neutral elements of $\mathbb{U}_1$ and $\mathbb{U}_2$, respectively. However, the characterization for $(\alpha, \mathbb{U}_2)$-migrative 2-uninorms $\mathbb{U}_1$ when $e_1 \neq e_2$ and $a_1 \neq a_2$ and $f_1 \neq f_2$ has not been resolved yet. Therefore, a complete characterization for $\alpha$-migrativity of 2-uninorms is necessary. This article will focus on this problem.

The rest of this article is organized as follows. In Section 2, we recall some necessary notions and one known result. In Section 3, we describes the migrative 2-uninorms completely. Conclusions are drawn in Section 4.

## 2 Preliminaries

This section just recalls some necessary definitions and results of 2-uninorms. Readers can refer to [4], [12],[24] and [31] for more information on t-norms, t-conorms, uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively.

**Definition 2.1** [1] *Let $\mathbb{U} : [0, 1]^2 \to [0, 1]$ be a commutative binary operator. Then $\{e, f\}_a$ with $0 \leq e \leq a \leq f \leq 1$ is called a 2-neutral element of $\mathbb{U}$ if $\mathbb{U}(e, x) = x$ for all $x \in [0, a]$ and $\mathbb{U}(f, x) = x$ for all $x \in [a, 1]$.*

**Definition 2.2** [1] *A binary operation $\mathbb{U} : [0, 1]^2 \to [0, 1]$ is called a 2-uninorm if it is commutative, associative, increasing in both variables and has a 2-neutral element $\{e, f\}_a$.*

**Definition 2.3** [14] *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0, 1]$ and $\alpha \in [0, 1]$. Then $\mathbb{U}_1$ is called $(\alpha, \mathbb{U}_2)$-migrative or $\alpha$-migrative over $\mathbb{U}_2$ if*

$$\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y)), \tag{3}$$

*for any $x, y \in [0, 1]$.*

Let $\lambda = \mathbb{U}_2(\alpha, e_1)$ and $\mu = \mathbb{U}_2(\alpha, f_1)$. If $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative, then $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(x, \lambda)$ for any $\mathbb{U}_2(\alpha, x) \leq a_1$ and $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(x, \mu)$ for any $\mathbb{U}_2(\alpha, x) \geq a_1$, which will be used repeatedly.

**Proposition 2.4** [14] *Let $\mathbb{U}$ be a 2-uninorm on the unit interval $[0, 1]$ with a 2-neutral element $\{e, f\}_a$. Then*

(i) $\mathbb{U}(x, y) = \mathbb{U}(x, a)$ *for all* $x \in [0, a]$ *and* $y \in [a, f]$.

(ii) $\mathbb{U}(x, y) = \mathbb{U}(a, y)$ *for all* $x \in [a, f]$ *and* $y \in [0, a]$.

(iii) $\mathbb{U}(x, y) = \mathbb{U}(x, a)$ *for all* $x \in [a, 1]$ *and* $y \in [e, a]$.

(iv) $\mathbb{U}(x, y) = \mathbb{U}(a, y)$ *for all* $x \in [e, a]$ *and* $y \in [a, 1]$.

(v) $\min(x, y) \leq \mathbb{U}(x, y) \leq a$ *for all* $(x, y) \in [0, e] \times [e, f] \cup [e, f] \times [0, e]$.

(vi) $a \leq \mathbb{U}(x, y) \leq \max(x, y)$ *for all* $(x, y) \in [e, f] \times [f, 1] \cup [f, 1] \times [e, f]$.

(vii) $\mathbb{U}(x, y) \in [0, e) \cup \{a\} \cup (f, 1]$ *for all* $(x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a]$.

## 3 $\alpha$-migrativity of 2-uninorms

In this section, we completely characterize the $\alpha$-migrativity of 2-uninorms with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively.

**Lemma 3.1** *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0,1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively.*

(i) *If $\mathbb{U}_2(\alpha, e_1) < e_1$, then $\mathbb{U}_2(\alpha, f_1) \le f_1$.*

(ii) *If $\mathbb{U}_2(\alpha, f_1) > f_1$, then $\mathbb{U}_2(\alpha, e_1) \ge e_1$.*

*Proof.* (i) We prove that $\mathbb{U}_2(\alpha, f_1) \le f_1$ by distinguishing two cases. If $e_1 \le a_2$, then from $\mathbb{U}_2(\alpha, e_1) < e_1$ we know $\alpha \le e_2$, and subsequently, $\mathbb{U}_2(\alpha, f_1) \le \mathbb{U}_2(e_2, f_1) = f_1$ for each $f_1 \le a_2$ and $\mathbb{U}_2(\alpha, f_1) \le \mathbb{U}_2(e_2, f_1) \le \mathbb{U}_2(f_2, f_1) = f_1$ for each $f_1 > a_2$, i.e., $\mathbb{U}_2(\alpha, f_1) \le f_1$. If $e_1 > a_2$, then by $\mathbb{U}_2(\alpha, e_1) < e_1$ we get $\alpha \le f_2$, showing that $\mathbb{U}_2(\alpha, f_1) \le \mathbb{U}_2(f_2, f_1) = f_1$.

(ii) We prove that $\mathbb{U}_2(\alpha, e_1) \ge e_1$ by the following two cases. If $f_1 \ge a_2$, then from $\mathbb{U}_2(\alpha, f_1) > f_1$ we obtain $\alpha \ge f_2$, and subsequently, $\mathbb{U}_2(\alpha, e_1) \ge \mathbb{U}_2(f_2, e_1) = e_1$ for each $e_1 \ge a_2$ and $\mathbb{U}_2(\alpha, e_1) \ge \mathbb{U}_2(f_2, e_1) \ge \mathbb{U}_2(e_2, e_1) = e_1$ for each $e_1 < a_2$. If $f_1 < a_2$, then by $\mathbb{U}_2(\alpha, f_1) > f_1$ we get $\alpha \ge e_2$, showing that $\mathbb{U}_2(\alpha, e_1) \ge \mathbb{U}_2(e_2, e_1) = e_1$.

**Lemma 3.2** *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0,1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively. Suppose that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative. The following two statements hold.*

(i) *If $\mathbb{U}_2(\alpha, e_1) \in [e_1, a_1]$, then $\mathbb{U}_2(\alpha, a_1) \ge a_1$.*

(ii) *If $\mathbb{U}_2(\alpha, f_1) \in [a_1, f_1]$, then $\mathbb{U}_2(\alpha, a_1) \le a_1$.*

*Proof.* Concerning (i), assuming that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative. If $\mathbb{U}_2(\alpha, a_1) < a_1$, then

$$\mathbb{U}_2(\alpha, a_1) = \mathbb{U}_1(\mathbb{U}_2(\alpha, a_1), e_1) = \mathbb{U}_1(a_1, \mathbb{U}_2(\alpha, e_1)) = a_1,$$

a contradiction. Thus $\mathbb{U}_2(\alpha, a_1) \ge a_1$.

To show (ii), assuming that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative. If $\mathbb{U}_2(\alpha, a_1) > a_1$, then

$$\mathbb{U}_2(\alpha, a_1) = \mathbb{U}_1(\mathbb{U}_2(\alpha, a_1), f_1) = \mathbb{U}_1(a_1, \mathbb{U}_2(\alpha, f_1)) = a_1,$$

a contradiction. Thus $\mathbb{U}_2(\alpha, a_1) \le a_1$.

**Proposition 3.3** *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0,1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively.*

(i) *If $\mathbb{U}_2(\alpha, e_1) \ge a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0,1]$.*

(ii) *If $\mathbb{U}_2(\alpha, f_1) \le a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\lambda, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0,1]$.*

*Proof.*

(i) We first prove $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\mu, x)$ for each $x \in [0,1]$ done by checking the subsequent two cases.

Case 1. There is at least one $k \in [0, e_1)$ such that $\mathbb{U}_2(\alpha, k) \le a_1$.

Let $\mathcal{B} = \{k \in [0, e_1) | \mathbb{U}_2(\alpha, k) \le a_1\}$ and $\eta = \sup \mathcal{B}$. In this case, the proof is split into two subcases as follows.

Subcase 1. If $\eta \in \mathcal{B}$, then $\mathbb{U}_2(\alpha, x) \le \mathbb{U}_2(\alpha, \eta) \le a_1$ for each $x \in [0, \eta]$ and $\mathbb{U}_2(\alpha, x) > a_1$ for each $x \in (\eta, 1]$, showing that

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, \eta], \\ \mathbb{U}_1(\mu, x), & x \in (\eta, 1]. \end{cases}$$

Moreover, since $\lambda = \mathbb{U}_2(\alpha, e_1) \ge a_1$, $\mu = \mathbb{U}_2(\alpha, f_1) \ge \mathbb{U}_2(\alpha, e_1) \ge a_1$, by (3) we have

$$\begin{aligned} \mathbb{U}_1(\mu, x) &= \mathbb{U}_1(\mathbb{U}_1(\mu, f_1), x) = \mathbb{U}_1(\mathbb{U}_2(\alpha, f_1), x) \\ &= \mathbb{U}_1(f_1, \mathbb{U}_2(\alpha, x)) = \mathbb{U}_1(f_1, \mathbb{U}_1(\lambda, x)) \\ &= \mathbb{U}_1(\mathbb{U}_1(f_1, \lambda), x) = \mathbb{U}_1(\lambda, x), \end{aligned}$$

for each $x \in [0, \eta]$ and, consequently, $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\mu, x)$ for each $x \in [0, 1]$.

Subcase 2. If $\eta \notin \mathcal{B}$, then $\mathbb{U}_2(\alpha, \eta) > a_1$. This follows from the monotonicity of $\mathbb{U}_2$ that $\mathbb{U}_2(\alpha, x) \geq \mathbb{U}_2(\alpha, \eta) > a_1$ for each $x \in [\eta, 1]$. Now, let $x \in [0, \eta)$. Then from the definition of $\eta$ there is a $k_0 \in \mathcal{B}$ such that $k_0 > x$, showing that $\mathbb{U}_2(\alpha, x) \leq \mathbb{U}_2(\alpha, k_0) \leq a_1$ from the monotonicity of $\mathbb{U}_2$. Therefore,

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, \eta), \\ \mathbb{U}_1(\mu, x), & x \in [\eta, 1]. \end{cases}$$

Moreover, similar to Subcase 1 we have $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(\lambda, x)$ for each $x \in [0, \eta)$ and consequently, $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\mu, x)$ for all $x \in [0, 1]$.

Case 2. There is no $k \in [0, e_1)$ such that $\mathbb{U}_2(\alpha, k) \leq a_1$.

Obviously, $\mathbb{U}_2(\alpha, x) \geq \mathbb{U}_2(\alpha, 0) > a_1$ for each $x \in [0, 1]$. Thus $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\mu, x)$ for each $x \in [0, 1]$.

Now, we prove that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative. In fact, if $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\mu, x)$ for each $x \in [0, 1]$, then since $\mathbb{U}_1$ is commutative and associative, we have

$$\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(\mathbb{U}_1(\mu, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\mu, y)) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y)).$$

(ii) Necessity. This will be done by checking the subsequent two cases.

**Case 1.** There is at least one $k \in (f_1, 1]$ such that $\mathbb{U}_2(\alpha, k) \geq a_1$.

Let $\mathcal{A} = \{k \in (f_1, 1] | \mathbb{U}_2(\alpha, k) \geq a_1\}$ and $\xi = \inf \mathcal{A}$. In this case, the proof is split into two subcases as follows.

**Subcase 1.** If $\xi \in \mathcal{A}$, then $\mathbb{U}_2(\alpha, x) \geq \mathbb{U}_2(\alpha, \xi) \geq a_1$ for each $x \in [\xi, 1]$ and $\mathbb{U}_2(\alpha, x) < a_1$ for $x \in [0, \xi)$, showing that

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, \xi), \\ \mathbb{U}_1(\mu, x), & x \in [\xi, 1]. \end{cases}$$

Moreover, since $\lambda = \mathbb{U}_2(\alpha, e_1) \leq \mathbb{U}_2(\alpha, f_1) \leq a_1$ and $\mu = \mathbb{U}_2(\alpha, f_1) \leq a_1$ we have

$$\begin{aligned} \mathbb{U}_1(\lambda, x) &= \mathbb{U}_1(\mathbb{U}_1(\lambda, e_1), x) = \mathbb{U}_1(\mathbb{U}_2(\alpha, e_1), x) \\ &= \mathbb{U}_1(e_1, \mathbb{U}_2(\alpha, x)) = \mathbb{U}_1(e_1, \mathbb{U}_1(\mu, x)) \\ &= \mathbb{U}_1(\mathbb{U}_1(e_1, \mu), x) = \mathbb{U}_1(\mu, x), \end{aligned}$$

for each $x \in [\xi, 1]$, and consequently, $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\lambda, x)$ for each $x \in [0, 1]$.

**Subcase 2.** If $\xi \notin \mathcal{A}$, then $\mathbb{U}_2(\alpha, \xi) < a_1$. This follows from the monotonicity of $\mathbb{U}_2$ that $\mathbb{U}_2(\alpha, x) \leq \mathbb{U}_2(\alpha, \xi) < a_1$ for each $x \in [0, \xi]$. Now, let $x \in (\xi, 1]$. Then from the definition of $\xi$ there exists a $k_0 \in \mathcal{A}$ such that $k_0 < x$. Thus from the monotonicity of $\mathbb{U}_2$ we have $\mathbb{U}_2(\alpha, x) \geq \mathbb{U}_2(\alpha, k_0) \geq a_1$. Therefore,

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, \xi], \\ \mathbb{U}_1(\mu, x), & x \in (\xi, 1]. \end{cases}$$

Moreover, similar to Subcase 1 we have $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(\mu, x)$ for each $x \in (\xi, 1]$ and consequently, $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\lambda, x)$ for each $x \in [0, 1]$.

**Case 2.** There is no $k \in (f_1, 1]$ such that $\mathbb{U}_2(\alpha, k) \geq a_1$.

Obviously, $\mathbb{U}_2(\alpha, x) \leq \mathbb{U}_2(\alpha, 1) < a_1$ for each $x \in [0, 1]$. Therefore, $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\lambda, x)$ for each $x \in [0, 1]$.

Sufficiency. If $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\lambda, x)$ for each $x \in [0, 1]$, then since $\mathbb{U}_1$ is commutative and associative, we have

$$\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(\mathbb{U}_1(\lambda, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\lambda, y)) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y)),$$

i.e., $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative.

**Proposition 3.4** *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0, 1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively.*

(i) *If* $\mathbb{U}_2(\alpha, e_1) \in [e_1, a_1]$, *then* $\mathbb{U}_1$ *is* $(\alpha, \mathbb{U}_2)$*-migrative iff*

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases}$$

*and* $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ *for each* $x \in [0, a_1]$.

(ii) *If* $\mathbb{U}_2(\alpha, f_1) \in [a_1, f_1]$, *then* $\mathbb{U}_1$ *is* $(\alpha, \mathbb{U}_2)$*-migrative iff*

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases}$$

*and* $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ *for each* $x \in [a_1, 1]$.

*Proof.*

(i) Assume that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative. Then by Lemma 3.2 (i) we get $\mathbb{U}_2(\alpha, a_1) \geq a_1$. Since $\mathbb{U}_2(\alpha, x) \geq \mathbb{U}_2(\alpha, a_1) \geq a_1$ for any $x \geq a_1$, we have $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(x, \mu)$ and $\mu = \mathbb{U}_2(\alpha, f_1) \geq \mathbb{U}_2(\alpha, a_1) \geq a_1$. From Proposition 2.4 (iii) we get $\mathbb{U}_2(\alpha, x) \leq \mathbb{U}_2(\alpha, a_1) = \mathbb{U}_1(\mu, a_1) = \mathbb{U}_1(\mu, e_1) = \mathbb{U}_1(\mathbb{U}_2(\alpha, f_1), e_1) = \mathbb{U}_1(f_1, \mathbb{U}_2(\alpha, e_1)) = a_1$ for any $x \leq a_1$. Consequently,

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1]. \end{cases}$$

Next, from (3) we have

$$\begin{aligned} \mathbb{U}_1(\mu, x) &= \mathbb{U}_1(\mathbb{U}_1(\mu, f_1), x) = \mathbb{U}_1(\mathbb{U}_2(\alpha, f_1), x) \\ &= \mathbb{U}_1(f_1, \mathbb{U}_2(\alpha, x)) = \mathbb{U}_1(f_1, \mathbb{U}_1(\lambda, x)) \\ &= \mathbb{U}_1(\mathbb{U}_1(f_1, \lambda), x) = \mathbb{U}_1(a_1, x), \end{aligned}$$

for each $x \in [0, a_1]$.

Conversely, we prove the $(\alpha, \mathbb{U}_2)$-migrativity of $\mathbb{U}_1$ by the following four cases.

**Case 1.** If $x, y \in [0, a_1]$, then $\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(\mathbb{U}_1(\lambda, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\lambda, y)) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y))$.

**Case 2.** If $x, y \in [a_1, 1]$, then $\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(\mathbb{U}_1(\mu, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\mu, y)) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y))$.

**Case 3.** If $x \in [0, a_1]$ and $y \in [a_1, 1]$, then from Proposition 2.4 (iv) and $\mathbb{U}_1(a_1, x) = \mathbb{U}_1(\mu, x)$ for each $x \in [0, a_1]$, we have

$$\begin{aligned} \mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) &= \mathbb{U}_1(\mathbb{U}_1(\lambda, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\lambda, y)) \\ &= \mathbb{U}_1(x, \mathbb{U}_1(a_1, y)) = \mathbb{U}_1(\mathbb{U}_1(x, a_1), y) \\ &= \mathbb{U}_1(\mathbb{U}_1(x, \mu), y) = \mathbb{U}_1(x, \mathbb{U}_1(\mu, y)) \\ &= \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y)). \end{aligned}$$

**Case 4.** If $x \in [a_1, 1]$ and $y \in [0, a_1]$, then similar to Case 3, we obtain (3).

(ii) Assume that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative. Then by Lemma 3.2 (ii) we get $\mathbb{U}_2(\alpha, a_1) \leq a_1$. Since $\mathbb{U}_2(\alpha, x) \leq \mathbb{U}_2(\alpha, a_1) \leq a_1$ for any $x \leq a_1$, we have $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(x, \lambda)$ and $\lambda = \mathbb{U}_2(\alpha, e_1) \leq \mathbb{U}_2(\alpha, a_1) \leq a_1$. From Proposition 2.4 (i) we get $\mathbb{U}_2(\alpha, x) \geq \mathbb{U}_2(\alpha, a_1) = \mathbb{U}_1(\lambda, a_1) = \mathbb{U}_1(\lambda, f_1) = \mathbb{U}_1(\mathbb{U}_2(\alpha, e_1), f_1) = \mathbb{U}_1(e_1, \mathbb{U}_2(\alpha, f_1)) = a_1$ for any $x \geq a_1$. Consequently,

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1]. \end{cases}$$

Next, from (3) we have

$$\begin{aligned} \mathbb{U}_1(\lambda, x) &= \mathbb{U}_1(\mathbb{U}_1(\lambda, e_1), x) = \mathbb{U}_1(\mathbb{U}_2(\alpha, e_1), x) \\ &= \mathbb{U}_1(e_1, \mathbb{U}_2(\alpha, x)) = \mathbb{U}_1(e_1, \mathbb{U}_1(\mu, x)) \\ &= \mathbb{U}_1(\mathbb{U}_1(e_1, \mu), x) = \mathbb{U}_1(a_1, x), \end{aligned}$$

for each $x \in [a_1, 1]$.

Conversely, we prove the $(\alpha, \mathbb{U}_2)$-migrativity of $\mathbb{U}_1$ by the following four cases.

**Case 1.** If $x, y \in [0, a_1]$, then $\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(\mathbb{U}_1(\lambda, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\lambda, y)) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y))$.

**Case 2.** If $x, y \in [a_1, 1]$, then $\mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) = \mathbb{U}_1(\mathbb{U}_1(\mu, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\mu, y)) = \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y))$.

**Case 3.** If $(x, y) \in [0, a_1] \times [a_1, 1]$, then from Proposition 2.4 (i) and $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ for each $x \in [a_1, 1]$,

$$\begin{aligned} \mathbb{U}_1(\mathbb{U}_2(\alpha, x), y) &= \mathbb{U}_1(\mathbb{U}_1(\lambda, x), y) = \mathbb{U}_1(x, \mathbb{U}_1(\lambda, y)) \\ &= \mathbb{U}_1(x, \mathbb{U}_1(a_1, y)) = \mathbb{U}_1(\mathbb{U}_1(x, a_1), y) \\ &= \mathbb{U}_1(\mathbb{U}_1(x, \mu), y) = \mathbb{U}_1(x, \mathbb{U}_1(\mu, y)) \\ &= \mathbb{U}_1(x, \mathbb{U}_2(\alpha, y)). \end{aligned}$$

**Case 4.** If $(x, y) \in [a_1, 1] \times [0, a_1]$, then the proof is in analogous to the Case 3.

Therefore, taking $\mathbb{U}_2(\alpha, e_1)$ as a classified point, we have the following theorem from Lemma 3.1, Propositions 3.3 and 3.4.

**Theorem 3.5** *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0, 1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively.*

(i) *If $\mathbb{U}_2(\alpha, e_1) < e_1$ and $\mathbb{U}_2(\alpha, f_1) \le a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\lambda, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0, 1]$.*

(ii) *If $\mathbb{U}_2(\alpha, e_1) < e_1$ and $\mathbb{U}_2(\alpha, f_1) \ge a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff*

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases}$$

*and $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ for each $x \in [a_1, 1]$.*

(iii) *If $\mathbb{U}_2(\alpha, e_1) \in [e_1, a_1]$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff*

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases} \tag{4}$$

*and $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ for each $x \in [0, a_1]$.*

(iv) *If $\mathbb{U}_2(\alpha, e_1) \ge a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0, 1]$.*

Taking $\mathbb{U}_2(\alpha, f_1)$ as a classified point, we have the following theorem from Lemma 3.1, Proposition 3.3 and 3.4.

**Theorem 3.6** *Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two 2-uninorms on the unit interval $[0, 1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_2}$, respectively.*

(i) *If $\mathbb{U}_2(\alpha, f_1) \le a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\lambda, x) = \mathbb{U}_2(\alpha, x)$ for any $x \in [0, 1]$.*

(ii) *If $\mathbb{U}_2(\alpha, f_1) \in [a_1, f_1]$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff*

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases} \tag{5}$$

*and $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ for any $x \in [a_1, 1]$.*

(iii) *If $\mathbb{U}_2(\alpha, f_1) > f_1$ and $\mathbb{U}_2(\alpha, e_1) \le a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff*

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases}$$

*and $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ for any $x \in [0, a_1]$.*

(iv) *If $\mathbb{U}_2(\alpha, f_1) > f_1$ and $\mathbb{U}_2(\alpha, e_1) \ge a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for any $x \in [0, 1]$.*

In particular, from Theorems 3.5 and 3.6, we have the following remark.

**Remark 3.7** (i) Let $e_1 = e_2 = a_1 = 1$ in Proposition 3.3 (ii). Then both $\mathbb{U}_1$ and $\mathbb{U}_2$ are t-norms. Therefore, we get Theorem 3 of [7].

(ii) Let $f_1 = f_2 = a_1 = 0$ in Proposition 3.3 (i). Then both $\mathbb{U}_1$ and $\mathbb{U}_2$ are t-conorms. Thus $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(\alpha, x)$ for each $x \in [0, 1]$.

(iii) Let $e_2 = a_1 = 1$ in Theorem 3.6. Then $\mathbb{U}_1$ is a uninorm and $\mathbb{U}_2$ is a t-norms. Therefore, Theorem 3.6 implies Theorem 3.1 of [38]. In particular, if $\mathbb{U}_1$ is locally internal on the boundary (i.e., $\mathbb{U}_1(1, x) \in \{1, x\}$ for each $x \in [0, 1]$ when $\mathbb{U}_1$ is a conjunctive uninorm [16]), then by $1 > \alpha = \mathbb{U}_2(\alpha, 1) = \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), e_1) = \mathbb{U}_1(1, \mathbb{U}_2(\alpha, e_1)) = \mathbb{U}_2(\alpha, e_1)$ for each $\alpha < 1$ we obtain Proposition 5 of [16].

(iv) Let $a_1 = f_2 = 0$ in Theorem 3.5. Then $\mathbb{U}_1$ is a uninorm and $\mathbb{U}_2$ is a t-conorm. Hence, we derive Theorem 3.3 of [38].

(v) Let $a_1 = a_2 = 1$ in Theorem 3.6. Then both $\mathbb{U}_1$ and $\mathbb{U}_2$ are uninorms and $\mathbb{U}_2(\alpha, 1) \leq 1$. This deduces Lemma 1 in [19].

(vi) Let $e_1 = e_2 = 0$ and $f_1 = f_2 = 1$ in Theorem 3.5. Then both $\mathbb{U}_1$ and $\mathbb{U}_2$ are nullnorms.

- Let $a_1 = a_2$. If $\alpha \leq a_1$, then $\mathbb{U}_2(\alpha, 0) \leq a_1$, $\mathbb{U}_1(\mathbb{U}_2(\alpha, 1), x) = \mathbb{U}_1(a_1, x) = a_1 = \mathbb{U}_1(\alpha, x)$ and $\mathbb{U}_1(\alpha, 0) = \alpha$. If $\alpha > a_1$, then $\mathbb{U}_2(\alpha, 0) = a_1$ and $\mathbb{U}_1(\alpha, 1) = \alpha$. Consequently, we have Theorem 3 of [36].
- Let $a_1 \neq a_2$ and $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative.
  (a) If $a_1 > a_2$, then we claim that $\alpha \geq a_1$. Indeed, if $\alpha < a_1$, then $\mathbb{U}_2(\alpha, 0) = \alpha < a_1$ for each $\alpha < a_2$ and $\mathbb{U}_2(\alpha, 0) = a_2 < a_1$ for each $\alpha \geq a_2$, i.e., $\mathbb{U}_2(\alpha, 0) < a_1$. Moreover, $a_2 = \mathbb{U}_2(\alpha, 1) \neq \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), 1) = a_1$ for each $\alpha < a_2$ and $\alpha = \mathbb{U}_2(\alpha, 1) \neq \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), 1) = a_1$ for each $\alpha > a_2$, i.e., $\mathbb{U}_2(\alpha, 1) \neq \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), 1)$. This contradicts the fact $\mathbb{U}_2(\alpha, 1) = \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), 1)$ obtained from Theorem 3.5 (iii).
  (b) If $a_1 < a_2$, then we claim that $\alpha \leq a_1$. Indeed, if $\alpha > a_1$, then $\mathbb{U}_2(\alpha, 0) = \alpha > a_1$ for each $\alpha < a_2$ and $\mathbb{U}_2(\alpha, 0) = a_2 > a_1$ for each $\alpha \geq a_2$, i.e., $\mathbb{U}_2(\alpha, 0) > a_1$. Analogously, one can prove that $\mathbb{U}_2(\alpha, 1) > a_1$. Thus $\mathbb{U}_2(\alpha, 0) > a_1 = \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), 0)$, contrary to the fact $\mathbb{U}_2(\alpha, 0) = \mathbb{U}_1(\mathbb{U}_2(\alpha, 1), 0)$ obtained from Theorem 3.5 (iv). Therefore, if both $\mathbb{U}_1$ and $\mathbb{U}_2$ are nullnorms with different absorbing elements, then we derive Theorem 7 in [36].

(vii) Let $e_1 = 0$, and $e_2 = a_2 = f_1 = 1$ in Theorem 3.6. Then $\mathbb{U}_1$ is a nullnorm and $\mathbb{U}_2$ is a t-norm. Since $\alpha \leq a_1$ yields $\mathbb{U}_2(\alpha, 1) \leq a_1$, and $\alpha > a_1$ leads to $\mathbb{U}_2(\alpha, 1) > a_1$, we obtain the following three results:

- If $\alpha \leq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(0, x)$ for any $x \in [0, 1]$.
- If $\alpha > a_1$, then since $\mathbb{U}_1(\lambda, y) = a_1 = \mathbb{U}_1(a_1, y)$ for any $y \in [a_1, 1]$, $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} x, & x \in [0, a_1], \\ \mathbb{U}_1(\alpha, x), & x \in [a_1, 1]. \end{cases}$$

- In particular, if $\alpha = a_1$, then we have Theorem 3.5 in [38].

(viii) If $e_1 = f_2 = 0$ and $f_1 = 1$ in Theorem 3.5, then $\mathbb{U}_1$ is a nullnorm with absorbing element $a_1$ and $\mathbb{U}_2$ is a t-conorm. Since $\alpha < a_1$ leads to $\mathbb{U}_2(\alpha, 0) < a_1$ and $\alpha \geq a_1$ yields $\mathbb{U}_2(\alpha, 0) \geq a_1$, we have the following three results:

- If $\alpha < a_1$, then from $\mathbb{U}_1(\mu, x) = a_1 = \mathbb{U}_1(a_1, x)$ for any $x \in [0, a_1]$, $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\alpha, x), & x \in [0, a_1], \\ x, & x \in [a_1, 1]. \end{cases}$$

- If $\alpha \geq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_2(\alpha, x) = \mathbb{U}_1(1, x)$ for any $x \in [0, 1]$.
- In particular, if $\alpha = a_1$, then we obtain Theorem 3.7 in [38].

(ix) Let $f_1 = f_2 = 1$ in Theorem 3.6. Then both $\mathbb{U}_1$ and $\mathbb{U}_2$ are uni-nullnorms with 2-neutral elements $\{e_1, 1\}_{a_1}$ and $\{e_2, 1\}_{a_2}$, respectively. If $\mathbb{U}_2(\alpha, 1) = \mu \geq a_1$, then from Lemma 3.2 (ii) we have

$$\lambda = \mathbb{U}_2(\alpha, e_1) \leq \mathbb{U}_2(\alpha, a_1) \leq a_1,$$

and subsequently, by Proposition 2.4 (i) we have $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(\lambda, a_1)$ for each $x \in [a_1, 1]$. Meanwhile, from Formula (5) we have $\mathbb{U}_2(\alpha, a_1) = \mathbb{U}_1(\lambda, a_1) = \mathbb{U}_1(\mu, a_1)$. Therefore, $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(\lambda, a_1) = \mathbb{U}_1(\mu, a_1) = a_1 = \mathbb{U}_1(x, a_1)$ for each $x \in [a_1, 1]$ and $\mathbb{U}_2(\alpha, 1) \geq a_1$. Consequently, by Theorem 3.6 (i) and (ii) we have all the results in Section 4 of [22].

(x) Let $e_1 = e_2 = 0$ in Theorem 3.5. Then both $\mathbb{U}_1$ and $\mathbb{U}_2$ are null-uninorms with 2-neutral elements $\{0, f_1\}_{a_1}$ and $\{0, f_2\}_{a_2}$, respectively. If $\mathbb{U}_2(\alpha, 0) = \lambda \leq a_1$, then from Lemma 3.2 (i) we have $\mu = \mathbb{U}_2(\alpha, f_1) \geq a_1$, and subsequently, by Proposition 2.4 (iii) we have $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(\mu, a_1)$ for any $x \in [0, a_1]$. Obviously, from Formula (4) we have $\mathbb{U}_2(\alpha, a_1) = \mathbb{U}_1(\lambda, a_1) = \mathbb{U}_1(\mu, a_1)$, thus $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(\mu, a_1) = \mathbb{U}_1(\lambda, a_1) = a_1 = \mathbb{U}_1(a_1, x)$ for any $x \in [0, a_1]$ and $\mathbb{U}_2(\alpha, 0) \leq a_1$. Consequently, by Theorem 3.5 (iii) and (iv) we obtain all the results in Section 4 of [22].

(xi) Let $a_1 = a_2$ in Theorems 3.5 and 3.6. Then $\mathbb{U}_1$ and $\mathbb{U}_2$ are two 2-uninorms on the unit interval $[0, 1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_2\}_{a_1}$, respectively. If $\alpha \in [0, e_2]$, then $\mathbb{U}_2(\alpha, f_1) \leq \mathbb{U}_2(f_2, f_1) = f_1$. If $\alpha \in [f_2, 1]$, then $\mathbb{U}_2(\alpha, e_1) \geq \mathbb{U}_2(e_2, e_1) = e_1$. If $\alpha \in [e_2, f_2]$, then $\mathbb{U}_1(a_1, x) = \mathbb{U}_1(\lambda, x)$ for any $x \in [a_1, 1]$ and $\mathbb{U}_1(a_1, x) = \mathbb{U}_1(\mu, x)$ for any $x \in [0, a_1]$. Therefore, we obtain Theorem 3.1 of [14].

(xii) Let $e_1 = e_2$ in Theorems 3.5 and 3.6. Then $\mathbb{U}_1$ and $\mathbb{U}_2$ are two 2-uninorms on the unit interval $[0, 1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_1, f_2\}_{a_2}$, respectively. Since $\lambda = \mathbb{U}_2(\alpha, e_1) = \alpha$ for any $\alpha \in [0, e_1]$, we have Theorems 3.2 and 3.3 in [14] in analogy to (xi).

(xiii) Let $f_1 = f_2$ in Theorems 3.5 and 3.6. Then $\mathbb{U}_1$ and $\mathbb{U}_2$ are two 2-uninorms on the unit interval $[0, 1]$ with 2-neutral elements $\{e_1, f_1\}_{a_1}$ and $\{e_2, f_1\}_{a_2}$, respectively. Since $\mu = \mathbb{U}_2(\alpha, f_1) = \alpha$ for any $\alpha \in [f_1, 1]$, we get Theorems 3.4 and 3.5 of [14] in analogy to (xi).

Further, we have the following corollary.

**Corollary 3.8** *Let $\mathbb{U}$ be a 2-uninorm with 2-neutral element $\{e_{\mathbb{U}}, f_{\mathbb{U}}\}_{a_{\mathbb{U}}}$, $T$ be a t-norm, $S$ be a t-conorm, $U$ be a uninorm with a neutral element $e_U$, $N$ be a nullnorm with an absorbing element $a_N$, $F$ be a uni-nullnorm with 2-neutral element $\{e_F, 1\}_{a_F}$ and $G$ be a null-uninorm with 2-neutral element $\{0, f_G\}_{a_G}$ on the unit interval $[0, 1]$, respectively.*

(i) *$T$ is $(\alpha, \mathbb{U})$-migrative iff $T(\mathbb{U}(\alpha, 1), x) = \mathbb{U}(\alpha, x)$ for each $x \in [0, 1]$.*

(ii)
- *If $T(\alpha, e_{\mathbb{U}}) < e_{\mathbb{U}}$ and $T(\alpha, f_{\mathbb{U}}) \leq a_{\mathbb{U}}$, then $\mathbb{U}$ is $(\alpha, T)$-migrative iff $\mathbb{U}(T(\alpha, e_{\mathbb{U}}), x) = T(\alpha, x)$ for each $x \in [0, 1]$.*
- *If $T(\alpha, e_{\mathbb{U}}) < e_{\mathbb{U}}$ and $T(\alpha, f_{\mathbb{U}}) \geq a_{\mathbb{U}}$, then $\mathbb{U}$ is $(\alpha, T)$-migrative iff*

$$T(\alpha, x) = \begin{cases} \mathbb{U}(T(\alpha, e_{\mathbb{U}}), x), & x \in [0, a_{\mathbb{U}}], \\ \mathbb{U}(T(\alpha, f_{\mathbb{U}}), x), & x \in [a_{\mathbb{U}}, 1], \end{cases}$$

*and $\mathbb{U}(T(\alpha, e_{\mathbb{U}}), x) = \mathbb{U}(a_{\mathbb{U}}, x)$ for any $x \in [a_{\mathbb{U}}, 1]$.*
- *If $T(\alpha, e_{\mathbb{U}}) = e_{\mathbb{U}}$, then $\mathbb{U}$ is $(\alpha, T)$-migrative iff*

$$T(\alpha, x) = \begin{cases} \mathbb{U}(T(\alpha, e_{\mathbb{U}}), x), & x \in [0, a_{\mathbb{U}}], \\ \mathbb{U}(T(\alpha, f_{\mathbb{U}}), x), & x \in [a_{\mathbb{U}}, 1] \end{cases}$$

*and $\mathbb{U}(T(\alpha, f_{\mathbb{U}}), x) = \mathbb{U}(a_{\mathbb{U}}, x)$ for each $x \in [0, a_{\mathbb{U}}]$.*

(iii) *$S$ is $(\alpha, \mathbb{U})$-migrative iff $S(\mathbb{U}(\alpha, 0), x) = \mathbb{U}(\alpha, x)$ for each $x \in [0, 1]$.*

(iv)
- *If $S(\alpha, f_{\mathbb{U}}) = f_{\mathbb{U}}$, then $\mathbb{U}$ is $(\alpha, S)$-migrative iff*

$$S(\alpha, x) = \begin{cases} \mathbb{U}(S(\alpha, e_{\mathbb{U}}), x), & x \in [0, a_{\mathbb{U}}], \\ \mathbb{U}(S(\alpha, f_{\mathbb{U}}), x), & x \in [a_{\mathbb{U}}, 1] \end{cases}$$

*and $\mathbb{U}(S(\alpha, e_{\mathbb{U}}), x) = \mathbb{U}(a_{\mathbb{U}}, x)$ for any $x \in [a_{\mathbb{U}}, 1]$.*
- *If $S(\alpha, f_{\mathbb{U}}) > f_{\mathbb{U}}$ and $S(\alpha, e_{\mathbb{U}}) \leq a_{\mathbb{U}}$, then $\mathbb{U}$ is $(\alpha, S)$-migrative iff*

$$S(\alpha, x) = \begin{cases} \mathbb{U}(S(\alpha, e_{\mathbb{U}}), x), & x \in [0, a_{\mathbb{U}}], \\ \mathbb{U}(S(\alpha, f_{\mathbb{U}}), x), & x \in [a_{\mathbb{U}}, 1], \end{cases}$$

*and $\mathbb{U}(S(\alpha, f_{\mathbb{U}}), x) = \mathbb{U}(a_{\mathbb{U}}, x)$ for any $x \in [0, a_{\mathbb{U}}]$.*

- *If* $S(\alpha, f_{\mathbb{U}}) > f_{\mathbb{U}}$ *and* $S(\alpha, e_{\mathbb{U}}) \geq a_{\mathbb{U}}$*, then* $\mathbb{U}$ *is* $(\alpha, S)$*-migrative iff* $\mathbb{U}(S(\alpha, f_{\mathbb{U}}), x) = S(\alpha, x)$ *for each* $x \in [0,1]$.

(v) $U$ *is* $(\alpha, \mathbb{U})$*-migrative iff* $U(\mathbb{U}(\alpha, e_U), x) = \mathbb{U}(\alpha, x)$ *for each* $x \in [0,1]$.

(vi)
- *If* $\mathbb{U}(\alpha, 0) \leq a_N$*, then* $N$ *is* $(\alpha, \mathbb{U})$*-migrative iff*

$$\mathbb{U}(\alpha, x) = \begin{cases} N(\mathbb{U}(\alpha, 0), x), & x \in [0, a_N], \\ N(\mathbb{U}(\alpha, 1), x), & x \in [a_N, 1]. \end{cases}$$

- *If* $\mathbb{U}(\alpha, 0) \geq a_N$*, then* $N$ *is* $(\alpha, \mathbb{U})$*-migrative iff* $N(\mathbb{U}(\alpha, 1), x) = \mathbb{U}(\alpha, x)$ *for each* $x \in [0,1]$.

(vii)
- *If* $\mathbb{U}(\alpha, 1) \leq a_F$*, then* $F$ *is* $(\alpha, \mathbb{U})$*-migrative iff* $F(\mathbb{U}(\alpha, e_F), x) = \mathbb{U}(\alpha, x)$ *for each* $x \in [0,1]$.
- *If* $\mathbb{U}(\alpha, 1) \geq a_F$*, then* $F$ *is* $(\alpha, \mathbb{U})$*-migrative iff*

$$\mathbb{U}(\alpha, x) = \begin{cases} F(\mathbb{U}(\alpha, e_F), x), & x \in [0, a_F], \\ F(\mathbb{U}(\alpha, 1), x), & x \in [a_F, 1]. \end{cases}$$

(viii)
- *If* $\mathbb{U}(\alpha, 0) \leq a_G$*, then* $G$ *is* $(\alpha, \mathbb{U})$*-migrative iff*

$$\mathbb{U}(\alpha, x) = \begin{cases} G(\mathbb{U}(\alpha, 0), x), & x \in [0, a_G], \\ G(\mathbb{U}(\alpha, f_G), x), & x \in [a_G, 1]. \end{cases}$$

- *If* $\mathbb{U}(\alpha, 0) \geq a_G$*, then* $G$ *is* $(\alpha, \mathbb{U})$*-migrative iff* $G(\mathbb{U}(\alpha, f_G), x) = \mathbb{U}(\alpha, x)$ *for each* $x \in [0,1]$.

*Proof.*

(i) Taking $e_1 = a_1 = f_1 = 1$ in Theorem 3.6, then $\mathbb{U}_2(\alpha, 1) \leq 1$ for any $\alpha \leq 1$, and subsequently, from Theorem 3.6 (i) we get that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\lambda, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0,1]$. Finally, let $\mathbb{U}_1 = T$ and $\mathbb{U}_2 = \mathbb{U}$. Then we get (i).

(ii) Taking $e_2 = a_2 = f_2 = 1$ in Theorem 3.5, then $\mathbb{U}_2(\alpha, e_1) \leq e_1$ for any $\alpha \leq 1$, and subsequently, from Theorem 3.5 (i), (ii) and (iii) we get that

- If $\mathbb{U}_2(\alpha, e_1) < e_1$ and $\mathbb{U}_2(\alpha, f_1) \leq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\lambda, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0,1]$.
- If $\mathbb{U}_2(\alpha, e_1) < e_1$ and $\mathbb{U}_2(\alpha, f_1) \geq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1], \end{cases}$$

  and $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ for any $x \in [a_1, 1]$.
- If $\mathbb{U}_2(\alpha, e_1) = e_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases}$$

  and $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ for each $x \in [0, a_1]$.

  Finally, let $\mathbb{U}_1 = \mathbb{U}$ and $\mathbb{U}_2 = T$. Then (ii) holds.

(iii) Taking $e_1 = a_1 = f_1 = 0$ in Theorem 3.5, then $\mathbb{U}_2(\alpha, 0) \geq 0$ for any $\alpha \geq 0$, and subsequently, from Theorem 3.5 (iv) we get that $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for any $x \in [0,1]$. Finally, let $\mathbb{U}_1 = S$ and $\mathbb{U}_2 = \mathbb{U}$. Then we get (iii).

(iv) Taking $a_2 = f_2 = 0$ in Theorem 3.6, then $\mathbb{U}_2(\alpha, f_1) \geq \mathbb{U}_2(0, f_1) = f_1$ for any $\alpha \geq 0$, and subsequently, from Theorem 3.6 (ii), (iii) and (iv) we get that

- If $\mathbb{U}_2(\alpha, f_1) = f_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1] \end{cases}$$

and $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ for any $x \in [a_1, 1]$.

- If $\mathbb{U}_2(\alpha, f_1) > f_1$ and $\mathbb{U}_2(\alpha, e_1) \leq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1], \end{cases}$$

and $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ for any $x \in [0, a_1]$.

- If $\mathbb{U}_2(\alpha, f_1) > f_1$ and $\mathbb{U}_2(\alpha, e_1) \geq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0, 1]$.

Finally, let $\mathbb{U}_1 = \mathbb{U}$ and $\mathbb{U}_2 = S$. Then (iv) holds.

(v) Taking $e_1 = a_1 = 0$ in Theorem 3.5, then $\mathbb{U}_2(\alpha, 0) \geq 0$ for any $\alpha \geq 0$, and subsequently, $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for any $x \in [0, 1]$. Finally, let $\mathbb{U}_1 = U$ and $\mathbb{U}_2 = \mathbb{U}$. Then (v) holds.

(vi) Taking $e_1 = 0$ and $f_1 = 1$ in Theorem 3.5, then from Theorem 3.5 (iii) and (iv) we get that

- If $\mathbb{U}_2(\alpha, e_1) \leq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1]. \end{cases}$$

In this case, $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ for each $x \in [0, a_1]$ is clearly established because of $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(\mu, a_1) = \mathbb{U}_1(\lambda, a_1) = a_1 = \mathbb{U}_1(a_1, x)$.

- If $\mathbb{U}_2(\alpha, e_1) \geq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0, 1]$.

Finally, let $\mathbb{U}_1 = N$ and $\mathbb{U}_2 = \mathbb{U}$. Then (vi) holds.

(vii) Taking $f_1 = 1$ in Theorem 3.6, then from Theorem 3.6 (i) and (ii) we get that

- If $\mathbb{U}_2(\alpha, f_1) \leq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\lambda, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0, 1]$.
- If $\mathbb{U}_2(\alpha, f_1) \geq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1]. \end{cases}$$

In this case, $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(a_1, x)$ for any $x \in [a_1, 1]$ is clearly established because of $\mathbb{U}_1(\lambda, x) = \mathbb{U}_1(\lambda, a_1) = \mathbb{U}_1(\mu, a_1) = a_1 = \mathbb{U}_1(a_1, x)$.

Finally, let $\mathbb{U}_1 = F$ and $\mathbb{U}_2 = \mathbb{U}$. Then (vii) holds.

(viii) Taking $e_1 = 0$ in Theorem 3.5, then from Theorem 3.5 (iii) and (iv) we get that

- If $\mathbb{U}_2(\alpha, e_1) \leq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff

$$\mathbb{U}_2(\alpha, x) = \begin{cases} \mathbb{U}_1(\lambda, x), & x \in [0, a_1], \\ \mathbb{U}_1(\mu, x), & x \in [a_1, 1]. \end{cases}$$

In this case, $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(a_1, x)$ for each $x \in [0, a_1]$ is clearly established because of $\mathbb{U}_1(\mu, x) = \mathbb{U}_1(\mu, a_1) = \mathbb{U}_1(\lambda, a_1) = a_1 = \mathbb{U}_1(a_1, x)$.

- If $\mathbb{U}_2(\alpha, e_1) \geq a_1$, then $\mathbb{U}_1$ is $(\alpha, \mathbb{U}_2)$-migrative iff $\mathbb{U}_1(\mu, x) = \mathbb{U}_2(\alpha, x)$ for each $x \in [0, 1]$.

Finally, let $\mathbb{U}_1 = G$ and $\mathbb{U}_2 = \mathbb{U}$. Then (viii) holds.

As a conclusion of this section, we give a simple example to help us understand the above characterizations.

**Example 3.9** Define $\mathbb{U}_1$ and $\mathbb{U}_2$ as follows:

$$\mathbb{U}_1(x,y)=\begin{cases}\min(x,y), & (x,y)\in[0,e_1]^2\cup[a_1,f_1]^2\\ a_1, & (x,y)\in[0,a_1]\times[a_1,f_1]\cup[a_1,f_1]\times[0,a_1],\\ \max(x,y), & \text{otherwise}\end{cases}$$

and

$$\mathbb{U}_2(x,y)=\begin{cases}\min(x,y), & (x,y)\in[0,e_2]^2\cup[a_2,f_2]^2\\ a_2, & (x,y)\in[0,a_2]\times[a_2,f_2]\cup[a_2,f_2]\times[0,a_2],\\ \max(x,y), & \text{otherwise.}\end{cases}$$

One can check that $\mathbb{U}_1$ and $\mathbb{U}_2$ are 2-uninorms with 2-neutral elements $\{e_1,f_1\}_{a_1}$ and $\{e_2,f_2\}_{a_2}$, respectively. Let $e_1=0.2$, $a_1=0.6$, $f_1=0.8$, $e_2=0.3$, $a_2=0.5$ and $f_2=0.7$.

- ◇ If $\alpha\in[0,0.3]$, then $\mathbb{U}_2(\alpha,e_1)\le e_1$, $\mu=\mathbb{U}_2(\alpha,f_1)=0.8=f_1$ and $\mathbb{U}_2(\alpha,0.6)=0.5\neq 0.6=\mathbb{U}_1(\mu,0.6)$, showing that $\mathbb{U}_1$ is not $(\alpha,\mathbb{U}_2)$-migrative by Proposition 3.4.
- ◇ If $\alpha\in(0.3,0.5]$, then $\mathbb{U}_2(\alpha,e_1)=\alpha\in[e_1,a_1]$, $\mu=\mathbb{U}_2(\alpha,f_1)=0.8$ and $\mathbb{U}_2(\alpha,0.6)=0.5\neq 0.6=\mathbb{U}_1(\mu,0.6)$, showing that $\mathbb{U}_1$ is not $(\alpha,\mathbb{U}_2)$-migrative by Proposition 3.4.
- ◇ If $\alpha\in(0.5,0.7)$, then $\mathbb{U}_2(\alpha,e_1)=0.5$, $\mu=\mathbb{U}_2(\alpha,f_1)=0.8=f_1$ and $\mathbb{U}_2(\alpha,0.7)=\alpha\neq 0.7=\mathbb{U}_1(\mu,0.7)$, showing that $\mathbb{U}_1$ is not $(\alpha,\mathbb{U}_2)$-migrative by Proposition 3.4.
- ◇ If $\alpha=0.7$, then $\mathbb{U}_2(\alpha,e_1)=0.5$, $\mu=\mathbb{U}_2(\alpha,f_1)=0.8=f_1$, $\mathbb{U}_2(\alpha,x)=0.5=\mathbb{U}_1(\lambda,x)$ for each $x\in[0,0.5]$, $\mathbb{U}_2(\alpha,x)=x=\mathbb{U}_1(\lambda,x)$ for each $x\in[0.5,0.6]$ and $\mathbb{U}_2(\alpha,x)=x=\mathbb{U}_1(\mu,x)$ for each $x\in[0.6,1]$, showing that $\mathbb{U}_1$ is $(\alpha,\mathbb{U}_2)$-migrative by Proposition 3.4.
- ◇ If $\alpha\in(0.7,0.8]$, then $\mathbb{U}_2(\alpha,e_1)=\alpha\ge a_1$, $\mu=\mathbb{U}_2(\alpha,f_1)=0.8$ and $\mathbb{U}_2(\alpha,0.7)=\alpha\neq 0.7=\mathbb{U}_1(\mu,0.7)$, showing that $\mathbb{U}_1$ is not $(\alpha,\mathbb{U}_2)$-migrative by Proposition 3.3.
- ◇ If $\alpha\in(0.8,1]$, then $\mathbb{U}_2(\alpha,e_1)=\alpha\ge a_1$, $\mu=\mathbb{U}_2(\alpha,f_1)=\alpha$ and $\mathbb{U}_2(\alpha,x)=\alpha\vee x=\mathbb{U}_1(\mu,x)$ for each $x\in[0,1]$, showing that $\mathbb{U}_1$ is $(\alpha,\mathbb{U}_2)$-migrative by Proposition 3.3.

# 4 Conclusions

This article provided a complete characterization for the $\alpha$-migrativity of 2-uninorms with $\alpha\in[0,1]$. Comparing with the results of Liang and Wang [14], we supplemented the migrativity of 2-uninorms $\mathbb{U}_1$ and $\mathbb{U}_2$ when $e_1\neq e_2$ and $a_1\neq a_2$ and $f_1\neq f_2$ where $\{e_1,f_1\}_{a_1}$ and $\{e_2,f_2\}_{a_2}$ are 2-neutral elements of $\mathbb{U}_1$ and $\mathbb{U}_2$, respectively. It is worth pointing out that our results are suitable for all 2-uninorms, unlike the results of Huang and Qin [10] and Wang et al. [27] which are for some special 2-uninorms. Furthermore, Corollary 3.8 presented some characterizations of the migrativity between 2-uninorm and other subclasses of 2-uninorms, which together Remark 3.7 shows us that our results unify and generalize some current existing characterizations for the $\alpha$-migrativity of t-norms, t-conorms, uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively.

# Acknowledgement

The authors are grateful to the anonymous referees for their valuable comments and suggestions, which help the authors to improve the earlier version. This work is supported by the National Natural Science Foundation of China (Nos.12071325,12471440).

# Declarations

# Conflict of interest

No potential conflict of interest was declared by the authors. All data generated or analysed during this study are included in this published article (and its supplementary information files).

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