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Original paper

Optimality in interval fractional programming problems using d-invexity

N. Pokharna [1] and I. P. Tripathi [2]

[1,2] *Department of Mathematics, Sardar Vallabhbhai National Institute of Technology, Surat*

ds19ma012@amhd.svnit.ac.in, ipd@amhd.svnit.ac.in

Abstract

In this paper, an interval fractional optimization problem with directionally differentiable functions is considered, and the d-invexity concept is introduced for interval-valued functions. Slater's constraint qualification and pre-invex directional derivative assumption are used to establish the necessary optimality conditions. Further, sufficient optimality conditions are derived under the d-invexity assumption, considering the LU-solution concept. As an application of interval fractional problems, a portfolio optimization problem with uncertain return and risk parameters subject to interval liquidity constraints is considered, and an optimal solution is obtained using the results developed in this paper. Also, the portfolio optimization problem is solved using the proposed global criteria method for interval optimization problems and two methods available in the literature. Moreover, to check the efficiency of the proposed method, a comparison between different methods is presented. Throughout the paper, non-trivial examples are presented at appropriate places to provide a better understanding of the results developed.

Keywords: Interval fractional problem, d-invexity, pre-invexity, optimality conditions, portfolio optimization.

1 Introduction

In an interval optimization problem, variables are uncertain, and this uncertainty may lie in the objective function coefficients [30], in the constraints, or in both [23]. Due to incomplete information, measurement error, prediction error, and a variety of other factors, real-world problems frequently have uncertain data. Therefore, dealing with these kinds of uncertainties in optimization problems is crucial. Interval optimization is used to address real-world problems in a number of fields, including image restoration problems, fuzzy logic, environment management, portfolio selection problems, robust optimization, fuzzy logic, traffic management and transportation, interval matrix games, interval difference equations [29] manufacturing operations, reservoir regulation problems etc.

The study of optimality conditions in optimization problems is greatly aided by the concept of convexity, as any local optima of a convex optimization problem is a global optima. Since convexity concept is restrictive, and most of the optimization problems are nonconvex in nature, there is a need of relaxing the convexity condition. For this several generalizations of convexity were introduced, one such generalization is d-invexity, introduced by Ye [32] which is more general concept than convexity and invexity. Antczak [2], Mishra et al. [21] and several researchers established optimality conditions for non-differentiable multiobjective problems under the d-invexity assumption.

For interval optimization problems, the optimality and duality results (Guo at el. [11–13]) are widely studied under the assumptions of generalized convexities. Under the setting of interval optimization, several variational control problems are explored and their relationship with different inequalities problems are discussed (For details refer: [6],[14–16],[24, 27, 28]). If the objective function is a ratio of two interval functions, the optimization problem is referred to as an interval fractional programming problem. Fractional programming problems are of interest due to their huge role in numerical analysis, information theory, decomposition algorithms for large linear systems, stochastic programming, etc. The study of fractional optimization problems with interval-valued functions was initiated by Debnath and Gupta [9].Thereafter, several researchers studied optimality conditions for both differentiable and non-differentiable interval

Corresponding Author: N. Pokharna

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fractional optimization problems under generalized convexity assumptions (e.g., Ahmad [1], Dar at el. [7], Debnath and Gupta [8], Chen [5], Su [22]).

Portfolio management is the process by which investors distribute a specific portion of their wealth among a variety of assets in order to diversify their risk and generate some predictable returns. Markowitz [20] presented modern portfolio theory using the mean-variance model. Later, based on the Markowitz mean-variance approach, Sharpe [26] and Mao [19] proposed simplified portfolio selection models. Faaland and Jacob [10] formulated a portfolio selection model as a linear fractional programming problem and discussed a relationship between the excess return to beta ratio of the optimal portfolio and individual security. Kumar and Bhurjee [17] discussed portfolio optimization problems involving interval variables by converting the original problem into a deterministic problem.

Since variables like risk, return, and turnover are all uncertain in the real world due to a number of circumstances, many researchers have suggested that it is preferable to address the portfolio selection problem under the assumption of interval parameters. To the best of our knowledge, there is no study available on the optimality conditions of the interval fractional optimization problem under the d-invexity assumption. Inspired and motivated by this, we have studied optimality conditions for an interval fractional optimization problem. Moreover, we have formulated a portfolio optimization problem as an interval fractional programming problem and utilized the established optimality conditions to obtain the solution of the considered portfolio optimization problem. We have treated interval optimization problem as multiobjective optimization problem with two objectives and applied global criteria method to solve the problem, further, the proposed approach is compared to the different methods available in the literature.

This paper is divided section-wise as follows: In Section 2, some basic definitions are recalled, and the notion of d-invexity for interval-valued functions is introduced. Section 3 is devoted to the formulation of an interval fractional programming problem and the study of necessary and sufficient optimality conditions under the d-invexity assumption. In Section 4, an interval fractional portfolio optimization problem is considered as an application of results derived in the paper. A comparison between the different methods available in the literature and the methodology used in this paper is presented in Section 5. Finally, in Section 6, we provide the conclusions of the paper.

2 Preliminaries and d-invexity

Throughout the paper, we shall assume that X is a non-empty open convex set in $\mathbb{R}^n$. For any two n-dimensional vectors $s = (s_1, s_2, \dots, s_n)$ and $\upsilon = (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$, we shall use the following relations:

$$\begin{aligned}
&\text{(i) } s = \upsilon \iff s_i = \upsilon_i, \forall i = 1, 2, \dots, n;\\
&\text{(ii) } s > \upsilon \iff s_i > \upsilon_i, \forall i = 1, 2, \dots, n;\\
&\text{(iii) } s \geqq \upsilon \iff s_i \geq \upsilon_i, \forall i = 1, 2, \dots, n;\\
&\text{(iv) } s \geq \upsilon \iff s \geqq \upsilon \text{ and } s \neq \upsilon.
\end{aligned}$$

Let J be a set of all closed and bounded real intervals and $J^n = J \times J \times \dots J$ (n times) be the product space. The set of intervals J is not totally ordered set and the following partial ordering will be used for $\mathcal{A} = [\alpha^L, \alpha^U]$ and $\mathcal{B} = [\beta^L, \beta^U]$:

$$\begin{aligned}
&\mathcal{A} \preceq \mathcal{B} \quad \text{iff} \quad \alpha^L \leq \beta^L \quad \text{and} \quad \alpha^U \leq \beta^U,\\
&\mathcal{A} \prec \mathcal{B} \quad \text{iff} \quad \mathcal{A} \preceq \mathcal{B} \quad \text{and} \quad \mathcal{A} \neq \mathcal{B}.
\end{aligned}$$

that is, one of the following holds:

$$\left.\begin{matrix} \alpha^L < \beta^L \\ \alpha^U < \beta^U \end{matrix}\right\} \text{or,} \quad \left.\begin{matrix} \alpha^L \leq \beta^L \\ \alpha^U < \beta^U \end{matrix}\right\} \text{or,} \quad \left.\begin{matrix} \alpha^L < \beta^L \\ \alpha^U \leq \beta^U \end{matrix}\right\}.$$

Consider the algebraic operations on $\mathcal{A}, \mathcal{B} \in J$ as follows ($* \in \{+, -, \otimes\}$)

$$\mathcal{A} * \mathcal{B} = \Big[\min_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (\alpha * \beta), \max_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (\alpha * \beta)\Big].$$

For $\delta \in \mathbb{R}$, $\delta \mathcal{A} = \begin{cases} [\delta\alpha^L, \delta\alpha^U], & \text{if } \delta \geq 0 \\ [\delta\alpha^U, \delta\alpha^L], & \text{if } \delta < 0 \end{cases}.$

In the following definitions, we recall pre-invexity and d-invexity concepts of functions. Throughout the paper, we assume $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ to be a vector-valued function with $\eta(u, \tilde{u}) = 0$ when $u = \tilde{u}$.

Definition 2.1. [31] *A function $f : X \to \mathbb{R}^n$ is said to be pre-invex with respect to a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ on X, if the following holds*

$$f(\tilde{u} + \lambda\eta(u, \tilde{u})) \leq \lambda f(u) + (1 - \lambda)f(\tilde{u}),$$

for every $u, \tilde{u} \in X$ and $\lambda \in [0, 1]$.

Definition 2.2. [32] *A directionally differentiable function $f : X \to \mathbb{R}^n$ is said to be d-invex at $\tilde{u}$ on X, if there exists a vector function $\eta(u, \tilde{u}) \in \mathbb{R}^n$ such that*

$$f(u) - f(\tilde{u}) \geq f'(u; \eta(u, \tilde{u})), \text{ for all } u \in X.$$

Let $\phi : X \to J$ be an interval-valued function defined as $\phi(u) = [\phi^L(u), \phi^U(u)]$, where $\phi^L, \phi^U : X \to \mathbb{R}$ are real valued functions and $\phi^L(u) \leq \phi^U(u)$, $\forall u \in X$. In the following definition, we introduce the notion of d-invexity for interval-valued function:

Definition 2.3. *Let $\phi : X \to J$ be defined as $\phi(u) = [\phi^L(u), \phi^U(u)]$ and ϕ^L, ϕ^U are directionally differentiable at $\tilde{u} \in X$. Then ϕ is said to be d-invex at $\tilde{u}$ on X with respect to $\eta : X \times X \to \mathbb{R}^n$, if ϕ^L and ϕ^U are d-invex function for any $u \in X$, that is*

$$\phi^L(u) - \phi^L(\tilde{u}) \geq {\phi'}^L(\tilde{u}, \eta(u, \tilde{u})),$$
$$\phi^U(u) - \phi^U(\tilde{u}) \geq {\phi'}^U(\tilde{u}, \eta(u, \tilde{u})),$$

where, ${\phi'}^L(\tilde{u}, \eta(u, \tilde{u})), {\phi'}^U(\tilde{u}, \eta(u, \tilde{u}))$ are the direction derivatives of ϕ^L and ϕ^U in the direction of $\eta(u, \tilde{u})$ respectively, defined as:

$${\phi'}^L(\tilde{u}, \eta(u, \tilde{u})) = \lim_{\lambda \to 0^+} \frac{\phi^L(\tilde{u} + \lambda\eta(u, \tilde{u})) - \phi^L(\tilde{u})}{\lambda},$$
$${\phi'}^U(\tilde{u}, \eta(u, \tilde{u})) = \lim_{\lambda \to 0^+} \frac{\phi^U(\tilde{u} + \lambda\eta(u, \tilde{u})) - \phi^U(\tilde{u})}{\lambda}.$$

2.1 An example of d-invex function

The following example illustrates the definition of an interval-valued d-invex function.

Example 2.4. *Let $X = \{(u, v) \in \mathbb{R}^2 : \pi/2 \leq u < \pi, 1 \leq v < 3\}$ and consider the function $\phi : X \to J$ as*

$$\phi(u, v) = [-sin(u) + v^2, cos(u) + v^2 + 3],$$

where $\phi^L(u, v) = -sin(u) + v^2$ and $\phi^U = cos(u) + v^2 + 3$. Let $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $\eta((u, v), (\tilde{u}, \tilde{v})) = \left(u - \tilde{u}, -\frac{1}{2}\sqrt{v - \tilde{v}}\right)$. From Definition 2.3 at $(\tilde{u}, \tilde{v}) = (\pi/2, 1)$ and $\forall (u, v) \in X$, we have

$$\phi^L(u, v) - \phi^L(\pi/2, 1) - {\phi'}^L\left((\pi/2, 1), \eta\big((u, v), (\pi/2, 1)\big)\right) = -sin(u) + v^2 + \sqrt{v - 1} = \Phi_1(u, v), (say).$$
$$\phi^U(u, v) - \phi^U(\pi/2, 1) - {\phi'}^U\left((\pi/2, 1), \eta\big((u, v), (\pi/2, 1)\big)\right) = cos(u) + v^2 - 1 + (u - \pi/2) + \sqrt{v - 1} = \Phi_2(u, v), (say).$$

The graphs of $\Phi_1(u, v)$ and $\Phi_2(u, v)$ are plotted in the Figure 1 and Figure 2, respectively.

Hence from Figures 1 and 2, ϕ is d-invex function with respect to η at $(\pi/2, 1)$.

3 Problem formulation and optimality conditions

In this section, the optimality conditions are derived under the d-invexity assumption for the following interval fractional programming problem:

$$(\mathrm{P}') \ \max \frac{[\phi^L(u), \phi^U(u)]}{[q^L(u), q^U(u)]}$$
$$\text{subject to } \zeta_k(u) \leq 0, k \in K,$$

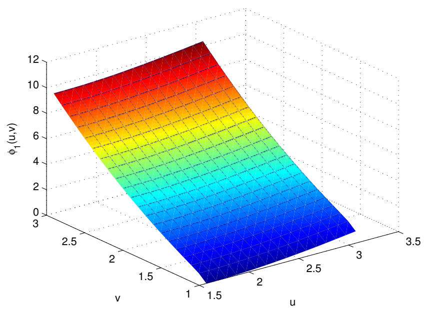

Figure 1: Graph of $\Phi_1(u, v)$

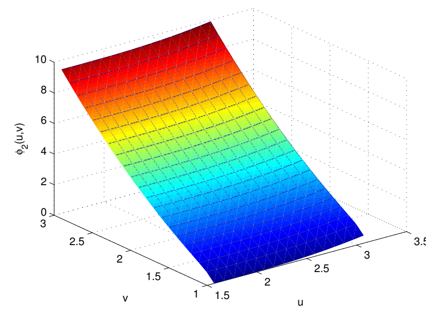

Figure 2: Graph of $\Phi_2(u, v)$

where $K = \{1, 2, \ldots, r\}$ is the index set and $K(u) = \{k \in K : \zeta_k(u) = 0\}$ is the set of active constraints. Assume that $\phi^L, \phi^U, q^L, q^U : X \to \mathbb{R}$ are real-valued directionally differentiable functions such that for every $u \in X$, $\phi^L(u) \geq 0$ and $q^L(u) > 0$. Also, $\zeta_k(u) : X \to \mathbb{R}, k \in K$ are real-valued directionally differentiable functions.
Now, let $q^L = \psi^U, q^U = \psi^L$ then (P$'$) can be formulated as

$$\text{(P)} \ \max \left[\frac{\phi^L(u)}{\psi^L(u)}, \frac{\phi^U(u)}{\psi^U(u)}\right]$$
$$\text{subject to } \zeta_k(u) \leq 0, k \in K.$$

Let $S = \{u \in X : \zeta_k(u) \leq 0, k \in K\}$ be the set of all feasible solutions of (P). Let for $\tilde{u} \in X$, $\gamma^L(\tilde{u}), \dfrac{\phi^L(\tilde{u})}{\psi^L(\tilde{u})}; \gamma^U(\tilde{u}) = \dfrac{\phi^U(\tilde{u})}{\psi^U(\tilde{u})}$ then, $\gamma^L(\tilde{u}) \geq 0, \gamma^U(\tilde{u}) \geq 0$.

Definition 3.1. *Let $\tilde{u}$ be a (P)-feasible solution, then $\tilde{u}$ is said to (P)-optimal if and only if, for every $u \in S$, the following holds*

$$\left[\frac{\phi^L(\tilde{u})}{\psi^L(\tilde{u})}, \frac{\phi^U(\tilde{u})}{\psi^U(\tilde{u})}\right] \succeq \left[\frac{\phi^L(u)}{\psi^L(u)}, \frac{\phi^U(u)}{\psi^U(u)}\right].$$

The following two lemmas will be used to prove the next results of the paper.

Lemma 3.2. *Let $\tilde{u}$ be (P)-optimal and ζ_k are continuous at $\tilde{u}$ for $k \notin K(\tilde{u})$, then the system*

$$\gamma^L(\tilde{u})\psi'^L(\tilde{u}, \eta(u, \tilde{u})) - \phi'^L(\tilde{u}, \eta(u, \tilde{u})) < 0, \tag{1}$$
$$\gamma^U(\tilde{u})\psi'^U(\tilde{u}, \eta(u, \tilde{u})) - \phi'^U(\tilde{u}, \eta(u, \tilde{u})) < 0, \tag{2}$$
$$\zeta'_k(\tilde{u}, \eta(u, \tilde{u})) < 0, \ k \in K, \tag{3}$$

has no solution for any $u \in X$.

Proof. Let (1) has a solution for some $\bar{u} \in X$, and

$$\theta^L(\tilde{u}, \bar{u}, \lambda) = \gamma^L(\tilde{u})\big(\psi^L(\tilde{u} + \lambda\eta(\bar{u}, \tilde{u})) - \psi^L(\tilde{u})\big) - \big(\phi^L(\tilde{u} + \lambda\eta(\bar{u}, \tilde{u})) - \phi^L(\tilde{u})\big). \tag{4}$$

Then for $\lambda = 0$, $\theta^L(\tilde{u}, \bar{u}, 0) = 0$. Consider

$$\begin{aligned}\lim_{\lambda \to 0^+} \frac{\theta^L(\tilde{u}, \bar{u}, \lambda) - \theta^L(\tilde{u}, \bar{u}, 0)}{\lambda} &= \lim_{\lambda \to 0^+} \left(\gamma^L(\tilde{u})\Big(\frac{\psi^L(\tilde{u} + \lambda\eta(\bar{u}, \tilde{u})) - \psi^L(\tilde{u})}{\lambda}\Big) - \frac{\phi^L(\tilde{u} + \lambda\eta(\bar{u}, \tilde{u})) - \phi^L(\tilde{u})}{\lambda}\right) \\ &= \gamma^L(\tilde{u})\psi'^L(\tilde{u}, \eta(\bar{u}, \tilde{u})) - \phi'^L(\tilde{u}, \eta(\bar{u}, \tilde{u})) < 0. \text{ (By (1))}\end{aligned}$$

Thus, for every $\epsilon^L > 0$ there exists some $\omega^L > 0$ with $|\lambda| < \omega^L$ such that

$$\left|\lim_{\lambda \to 0^+} \frac{\theta^L(\tilde{u}, \bar{u}, \lambda) - \theta^L(\tilde{u}, \bar{u}, 0)}{\lambda} - \Big(\gamma^L(\tilde{u})\psi'^L(\tilde{u}, \eta(\bar{u}, \tilde{u})) - \phi'^L(\tilde{u}, \eta(\bar{u}, \tilde{u}))\Big)\right| < \epsilon^L.$$

By assuming $\epsilon^L = -\frac{1}{2}\Big(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u},\eta(\bar{u},\tilde{u})) - {\phi'}^L(\tilde{u},\eta(\bar{u},\tilde{u}))\Big)$ and $|\lambda| < \omega^L$, we get

$$\lim_{\lambda\to 0^+}\frac{\theta^L(\tilde{u},\bar{u},\lambda)-\theta^L(\tilde{u},\bar{u},0)}{\lambda} < \Big(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u},\eta(\bar{u},\tilde{u})) - {\phi'}^L(\tilde{u},\eta(\bar{u},\tilde{u}))\Big) + \epsilon^L = \frac{1}{2}\Big(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u},\eta(\bar{u},\tilde{u})) - {\phi'}^L(\tilde{u},\eta(\bar{u},\tilde{u}))\Big).$$

From (4), the following holds

$$\gamma^L(\tilde{u})\big(\psi^L(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) - \psi^L(\tilde{u})\big) - \big(\phi^L(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) - \phi^L(\tilde{u})\big) < 0. \tag{5}$$

Let $u^* = \tilde{u} + \lambda\eta(\bar{u},\tilde{u})$, then from (5) we get

$$\phi^L(\tilde{u}) - \gamma^L(\tilde{u})\psi^L(\tilde{u}) < \phi^L(u^*) - \gamma^L(\tilde{u})\psi^L(u^*). \tag{6}$$

Assume that (2) has a solution for some $\bar{u} \in X$ and consider the following function

$$\theta^U(\tilde{u},\bar{u},\lambda) = \gamma^U(\tilde{u})\big(\psi^U(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) - \psi^U(\tilde{u})\big) - \big(\phi^U(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) - \phi^U(\tilde{u})\big).$$

Then in the similar lines as above, for every $\epsilon^U > 0$ there exists some $\omega^U > 0$ with $|\lambda| < \omega^U$ such that, we have

$$\phi^U(\tilde{u}) - \gamma^U(\tilde{u})\psi^U(\tilde{u}) < \phi^U(u^*) - \gamma^U(\tilde{u})\psi^U(u^*). \tag{7}$$

Assume that (3) has a solution for some $\bar{u} \in X$, then $\zeta'_k(\tilde{u},\eta(\bar{u},\tilde{u})) < 0$. Consider the following function

$$\theta^\zeta_k(\tilde{u},\bar{u},\lambda) = \zeta_k(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) - \zeta_k(\tilde{u}). \tag{8}$$

Then for $\lambda = 0$, $\theta^\zeta_k(\tilde{u},\bar{u},0) = 0$ and consider

$$\lim_{\lambda\to 0^+}\frac{\theta^\zeta_k(\tilde{u},\bar{u},\lambda)-\theta^\zeta_k(\tilde{u},\bar{u},0)}{\lambda} = \lim_{\lambda\to 0^+}\Big(\frac{\zeta_k(\tilde{u}+\lambda\eta(\bar{u},\tilde{u}))-\zeta_k(\tilde{u})}{\lambda}\Big) < 0.$$

Then for every $\epsilon_k > 0$ there exists some $\omega_k > 0$ with $|\lambda| < \omega_k$ such that $\theta^\zeta_k(\tilde{u},\bar{u},\lambda) < 0$. From (8), we have

$$\zeta_k(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) < \zeta_k(\tilde{u}), \forall k \in K.$$

By definition of $K(\tilde{u})$, it implies that

$$\zeta_k(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) < 0, k \in K(\tilde{u}). \tag{9}$$

Since, ζ_k is continuous at $k \notin K(\tilde{u})$, then

$$\zeta_k(\tilde{u}+\lambda\eta(\bar{u},\tilde{u})) < 0, k \notin K(\tilde{u}). \tag{10}$$

From (9) and (10), it follows that ζ_k satisfies feasibility condition at $u^* = \tilde{u} + \lambda\eta(\bar{u},\tilde{u})$. Hence, u^* is a feasible solution of the problem (P) and from (6) and (7) we get the contradiction that $\tilde{u}$ is (P)-optimal. Thus, the system (1)-(3) has no solution in X. Hence, the proof. □

The following lemma and constraint qualification will be used to derive the necessary conditions.

Lemma 3.3. [31] *Let X be a nonempty set in $\mathbb{R}^n$ and $\delta : X \to \mathbb{R}^n$ be a pre-invex function on X. Then either $\delta(u) < 0$ has a solution $u \in X$, or $d^T\delta(u) \geqq 0$ for all $u \in X$, for some $d \in \mathbb{R}^n_+$, but both alternatives are never true.*

Slater's constraint qualification: The Slater's constraint qualification is said to satisfy at a feasible solution $\tilde{u}$, if ζ_k is d-invex at $\tilde{u}$ for every $k \in K$ and there exists $u \in S$ such that $\gamma^L(\tilde{u})\psi^L(u) - \phi^L(u) < 0, \gamma^U(\tilde{u})\psi^U(u) - \phi^U(u) < 0$, and $\zeta_k(u) < 0,\ k \in K(\tilde{u})$.

Theorem 3.4. *(Necessary conditions) Let $\tilde{u}$ be (P)-optimal and $\phi^L, \phi^U, \psi^L, \psi^U$ and ζ_k, $k \in K$ are directionally differentiable function at $\tilde{u}$ and $\zeta_k, k \in K(\tilde{u})$ are continuous at $\tilde{u}$. Moreover, assume that $-{\phi'}^L(\tilde{u},\eta(u,\tilde{u}))$, $-{\phi'}^U(\tilde{u},\eta(u,\tilde{u}))$,*

${\psi'}^L(\tilde{u},\eta(u,\tilde{u}))$, ${\psi'}^U(\tilde{u},\eta(u,\tilde{u}))$, $\zeta'_k(\tilde{u},\eta(u,\tilde{u}))$, $k\in K$ are pre-invex functions of u on $\mathcal{S}$. Further, if Slater's constraint qualification is satisfied at $\tilde{u}$, then there exists $w_1>0$, $w_2>0$, $\mu=(\mu_1,\dots,\mu_r)\geqq 0$ such that

$$w_1 z_1(\tilde{u}) + w_2 z_2(\tilde{u}) + \sum_{k=1}^{r}\mu_k\zeta'_k(\tilde{u},\eta(u,\tilde{u}))\geq 0, \tag{11}$$

$$\mu_k\zeta_k(\tilde{u})=0,\quad k\in K, \tag{12}$$

$$w_1>0, w_2>0, \mu\geqq 0, \tag{13}$$

where

$$z_1(\tilde{u}) = \frac{1}{\psi^L(\tilde{u})}\big(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^L(\tilde{u},\eta(u,\tilde{u}))\big),$$
$$z_2(\tilde{u}) = \frac{1}{\psi^U(\tilde{u})}\big(\gamma^U(\tilde{u}){\psi'}^U(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^U(\tilde{u},\eta(u,\tilde{u}))\big).$$

Proof. By assumption $\tilde{u}$ is (P)-optimal, then by Lemma 3.2 the system

$$\gamma^L(\tilde{u}){\psi'}^L(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^L(\tilde{u},\eta(u,\tilde{u}) < 0,$$
$$\gamma^U(\tilde{u}){\psi'}^U(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^U(\tilde{u},\eta(u,\tilde{u}) < 0,$$
$$\zeta'_k(\tilde{u},\eta(u,\tilde{u}))<0,\ k\in K,$$

has no solution for any $u\in S$. That is,

$$\frac{1}{\psi^L(\tilde{u})}\big(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^L(\tilde{u},\eta(u,\tilde{u})\big) < 0,$$
$$\frac{1}{\psi^U(\tilde{u})}\big(\gamma^U(\tilde{u}){\psi'}^U(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^U(\tilde{u},\eta(u,\tilde{u})\big) < 0,$$
$$\zeta'_k(\tilde{u},\eta(u,\tilde{u}))<0,\ k\in K,$$

has no solution for any $u\in S$.

Since -${\phi'}^L(\tilde{u},\eta(u,\tilde{u})), -{\phi'}^U(\tilde{u},\eta(u,\tilde{u})), {\psi'}^L(\tilde{u},\eta(u,\tilde{u})),{\psi'}^U(\tilde{u},\eta(u,\tilde{u}))$ and $\zeta'_k(\tilde{u},\eta(u,\tilde{u}))$, $k\in K$ are pre-invex functions of u on S, by Lemma 3.3, there exist $w_1\geq 0$, $w_2\geq 0$,$\mu\geqq 0$ such that (11) holds. Further, let $\mu_k=0,\forall k\notin K(\tilde{u})$, then (12) follows.

We have to prove that $w_1\neq 0$ and $w_2\neq 0$. We begin with the contradiction, then either $w_1=0$, or $w_2=0$, or, $w_1=0, w_2=0$. We first consider the case when $w_1=0$ and $w_2>0$. Then from (11), we have

$$w_2 z_2(\tilde{u}) + \sum_{k=1}^{r}\mu_k\zeta'_k(\tilde{u},\eta(u,\tilde{u}))\geq 0. \tag{14}$$

Since Slater's constraint qualification is satisfied at $\tilde{u}$, there exists $u\in S$ such that we have

$$\gamma^U(\tilde{u})\psi^U(u) - \phi^U(u) < 0 \implies \gamma^U(\tilde{u})\psi^U(u) - \phi^U(u) < \gamma^U(\tilde{u})\psi^U(\tilde{u}) - \phi^U(\tilde{u}).$$

Let $u=\tilde{u}+\lambda\eta(u,\tilde{u})$, then we have

$$\gamma^U(\tilde{u})\psi^U(\tilde{u}+\lambda\eta(u,\tilde{u})) - \phi^U(\tilde{u}+\lambda\eta(u,\tilde{u})) < \gamma^U(\tilde{u})\psi^U(\tilde{u}) - \phi^U(\tilde{u}). \tag{15}$$

Since the functions ϕ^U,ψ^U are directionally differentiable function at $\tilde{u}$, from (15), we have

$$\gamma^U(\tilde{u}){\psi'}^U(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^U(\tilde{u},\eta(u,\tilde{u}) < 0. \implies \frac{1}{\psi^U(\tilde{u})}\big(\gamma^U(\tilde{u}){\psi'}^U(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^L(\tilde{u},\eta(u,\tilde{u})\big) < 0. \tag{16}$$

Since $w_2>0$, from (16), we have

$$w_2\left(\frac{1}{\psi^U(\tilde{u})}\big(\gamma^U(\tilde{u}){\psi'}^U(\tilde{u},\eta(u,\tilde{u})) - {\phi'}^L(\tilde{u},\eta(u,\tilde{u}))\big)\right) < 0. \implies w_2 z_2(\tilde{u}) < 0. \tag{17}$$

From (12) and the constraint of (P), we have

$$\sum_{k=1}^{r} \mu_k \zeta_k(u) \leq \sum_{k=1}^{r} \mu_k \zeta_k(\tilde{u}). \tag{18}$$

Since ζ_k are d-invex functions and $\mu \geqq 0$, from (18), we have

$$\sum_{k=1}^{r} \mu_k \zeta'(\tilde{u}, \eta(u, \tilde{u})) \leq 0. \tag{19}$$

On adding (17) and (19) we get contradiction to (14), thus $w_1 = 0$ can not hold. Similarly, for the case (ii), we prove that $w_2 = 0$ can not hold.
Next, we assume that $w_1 = 0, w_2 = 0$, then from (11), we have

$$\sum_{k=1}^{r} \mu_k \zeta_k'(\tilde{u}, \eta(u, \tilde{u})) \geq 0. \tag{20}$$

By the Slater's constraint qualification there exists $u \in S$ such that

$$\zeta_k(u) < 0. \tag{21}$$

Since ζ_k is d-invex at $\tilde{u}$, we have

$$\zeta_k(u) - \zeta_k(\tilde{u}) \geq \zeta_k'(\tilde{u}, \eta(u, \tilde{u})).$$

By assumption $\mu \geqq 0$, then from the above inequality, we get

$$\sum_{k=1}^{r} \mu_k \zeta_k(u) - \sum_{k=1}^{r} \mu_k \zeta_k(\tilde{u}) \geq \sum_{k=1}^{r} \mu_k \zeta_k'(\tilde{u}, \eta(u, \tilde{u})). \tag{22}$$

Using (20)-(21) in (22), the following holds

$$\sum_{k=1}^{r} \mu_k \zeta_k(u) \geq \sum_{k=1}^{r} \mu_k \zeta_k(\tilde{u}) = 0,$$

which is contradiction to the Slater's constraint qualification. Hence $w_1 \neq 0, w_2 \neq 0$, and this completes the proof. $\square$

Theorem 3.5. *(Sufficient conditions) Let $\tilde{u}$ be a (P)-feasible solution such that* (11)-(13) *hold. Assume that the functions $-\phi^L, -\phi^U, \psi^L, \psi^U$ and $\zeta_k, k \in K$ are d-invex with respect to η at $\tilde{u}$, then $\tilde{u}$ is (P)-optimal.*

Proof. We begin with a contradiction, that $\tilde{u}$ is not (P)-optimal, then for any feasible solution $u \in S$, we have

$$\left[\frac{\phi^L(\tilde{u})}{\psi^L(\tilde{u})}, \frac{\phi^U(\tilde{u})}{\psi^U(\tilde{u})}\right] \prec \left[\frac{\phi^L(u)}{\psi^L(u)}, \frac{\phi^U(u)}{\psi^U(u)}\right],$$

which implies one of the following relation hold

$$\left.\begin{array}{rl} & \phi^L(u) - \gamma^L(\tilde{u})\psi^L(u) \geq 0, \phi^U(u) - \gamma^U(\tilde{u})\psi^U(u) > 0 \\ \text{or,} & \phi^L(u) - \gamma^L(\tilde{u})\psi^L(u) > 0, \phi^U(u) - \gamma^U(\tilde{u})\psi^U(u) \geq 0 \\ \text{or,} & \phi^L(u) - \gamma^L(\tilde{u})\psi^L(u) > 0, \phi^U(u) - \gamma^U(\tilde{u})\psi^U(u) > 0. \end{array}\right\} \tag{23}$$

Since $\dfrac{w_1}{\psi^L(\tilde{u})} > 0$ and $\dfrac{w_2}{\psi^U(\tilde{u})} > 0$, from (23), we have

$$\frac{w_1}{\psi^L(\tilde{u})}\left(\phi^L(u) - \gamma^L(\tilde{u})\psi^L(u)\right) + \frac{w_2}{\psi^U(\tilde{u})}\left(\phi^U(u) - \gamma^U(\tilde{u})\psi^U(u)\right) > 0. \tag{24}$$

By assumption, the functions $-\phi^L, -\phi^U, \psi^L, \psi^U$ are d-invex with respect to η at $\tilde{u}$. Then by Definition 2.3, we have

$$-\phi^L(u) + \phi^L(\tilde{u}) \geq -{\phi'}^L(\tilde{u}, \eta(u, \tilde{u})), \tag{25}$$

$$-\phi^U(u) + \phi^U(\tilde{u}) \geq -{\phi'}^U(\tilde{u}, \eta(u, \tilde{u})), \tag{26}$$

$$\psi^L(u) - \psi^L(\tilde{u}) \geq {\psi'}^L(\tilde{u}, \eta(u, \tilde{u})), \tag{27}$$

$$\psi^U(u) - \psi^U(\tilde{u}) \geq {\psi'}^U(\tilde{u}, \eta(u, \tilde{u})). \tag{28}$$

Multiplying (27) by $\gamma^L(\tilde{u}) \geq 0$ and adding to (25), we have

$$\gamma^L(\tilde{u})\psi^L(u) - \phi^L(u) \geq \gamma^L(\tilde{u}){\psi'}^L(\tilde{u}, \eta(u, \tilde{u})) - {\phi'}^L(\tilde{u}, \eta(u, \tilde{u})). \tag{29}$$

Multiplying (28) by $\gamma^U(\tilde{u}) \geq 0$ and adding to (26), the following holds

$$\gamma^U(\tilde{u})\psi^U(u) - \phi^U(u) \geq \gamma^U(\tilde{u}){\psi'}^U(\tilde{u}, \eta(u, \tilde{u})) - {\phi'}^U(\tilde{u}, \eta(u, \tilde{u})). \tag{30}$$

Multiplying $\dfrac{w_1}{\psi^L(\tilde{u})} > 0$ by (29), $\dfrac{w_2}{\psi^U(\tilde{u})} > 0$ by (30), and then adding, we have

$$\begin{aligned}\frac{w_1}{\psi^L(\tilde{u})}\left(\gamma^U(\tilde{u})\psi^U(u) - \phi^U(u)\right) + \frac{w_2}{\psi^U(\tilde{u})}\left(\gamma^U(\tilde{u})\psi^U(u) - \phi^U(u)\right) \geq{}& \frac{w_1}{\psi^L(\tilde{u})}\left(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u}, \eta(u, \tilde{u})) - {\phi'}^L(\tilde{u}, \eta(u, \tilde{u}))\right) \\ &+ \frac{w_2}{\psi^U(\tilde{u})}\left(\gamma^U(\tilde{u}){\psi'}^U(\tilde{u}, \eta(u, \tilde{u})) - {\phi'}^U(\tilde{u}, \eta(u, \tilde{u}))\right).\end{aligned}$$

Using (24) in the above inequality, we get

$$\frac{w_1}{\psi^L(\tilde{u})}\left(\gamma^L(\tilde{u}){\psi'}^L(\tilde{u}, \eta(u, \tilde{u})) - {\phi'}^L(\tilde{u}, \eta(u, \tilde{u}))\right) + \frac{w_2}{\psi^U(\tilde{u})}\left(\gamma^U(\tilde{u}){\psi'}^U(\tilde{u}, \eta(u, \tilde{u})) - {\phi'}^U(\tilde{u}, \eta(u, \tilde{u}))\right) < 0,$$

which is equal to

$$w_1 z_1(\tilde{u}) + w_2 z_2(\tilde{u}) < 0. \tag{31}$$

Since $\zeta_k, k \in K$ are d-invex functions with respect to η at $\tilde{u}$, we get

$$\zeta_k(u) - \zeta_k(\tilde{u}) \geq \zeta_k'(\tilde{u}, \eta(u, \tilde{u})). \tag{32}$$

Multiplying (32) by $\mu \geqq 0$, we have

$$\sum_{k=1}^{r} \mu_k \zeta_k(u) - \sum_{k=1}^{r} \mu_k \zeta_k(\tilde{u}) \geq \sum_{k=1}^{r} \mu_k \zeta_k'(\tilde{u}, \eta(u, \tilde{u})). \tag{33}$$

Using the feasibility of (P) and (12) in (33), we have

$$\sum_{k=1}^{r} \mu_k \zeta_k'(\tilde{u}, \eta(u, \tilde{u})) \leq 0. \tag{34}$$

On adding (31) and (34), the following holds

$$w_1 z_1(\tilde{u}) + w_2 z_2(\tilde{u}) + \sum_{k=1}^{r} \mu_k \zeta_k'(\tilde{u}, \eta(u, \tilde{u})) < 0,$$

which is a contradiction to (11), hence $\tilde{u}$ is a (P)-optimal solution. This completes the proof. □

The following example is presented to illustrate the results developed in Section 3.

3.1 Numerical example

Let $X = \{u \in \mathbb{R} : 0 < u < 3\}$ be a non-empty subset of $\mathbb{R}$ and consider the following functions

$$\phi^L(u) = \begin{cases} 4 - e^u & \text{if } u \in (0,1) \\ sin(1) + cos(1) & \text{if } u \in [1,3) \end{cases}, q^L(u) = \begin{cases} u^2 + 1 & \text{if } u \in (0,1) \\ u^2 + 2 & \text{if } u \in [1,3) \end{cases},$$

$$\phi^U(u) = \begin{cases} u + 4 - e^u & \text{if } u \in (0,1) \\ \log(u) - u + 5 & \text{if } u \in [1,3) \end{cases}, \zeta(u) = \begin{cases} u^2 - 1 & \text{if } u \in (0,1) \\ 0 & \text{if } u \in [1,3) \end{cases},$$

$$q^U(u) = \begin{cases} u^2 + \sin(u) + 1 & \text{if } u \in (0,1) \\ u^2 + 3 & \text{if } u \in [1,3) \end{cases}.$$

In view of the problem (P) with the above defined functions, consider the following interval fractional programming problem

$$\text{(P1) } \max \left[\frac{\phi^L(u)}{\psi^L(u)}, \frac{\phi^U(u)}{\psi^U(u)}\right] = \max \begin{cases} \left[\dfrac{4 - e^u}{u^2 + \sin(u) + 1}, \dfrac{u + 4 - e^u}{u^2 + 1}\right] & \text{if } u \in (0,1) \\ \left[\dfrac{sin(1) + cos(1)}{u^2 + 3}, \dfrac{\log(u) - u + 5}{u^2 + 2}\right] & \text{if } u \in [1,3) \end{cases}$$

$$\text{subject to } \zeta(u) = \begin{cases} u^2 - 1 & \text{if } u \in (0,1) \\ 0 & \text{if } u \in [1,3) \end{cases} \leq 0.$$

Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as $\eta(u, \tilde{u}) = \begin{cases} (u - \tilde{u}) & \text{if } u \in (0,1) \\ 0.5(1 - e^{\tilde{u} - u}) & \text{if } u \in [1,3) \end{cases}$.

The set X is invex with respect to the above defined η and $\tilde{u} = 1$ is a feasible solution of (P1). Let $\mathcal{F} = \{\phi^L, \phi^U, \psi^L, \psi^U, \zeta\}$ be the set of functions involved in (P1) and for any $f \in \mathcal{F}$, the following notation is used to denote the directional derivative of f at $\tilde{u} = 1$.

$$f'(1, \eta(u,1)) = \begin{cases} p_1^f(u) & \text{if } u \in (0,1) \\ p_2^f(u) & \text{if } u \in [1,3). \end{cases}$$

The directional derivative of $\phi^L(u)$ in the direction of $\eta(u,1)$ is calculated as follows:

$$u \in (0,1),\ p^{\phi^L}{}_1(u) = \lim_{\lambda \to 0^+} \frac{\phi^L(1 + \lambda\eta(u,1)) - \phi^L(1)}{\lambda} = -(u-1)e.$$

$$u \in [1,3),\ p^{\phi^L}{}_2(u) = \lim_{\lambda \to 0^+} \frac{\phi^L(1 + \lambda\eta(u,1)) - \phi^L(1)}{\lambda} = 0.$$

Similarly, the directional derivatives of the functions ϕ^U, ψ^L, ψ^U and ζ have been calculated and given in Table 1 below. The following notations is used to check the pre-invexity of the directional derivatives listed in Table 1:

Table 1: Directional derivatives

	$u \in (0,1)$		$u \in [1,3)$
$p^{\phi^L}{}_1(u)$	$-(u-1)e$	$p^{\phi^L}{}_2(u)$	0
$p^{\phi^U}{}_1(u)$	$(u-1)(1-e)$	$p^{\phi^U}{}_2(u)$	0
$p^{\psi^L}{}_1(u)$	$(u-1)(2+cos(1))$	$p^{\psi^L}{}_2(u)$	$1 - e^{(1-u)}$
$p^{\psi^U}{}_1(u)$	$2(u-1)$	$p^{\psi^U}{}_2(u)$	$1 - e^{(1-u)}$
$p_1^{\zeta}(u)$	$2(u-1)$	$p_2^{\zeta}(u)$	0

$$\begin{cases} p_1^f(1 + \lambda\eta(u,1)) - \lambda p_1^f(u) + (1-\lambda)p_1^f(1) = \Phi_1^f(u) & \text{if } u \in (0,1) \\ p_2^f(1 + \lambda\eta(u,1)) - \lambda p_2^f(u) + (1-\lambda)p_2^f(1) = \Phi_2^f(u) & \text{if } u \in [1,3). \end{cases}$$

For $-\phi'^L(1,\eta(u,1))$, pre-invexity is checked for any $u,\tilde{u}\in X$ and $\lambda\in[0,1], u\in(0,1)$, in the following steps:

$$-\Phi^{\phi^L_1}(u) = -p^{\phi^L_1}(1+\lambda\eta(u,1)) + \lambda p^{\phi^L_1}(u) + (1-\lambda)p^{\phi^L_1}(1) = (1+\lambda(u-1)-1)e + \lambda(u-1)e = 0.$$

When $u\in[1,3)$, $-\Phi^{\phi^L_2}(u) = -p^{\phi^L_2}(1+\lambda\eta(u,1))+\lambda p^{\phi^L_2}(u)+(1-\lambda)p^{\phi^L_2}(1) = 0$. The values of $-\Phi^{\phi^U_1}, \Phi^{\psi^L_1}, \Phi^{\psi^U_1}, \Phi^{\zeta}_1, -\Phi^{\phi^U_2}$, $\Phi^{\psi^L_2}, \Phi^{\psi^U_2}, \Phi^{\zeta}_2$ are calculated using Table 1 and are listed in Table 2 below. From Table 2 and Figure 3, the functions

Table 2: Pre-invexity of directional derivatives

	$u\in(0,1)$		$u\in[1,3)$
$-\Phi^{\phi^L_1}(u)$	0	$-\Phi^{\phi^L_2}(u)$	0
$-\Phi^{\phi^U_1}(u)$	0	$-\Phi^{\phi^U_2}(u)$	0
$\Phi^{\psi^L_1}(u)$	0	$\Phi^{\psi^L_2}(u)$	$1-e^{(-0.5\lambda(1-e^{(1-u)}))}-\lambda(1-e^{(1-u)})$
$\Phi^{\psi^U_1}(u)$	0	$\Phi^{\psi^U_2}(u)$	$1-e^{(-0.5\lambda(1-e^{(1-u)}))}-\lambda(1-e^{(1-u)})$
$\Phi^{\zeta}_1(u)$	0	$\Phi^{\zeta}_2(u)$	0

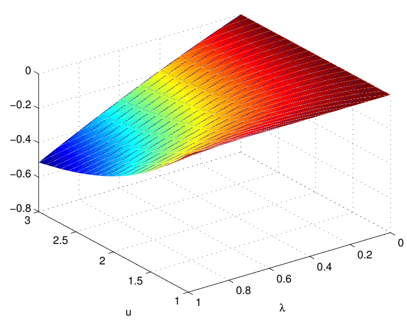

Figure 3: Graph of $\Phi^{\psi^U_2}(u)$

$-\phi'^L(1,\eta(u,1))$, $-\phi'^U(1,\eta(u,1)), \psi'^L(1,\eta(u,1)), \psi'^U(1,\eta(u,1))$ and $\zeta(1,\eta(u,1))$ are pre-invex on X. Further, from Table 1m it can be seen that the functions $-\phi'^L(1,\eta(u,1))$, $-\phi'^U(1,\eta(u,1))$, $\psi'^L(1,\eta(u,1)), \psi'^U(1,\eta(u,1))$ vanish at $\tilde{u}=1$. Then

$$z_1(1) = \frac{1}{\psi^L(1)}\big(\gamma^L(1)\psi'^L(1,\eta(u,1)) - \phi'^L(1,\eta(u,1))\big) = 0,$$
$$z_2(1) = \frac{1}{\psi^U(1)}\big(\gamma^U(1)\psi'^U(1,\eta(u,1)) - \phi'^U(1,\eta(u,1))\big) = 0.$$

Also, $\zeta'(1,\eta(u,1)) = 0$, which implies $w_1z_1(1) + w_2z_2(1) + \mu\zeta'(1,\eta(u,1)) = 0$. Hence, at feasible solution $\tilde{u}=1$, the condition (11) holds. Again (12) is obvious and we can choose w_1, w_2 and λ such that (13) holds.
For every $f\in\mathcal{F}$, the following notations will be used to check the d-invexity of the functions $-\phi^L, -\phi^U, \psi^L, \psi^U$ and ζ:

$$\left.\begin{array}{ll} f(u)-f(1)-f'(1,\eta(u,1)) = \delta^f_1(u) & \text{if } u\in(0,1) \\ f(u)-f(1)-f'(1,\eta(u,1)) = \delta^f_2(u) & \text{if } u\in[1,3). \end{array}\right\}$$

For the function $-\phi^L$ is d-invexity is checked in the following steps:

$$\text{When } u\in(0,1),\quad -\delta^{\phi^L_1}(u) = -\phi^L(u)+\phi^L(1)+\phi'^L(1,\eta(u,1)) = -(4-e^u)+(4-e)-(u-1)e = e^u - eu.$$
$$\text{When } u\in[1,3),\quad -\delta^{\phi^L_2}(u) = -\phi^L(u)+\phi^L(1)+\phi'^L(1,\eta(u,1)) = 0.$$

Similarly, $-\delta^{\phi^U_1}(u), \delta^{\psi^L_1}(u), \delta^{\psi^U_1}(u), \delta^{\zeta}_1(u), -\delta^{\phi^U_2}(u), \delta^{\psi^L_2}(u), \delta^{\psi^U_2}(u), \delta^{\zeta}_2(u)$ are calculated and listed in Table 3 below. From Figures 4 and 5, it is clear that the functions $-\phi^L, -\phi^U, \psi^L, \psi^U$ and ζ are d-invex function with respect to $\eta(u,1)$ at $\tilde{u}=1$. Consequently, from Theorem 3.5, $\tilde{u}=1$ is an optimal solution of (P1).

Table 3: d-invexity of functions

	$u \in (0,1)$		$u \in [1,3)$
$-\delta\phi^{L}_{1}(u)$	$e^u - eu$	$-\delta\phi^{L}_{2}(u)$	0
$-\delta\phi^{U}_{1}(u)$	$e^u - eu$	$-\delta\phi^{U}_{2}(u)$	$u - \log(u) - 1$
$\delta\psi^{L}_{1}(u)$	$u^2 + \sin(u) - 1 - sin(1) - (u-1)(2 + cos(1))$	$\delta\psi^{L}_{2}(u)$	$u^2 - 2 + e^{(1-u)}$
$\delta\psi^{U}_{1}(u)$	$(u-1)^2$	$\delta\psi^{U}_{2}(u)$	$u^2 - 2 + e^{(1-u)}$
$\delta^{\zeta}_{1}(u)$	$(u-1)^2$	$\delta^{\zeta}_{2}(u)$	0

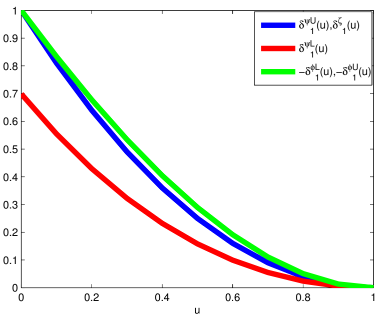

Figure 4: Graphs of $-\delta\phi^{L}_{1}, \delta\psi^{L}_{1}, \delta^{\zeta}_{1}$

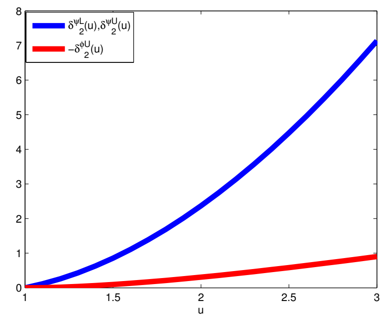

Figure 5: Graphs of $-\delta\phi^{U}_{2}, \delta\psi^{L}_{2}, \delta\psi^{U}_{2}$

4 Application to portfolio optimization

In this section, we present an application of the interval fractional programming problem in portfolio optimization. This section shows the utility of Theorem 3.5, derived for the interval fractional programming problem (P), in portfolio optimization. Here, we have taken Mao's model [10] as a linear fractional programming problem, where the expected return, liquidity, systematic risk (beta), and proportions of total investment are taken as interval numbers. A portfolio optimization problem is modelled as an interval fractional optimization problem, and the solution is obtained by applying the global criteria method (GCM), discussed as follows:

Global criteria method [33]

The global criteria method (GCM) is a technique for obtaining solutions to multiobjective optimization problems by converting them into single-objective optimization problems. Consider the following interval optimization problem:

$$\begin{aligned}(\mathrm{OP}) \quad & \max\ [\phi_1(u), \phi_2(u)] \\ & \text{subject to } \zeta_k(u) \leq 0, \forall k \in K.\end{aligned}$$

where $\phi_1, \phi_2 : X \to \mathbb{R}$ are given functions and ζ_k are given functions for $k \in K$. Then, this problem can be formulated as following optimization problems [9]:

$$\begin{aligned}(\mathrm{OP}_1) \quad & \max\ \phi_1(u) \\ & \text{subject to } \zeta_k(u) \leq 0, \forall k \in K,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{OP}_2) \quad & \max\ \phi_2(u) \\ & \text{subject to } \zeta_k(u) \leq 0, \forall k \in K.\end{aligned}$$

Let $\phi_1(\tilde{u})$ and $\phi_2(\tilde{u})$ be the optimal solutions of (OP_1) and (OP_2), respectively. Then, by using the global criteria method, we convert problem (OP) as follows:

$$(\mathrm{OP}_G) \quad \min \left\{ \sum_{i=1}^{2} \left(\frac{\phi_i(\tilde{u}) - \phi_i(u)}{\phi_i(\tilde{u})} \right)^p \right\}^{1/p}$$

$$\text{subject to } \zeta_k(u) \leq 0, \forall k \in K,$$

where $1 \leq p \leq \infty$. Boychuk and Ovchinnikov [4] recommended $p = 1$ for global criteria, which means giving equal importance to all the deviations. Salukvadze [25] recommended $p = 2$ for global criteria, which means that these deviations are weighted proportionately, with the largest deviations having the largest weight.

4.1 Model description

Suppose an investor wants to invest in different risky stocks in such a way that the expected excess return to beta ratio is maximum, subject to market liquidity constraints. We have taken monthly historical data of five Bombay Stock Exchange (BSE) listed companies for the period Jan-2018 to Jan-2021 from the Bombay Stock Exchange, India website. The names of these companies, along with their scrip codes, are listed in Table 4 below.

Table 4: BSE stock list

Stock i	Company name	Scrip code
Stock 1	HCL	532281
Stock 2	TATA	532540
Stock 3	INFOSYS	500209
Stock 4	ULTRATECH	532538
Stock 5	NESTLE	500790

Rate of return:

Let $\mu_i = [\mu_i^L, \mu_i^U], i = \{1, 2, \ldots, 5\}$ be the interval expected rate of return for i^{th} stock. Assume that μ_{it}^{open}, μ_{it}^{close}, μ_{it}^{high}, μ_{it}^{low} are the monthly rate of return corresponding to the open, close, high, and low price of i^{th} stock obtained from the historical data at time t, respectively. Then, we calculate the expected rate of return corresponding to these prices in [18] as follows:

$$\bar{\mu}_i^{open} = \frac{1}{T}\sum_{i=1}^{T}\mu_{it}^{open}, \quad \bar{\mu}_i^{close} = \frac{1}{T}\sum_{i=1}^{T}\mu_{it}^{close}, \bar{\mu}_i^{high} = \frac{1}{T}\sum_{i=1}^{T}\mu_{it}^{high}, \quad \bar{\mu}_i^{low} = \frac{1}{T}\sum_{i=1}^{T}\mu_{it}^{low},$$

where, T-denote the total number of period.
For the i^{th} stock, the lower bound μ_i^L and the upper bound μ_i^U of the expected rate of return are calculated by the following formula:

$$\mu_i^L = min\left\{\bar{\mu}_i^{open}, \bar{\mu}_i^{close}, \bar{\mu}_i^{high}, \bar{\mu}_i^{low}\right\}, \quad \mu_i^U = max\left\{\bar{\mu}_i^{open}, \bar{\mu}_i^{close}, \bar{\mu}_i^{high}, \bar{\mu}_i^{low}\right\}. \tag{35}$$

For the considered stocks, the expected rate of return bounds are computed using (35) and are presented in Table 5.

Turnover rate:

Turnover is a measure of liquidity, and the liquidity of stock indicates the ease of conversion from stock to currency or vice versa without affecting its market price. The easier the conversion, the more liquid the stock. Let liquidity $l_i = [l_i^L, i_i^U]$, where l_i^L, l_i^U are the lower and upper bounds of the turnover rate, respectively. The turnover rate of i^{th} stock at time t is calculated by the following formula:

$$\text{Turnover rate} = l_{it} = \frac{\text{Number of trades}}{\text{Total share}},$$

where the number of trades and total number of shares of i^{th} stock are taken from historical data. The lower and upper bounds of the turnover rate of i^{th} stock are obtained as follows:

$$l_i = [l_i^L, l_i^U] = \left[\min_{t\in T} l_{it}, \max_{t\in T} l_{it}\right]. \tag{36}$$

The bounds of turnover rate for the five stocks are calculated using (36) and listed in Table 5.

Systematic risk:

Beta (b_i) is a measure of the systematic risk of the i^{th} stock compared to the market as a whole. The formula to calculate beta (b_i) for i^{th} stock is as follows:

$$b_i = \frac{\text{Covariance between the rate of return of the s}i^{th}\text{ stock and the market}}{\text{Variance of the market}}.$$

In this example, we have taken S&P BSE SENSEX as the market index. Let $m_t^{open}, m_t^{close}, m_t^{high}, m_t^{low}$ be the monthly rate of return of market index corresponding to open, close, high, and low price of market index at time t, respectively. Now, the lower bound m^L and the upper bound m^U of the rate of return of the market index are calculated by using the following formulas:

$$m^L = min\{\bar{m}^{open}, \bar{m}^{close}, \bar{m}^{high}, \bar{m}^{low}\}; \quad m^U = max\{\bar{m}^{open}, \bar{m}^{close}, \bar{m}^{high}, \bar{m}^{low}\}, \tag{37}$$

where $\bar{m}^{open}, \bar{m}^{close}, \bar{m}^{high}$ and $\bar{m}^{low}$ are the average of the monthly rate of return of the market index corresponding to the open, close, high, and low prices of the market index over a T time period, respectively. We consider b_i as interval data and denote as $b_i = [b_i^L, b_i^U]$ and the bounds of b_i for i^{th} stock is obtained by following formula:

$$b_i = [b_i^L, b_i^U] = \left[\frac{cov(\mu_i^L, m^L)}{var(m^L)}, \frac{cov(\mu_i^U, m^U)}{var(m^U)}\right]. \tag{38}$$

The lower and upper bounds of beta for the stocks are calculated using formula (38) and summarized in Table 5.

Table 5: Rate of return, turnover rate and systematic risk of stocks

Stock no.	Rate of return		Turnover rate		Systematic risk	
	Lower(μ_i^L)	Upper(μ_i^U)	Lower(l_i^L)	Upper(l_i^U)	Lower(b_i^L)	Upper(b_i^L)
Stock 1	0.0072	0.0112	0.0024	0.0400	0.8188	0.9320
Stock 2	0.0075	0.0101	0.0183	0.0766	0.0973	0.1093
Stock 3	0.0104	0.0152	0.0116	0.0332	0.8654	0.9463
Stock 4	0.0086	0.0099	0.0433	0.1560	0.2117	0.8827
Stock 5	0.0249	0.0256	0.0690	0.4516	0.2639	0.5376

Model formulation:

Let $u_i = [u_i^L, u_i^U]$ be the amount of investment in the i^{th} stock, where u_i^L and u_i^U denote the minimum and maximum amount of investment in the i^{th} stock, respectively. We assume $l_0 = [l_0^L, l_0^U]$ be the minimum liquidity of stocks. Also, we assume that the total amount of investment cannot exceed the budget (we have assumed the total budget as 1) and that all the investments are non-negative. The objective of this problem is to find an optimal investment $(u_1, u_2, \ldots, u_5)$ that maximizes the expected excess return to beta ratio subject to the above-mentioned constraints. We can formulate this problem as an interval fractional programming problem as follows:

$$\text{(IFP}'\text{)} \quad \max \frac{\sum_{i=1}^5 \mu_i \otimes u_i}{\sum_{i=1}^5 b_i \otimes u_i} = \max \frac{\sum_{i=1}^5 [\mu_i^L, \mu_i^U] \otimes [u_i^L, u_i^U]}{\sum_{i=1}^5 [b_i^L, b_i^U] \otimes [u_i^L, u_i^U]} = \max \left[\frac{\sum_{i=1}^5 \mu_i^L u_i^L}{\sum_{i=1}^5 b_i^U u_i^U}, \frac{\sum_{i=1}^5 \mu_i^U u_i^U}{\sum_{i=1}^5 b_i^L u_i^L}\right]$$

subject to

$$\sum_{i=1}^5 l_i \otimes u_i \succeq l_0,$$

$$\sum_{i=1}^5 u_i \preceq [1, 1],$$

$$u_i^L, u_i^U \geq 0, i = 1, 2, ..., 5.$$

4.2 Solving (IFP′) by global criteria method

In view of Table 5, (IFP′) can be written as the following problems:

$$(\text{IFP}_1)\ \max\ \theta_1(u) = \max\ \frac{0.0072u_1^L + 0.0075u_2^L + 0.0104u_3^L + 0.0086u_4^L + 0.0249u_5^L}{0.9320u_1^U + 0.1093u_2^U + 0.9463u_3^U + 0.8827u_4^U + 0.5376u_5^U}$$

subject to

$$\begin{aligned}
&0.0024u_1^L + 0.0183u_2^L + 0.0116u_3^L + 0.0433u_4^L + 0.0690u_5^L \geq l_0^L,\\
&0.0400u_1^U + 0.0766u_2^U + 0.0332u_3^U + 0.1560u_4^U + 0.4516u_5^U \geq l_0^U,\\
&u_1^L + u_2^L + u_3^L + u_4^L + u_5^L \leq 1,\\
&u_1^U + u_2^U + u_3^U + u_4^U + u_5^U \leq 1,\\
&u_i^L \leq u_i^U, i = 1, 2, ..., 5,\\
&u_i^L, u_i^U \geq 0, i = 1, 2, ..., 5.
\end{aligned}$$

$$(\text{IFP}_2)\ \max\ \theta_2(u) = \max\ \frac{0.0112u_1^U + 0.0101u_2^U + 0.0152u_3^U + 0.0099u_4^U + 0.0256u_5^U}{0.8188u_1^L + 0.0973u_2^L + 0.8654u_3^L + 0.2117u_4^L + 0.2639u_5^L}$$

subject to

$$\begin{aligned}
&0.0024u_1^L + 0.0183u_2^L + 0.0116u_3^L + 0.0433u_4^L + 0.0690u_5^L \geq l_0^L,\\
&0.0400u_1^U + 0.0766u_2^U + 0.0332u_3^U + 0.1560u_4^U + 0.4516u_5^U \geq l_0^U,\\
&u_1^L + u_2^L + u_3^L + u_4^L + u_5^L \leq 1,\\
&u_1^U + u_2^U + u_3^U + u_4^U + u_5^U \leq 1,\\
&u_i^L \leq u_i^U, i = 1, 2, ..., 5,\\
&u_i^L, u_i^U \geq 0, i = 1, 2, ..., 5.
\end{aligned}$$

Let $\theta_1(\tilde{u})$ and $\theta_2(\tilde{u})$ be the optimal solutions of the problems (IFP_1) and (IFP_2), respectively. Using the global criteria method, the following problem is formulated:

$$(\text{IFP}_G)\quad \min\quad \left\{\left(\frac{\phi_1(\tilde{u}) - \phi_1(u)}{\phi_1(\tilde{u})}\right)^2 + \left(\frac{\phi_2(\tilde{u}) - \phi_2(u)}{\phi_2(\tilde{u})}\right)^2\right\}^{1/2}$$

subject to

$$\begin{aligned}
&0.0024u_1^L + 0.0183u_2^L + 0.0116u_3^L + 0.0433u_4^L + 0.0690u_5^L \geq l_0^L,\\
&0.0400u_1^U + 0.0766u_2^U + 0.0332u_3^U + 0.1560u_4^U + 0.4516u_5^U \geq l_0^U,\\
&u_1^L + u_2^L + u_3^L + u_4^L + u_5^L \leq 1,\\
&u_1^U + u_2^U + u_3^U + u_4^U + u_5^U \leq 1,\\
&u_i^L \leq u_i^U, i = 1, 2, ..., 5,\\
&u_i^L, u_i^U \geq 0, i = 1, 2, ..., 5.
\end{aligned}$$

We solve (IFP_G) for the liquidity ranges $[l_0^L, l_0^U] = [0.0200, 0.0250]$,$[0.0250, 0.0300]$, $[0.0300, 0.0350]$,$[0.0350, 0.0400]$, $[0.0400, 0.0450]$,$[0.0450, 0.0500]$,$[0.0500, 0.0550]$,$[0.0550, 0.0600]$,$[0.0600, 0.0650]$,$[0.0650, 0.0700]$ by using WOLFRAM MATHEMATICA 12 software and the optimal solutions are listed in Table 6 and shown in Figure 6. From Figure 6, for the liquidity range $[0.0200, 0.0250]$ we get the optimal solution range $[0.0308, 0.1895]$ which is the largest interval of optimal solution range. As liquidity increases, the interval of the optimal solution starts to contract. With the increase in liquidity, the upper bound of the optimal solution decreases rapidly. However, the lower bound of the optimal solution increases gradually with an increase in liquidity but, it improves the lower bound or minimum expected excess return to beta ratio. As it can be seen that for liquidity range $[0.0200, 0.0250]$ the minimum expected excess return to beta ratio was 0.0308 and at liquidity range $[0.0650, 0.0700]$ it increases to the value 0.0457.

4.2.1 Verification of Theorem 3.5

Let $\tilde{u} = (\tilde{u}_1^L, \tilde{u}_2^L, \tilde{u}_3^L, \tilde{u}_4^L, \tilde{u}_5^L, \tilde{u}_1^U, \tilde{u}_2^U, \tilde{u}_3^U, \tilde{u}_4^U, \tilde{u}_5^U) = (0.0000, 0.0000, 0.0000, 0.0000, 0.2899, 0.0000, 0.7101, 0.0000, 0.0000,$ $0.2899)$ be a feasible solution of the problem IFP. Using the Theorem 3.5, we check whether the feasible solution $\tilde{u}$ is

Table 6: Optimal solutions of (IFP_G) by Global criteria method

	Liquidity ($l_0 = [l_0^L, l_0^U]$)	u_1	u_2	u_3	u_4	u_5	Optimal
Lower	0.0200	0.0000	0.0000	0.0000	0.0000	0.2899	0.0308
Upper	0.0250	0.0000	0.7101	0.0000	0.0000	0.2899	0.1895
Lower	0.0250	0.0000	0.0001	0.0000	0.0000	0.3623	0.0340
Upper	0.0300	0.0000	0.6377	0.0000	0.0000	0.3623	0.1653
Lower	0.0300	0.0000	0.0000	0.0000	0.0000	0.4348	0.0365
Upper	0.0350	0.0000	0.5652	0.0000	0.0000	0.4348	0.1465
Lower	0.0350	0.0000	0.0000	0.0000	0.0000	0.5072	0.0386
Upper	0.0400	0.0000	0.4928	0.0000	0.0000	0.5072	0.1404
Lower	0.0400	0.0000	0.0000	0.0000	0.0000	0.5797	0.0414
Upper	0.0450	0.0000	0.4203	0.0000	0.0000	0.5797	0.1242
Lower	0.0450	0.0000	0.0000	0.0000	0.0000	0.6522	0.0417
Upper	0.0500	0.0000	0.3478	0.0000	0.0000	0.6522	0.1174
Lower	0.0500	0.0000	0.0000	0.0000	0.0000	0.7246	0.0429
Upper	0.0550	0.0000	0.2754	0.0000	0.0000	0.7246	0.1114
Lower	0.0550	0.0000	0.0000	0.0000	0.0000	0.7971	0.0438
Upper	0.0600	0.0000	0.1992	0.0000	0.0000	0.8008	0.1069
Lower	0.0600	0.0000	0.0000	0.0000	0.0000	0.8696	0.0448
Upper	0.0650	0.0000	0.1237	0.0000	0.0000	0.8763	0.1028
Lower	0.0650	0.0000	0.0000	0.0000	0.0000	0.9420	0.0457
Upper	0.0700	0.0000	0.0534	0.0000	0.0000	0.9466	0.0994

an optimal solution of IFP. For the problem (IFP), we have

$$\left.\begin{aligned}
\phi^L(u) &= 0.0072u_1^L + 0.0075u_2^L + 0.0104u_3^L + 0.0086u_4^L + 0.0249u_5^L,\\
\phi^U(u) &= 0.0112u_1^U + 0.0101u_2^U + 0.0152u_3^U + 0.0099u_4^U + 0.0256u_5^U,\\
\psi^L(u) &= 0.9320u_1^U + 0.1093u_2^U + 0.9463u_3^U + 0.8827u_4^U + 0.5376u_5^U,\\
\psi^U(u) &= 0.8188u_1^L + 0.0973u_2^L + 0.8654u_3^L + 0.2117u_4^L + 0.2639u_5^L.
\end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned}
&\zeta_1(u) = l_0^L - (0.0024u_1^L + 0.0183u_2^L + 0.0116u_3^L + 0.0433u_4^L + 0.0690u_5^L),\\
&\zeta_2(u) = l_0^U - (0.0400u_1^U + 0.0766u_2^U + 0.0332u_3^U + 0.1560u_4^U + 0.4516u_5^U),\\
&\zeta_3(u) = u_1^L + u_2^L + u_3^L + u_4^L + u_5^L - 1,\\
&\zeta_4(u) = u_1^U + u_2^U + u_3^U + u_4^U + u_5^U - 1,\\
&\zeta_5(u) = u_1^L - u_1^U, \quad \zeta_6(u) = u_2^L - u_2^U,\\
&\zeta_7(u) = u_3^L - u_3^U, \quad \zeta_8(u) = u_4^L - u_4^U,\\
&\zeta_9(u) = u_5^L - u_5^U,\\
&u_i^L, u_i^U \geq 0, i = 1, 2, ..., 5.
\end{aligned}\right\} \tag{40}$$

Let us define $\eta : \mathbb{R}^{10} \times \mathbb{R}^{10} \to \mathbb{R}^{10}$ as $\eta(u, \tilde{u}) = (-6, -2, -2, 0, 4, -5, -1, -1, 1, 5)$. We check whether the conditions (11)-(13) hold at the feasible solution $\tilde{u}$, By solving the inequality (11) for the liquidity range $[0.0200, 0.0250]$, we get the following equations:

$$
\begin{aligned}
0.1848w_1 - 12.2508w_2 + 0.0144\mu_1 - 6\mu_3 - 6\mu_5 &= 0,\\
0.0642w_1 - 0.4862w_2 + 0.0366\mu_1 - 2\mu_3 - 2\mu_6 &= 0,\\
0.0890w_1 - 4.3164w_2 + 0.0232\mu_1 - 2\mu_3 - 2\mu_7 &= 0,\\
-0.4264w_1 + 2.6352w_2 - 0.2760\mu_1 + 4\mu_3 + 4\mu_9 &= 0,\\
-0.6145w_1 + 0.7320w_2 + 0.2000\mu_2 - 5\mu_4 + 5\mu_5 &= 0,\\
-0.0146w_1 + 0.1320w_2 + 0.0766\mu_2 - \mu_4 + \mu_6 &= 0,\\
-0.1246w_1 + 0.1987w_2 + 0.0332\mu_2 - \mu_4 + \mu_7 &= 0,\\
0.1165w_1 - 0.1294w_2 - 0.1560\mu_2 + \mu_4 - \mu_8 &= 0,\\
0.3555w_1 - 1.6730w_2 - 2.2580\mu_2 + 5\mu_4 - 5\mu_9 &= 0.
\end{aligned}
$$

By the condition (12) it implies that, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0$. Solving the above equations along with the values of $\mu_1, \mu_2, \mu_3, \mu_4, \mu_6$, we get $w_1 = 31.1526, w_2 = 4.1135, \mu_5 = 3.2265, \mu_7 = 3.0643, \mu_8 = 3.0970, \mu_9 = 0.8385$. Hence, it implies that the conditions (11)-(13) hold at the feasible solution $\tilde{u}$. The directional derivatives of the functions $-\phi^L, -\phi^U, \psi^L, \psi^U, \zeta_k, k = 1, 2, \ldots, 9$ are constants. Thus, $-{\phi'}^L(\tilde{u}, \eta(u, \tilde{u})), -{\phi'}^U(\tilde{u}, \eta(u, \tilde{u}))$, ${\psi'}^L(\tilde{u}, \eta(u, \tilde{u}))$, ${\psi'}^U(\tilde{u}, \eta(u, \tilde{u})), \zeta'_k(\tilde{u}, \eta(u, \tilde{u})), k = 1, 2, \ldots, 9$ are pre-invex functions with respect to the above defined η. Further, the functions $-\phi^L, -\phi^U, \psi^L, \psi^U, \zeta_k, k = 1, 2, \ldots, 9$ given in (39) and (40) are d-invex functions with respect to $\eta = (-6, -2, -2, 0, 4, -5, -1, -1, 1, 5)$. Consequently, all the assumptions of the Theorem 3.5 are satisfied at feasible solution $\tilde{u}$. Hence, we conclude that $\tilde{u}$ is an optimal solution of IFP.

4.3 Solving (IFP$'$) by deterministic approaches

4.3.1 For enhanced interval solution [17]

Kumar and Bhurjee [17] considered a multiobjective optimization problem with interval decision variables and parameters and studied the existence of solution by parametrizing the intervals. Now, we solve the problem (IFP$'$) by converting the interval input data and the interval decision variables into their parametric form. Let $u = [u^L, u^U]$ be a non-empty closed interval with u^L and u^U be its lower and upper bounds, respectively. Then, this interval can be written in parametric form as $u = u^L + c(u^U - u^L)$, where $c^T \in [0, 1]$. By using the Table 5 and the parametric form, we have

$$
\begin{aligned}
\mu U = \sum_{i=1}^{5} \mu_i u_i =& (0.0072 + 0.0040t_1)\left(u_1^L + c_1(u_1^U - u_1^L)\right) + (0.0075 + 0.0026t_2)(u_2^L + c_2(u_2^U - u_2^L)) + (0.0104 + 0.0048t_3)\\
&\left(u_3^L + c_3(u_3^U - u_3^L)\right) + (0.0086 + 0.0013t_4)\left(u_4^L + c_4(u_4^U - u_4^L)\right) + (0.0249 + 0.0007t_5)\left(u_5^L + c_5(u_5^U - u_5^L)\right),\\
bU = \sum_{i=1}^{5} b_i u_i =& (0.8188 + 0.1132t_1)\left(u_1^L + c_1(u_1^U - u_1^L)\right) + (0.0973 + 0.0120t_2)(u_2^L + c_2(u_2^U - u_2^L)) + (0.8654 + 0.0809t_3)\\
&\left(u_3^L + c_3(u_3^U - u_3^L)\right) + (0.2117 + 0.6710t_4)\left(u_4^L + c_4(u_4^U - u_4^L)\right) + (0.2639 + 0.2737t_5)\left(u_5^L + c_5(u_5^U - u_5^L)\right).
\end{aligned}
$$

where, $t = (t_1, t_2, t_3, t_4, t_5)^T \in [0, 1]^5$,$c = (c_1, c_2, c_3, c_4, c_5)^T \in [0, 1]^5$. Further, let $\mathcal{U}_i = (u_i^L + u_i^U)$ then for $w_1 : [0, 1]^5 \to \mathbb{R}_+, w_1(t) = 1$ and $w_2 : [0, 1]^5 \to \mathbb{R}_+, w_2(c) = 1$

$$
\underbrace{\int \cdots \int}_{10} w_1(t) w_2(c) \mu U \, dt dc = 0.0046\,\mathcal{U}_1 + 0.0044\,\mathcal{U}_2 + 0.0064\,\mathcal{U}_3 + 0.0046\,\mathcal{U}_4 + 0.0126\,\mathcal{U}_5. \tag{41}
$$

$$
\underbrace{\int \cdots \int}_{10} w_1(t) w_2(c) bU \, dt dc = 0.4377\,\mathcal{U}_1 + 0.0517\,\mathcal{U}_2 + 0.4529\,\mathcal{U}_3 + 0.2736\,\mathcal{U}_4 + 0.2004\,\mathcal{U}_5. \tag{42}
$$

Now, using (41)-(42) and Table 5, the problem (IFP′) can be written as follows:

$$(\text{IFP}_3) \quad \max \quad \frac{0.0046\,\mathcal{U}_1 + 0.0044\,\mathcal{U}_2 + 0.0064\,\mathcal{U}_3 + 0.0046\,\mathcal{U}_4 + 0.0126\,\mathcal{U}_5}{0.4377\,\mathcal{U}_1 + 0.0517\,\mathcal{U}_2 + 0.4529\,\mathcal{U}_3 + 0.2736\,\mathcal{U}_4 + 0.2004\,\mathcal{U}_5}$$

subject to

$$\begin{aligned}
&0.0024u_1^L + 0.0183u_2^L + 0.0116u_3^L + 0.0433u_4^L + 0.0690xu_5^L \geq l_0^L,\\
&0.0400u_1^U + 0.0766u_2^U + 0.0332u_3^U + 0.1560u_4^U + 0.4516u_5^U \geq l_0^U,\\
&u_1^L + u_2^L + u_3^L + u_4^L + u_5^L \leq 1,\\
&u_1^U + u_2^U + u_3^U + u_4^U + u_5^U \leq 1,\\
&u_i^L \leq u_i^U, i = 1, 2, ..., 5,\\
&u_i \geq 0, i = 1, 2, ..., 5.
\end{aligned}$$

We solve the above problem using WOLFRAM MATHEMATICA 12 software for different liquidity ranges and the solutions obtained are summarized in below Table 7 and shown in Figure 7.

Table 7: Optimal solutions of (IFP_3) by deterministic approach [17]

	Liquidity ($l_0 = [l_0^L, l_0^U]$)	u_1	u_2	u_3	u_4	u_5	Optimal
Lower	0.0200	0.0000	0.9665	0.0000	0.0000	0.0335	0.0647
Upper	0.0250	0.0000	0.9665	0.0000	0.0000	0.0335	0.1041
Lower	0.0250	0.0000	0.8679	0.0000	0.0000	0.1322	0.0590
Upper	0.0300	0.0000	0.8679	0.0000	0.0000	0.1322	0.1023
Lower	0.0300	0.0000	0.7692	0.0000	0.0000	0.2308	0.0552
Upper	0.0350	0.0000	0.7692	0.0000	0.0000	0.2308	0.1010
Lower	0.0350	0.0000	0.6706	0.0000	0.0000	0.3294	0.0527
Upper	0.0400	0.0000	0.6706	0.0000	0.0000	0.3294	0.0999
Lower	0.0400	0.0000	0.5720	0.0000	0.0000	0.4280	0.0513
Upper	0.0450	0.0000	0.5720	0.0000	0.0000	0.4280	0.0996
Lower	0.0450	0.0000	0.4734	0.0000	0.0000	0.5266	0.0499
Upper	0.0500	0.0000	0.4734	0.0000	0.0000	0.5266	0.0978
Lower	0.0500	0.0000	0.3748	0.0000	0.0000	0.6252	0.0488
Upper	0.0550	0.0000	0.3748	0.0000	0.0000	0.6252	0.0971
Lower	0.0550	0.0000	0.2761	0.0000	0.0000	0.7239	0.0479
Upper	0.0600	0.0000	0.2761	0.0000	0.0000	0.7239	0.0965
Lower	0.0600	0.0000	0.1775	0.0000	0.0000	0.8225	0.0472
Upper	0.0650	0.0000	0.1775	0.0000	0.0000	0.8225	0.0964
Lower	0.0650	0.0000	0.0789	0.0000	0.0000	0.9211	0.0466
Upper	0.0700	0.0000	0.0789	0.0000	0.0000	0.9211	0.0959

4.3.2 For degenerate interval solution [3]

We solve (IFP′) by writing the interval input data in their parametric form. In this case, the decision variables u_i are degenerate intervals that is, $u_i = [u_i^L, u_i^U]$ where $u_i^L = u_i^U$. By using the Tables 5 and parametric form, we have

$$\begin{aligned}
\mu U =&(0.0072 + 0.0040t_1)u_1 + (0.0075 + 0.0026t_2)u_2 + (0.0104 + 0.0048t_3)u_3 + (0.0086 + 0.0013t_4)u_4 + (0.0249 + 0.0007t_5)u_5.\\
bU =&(0.8188 + 0.1132t_1)u_1 + (0.0973 + 0.0120t_2)u_2 + (0.865 + 0.0809t_3)u_3 + (0.2117 + 0.6710t_4)u_4 + (0.2639 + 0.2737t_5)u_5.
\end{aligned}$$

where, $t = (t_1, t_2, t_3, t_4, t_5)^T \in [0,1]^5$ then for $w : [0,1]^5 \to \mathbb{R}_+, w(t) = 1$,

$$\underbrace{\int \dots \int}_{5} w(t)\mu U dt = 0.0092u_1 + 0.0088u_2 + 0.0128u_3 + 0.0092u_4 + 0.0252u_5. \tag{43}$$

$$\underbrace{\int \dots \int}_{5} w(t) b U dt = 0.8754u_1 + 0.1034u_2 + 0.9058u_3 + 0.5472u_4 + 0.4008u_5. \tag{44}$$

Using the deterministic objective function from (43)-(44) and Table 5 the problem (IFP$'$) can be written as follows:

$$\begin{aligned}
(\text{IFP}_4) \quad \max \quad & \frac{0.0092u_1 + 0.0088u_2 + 0.0128u_3 + 0.0092u_4 + 0.0252u_5}{0.8754u_1 + 0.1034u_2 + 0.9058u_3 + 0.5472u_4 + 0.4008u_5} \\
\text{subject to} \quad & \\
& -0.0024u_1 - 0.0183u_2 - 0.0116u_3 - 0.0433u_4 - 0.690u_5 \leq -l_0^U, \\
& u_1 + u_2 + u_3 + u_4 + u_5 \leq 1, \\
& u_i \geq 0, i = 1, 2, ..., 5.
\end{aligned}$$

We solve the above problem using WOLFRAM MATHEMATICA 12 for the liquidity ranges and the solution obtained is listed in the Table 8 and shown in Figure 8.

Table 8: Optimal solutions of (IFP$_4$) by deterministic approach [3]

	Liquidity ($l_0 = [l_0^L, l_0^U]$)	u_1	u_2	u_3	u_4	u_5	Optimal
Lower Upper	0.0200 0.0250	0.0000	0.8679	0.0000	0.0000	0.1321	0.0764
Lower Upper	0.0250 0.0300	0.0000	0.7692	0.0000	0.0000	0.2308	0.0733
Lower Upper	0.0300 0.0350	0.0000	0.6706	0.0000	0.0000	0.3294	0.0705
Lower Upper	0.0350 0.0400	0.0000	0.5720	0.0000	0.0000	0.4280	0.0685
Lower Upper	0.0400 0.0450	0.0000	0.4734	0.0000	0.0000	0.5266	0.0673
Lower Upper	0.0450 0.0500	0.0000	0.3748	0.0000	0.0000	0.6252	0.0660
Lower Upper	0.0500 0.0550	0.0000	0.2761	0.0000	0.0000	0.7239	0.0647
Lower Upper	0.0550 0.0600	0.0000	0.1775	0.0000	0.0000	0.8225	0.0641
Lower Upper	0.0600 0.0650	0.0000	0.0789	0.0000	0.0000	0.9211	0.0633

5 Results and discussion

In this section, we present a comparative study of the optimal solutions of the problem (IFP$'$) solved using the Global criteria method (Section 4) with two deterministic methods discussed in Subsections 4.3.1 and 4.3.2. The problem (IFP$'$) is solved by these methods for similar input data given in Tables 5 and the solutions by these methods are given in Tables 6-8. A comparison of solutions is given in the Figures 6-9 below where, X-axis and Y-axis represent liquidity and expected excess return to beta ratio, respectively.

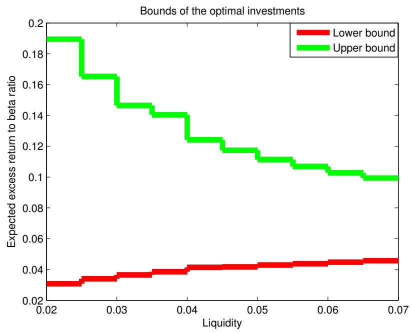

Figure 6: Optimal solutions of (IFP$_G$)

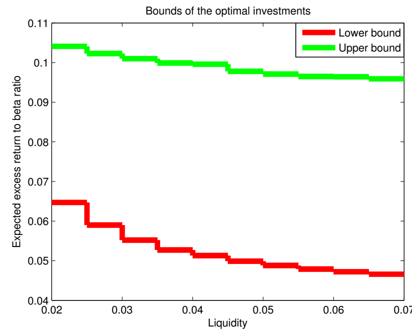

Figure 7: Optimal solutions of (IFP$_3$)

5.1 Comparison of optimal solutions of (IFP$_G$),(IFP$_3$) and (IFP$_4$)

We compare the upper bounds of the optimal solutions of (IFP$_G$),(IFP$_3$) and (IFP$_4$). From Tables 6-8, for the liquidity range $[0.0200, 0.0250]$, the upper bounds of the optimal solutions of (IFP$_G$), (IFP$_3$) and (IFP$_4$) are 0.1895,0.1041 and 0.0764 respectively. It can be observed from Figure 9 that for every liquidity range the optimal solution obtained from

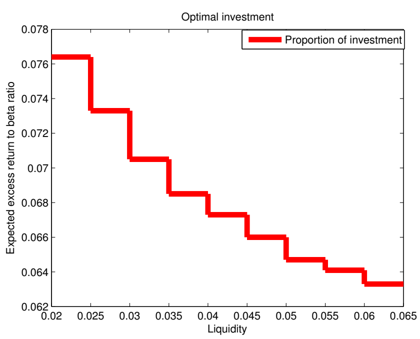

Figure 8: Optimal solution of (IFP$_4$)

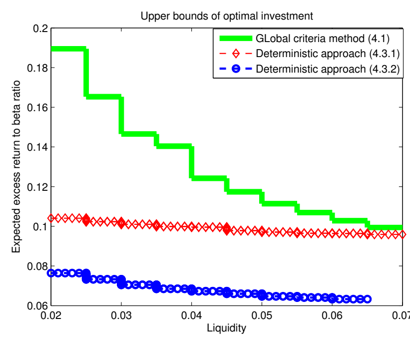

Figure 9: Optimal solutions of (IFP$_G$),(IFP$_3$) and (IFP$_4$)

the approach discussed in this paper for (IFP$_G$) is far better than the approaches given in [17] for (IFP$_3$) and in [3] for (IFP$_4$).

6 Conclusion

In this paper, we have extended the definition of d-invexity to the class of interval optimization problems. For the interval fractional programming problem, we have derived the necessary and sufficient optimality conditions under the d-invexity assumption. Several numerical examples are presented at suitable places to provide a clear understanding of the results established in the paper. An application of the results derived in the paper is presented in the context of a real-world portfolio optimization problem. Further, a comparative study between the optimal solutions of the problem (IFP$_G$) obtained by the global criteria method and the optimal solutions of the problems (IFP$_3$) and (IFP$_4$) obtained using different approaches in [17] and [3], respectively, is illustrated. Consequently, these comparisons show that the maximum expected excess return to beta ratio for the portfolio optimization problem (IFP$'$) obtained by the global criteria method discussed in this paper dominates the maximum expected excess return to beta ratio obtained by the deterministic approaches discussed in [17] and [3]. This shows the significance of the results developed in the paper.

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