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Construction of $\bar{X} - R$ control charts using beta distribution for triangular fuzzy quality

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Abstract

The primary objective of statistical quality control is to ensure that products or services meet predetermined standards while minimizing variation and instability in processes. Control charts are indispensable tools in quality control, playing a crucial role in enhancing and improving process quality. Given their significance, incorporating fuzzy quality into control charts introduces flexibility into product quality assessment. In this paper, we design mean ($\bar{X}$) and range (R) control charts based on fuzzy quality, as opposed to traditional crisp or precise quality. We construct quantile-based control charts for the degree of membership of observations to triangular fuzzy quality using parameter estimation in the beta distribution. Specifically, we propose two novel methods for constructing $\bar{X}$ and R control charts, namely, the method of moments and maximum likelihood estimation, both based on triangular fuzzy quality.

Keywords: Quantile-based control chart, fuzzy quality, beta distribution, maximum likelihood estimation, method of moments estimation.

1 Introduction and preliminaries

A situation in statistical quality control where quality is described by a fuzzy set over the interval $[LSL, USL]$, rather than a traditional crisp set, is termed "fuzzy quality". Here, LSL and USL denote the lower and upper specification limits, respectively. Fuzzy quality enables the representation and visualization of information in a way that allows experts, engineers, and product designers to account for the ambiguity and fuzziness inherent in real-world situations concerning a product's quality characteristics. Instead of insisting on precise values (i.e., either zero or one) for quality measures, fuzzy quality encompasses a spectrum of values with membership degrees between zero and one, thereby offering a more flexible approach to modeling and visualizing quality.

Yongting [29] introduced the concept of fuzzy quality by replacing the indicator function $I_{\{x:x\in[LSL,USL]\}}$ with the membership function of a fuzzy set $\tilde{Q}$. In this context, $\tilde{Q}$ represents the membership function associated with fuzzy quality, where $\tilde{Q}(x)$ denotes the degree of conformity with standard quality (i.e., the quality degree) when the measured quality characteristic is x [22]. Parchami et al. [22] discussed the motivations and advantages of employing fuzzy quality. Furthermore, Parchami et al. [21] proposed testing the capability of manufacturing processes based on fuzzy quality using Yongting's index and employed Monte Carlo simulations to evaluate process quality in the pipe manufacturing industry [20]. Additionally, Iranmanesh et al. [11] explored quality testing using fuzzy specification limits.

Statistical quality control aims to monitor processes, identify root causes of variations, and implement corrective actions. Control charts are among the most widely used tools in statistical quality control, playing a pivotal role in improving product quality. Given their significance in enhancing and optimizing process quality, integrating fuzzy quality into control charts becomes essential to add flexibility to product quality assessments.

The primary contribution of this paper lies in designing mean ($\bar{X}$) and range (R) control charts based on fuzzy quality rather than crisp quality. The application of fuzzy set theory to control chart construction has been explored

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in various works [8, 9, 10, 12, 13, 15, 23, 25, 26, 28] using different approaches. Parchami et al. [19] proposed $\bar{X}$ and R control charts based on triangular fuzzy quality using quantiles derived from kernel density estimation for quality degrees. Amirzadeh et al. [2] developed a fuzzy p-chart based on the mean degree of nonconformity under the assumption that quality characteristics follow a normal distribution, demonstrating that their fuzzy p-chart outperforms traditional p-charts in terms of operating characteristic and average run length (ARL) curves. Furthermore, Amirzadeh et al. [3] introduced a fuzzy multinomial control chart for linguistic variables.

Alizadeh and Fatemi Ghomi [1] developed $\bar{X}$ and R control charts in a fuzzy environment using various transformation techniques. Bai and Choi [4] designed mean and range control charts using a weighted variance method tailored to skewed distributions. Chan and Cui [5] proposed $\bar{X}$ and R control charts based on a skewness correction method. Chang and Bai [6] designed $\bar{X}$ control charts using weighted standard deviations, while Chen and Kuo [7] compared symmetric and asymmetric control limits for mean and range charts. Koyuncu and Karagoz [14] constructed robust R control charts using ranked set sampling and median ranked set sampling for asymmetric distributions. Veljkovic [27] focused on R control charts for positively skewed distributions.

Let X be a univariate quality characteristic. In crisp quality assessments, if the measured characteristic lies within the specification limits $[LSL, USL]$, the product is classified as non-defective; otherwise, it is classified as defective. Similar to any fuzzy set defined by a membership function, fuzzy quality is characterized by its membership function. Here, LSL and USL represent the minimum and maximum allowable limits based on process standards, while $T \in [LSL, USL]$ denotes the target value of the quality characteristic in the production process.

We consider a triangular fuzzy number $\tilde{Q}_\Delta$, defined by LSL, T, and USL, and denoted as $Tr(LSL, T, USL)$, with a membership function given by

$$\tilde{Q}_\Delta(x) = \begin{cases} \frac{x-LSL}{T-LSL}, & \text{if } LSL \le x < T, \\ \frac{USL-x}{USL-T}, & \text{if } T \le x < USL, \\ 0, & \text{otherwise,} \end{cases}$$

where the target value T and the interval $[LSL, USL]$ serve as the core and support of the triangular fuzzy quality, respectively.

In this study, real-valued data are utilized for the design of control charts, with quality treated as a fuzzy set. The main objective is to develop $\bar{X}$ and R control charts for the degrees of membership of observations to the triangular fuzzy quality $\tilde{Q}_\Delta$, using quantiles from the beta distribution. This approach enables decision-making based on fuzzy quality membership functions, offering greater adaptability compared to traditional crisp quality assessments.

This paper is structured as follows. Section 2 reviews two methods for parameter estimation in the beta distribution. In Section 3, we develop $\bar{X}$ and R control charts based on triangular fuzzy quality using quantiles from the fitted beta distribution through two distinct methods. A case study is presented in Section 4 to demonstrate the performance of the proposed control charts. In Section 5, we compare the proposed charts through simulation studies. Finally, conclusions and future research directions are discussed in the concluding section.

2 Parameter estimation in the beta distribution

The beta distribution is a family of continuous distributions defined on the interval $[0, 1]$. A random variable X is said to follow a beta distribution with parameters $a > 0$ and $b > 0$, denoted by $X \sim Beta(a, b)$, if its probability density function (pdf) is given by

$$f(x; a, b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 \le x \le 1, \tag{1}$$

where $B(a, b)$ is the beta function, and $\Gamma(\cdot)$ is the gamma function.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from $Beta(a, b)$, and let $\mathbf{x} = (x_1, \ldots, x_n)$ represent the observed sample. Since the quality degrees of observations lie between zero and one, they can be effectively modeled using a beta distribution. This section provides an overview of the maximum likelihood estimators (MLEs) and method of moments estimators (MMEs) for the unknown parameters a and b [17, 18].

2.1 Maximum likelihood estimators (MLEs) for the beta distribution

The likelihood function based on the observations $\mathbf{x}$ is given by

$$L(a, b; \mathbf{x}) = \prod_{i=1}^{n} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x_i^{a-1}(1-x_i)^{b-1} = \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{a-1} \left(\prod_{i=1}^{n}(1-x_i)\right)^{b-1}. \tag{2}$$

The corresponding log-likelihood function is

$$l(a,b;\mathbf{x}) = n\ln(\Gamma(a+b)) - n\ln(\Gamma(a)) - n\ln(\Gamma(b)) + (a-1)\sum_{i=1}^{n}\ln(x_i) + (b-1)\sum_{i=1}^{n}\ln(1-x_i). \tag{3}$$

The MLEs of the parameters a and b are the values that maximize the log-likelihood function. To find these estimators, we compute the partial derivatives of the log-likelihood function with respect to each parameter and set them equal to zero

$$\frac{\partial}{\partial a}l(a,b;\mathbf{x}) = \frac{n\frac{\partial\Gamma(a+b)}{\partial a}}{\Gamma(a+b)} - \frac{n\frac{\partial\Gamma(a)}{\partial a}}{\Gamma(a)} + \sum_{i=1}^{n}\ln(x_i) = 0, \tag{4}$$

$$\frac{\partial}{\partial b}l(a,b;\mathbf{x}) = \frac{n\frac{\partial\Gamma(a+b)}{\partial b}}{\Gamma(a+b)} - \frac{n\frac{\partial\Gamma(b)}{\partial b}}{\Gamma(b)} + \sum_{i=1}^{n}\ln(1-x_i) = 0. \tag{5}$$

The MLEs for the parameters a and b are obtained by solving Eqs. (4) and (5). However, these equations do not have closed-form solutions. Therefore, numerical methods are required to solve them. The MLEs, denoted by $\widehat{a^{\mathbf{x}}_{ML}}$ and $\widehat{b^{\mathbf{x}}_{ML}}$, are computed using numerical techniques, such as the Newton-Raphson method or other iterative optimization algorithms.

2.2 Method of moments estimators (MMEs) for the beta distribution

The method of moments estimation involves equating sample moments to the corresponding population moments. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i$ denote the sample mean, and let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ denote the sample variance. For a beta distribution, the population mean and variance are given by

$$\mathbb{E}[X] = \frac{a}{a+b}, \tag{6}$$

$$\mathrm{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}. \tag{7}$$

By equating the sample mean and variance to the population mean and variance, and solving Eqs. (6) and (7) for a and b, we obtain the method of moments estimators as follows:

$$\widehat{a^{\mathbf{x}}_{MM}} = \bar{X}\left(\frac{\bar{X}(1-\bar{X})}{S^2} - 1\right), \tag{8}$$

$$\widehat{b^{\mathbf{x}}_{MM}} = (1-\bar{X})\left(\frac{\bar{X}(1-\bar{X})}{S^2} - 1\right), \tag{9}$$

where the superscript "$\mathbf{x}$" indicates that the estimators are based on the observed sample, and the subscript "MM" signifies the use of the method of moments for parameter estimation.

3 Construction of $\bar{X}$ and R control charts using quantiles of the beta distribution based on triangular fuzzy quality

Let f be the probability density function of a one-dimensional continuous quality characteristic X, which is assumed to be unknown in this study. Suppose we have m samples of size n available from the quality characteristic X. Typically, in quality control, sample sizes are kept small due to the high costs associated with sampling and inspection. The main focus of this paper is the construction of $\bar{X}$ and R control charts for the quality degree of observations x_{ij}, denoted by $\tilde{Q}_{\Delta}(x_{ij})$ (rather than the observations x_{ij} themselves).

The distribution of the quality degrees $\tilde{Q}_{\Delta}(x_{ij})$ is unknown since it depends on both the underlying distribution of X_{ij} and the membership function of the fuzzy quality. It is important to note that the quality degrees $\tilde{Q}_{\Delta}(x_{ij})$ exhibit a probabilistic-possibilistic nature because $\tilde{Q}_{\Delta}(X_{ij})$, as a function of the random variable X_{ij}, is itself a random variable. Consequently, the mean and range of these quality degrees must also follow some distribution, which we model using a beta distribution. As illustrated in Figure 1, the beta distribution is capable of modeling various symmetric and asymmetric unimodal distributions over the interval $[0,1]$.

Let $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{nj})$ denote the j-th sample for $j = 1, \ldots, m$. Thus, $\tilde{Q}_{\Delta j} = (\tilde{Q}_\Delta(x_{1j}), \tilde{Q}_\Delta(x_{2j}), \ldots, \tilde{Q}_\Delta(x_{nj}))$ represents the j-th sample of quality degrees, where $\tilde{Q}_\Delta(x_{ij}) \in [0,1]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Consequently, the mean and range of the j-th sample of quality degrees are computed as follows:

$$\overline{\tilde{Q}}_{\Delta j} = \frac{1}{n} \sum_{i=1}^{n} \tilde{Q}_\Delta(x_{ij}), \tag{10}$$

and

$$R_j = \max_{i=1,\ldots,n} \tilde{Q}_\Delta(x_{ij}) - \min_{i=1,\ldots,n} \tilde{Q}_\Delta(x_{ij}). \tag{11}$$

Additionally, we define the vectors of means and ranges of the quality degrees as:

$$\overline{\tilde{\mathbf{Q}}}_\Delta = (\overline{\tilde{Q}}_{\Delta 1}, \ldots, \overline{\tilde{Q}}_{\Delta m}), \tag{12}$$

and

$$\mathbf{R} = (R_1, \ldots, R_m), \tag{13}$$

respectively. These vectors serve as the basis for plotting $\bar{X}$ and R control charts for fuzzy quality based on sample numbers.

In certain cases, the results of goodness-of-fit (GOF) tests confirm that a beta distribution is a suitable model for the distribution of the quality degrees. This motivates the use of the beta distribution in the design of control charts in this study. The beta distribution, a family of continuous probability distributions defined on the interval $[0, 1]$, can exhibit various shapes on this range (as illustrated in Figure 1). By adjusting its two parameters, the beta distribution can represent a wide range of symmetric and asymmetric unimodal distributions, making it an appropriate model for the statistics of quality degrees $\overline{\tilde{Q}}_{\Delta j}$ and R_j, which are unimodal continuous random variables constrained to the interval $[0, 1]$.

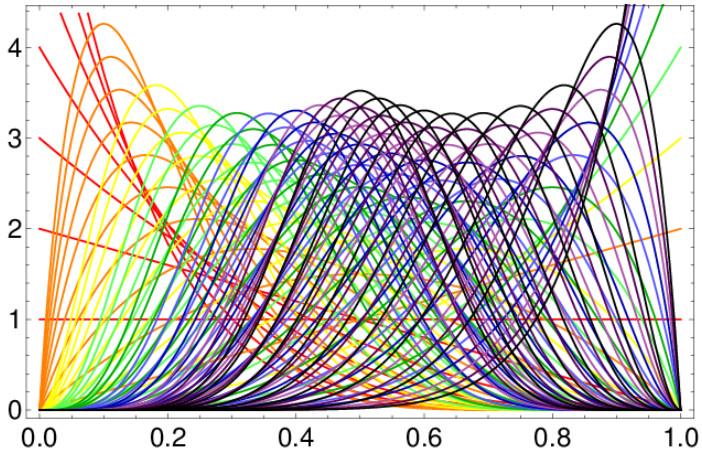

Figure 1: Various forms of the beta distribution over the range [0, 1].

Assume that the means $\overline{\tilde{Q}}_{\Delta j}$ follow a beta distribution with unknown parameters $a^{\overline{\tilde{\mathbf{Q}}}_\Delta}$ and $b^{\overline{\tilde{\mathbf{Q}}}_\Delta}$, and similarly, that the ranges R_j follow a beta distribution with parameters $a^{\mathbf{R}}$ and $b^{\mathbf{R}}$. In this section, we investigate two distinct methods for designing $\bar{X}$ and R control charts based on the quality degrees $\tilde{Q}_\Delta(x_{ij})$.

3.1 MLE-based method

In this approach, the parameters of the distributions for $\overline{\tilde{Q}}_{\Delta j}$ and R_j are estimated using the Maximum Likelihood Estimation (MLE) method. The center line (CL) and control limits (LCL, UCL) for the $\bar{X}$ control chart, based on quantiles of the beta distribution, are computed as follows:

$$\begin{cases} LCL_{\bar{X}}^{MLE} = F_{p/2}^{-1}\left(\widehat{a_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}, \widehat{b_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}\right), \\ CL_{\bar{X}}^{MLE} = F_{1/2}^{-1}\left(\widehat{a_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}, \widehat{b_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}\right), \\ UCL_{\bar{X}}^{MLE} = F_{1-p/2}^{-1}\left(\widehat{a_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}, \widehat{b_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}\right), \end{cases} \tag{14}$$

where F_p^{-1} represents the p-th quantile of the beta distribution with parameters $\widehat{a_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ and $\widehat{b_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$. The terms $\widehat{a_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ and $\widehat{b_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ are the MLEs of the parameters $a^{\overline{\tilde{\mathbf{Q}}}_\Delta}$ and $b^{\overline{\tilde{\mathbf{Q}}}_\Delta}$, as obtained from Eqs. (4) and (5). In other words, the center line, lower control limit, and upper control limit correspond to the median, $\frac{p}{2}$-th quantile, and $(1-\frac{p}{2})$-th quantile of the beta distribution with the estimated parameters $\widehat{a_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ and $\widehat{b_{ML}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$. Typically, $p = 0.0027$ is chosen since $P(-3 \le Z \le 3) = 1 - p = 0.9973$, where Z is a standard normal random variable [16].

Similarly, the center line and control limits for the R control chart, based on quantiles of the beta distribution, are given by

$$\begin{cases} LCL_R^{MLE} = F_{p/2}^{-1}\left(\widehat{a_{ML}^{\mathbf{R}}}, \widehat{b_{ML}^{\mathbf{R}}}\right), \\ CL_R^{MLE} = F_{1/2}^{-1}\left(\widehat{a_{ML}^{\mathbf{R}}}, \widehat{b_{ML}^{\mathbf{R}}}\right), \\ UCL_R^{MLE} = F_{1-p/2}^{-1}\left(\widehat{a_{ML}^{\mathbf{R}}}, \widehat{b_{ML}^{\mathbf{R}}}\right), \end{cases} \tag{15}$$

where F_p^{-1} denotes the p-th quantile of the beta distribution with parameters $\widehat{a_{ML}^{\mathbf{R}}}$ and $\widehat{b_{ML}^{\mathbf{R}}}$. The MLEs $\widehat{a_{ML}^{\mathbf{R}}}$ and $\widehat{b_{ML}^{\mathbf{R}}}$ are the estimates for the parameters $a^{\mathbf{R}}$ and $b^{\mathbf{R}}$, derived using Eqs. (4) and (5).

In summary, the construction of control charts using MLE-based quantiles of the beta distribution provides a flexible approach for monitoring the quality degrees based on triangular fuzzy sets. This method allows for the incorporation of uncertainty inherent in the process, thus enhancing the robustness of quality control monitoring.

3.2 MME-based method

In this method, the Method of Moments Estimators (MMEs) for the parameters of $\overline{\tilde{Q}}_{\Delta j}$ and R_j are employed to construct an alternative set of control limits for the $\bar{X}$-R control charts, based on the degree of membership to fuzzy quality. The center line (CL) and control limits (LCL, UCL) for the $\bar{X}$ control chart are calculated as follows:

$$\begin{cases} LCL_{\bar{X}}^{MME} = F_{p/2}^{-1}\left(\widehat{a_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}, \widehat{b_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}\right), \\ CL_{\bar{X}}^{MME} = F_{1/2}^{-1}\left(\widehat{a_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}, \widehat{b_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}\right), \\ UCL_{\bar{X}}^{MME} = F_{1-p/2}^{-1}\left(\widehat{a_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}, \widehat{b_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}\right), \end{cases} \tag{16}$$

where F_p^{-1} denotes the p-th quantile of the beta distribution with parameters $\widehat{a_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ and $\widehat{b_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$. Here, $\widehat{a_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ and $\widehat{b_{MM}^{\overline{\tilde{\mathbf{Q}}}_\Delta}}$ are the MMEs of the parameters $a^{\overline{\tilde{\mathbf{Q}}}_\Delta}$ and $b^{\overline{\tilde{\mathbf{Q}}}_\Delta}$, obtained using Eqs. (8) and (9).

Similarly, the control limits for the R control chart are given by

$$\begin{cases} LCL_R^{MME} = F_{p/2}^{-1}\left(\widehat{a_{MM}^{\mathbf{R}}}, \widehat{b_{MM}^{\mathbf{R}}}\right), \\ CL_R^{MME} = F_{1/2}^{-1}\left(\widehat{a_{MM}^{\mathbf{R}}}, \widehat{b_{MM}^{\mathbf{R}}}\right), \\ UCL_R^{MME} = F_{1-p/2}^{-1}\left(\widehat{a_{MM}^{\mathbf{R}}}, \widehat{b_{MM}^{\mathbf{R}}}\right), \end{cases} \tag{17}$$

where F_p^{-1} is the p-th quantile of the beta distribution with parameters $\widehat{a_{MM}^{\mathbf{R}}}$ and $\widehat{b_{MM}^{\mathbf{R}}}$. These parameters are the MMEs of $a^{\mathbf{R}}$ and $b^{\mathbf{R}}$, calculated using Eqs. (8) and (9).

Remark 3.1. *Points that appear above the UCL in the control chart should be interpreted with caution, even if they are seemingly satisfactory. In many cases, such points do not necessarily indicate an actual improvement in process quality. They might arise due to errors during inspection or, potentially, due to operators reporting inflated results to indicate a higher process quality. Thus, not all upward deviations in $\tilde{Q}$ signify an enhancement in process performance.*

Remark 3.2. *Parchami et al.* [19] *constructed $\bar{X}$ and R control charts using quantiles derived from data and applying a kernel density estimation approach. In contrast, this paper presents control limits for $\bar{X}$ and R control charts by fitting a suitable beta distribution based on both maximum likelihood and moments estimation methods. The most innovative aspect and advantage of the proposed control charts is their accuracy in reflecting the distribution of the quality degree. Additionally, compared to traditional crisp quality assessments, these quantile-based control charts offer users greater flexibility in designing the fuzzy quality membership function, allowing for a more adaptable and robust quality control process.*

4 Case study

The hard-bake process is a critical step used alongside photolithography in the production of semiconductors. One key quality characteristic in this process is the flow width of the resist, which indicates the extent of its expansion during the baking stage. To monitor this characteristic, 45 samples, each containing five wafers, were collected, with measurements recorded in microns [16]. The objective is to establish the proposed $\bar{X}$ and R control charts for monitoring the flow width of the resist in this manufacturing process.

For the flow width measurements, we assume the specification limits $[1, 2]$ and a target value $T = 1.5$. Following the approach described in Section 1, the triangular fuzzy quality is defined as $\tilde{Q}_\Delta = Tr(1, 1.5, 2)$, with the corresponding membership function given by:

$$\tilde{Q}_\Delta(x) = \begin{cases} \frac{x-1}{0.5}, & \text{if } 1 \le x < 1.5, \\ \frac{2-x}{0.5}, & \text{if } 1.5 \le x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

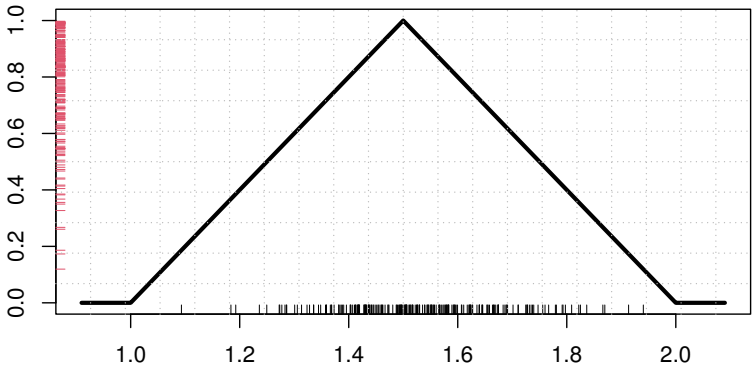

Figure 2: Membership function of the triangular fuzzy quality $\tilde{Q}_\Delta$ used in the hard-bake process.

The flow width measurements and their corresponding quality degrees, based on the triangular fuzzy quality $\tilde{Q}_\Delta$, are provided in Table 1. Additionally, the mean and range of the quality degrees for each sample are computed and shown in the last two columns of Table 1.

As an illustrative example, consider the first sample and its flow width measurements. Using the triangular fuzzy quality $\tilde{Q}_\Delta$, the quality degrees for this sample are calculated as:

$$\tilde{Q}_{\Delta 1} = (0.6470, 0.8256, 0.6512, 0.9146, 0.6172).$$

The mean and range of the quality degrees for the first sample are computed as:

$$\overline{\tilde{Q}}_{\Delta 1} = \frac{0.6470 + 0.8256 + 0.6512 + 0.9146 + 0.6172}{5} = 0.7311,$$

$$R_1 = \max\{0.6470, 0.8256, 0.6512, 0.9146, 0.6172\} - \min\{0.6470, 0.8256, 0.6512, 0.9146, 0.6172\} = 0.2974.$$

These calculated values will be used to construct the proposed $\bar{X}$ and R control charts based on the fuzzy quality degrees.

Table 1: Forty-five samples of the flow width measurements with their quality degrees in the case study (25 samples for phase I and 20 samples for phase II).

j	x_{ij}					$\tilde{Q}_\Delta(x_{ij})$					$\overline{\tilde{Q}}_{\Delta j}$	R_j
1	1.3235	1.4128	1.6744	1.4573	1.6914	0.6470	0.8256	0.6512	0.9146	0.6172	0.7311	0.2974
2	1.4314	1.3592	1.6075	1.4666	1.6109	0.8628	0.7184	0.7850	0.9332	0.7782	0.8155	0.2148
3	1.4284	1.4871	1.4932	1.4324	1.5674	0.8568	0.9742	0.9864	0.8648	0.8652	0.9095	0.1296
4	1.5028	1.6352	1.3841	1.2831	1.5507	0.9944	0.7296	0.7682	0.5662	0.8986	0.7914	0.4282
5	1.5604	1.2735	1.5265	1.4363	1.6441	0.8792	0.5470	0.9470	0.8726	0.7118	0.7915	0.4000
6	1.5955	1.5451	1.3574	1.3281	1.4198	0.8090	0.9098	0.7148	0.6562	0.8396	0.7859	0.2536
7	1.6274	1.5064	1.8366	1.4177	1.5144	0.7452	0.9872	0.3268	0.8354	0.9712	0.7732	0.6604
8	1.4190	1.4303	1.6637	1.6067	1.5519	0.8380	0.8606	0.6726	0.7866	0.8962	0.8108	0.2236
9	1.3884	1.7277	1.5355	1.5176	1.3688	0.7768	0.5446	0.9290	0.9648	0.7376	0.7906	0.4202
10	1.4039	1.6697	1.5089	1.4627	1.5220	0.8078	0.6606	0.9822	0.9254	0.9560	0.8664	0.3216
11	1.4158	1.7667	1.4278	1.5928	1.4181	0.8316	0.4666	0.8556	0.8144	0.8362	0.7609	0.3890
12	1.5821	1.3355	1.5777	1.3908	1.7559	0.8358	0.6710	0.8446	0.7816	0.4882	0.7242	0.3564
13	1.2856	1.4106	1.4447	1.6398	1.1928	0.5712	0.8212	0.8894	0.7204	0.3856	0.6776	0.5038
14	1.4951	1.4036	1.5893	1.6458	1.4969	0.9902	0.8072	0.8214	0.7084	0.9938	0.8642	0.2854
15	1.3589	1.2863	1.5996	1.2497	1.5471	0.7178	0.5726	0.8008	0.4994	0.9058	0.6993	0.4064
16	1.5747	1.5301	1.5171	1.1839	1.8662	0.8506	0.9398	0.9658	0.3678	0.2676	0.6783	0.6982
17	1.3680	1.7269	1.3957	1.5014	1.4449	0.7360	0.5462	0.7914	0.9972	0.8898	0.7921	0.4510
18	1.4163	1.3864	1.3057	1.6210	1.5573	0.8326	0.7728	0.6114	0.7580	0.8854	0.7720	0.2740
19	1.5796	1.4185	1.6541	1.5116	1.7247	0.8408	0.8370	0.6918	0.9768	0.5506	0.7794	0.4262
20	1.7106	1.4412	1.2361	1.3820	1.7601	0.5788	0.8824	0.4722	0.7640	0.4798	0.6354	0.4102
21	1.4371	1.5051	1.3485	1.5670	1.4880	0.8742	0.9898	0.6970	0.8660	0.9760	0.8806	0.2928
22	1.4738	1.5936	1.6583	1.4973	1.4720	0.9476	0.8128	0.6834	0.9946	0.9440	0.8765	0.3112
23	1.5917	1.4333	1.5551	1.5295	1.6866	0.8166	0.8666	0.8898	0.9410	0.6268	0.8282	0.3142
24	1.6399	1.5243	1.5705	1.5563	1.5530	0.7202	0.9514	0.8590	0.8874	0.8940	0.8624	0.2312
25	1.5797	1.3663	1.6240	1.3732	1.6887	0.8406	0.7326	0.7520	0.7464	0.6226	0.7388	0.2180
26	1.4483	1.5458	1.4538	1.4303	1.6206	0.8966	0.9084	0.9076	0.8606	0.7588	0.8664	0.1496
27	1.5435	1.6899	1.5830	1.3358	1.4187	0.9130	0.6202	0.8340	0.6716	0.8374	0.7752	0.2928
28	1.5175	1.3446	1.4723	1.6657	1.6661	0.9650	0.6892	0.9446	0.6686	0.6678	0.7870	0.2972
29	1.5454	1.0931	1.4072	1.5039	1.5264	0.9092	0.1862	0.8144	0.9922	0.9472	0.7698	0.8060
30	1.4418	1.5059	1.5124	1.4620	1.6263	0.8836	0.9882	0.9752	0.9240	0.7474	0.9037	0.2408
31	1.4301	1.2725	1.5945	1.5397	1.5252	0.8602	0.5450	0.8110	0.9206	0.9496	0.8173	0.4046
32	1.4981	1.4506	1.6174	1.5837	1.4962	0.9962	0.9012	0.7652	0.8326	0.9924	0.8975	0.2310
33	1.3009	1.5060	1.6231	1.5831	1.6454	0.6018	0.9880	0.7538	0.8338	0.7092	0.7773	0.3862
34	1.4132	1.4603	1.5808	1.7111	1.7313	0.8264	0.9206	0.8384	0.5778	0.5374	0.7401	0.3832
35	1.3817	1.3135	1.4953	1.4894	1.4596	0.7634	0.6270	0.9906	0.9788	0.9192	0.8558	0.3636
36	1.5765	1.7014	1.4026	1.2773	1.4541	0.8470	0.5972	0.8052	0.5546	0.9082	0.7424	0.3536
37	1.4936	1.4373	1.5139	1.4808	1.5293	0.9872	0.8746	0.9722	0.9616	0.9414	0.9474	0.1126
38	1.5729	1.6738	1.5048	1.5651	1.7473	0.8542	0.6524	0.9904	0.8698	0.5054	0.7744	0.4850
39	1.8089	1.5513	1.8250	1.4389	1.6558	0.3822	0.8974	0.3500	0.8778	0.6884	0.6392	0.5474
40	1.6236	1.5393	1.6738	1.8698	1.5036	0.7528	0.9214	0.6524	0.2604	0.9928	0.7160	0.7324
41	1.4120	1.7931	1.7345	1.6391	1.7791	0.8240	0.4138	0.5310	0.7218	0.4418	0.5865	0.4102
42	1.7372	1.5663	1.4910	1.7809	1.5504	0.5256	0.8674	0.9820	0.4382	0.8992	0.7425	0.5438
43	1.5971	1.7394	1.6832	1.6677	1.7974	0.8058	0.5212	0.6336	0.6646	0.4052	0.6061	0.4006
44	1.4295	1.6536	1.9134	1.7272	1.4370	0.8590	0.6928	0.1732	0.5456	0.8740	0.6289	0.7008
45	1.6217	1.8220	1.7915	1.6744	1.9404	0.7566	0.3560	0.4170	0.6512	0.1192	0.4600	0.6374

The appropriateness of the beta distribution is checked using the GOF test for the following hypotheses based on the moments estimation and maximum likelihood methods, as shown in Table 2. It must be mentioned that the decisions

in Table 2 are for common significance levels 1% and 5%.

$$\begin{cases} H_0 : R_1, \ldots, R_{25} \sim \textit{Beta(first estimated parameter, second estimated parameter)} \\ H_1 : \text{not of } H_0 \end{cases} \tag{18}$$

$$\begin{cases} H_0 : \bar{\tilde{Q}}_{\Delta 1}, \ldots, \bar{\tilde{Q}}_{\Delta 25} \sim \textit{Beta(first estimated parameter, second estimated parameter)} \\ H_1 : \text{not of } H_0 \end{cases} \tag{19}$$

Table 2: Decisions and p-value's of the Kolmogorov-Smirnov test for fitting the beta distribution in the case study.

Data set	Statistic of test based on ML method	p-value based on ML method	Statistic of test based on MM method	p-value based on MM method	Decision
$\bar{\tilde{Q}}_{\Delta j}$	0.1344	0.708	0.1347	0.7049	The beta distribution is not rejected
R_j	0.1285	0.7567	0.1298	0.7457	The beta distribution is not rejected

It can be seen from Table 2 that the beta distribution is suitable for fitting the mean and range of the quality degrees.

Now, we can fit four different beta distributions to R_j's and $\bar{\tilde{Q}}_{\Delta j}$'s using maximum likelihood and moments estimation methods which are listed in Table 3. The 3D surface and contour plots of the log-likelihood function for the mean and range of the quality degrees are shown in Figures 3 and 4, respectively. As can be seen in Figure 3, the values of parameters $a^{\bar{\tilde{Q}}_\Delta}$ and $b^{\bar{\tilde{Q}}_\Delta}$ which for them the log-likelihood function is maximized, are indicated by white dotted lines in the contour plot and the intersection of these lines, which represents the maximum value of the log-likelihood function, is shown by a black bold point. In addition, the white dotted lines in the contour plot of Figure 4 show the values of parameters a^R and b^R that maximize the log-likelihood function and the intersection of these lines, which represents the maximum value of the log-likelihood function, is indicated by a black bold point. The maximum values of the log-likelihood functions in Figures 3 and 4 are 31.8264 and 16.3984, respectively. Also, parameter values that maximize the log-likelihood function – i.e., the MLEs of parameters in the beta distribution – are provided in Table 3.

Table 3: The MLEs and MMEs of parameters in the case study.

Type of control chart	Type of estimator	Parameters $(a,\ b)$
$\bar{X}$	MLE	(26.8868, 7.3408)
R	MLE	(4.7311, 8.4527)
$\bar{X}$	MME	(26.1824, 7.1526)
R	MME	(4.3061, 7.7661)

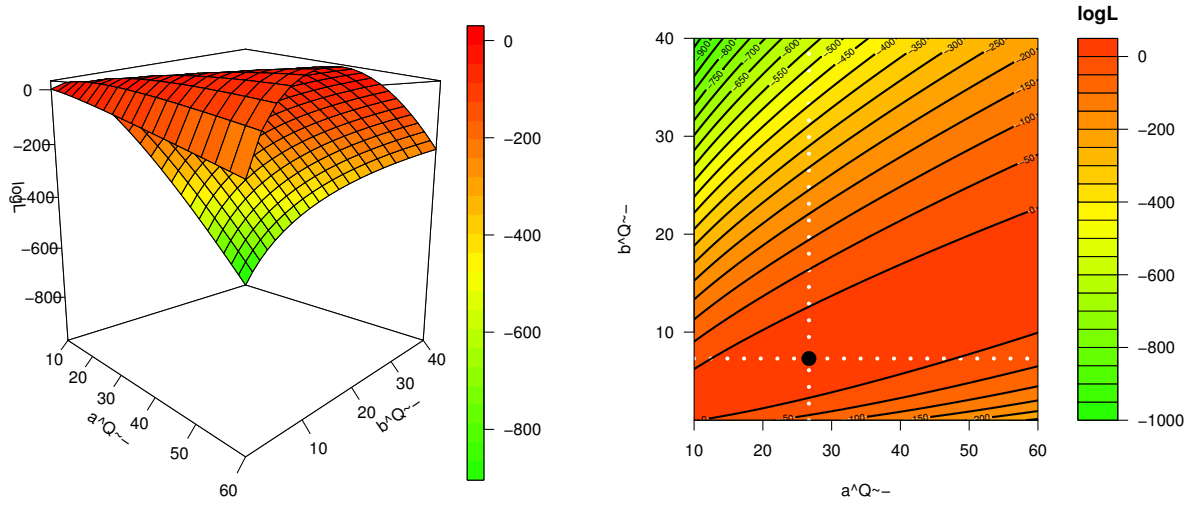

Figure 3: The 3D surface and contour plots of the log-likelihood function for the mean on the quality degrees.

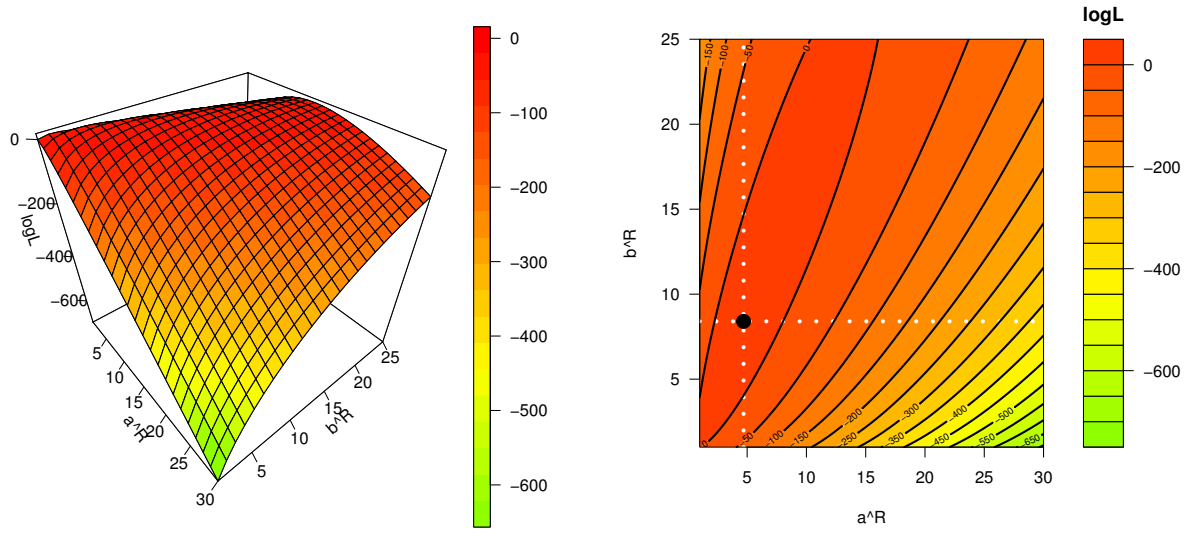

Figure 4: The 3D surface and contour plots of the log-likelihood function for the range on the quality degrees.

The center line and control limits of the proposed $\bar{X}$ and R control charts in Section 3 are computed as follows using Eqs. (14)-(17) and the results are presented in Table 4.

$$\begin{cases} LCL_{\bar{X}}^{MLE} = F_{p/2}^{-1}(26.8868, 7.3408) = 0.5440, \\ CL_{\bar{X}}^{MLE} = F_{1/2}^{-1}(26.8868, 7.3408) = 0.7911, \\ UCL_{\bar{X}}^{MLE} = F_{1-p/2}^{-1}(26.8868, 7.3408) = 0.9430, \end{cases} \tag{20}$$

$$\begin{cases} LCL_{R}^{MLE} = F_{p/2}^{-1}(4.7311, 8.4527) = 0.0660, \\ CL_{R}^{MLE} = F_{1/2}^{-1}(4.7311, 8.4527) = 0.3515, \\ UCL_{R}^{MLE} = F_{1-p/2}^{-1}(4.7311, 8.4527) = 0.7512, \end{cases} \tag{21}$$

$$\begin{cases} LCL_{\bar{X}}^{MME} = F_{p/2}^{-1}(26.1824, 7.1526) = 0.5404, \\ CL_{\bar{X}}^{MME} = F_{1/2}^{-1}(26.1824, 7.1526) = 0.7912, \\ UCL_{\bar{X}}^{MME} = F_{1-p/2}^{-1}(26.1824, 7.1526) = 0.9443, \end{cases} \tag{22}$$

$$\begin{cases} LCL_{R}^{MME} = F_{p/2}^{-1}(4.3061, 7.7661) = 0.0581, \\ CL_{R}^{MME} = F_{1/2}^{-1}(4.3061, 7.7661) = 0.3485, \\ UCL_{R}^{MME} = F_{1-p/2}^{-1}(4.3061, 7.7661) = 0.7642. \end{cases} \tag{23}$$

Table 4: Mean and range control limits based on both the proposed control charts in Section 3.

Type of control chart	LCL	CL	UCL
MLE-based X control chart	0.5440	0.7911	0.9430
MLE-based R control chart	0.0660	0.3515	0.7512
MME-based $\bar{X}$ control chart	0.5404	0.7912	0.9443
MME-based R control chart	0.0581	0.3485	0.7642

Also, the proposed $\bar{X}$ and R control charts are respectively plotted in Figures 5 and 6 using package qcc in R software [24].

It must be mentioned that the initial 25 subgroup used at phase I to estimate control limits were found to be in control. Similar to the conventional control charts, if any of these points had been out-of-control, they would have been eliminated and a new control chart would have been plotted with the remaining points. If all points on the new chart were in control, they would have been used to estimate control limits once again. If any points were out-of-control, they would have been eliminated and the process repeated until all sample points fell within the control limits. Phase II begins with 20 additional samples with size 5. Any sample point falling outside the control limits will be considered

out-of-control [2, 16]. Even if all points fall within the control limits, if they exhibit a systematic or nonrandom behavior, this indicates that the process is out-of-control. The red and yellow points on the $\bar{X}$ and R control charts in Figures 5 and 6 signify points that are out-of-control and exhibit a systematic or nonrandom pattern, respectively. When such patterns appear on control charts, it is typically indicative of a specific reason behind them and corrective actions are needed to determine and eliminate the source of deviations and assignable changes. By identifying and eliminating this reason, process performance can be improved in this case study.

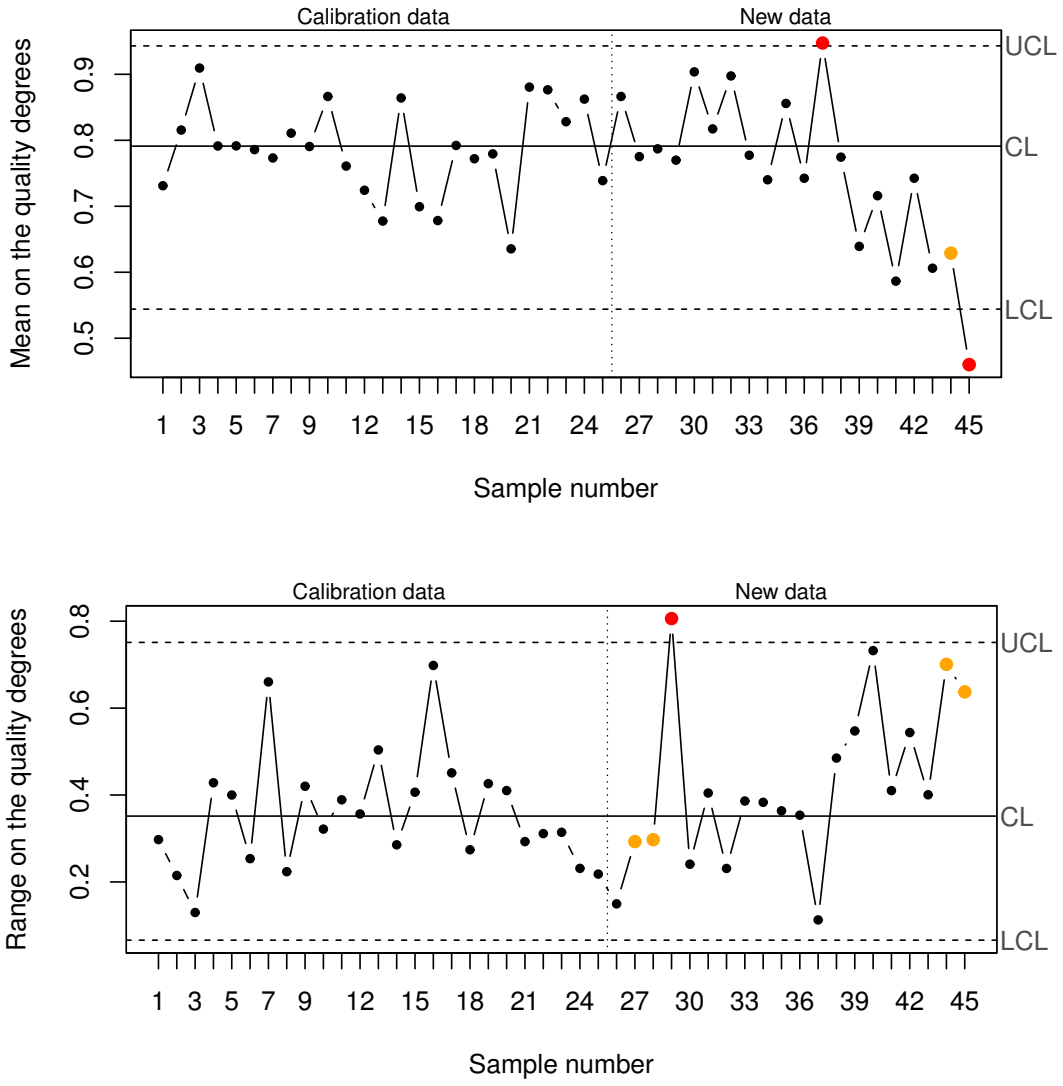

Figure 5: MLE-based $\bar{X}$ and R control charts in the case study.

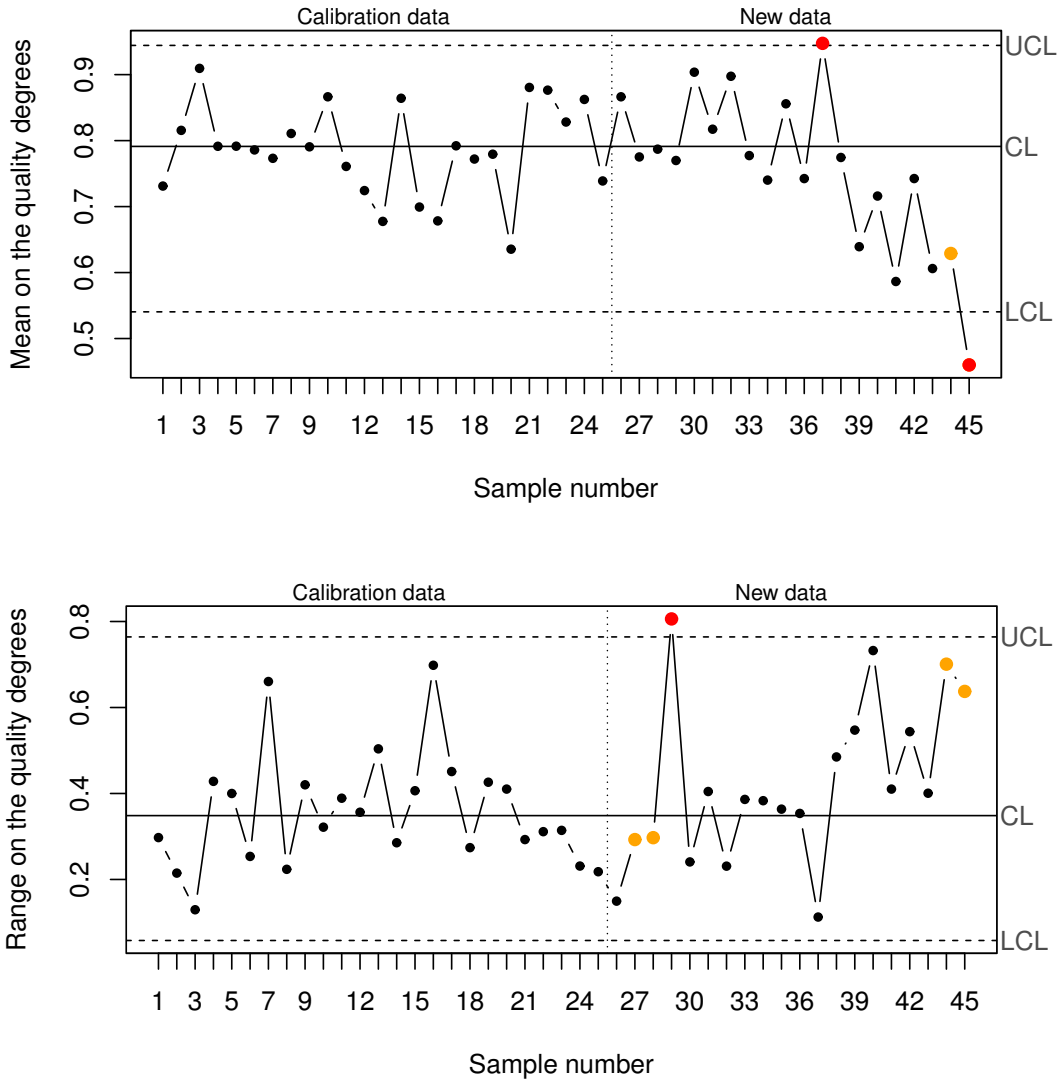

Figure 6: MME-based $\bar{X}$ and R control charts in the case study.

5 Comparison of control charts by simulation

This section is divided into two subsections. In Subsection 5.1, we compare the performance of the proposed control charts, while in Subsection 5.2, we contrast their performance with the percentile-based control charts presented by Parchami et al. [19].

5.1 Comparison of the proposed control charts

In this subsection, we compare the MLE-based and MME-based control charts using simulation, based on the flow width measurements provided in Table 1. Additionally, the Average Run Length (ARL) curves are plotted for out-of-control processes when the process mean and standard deviation change. The process mean and standard deviation are assumed to be 1.5 microns and 0.15 microns, respectively.

Initially, the control limits are computed using the first 25 samples from Table 1 for both the MLE-based and MME-based methods, as described in Section 3 using the triangular fuzzy quality $\tilde{Q}_{\Delta} = Tr(1, 1.5, 2)$. These results are presented in Table 4.

When the process is in control, the probability of a point falling between the control limits should be approximately

$1-p=0.9973$. To verify this, we simulated $m=10^6$ samples of size 5 independently from a normal distribution with a mean of 1.5 and a standard deviation of 0.15. Using Monte Carlo simulation, we estimated the probabilities of a point being below the LCL, above the UCL, or between the control limits. The results are summarized in Table 5, where the symbol $\bullet$ denotes a random point on the control chart. For example, the probability of a random point R being between the control limits of the MLE-based R control chart was estimated using the formula:

$$\hat{P}\left(LCL_R^{MLE} < R < UCL_R^{MLE}\right) = \frac{1}{10^6}\sum_{j=1}^{10^6} I\left(LCL_R^{MLE} < R_j < UCL_R^{MLE}\right),$$

where UCL_R^{MLE} and LCL_R^{MLE} are the control limits obtained in Table 4, and $R_1,\ldots,R_{10^6}$ are simulated independently.

Table 5: Simulation results of the proposed control charts in Section 3 based on triangular fuzzy quality $\tilde{Q}_\Delta$.

Type of Method and Chart	$\hat{P}(LCL \geq \bullet)$	$\hat{P}(LCL < \bullet < UCL)$	$\hat{P}(\bullet \geq UCL)$	$\hat{p}$
MLE-based $\bar{X}$ Control Chart	0.008709	0.989448	0.001843	0.010552
MLE-based R Control Chart	0.001851	0.96508	0.033069	0.03492
MME-based $\bar{X}$ Control Chart	0.007979	0.990362	0.001659	0.009638
MME-based R Control Chart	0.001144	0.969854	0.029002	0.030146

From the probabilities in the last column of Table 5, it is evident that the MME-based control charts yield probabilities closer to the desired value of 0.0027, indicating superior performance compared to the MLE-based control charts in this simulation study.

Next, we compare the performance of the proposed $\bar{X}$-R control charts using ARL curves for out-of-control processes when the process mean and standard deviation change. The ARL is defined as the expected number of points plotted within the control limits before an out-of-control point is detected, and it is computed using the formula:

$$ARL = \frac{1}{\text{probability that one point plots out-of-control}} = \frac{1}{1-\beta},$$

where β is the probability of a point being within the control limits for an out-of-control process (i.e., the probability of a type II error) [16].

Assuming that the process mean shifts from the in-control value $\mu_0=1.5$ to an out-of-control value $\mu_1=\mu_0+\delta$, where δ denotes the change in the process mean, we simulated $m=10^6$ samples of size 5 from a normal distribution with a mean of $1.5+\delta$ and a standard deviation of 0.15. The ARL values for various δ values are presented in Table 6, and the normal curves for both in-control and out-of-control processes are shown in Figure 7.

Table 6: The ARL values for MLE-based and MME-based proposed $\bar{X}$ control charts when the process is out-of-control for different mean changes.

δ	$ARL_{\bar{X}_{MLE}}$	$ARL_{\bar{X}_{MME}}$
0.05	58.899753	64.053292
0.07	39.869229	43.105306
0.09	24.966296	26.755137
0.11	15.377755	16.381626
0.13	9.665198	10.200231
0.15	6.301833	6.606460
0.17	4.278313	4.455077
0.19	3.051181	3.155132
0.21	2.303787	2.368142
0.23	1.826641	1.866873
0.25	1.521454	1.546915
0.27	1.322452	1.338857
0.29	1.194033	1.204719
0.31	1.112149	1.118494
0.33	1.062278	1.066308
0.35	1.032190	1.034421

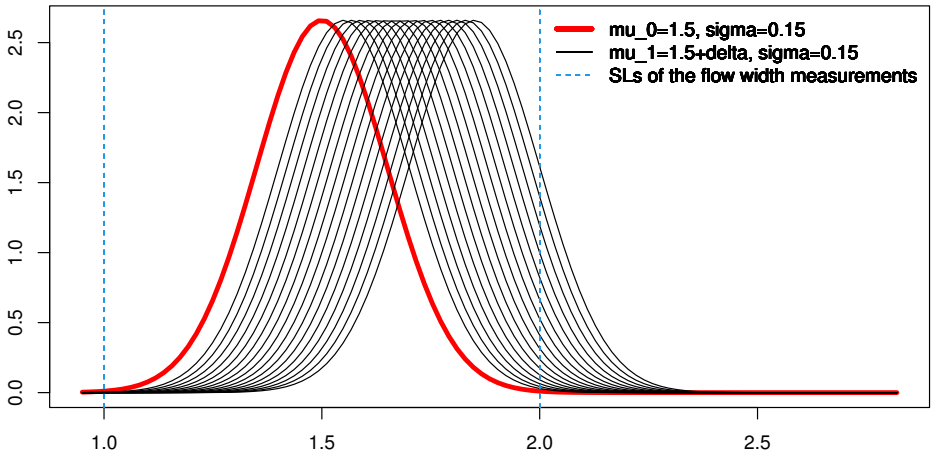

Figure 7: Normal curves of in-control process and out-of-control process for different listed values of δ in Table 6 when the mean changes.

The results in Table 6 indicate that the ARL values for the MLE-based $\bar{X}$ control chart are generally lower than those for the MME-based $\bar{X}$ control chart (see Figure 8). This suggests that the MLE-based $\bar{X}$ chart is more effective in detecting shifts in the process mean. For instance, when the mean shifts from $\mu_0 = 1.5$ to $\mu_1 = \mu_0 + 0.05$, the MLE-based chart requires approximately 59 points, on average, to detect the shift, while the MME-based chart requires about 64 points.

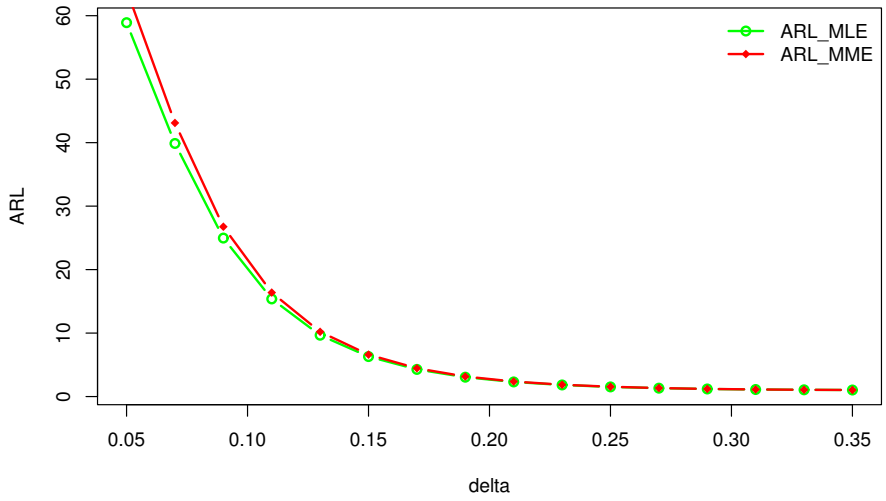

Figure 8: The ARL curves for two MLE-based and MME-based $\bar{X}$ control charts in out-of-control process when the mean changes.

Similarly, when the process standard deviation changes from the in-control value $\sigma_0 = 0.15$ to $\sigma_1 = \lambda\sigma_0$, where λ denotes the change factor, $m = 10^6$ samples of size 5 were simulated from a normal distribution with mean 1.5 and standard deviation 0.15λ. The ARL values for various λ values are shown in Table 7, with the corresponding normal curves illustrated in Figure 9. The MLE-based R control chart demonstrates better performance in detecting shifts in process variability compared to the MME-based R control chart, as shown in Figure 10. For example, when $\sigma_1 = 1.1\sigma_0$, the MLE-based R control chart detects the change after approximately 16 points on average, whereas the MME-based R control chart requires about 18 points.

Table 7: The ARL values for MLE-based and MME-based proposed R control charts when the process is out-of-control for different standard deviation changes.

λ	$ARL_{R_{MLE}}$	$ARL_{R_{MME}}$
1.1	15.621095	17.533401
1.2	9.757906	10.738024
1.3	6.700482	7.275690
1.4	4.998425	5.371204
1.5	3.951195	4.220656
1.6	3.276561	3.476894
1.7	2.819753	2.978069
1.8	2.507378	2.639846
1.9	2.282990	2.397363
2	2.115296	2.215070

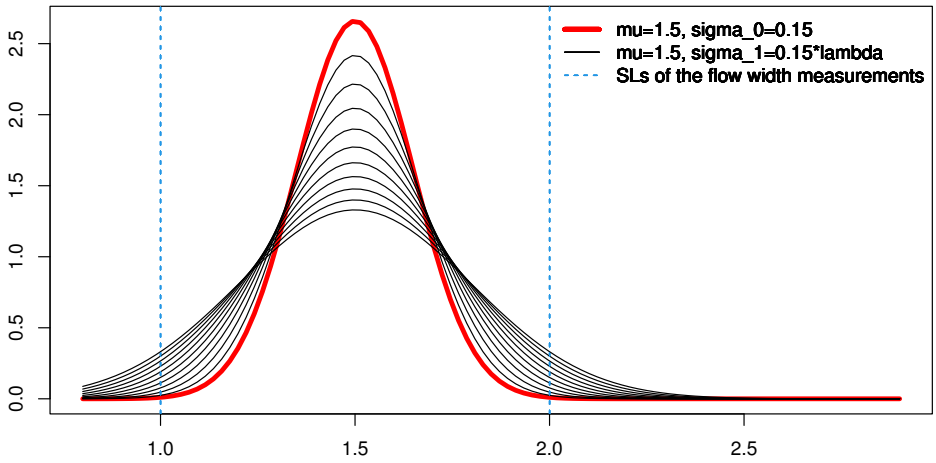

Figure 9: Normal curves of in-control process and out-of-control process for different listed values of λ in Table 7 when the standard deviation changes.

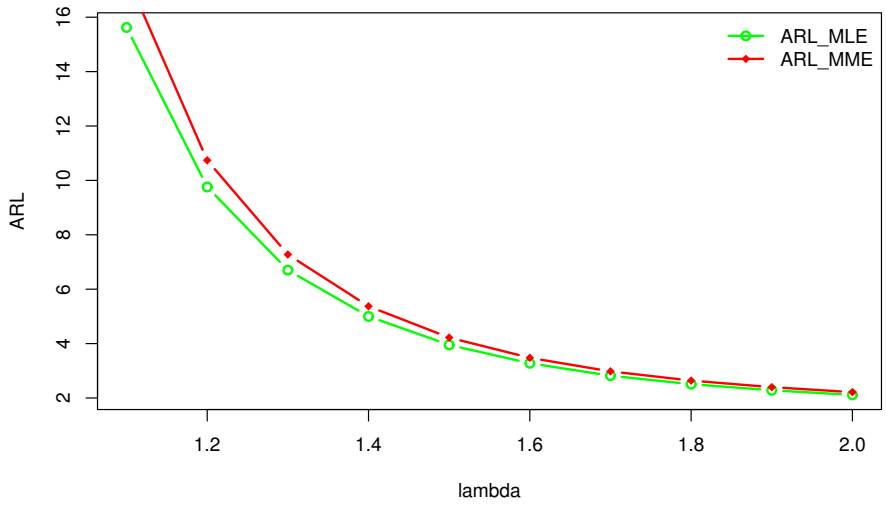

Figure 10: The ARL curves for two MLE-based and MME-based R control charts in out-of-control process when the standard deviation changes.

5.2 Comparison with percentile-based control charts

In this section, we compare the proposed control charts with the percentile-based control charts presented by Parchami et al. [19], using the flow width measurements in Table 1. The control limits for the percentile-based charts are calculated using the first 25 samples of flow width measurements based on the triangular fuzzy quality $\tilde{Q}_\Delta = Tr(1, 1.5, 2)$. It should be noted that the triangular kernel function was utilized to compute the control limits for the density-based control charts. As in Subsection 5.1, Monte Carlo simulations were conducted to estimate the probabilities of a random point being below the LCL, above the UCL, or within the control limits when the process is in control. The results are shown in Table 8.

Table 8: Simulation results for the percentile-based control charts presented in [19] based on triangular fuzzy quality $\tilde{Q}_\Delta$.

Type of Method and Chart	LCL	UCL	$\hat{P}(LCL \geq \bullet)$	$\hat{P}(LCL < \bullet < UCL)$	$\hat{P}(\bullet \geq UCL)$	$\hat{p}$
Data-based $\bar{X}$ Control Chart	0.6354	0.9095	0.071242	0.912661	0.016097	0.087339
Data-based R Control Chart	0.1296	0.6982	0.021043	0.924218	0.054739	0.075782
Density-based $\bar{X}$ Control Chart	0.6121	0.9328	0.044251	0.951752	0.003997	0.048248
Density-based R Control Chart	0.0914	0.7364	0.0061	0.955538	0.038362	0.044462

A comparison of the results in Table 8 with those in Table 5 shows that the MME-based control charts exhibit a lower probability of a type I error (i.e., a random point falling outside the control limits when the process is in control), with values closer to the desired probability $p = 0.0027$. Thus, MME-based charts demonstrate slightly better performance than the percentile-based control charts for the in-control process in this simulation.

Next, we examine the ARL values for out-of-control processes, simulated when the process mean shifts from $\mu_0 = 1.5$ to $\mu_1 = \mu_0 + \delta$, for $\delta = 0.05(0.02)0.35$. Additionally, the ARL values are computed for cases where the process standard deviation shifts from $\sigma_0 = 0.15$ to $\sigma_1 = \lambda\sigma_0$ for $\lambda = 1.1(0.1)2$. The results are presented in Tables 9 and 10. The ARL curves for different values of δ and λ, representing changes in the process mean and standard deviation, are plotted in Figures 11 and 12, respectively.

Table 9: ARL values for data-based and density-based $\bar{X}$ control charts when the process is out-of-control for various mean shifts.

δ	$ARL_{\bar{X}_{\text{Data}}}$	$ARL_{\bar{X}_{\text{Density}}}$
0.05	8.799	14.437
0.07	6.911	10.645
0.09	5.272	7.607
0.11	3.972	5.417
0.13	3.012	3.896
0.15	2.346	2.902
0.17	1.888	2.241
0.19	1.577	1.805
0.21	1.367	1.514
0.23	1.227	1.322
0.25	1.136	1.197
0.27	1.078	1.116
0.29	1.042	1.064
0.31	1.021	1.034
0.33	1.010	1.017
0.35	1.005	1.008

Table 10: The ARL values for data-based and density-based R control charts when the process is out-of-control for different standard deviation changes.

λ	$ARL_{R_{Data}}$	$ARL_{R_{Density}}$
1.1	9.031384	13.298403
1.2	6.381132	8.642744
1.3	4.736373	6.063436
1.4	3.717417	4.582489
1.5	3.056496	3.667988
1.6	2.605035	3.062872
1.7	2.293625	2.653611
1.8	2.072376	2.367934
1.9	1.911973	2.164474
2	1.789463	2.010960

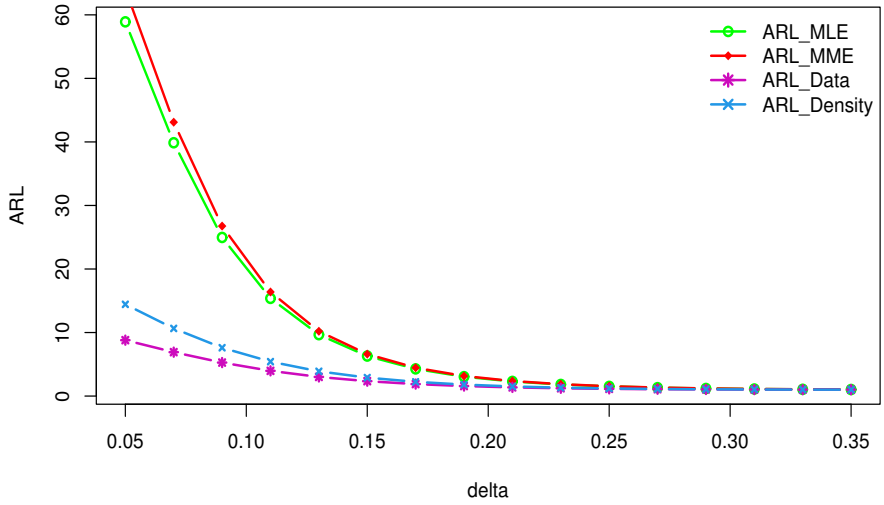

Figure 11: The ARL curves for two data-based and density-based $\bar{X}$ control charts in out-of-control process when the mean changes.

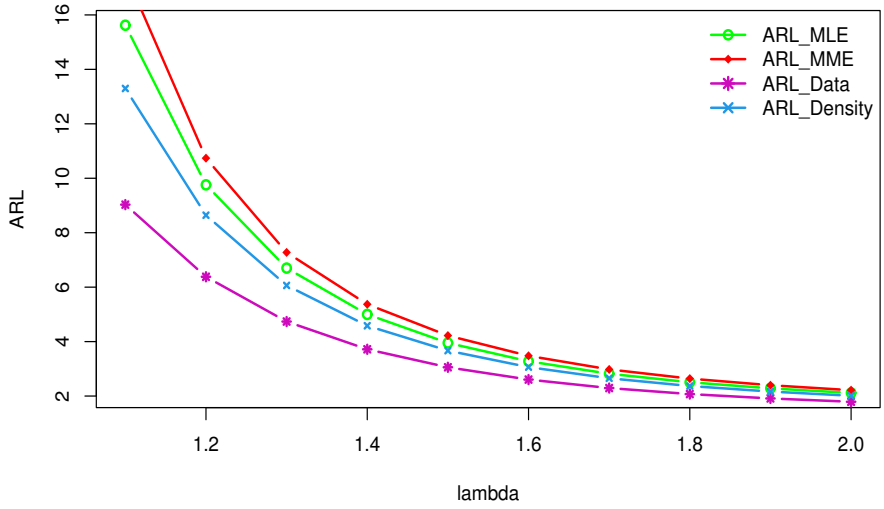

Figure 12: The ARL curves for two data-based and density-based R control charts in out-of-control process when the standard deviation changes.

A comparison of Tables 6 and 9 reveals that the *ARL* values for the data-based $\bar{X}$ control chart are generally lower than those for other types of $\bar{X}$ control charts, indicating better performance when the process mean shifts. Similarly, the *ARL* values in Tables 7 and 10 show that the data-based R control chart performs better when there is a change in process standard deviation.

These results demonstrate that the data-based control charts, particularly the $\bar{X}$ and R charts, are more effective in detecting shifts in process parameters compared to both density-based and the proposed MLE- and MME-based control charts.

Remark 5.1. *According to the simulation results in Section 5, the proposed control charts demonstrate superior performance in identifying out-of-control points when the process is in control. However, for processes that are already out-of-control, the control charts presented by Parchami et al.* [19] *tend to perform better in terms of detection sensitivity.*

6 Conclusions and future works

When the quality characteristic is a continuous variable, it is essential to monitor both its mean and variability. $\bar{X}$ and R control charts are commonly employed for the simultaneous monitoring of mean and variation in manufacturing processes. In this study, we developed $\bar{X}$ and R control charts based on triangular fuzzy quality. A beta distribution was fitted to the quality degrees using both the Method of Moments and Maximum Likelihood Estimations, and the quantiles of these fitted beta distributions were used to construct the control charts.

Although the proposed control charts differ from those presented by Parchami et al. [19], the main advantage of our quantile-based approach is its higher accuracy with respect to the distribution of quality degrees. Additionally, the proposed control charts offer greater flexibility in designing fuzzy quality membership functions. The case study illustrated the effectiveness of these control charts, and their performance was rigorously evaluated through simulation.

Future research could explore the development of other types of control charts using similar methodologies based on fuzzy quality concepts. Investigating the design of fuzzy quality control charts using trapezoidal or normal fuzzy quality, instead of triangular fuzzy quality, is a promising direction for further study.

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Appendix

Some parts of the written R codes are provided in three following subsections.

Data entry and necessary preliminary information to the R software:

```
library(FuzzyNumbers)
library(qcc)
flow_width = data.frame(measurements = c(
1.3235, 1.4128, 1.6744, 1.4573, 1.6914,
1.4314, 1.3592, 1.6075, 1.4666, 1.6109,
1.4284, 1.4871, 1.4932, 1.4324, 1.5674,
1.5028, 1.6352, 1.3841, 1.2831, 1.5507,
1.5604, 1.2735, 1.5265, 1.4363, 1.6441,
1.5955, 1.5451, 1.3574, 1.3281, 1.4198,
1.6274, 1.5064, 1.8366, 1.4177, 1.5144,
1.4190, 1.4303, 1.6637, 1.6067, 1.5519,
1.3884, 1.7277, 1.5355, 1.5176, 1.3688,
1.4039, 1.6697, 1.5089, 1.4627, 1.5220,
1.4158, 1.7667, 1.4278, 1.5928, 1.4181,
1.5821, 1.3355, 1.5777, 1.3908, 1.7559,
1.2856, 1.4106, 1.4447, 1.6398, 1.1928,
1.4951, 1.4036, 1.5893, 1.6458, 1.4969,
1.3589, 1.2863, 1.5996, 1.2497, 1.5471,
1.5747, 1.5301, 1.5171, 1.1839, 1.8662,
1.3680, 1.7269, 1.3957, 1.5014, 1.4449,
1.4163, 1.3864, 1.3057, 1.6210, 1.5573,
1.5796, 1.4185, 1.6541, 1.5116, 1.7247,
1.7106, 1.4412, 1.2361, 1.3820, 1.7601,
1.4371, 1.5051, 1.3485, 1.5670, 1.4880,
1.4738, 1.5936, 1.6583, 1.4973, 1.4720,
1.5917, 1.4333, 1.5551, 1.5295, 1.6866,
1.6399, 1.5243, 1.5705, 1.5563, 1.5530,
1.5797, 1.3663, 1.6240, 1.3732, 1.6887,
1.4483, 1.5458, 1.4538, 1.4303, 1.6206,
1.5435, 1.6899, 1.5830, 1.3358, 1.4187,
1.5175, 1.3446, 1.4723, 1.6657, 1.6661,
1.5454, 1.0931, 1.4072, 1.5039, 1.5264,
1.4418, 1.5059, 1.5124, 1.4620, 1.6263,
1.4301, 1.2725, 1.5945, 1.5397, 1.5252,
1.4981, 1.4506, 1.6174, 1.5837, 1.4962,
1.3009, 1.5060, 1.6231, 1.5831, 1.6454,
1.4132, 1.4603, 1.5808, 1.7111, 1.7313,
1.3817, 1.3135, 1.4953, 1.4894, 1.4596,
1.5765, 1.7014, 1.4026, 1.2773, 1.4541,
1.4936, 1.4373, 1.5139, 1.4808, 1.5293,
1.5729, 1.6738, 1.5048, 1.5651, 1.7473,
1.8089, 1.5513, 1.8250, 1.4389, 1.6558,
1.6236, 1.5393, 1.6738, 1.8698, 1.5036,
1.4120, 1.7931, 1.7345, 1.6391, 1.7791,
1.7372, 1.5663, 1.4910, 1.7809, 1.5504,
1.5971, 1.7394, 1.6832, 1.6677, 1.7974,
1.4295, 1.6536, 1.9134, 1.7272, 1.4370,
```

```
1.6217, 1.8220, 1.7915, 1.6744, 1.9404), sample=c(rep(1:45,each=5)))

#For Triangular fuzzy quality
a = 1
b = 1.5
c = 1.5
d = 2
Q <- TrapezoidalFuzzyNumber(a, b, c, d)
measurements = qcc.groups(flow_width$measurements, flow_width$sample)
Q.diam = matrix(, nrow=dim(measurements)[1], ncol=dim(measurements)[2])
for (i in 1:dim(measurements)[1])
for (j in 1:dim(measurements)[2]){
 Q.diam[i,j] <- evaluate(Q, measurements[i,j])
}
Q.means = apply(Q.diam[1:25,], 1, mean)
Ran = function(vec) max(vec)-min(vec)
Q.ranges = apply(Q.diam[1:25,], 1, Ran)
```

The R code for Figure 4 (3D surface plot):

```
library(plot3D)
Log.likelihood = function(a,b){
 n = 25
 x = Q.ranges
 n*log(gamma(a+b))-n*log(gamma(a))-n*log(gamma(b))+(a-1)*sum(log(x))+(b-1)*sum(log(1-x))
}
a = seq(1, 30, length=25)
b = seq(1, 25, length=25)
logL = outer(a, b,  Log.likelihood)
par(mfcol=c(1,1), mar=c(0,0.8,0,2.5))

persp3D(a, b, logL, theta=50, border="black", col=rainbow(75)[19:1],
ticktype="detailed", xlab="a^R", ylab="b^R", zlab="logL")

#Contour plot
max = which(logL==max(logL), arr.ind=TRUE)
par(mfcol=c(1,1), mar=c(4,4,2.5,1))
filled.contour(a, b, logL, nevels=20, col=rainbow(75)[19:1],
key.title=title(main="logL"), plot.title=title(xlab="a^R",ylab="b^R"),
plot.axes = {
 axis(1)
 axis(2)
 abline(v=a[max[1]], h=b[max[2]], col="white", lwd=4, lty=3)
 points(a[max[1]], b[max[2]], col="black", pch=16, cex=2)
 contour(a, b, logL, add = TRUE, lwd = 2, nlevels=20)
})
```

The R code for Figure 6 (MME-based Control charts):

```
s2.Q.means = var(Q.means)
mean.Q.means = mean(Q.means)
ahat.Q.means =  mean.Q.means*((mean.Q.means*(1-mean.Q.means))/s2.Q.means-1)
bhat.Q.means = (1-mean.Q.means)*((mean.Q.means*(1-mean.Q.means))/s2.Q.means-1)
```

```
s2.Q.ranges = var(Q.ranges)
mean.Q.ranges = mean(Q.ranges)
ahat.Q.ranges = mean.Q.ranges*((mean.Q.ranges*(1-mean.Q.ranges))/s2.Q.ranges-1)
bhat.Q.ranges = (1-mean.Q.ranges)*((mean.Q.ranges*(1-mean.Q.ranges))/s2.Q.ranges-1)

# Xbar control chart
p = 0.0027
LCL.X = qbeta(p/2, ahat.Q.means, bhat.Q.means); LCL.X
CL.X = qbeta(1/2, ahat.Q.means, bhat.Q.means); CL.X
UCL.X = qbeta(1-p/2, ahat.Q.means, bhat.Q.means); UCL.X

Q.xbar = qcc(Q.diam[1:25,], type="xbar", center=CL.X,
limits=c(LCL.X,UCL.X), newdata=Q.diam[26:45,])
plot(Q.xbar, chart.all=FALSE)
plot(Q.xbar, add.stats=F, title='', xlab='Sample number', ylab='Mean on the quality degrees')

# R control chart
LCL.R = qbeta(p/2, ahat.Q.ranges, bhat.Q.ranges); LCL.R
CL.R = qbeta(1/2,  ahat.Q.ranges, bhat.Q.ranges); CL.R
UCL.R = qbeta(1-p/2,  ahat.Q.ranges, bhat.Q.ranges); UCL.R

Q.R = qcc(Q.diam[1:25,], type="R", center=CL.R,
limits=c(LCL.R,UCL.R), newdata=Q.diam[26:45,])
plot(Q.R, add.stats=F, title='', xlab='Sample number', ylab='Range on the quality degrees')
```