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Modeling directional monotonicity with copulas

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Abstract

The purpose of this paper is to characterize the concept of monotonicity according to a direction related to a set of n random variables in terms of its associated n-copula. We start establishing relationships in the bivariate and trivariate cases, which will help to understand the extension to the multivariate case. Several examples are provided.

Keywords: Copula, directional monotonicity, random variable.

1 Introduction

Dependence among random variables is a widely field of research in statistics and probability. Analyzing this dependence structure is crucial when we want to figure out the behaviour of a complex model components. Therefore, a deep research of dependence relations could bring us comprehensively information about the model of interest.

Random variables can be related in several ways, presenting different relationships of dependence. One of the most important in literature is positive dependence, which can be characterize as the inclination of components within a random vector to assume concordant values. Negative dependence can be defined similarly, but now the variables values moves in opposite directions. Both concepts may not be enough to express all the dependence that the variables show. For that reason, in this paper, we will focus on the concept of monotonicity according to a direction $\alpha \in \mathbb{R}^n$ defined in [14], whose positive (respectively, negative) dependence concept is denoted by $\mathrm{I}(\alpha)$ (respectively, $\mathrm{D}(\alpha)$).

The concept of directional monotonicity presents significant advances in modeling complex dependencies within fuzzy systems. Traditional copulas generally express standard forms of dependence, such as simple positive or negative relationships, but may fall short for capturing directional trends in multivariate data. By extending the traditional copula approach to account for directional monotonicity, this framework provides a refined mechanism for depicting dependencies with specific directional tendencies among variables. This enhancement is particularly relevant to fields such as finance, environmental sciences, and risk management, where variables relationships may shift in response to economic trends or environmental changes. One key application of directional monotonicity is in multivariate risk analysis, especially for understanding dependencies between variables such as asset returns or environmental factors. Capturing both the direction and strength of these dependencies improves the accuracy of risk assessments. For instance, by incorporating directionally monotonic copulas into portfolio risk models, more nuanced evaluations of risk across economic or stress scenarios can be achieved, enhancing the robustness of portfolio management.

Copulas –multivariate distribution functions with univariate uniform marginals– serve as a valuable tool for examining the positive dependence characteristics of a random vector. This is because they capture the dependence structure of the corresponding multivariate distribution function, regardless of the individual marginal distribution functions [13]. Additionally, they provide scale-free measures of dependence and serve as a foundation for constructing families of distribution functions [6]. Copulas have attracted an increasing interest by researchers in some topics of fuzzy sets theory, such as preference modeling, similarities and fuzzy logics (see, for instance, [7, 8]).

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Our objective in this paper, as we have already mentioned, is to explore multivariate copulas associated to random vectors that exhibit the I(α) property–we will refer to these as I(α) copulas. However, for simplicity–and for a better understanding of the general case–, we will start with the bivariate and trivariate cases, and then we will extrapolate our results to the multivariate case.

This paper is organized as follows: we begin with some preliminary concepts and results concerning multivariate dependence, specifically monotonicity according to a direction, and copula theory (Section 2). Straightaway, in Section 3, we will characterize the concept of I(α) for the bivariate case by using copulas, providing several examples of copulas that have this kind of dependence for different directions $\alpha \in \mathbb{R}^2$. Analogously, we will carry out a similar study for the trivariate case. Once we have used this concepts for lower dimensions, we will extrapolate the characterization by using copulas in the multivariate case. Then, a general result is stated, which allows us to obtain an inequality, for every direction $\alpha \in \mathbb{R}^n$, in terms of the n-copula associated with the random vector, and its two-dimensional marginal copulas. Finally, Section 4 is devoted to discuss the conclusions derived from our research.

2 Preliminaries

Let $n \geq 2$ be a natural number, and $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ an n-dimensional random vector. In the sequel, the expression "nonincreasing in $\mathbf{x}$"–and similarly for nondecreasing–will mean that it is nonincreasing–nondecreasing–in each of the components of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $\mathbf{X} \leq \mathbf{x}$ will mean $X_i \leq x_i$ for all $i = 1, 2, \ldots, n$.

Regarding multivariate dependence, the following two positive dependence notions, introduced in [9], are widely known:

i) $\mathbf{X}$ is *left corner set decreasing*, denoted by LCSD($\mathbf{X}$), if

$$\mathbb{P}[\mathbf{X} \leq \mathbf{x} | \mathbf{X} \leq \mathbf{x}'] \text{ is nonincreasing in } \mathbf{x}' \text{ for all } \mathbf{x}.$$

ii) $\mathbf{X}$ is *right corner set increasing*, denoted by RCSI($\mathbf{X}$), if

$$\mathbb{P}[\mathbf{X} > \mathbf{x} | \mathbf{X} > \mathbf{x}'] \text{ is nondecreasing in } \mathbf{x}' \text{ for all } \mathbf{x}.$$

The equivalent negative dependence concepts LCSI($\mathbf{X}$) *(left corner set increasing)* and RCSD($\mathbf{X}$) *(right corner set decreasing)* are defined exchanging "nondecreasing" and "nonincreasing" in their respective expressions.

These concepts of multivariate positive dependence can be extended to the concept of monotonicity according to a direction, defined bellow, which allow us to capture new dependence structures among random variables.

Definition 2.1. [14] *Let $\mathbf{X}$ be a n-dimensional random vector and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, ..., n$. The random vector $\mathbf{X}$, or its joint distribution function, is said to be increasing (respectively, decreasing) according to the direction α, denoted by $I(\alpha)$ (respectively, $D(\alpha)$), if*

$$\mathbb{P}[\alpha\mathbf{X} > \mathbf{x} | \alpha\mathbf{X} > \mathbf{x}'],$$

is nondecreasing (respectively, nonincreasing) in $\mathbf{x}'$ for all $\mathbf{x}$.

In this paper we will focus on the I(α) dependence concept. Similar results can be obtain for D(α). Note that the concept I(α) generalizes the RCSI and LCSD concepts mentioned above; namely, LCSD corresponds to I($-\mathbf{1}$) and RCSI corresponds to I($\mathbf{1}$), where $-\mathbf{1} = (-1, -1, ..., -1)$ and $\mathbf{1} = (1, 1, \ldots, 1)$.

Now we recall some notions related to copulas. For $n \geq 2$, an n-dimensional *copula* (n-copula, for short) is the restriction to $[0, 1]^n$ of a continuous n-dimensional distribution function whose univariate marginals are uniform on $[0, 1]$. The importance of copulas in statistics is described in the following result due to Abe Sklar [15]: Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with joint distribution function F and one-dimensional marginal distribution functions $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C, which is uniquely determined on $\times_{i=1}^n \text{Range} F_i$, such that

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \quad \text{for all } \mathbf{x} \in [-\infty, +\infty]^{\text{n}},$$

(for a complete proof of this result, see [16]). Thus, copulas link joint distribution functions to their one-dimensional marginals. For a survey on copulas, see [4, 13]; for some recent applications, see [2, 3, 5]; and for some results about positive dependence properties by using copulas can be seen, for instance, [10, 11, 12, 13, 17].

Let Π^n denote the n-copula for independent random variables (or product n-copula), i.e., $\Pi^n(\mathbf{u}) = \prod_{i=1}^n u_i$ for all $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in [0, 1]^n$.

For any n-copula C we have

$$W^n(\mathbf{u}) = \max\left\{0, \sum_{i=1}^{n} u_i - n + 1\right\} \le C(\mathbf{u}) \le \min\{u_1, u_2, \ldots, u_n\} = M^n(\mathbf{u}),$$

for all $\mathbf{u}$ in $[0,1]^n$. M^n is an n-copula for all $n \ge 2$; however, W^n is an n-copula only when $n = 2$.

The definition of d-marginal of an n-copula C, where $1 \le d < n$, which will be useful in the study of dependence in higher dimensions, is as follows. Let $\mathbf{X}$ be an n-dimensional random vector with associated n-copula C and let $\sigma = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ such that $1 \le m \le n-1$. The σ-marginal copula of C, $C_\sigma : [0,1]^m \longrightarrow [0,1]$, is defined by setting $n - m$ arguments of C equal to 1, i.e.,

$$C_\sigma(u_1, \ldots, u_m) = C(v_1, \ldots, v_n),$$

where $v_i = u_i$ if $i \in \{i_1, \ldots, i_m\}$, and $v_i = 1$ otherwise.

Finally, given a random vector $(X_1, X_2, \ldots, X_n)$ with n-copula C, the *survival n-copula associated with C*, which we denote by $\widehat{C}$, is given by

$$\widehat{C}(\mathbf{u}) = \mathbb{P}\left[X_1 \ge 1 - u_1, X_2 \ge 1 - u_2, \ldots, X_n \ge 1 - u_n\right],$$

for all $\mathbf{u} \in [0,1]^n$.

3 Monotonicity according to a direction and copulas

In this section, we will characterize the I(α) concept for the bivariate case by using copulas, providing various examples that exhibit this type of dependence for different directions. Similarly, we will conduct an analogous analysis for the trivariate case. Once we become familiar with these concepts in lower dimensions, we will extend the description using n-copulas to the multivariate case, establishing a general result that enables us to derive a characterization for any direction $\alpha \in \mathbb{R}^n$ in terms of the n-copula associated with the random vector and its marginal d-copulas.

3.1 The bivariate case

We start our study on the I(α) dependence concept for bivariate copulas.

Theorem 3.1. *Let (U, V) be a random pair with associated 2-copula C and uniform marginals on $[0,1]$. Then C is:*

i. *I$(-1,-1)$ if, and only if, $C(u,v)C(u',v') \ge C(u,v')C(u',v)$ for all u, v, u', v' in $[0,1]$ such that $u \le u'$ and $v \le v'$;*

ii. *I$(1,-1)$ if, and only if, $[v - C(u,v)][v' - C(u',v')] \le [v - C(u',v)][v' - C(u,v')]$ for all u, v, u', v' in $[0,1]$ such that $u \le u'$ and $v \le v'$;*

iii. *I$(-1,1)$ if, and only if, $[u - C(u,v)][u' - C(u',v')] \le [u' - C(u',v)][u - C(u,v')]$ for all u, v, u', v' in $[0,1]$ such that $u \le u'$ and $v \le v'$;*

iv. *I$(1,1)$ if, and only if, $\widehat{C}(u,v)\widehat{C}(u',v') \ge \widehat{C}(u,v')\widehat{C}(u',v)$ for all u, v, u', v' in $[0,1]$ such that $u \le u'$ and $v \le v'$.*

Proof. The proof of parts i. and iv. can be found in [13, Theorem 5.2.15 and Corollary 5.2.17]. Now we prove part ii.—the proof of part iii. is similar, and we omit it.

Let (U, V) be a pair of random variables with associated 2-copula C and uniform marginals on $[0,1]$, and assume that C is I$(1,-1)$. Then we have that $\mathbb{P}[U > u, -V > w | U > u', -V > w']$ is nondecreasing in (u', w') for all (u, w), i.e.,

$$\mathbb{P}[U > u, -V > w | U > u', -V > w'] \le \mathbb{P}[U > u, -V > w | U > u'', -V > w''], \tag{1}$$

for all $u, u', u'' \in [0,1]$ and $w, w', w'' \in [-1, 0]$ such that $u' \le u''$ and $w' \le w''$. Condition (1) is equivalent to

$$\frac{\mathbb{P}[U > u, -V > w, U > u', -V > w']}{\mathbb{P}[U > u', -V > w']} \le \frac{\mathbb{P}[U > u, -V > w, U > u'', -V > w'']}{\mathbb{P}[U > u'', -V > w'']},$$

i.e.,

$$\frac{\mathbb{P}[U > u, V < -w, U > u', V < -w']}{\mathbb{P}[U > u', V < -w']} \le \frac{\mathbb{P}[U > u, V < -w, U > u'', V < -w'']}{\mathbb{P}[U > u'', V < -w'']},$$

therefore,

$$\frac{\mathbb{P}[U > \max\{u, u'\}, V < \min\{-w, -w'\}]}{\mathbb{P}[U > u', V < -w']} \leq \frac{\mathbb{P}[U > \max\{u, u''\}, V < \min\{-w, -w''\}]}{\mathbb{P}[U > u'', V < -w'']},$$

that is,

$$\frac{\min\{-w, -w'\} - C(\max\{u, u'\}, \min\{-w, -w'\})}{-w' - C(u', -w')} \leq \frac{\min\{-w, -w''\} - C(\max\{u, u''\}, \min\{-w, -w''\})}{-w'' - C(u'', -w'')}.$$

Taking $v = -w$, $v' = -w'$ and $v'' = -w''$, we obtain

$$\begin{aligned}[v'' - C(u'', v'')] \cdot [\min\{v, v'\} \; &- \; C(\max\{u, u'\}, \min\{v, v'\})] \\ &\leq \; [v' - C(u', v')] \cdot [\min\{v, v''\} - C(\max\{u, u''\}, \min\{v, v''\})],\end{aligned}$$

for all u, v, u', v', u'', v'' in $[0, 1]$ such that $u' \leq u''$ and $v'' \leq v'$. By setting $u = u' \leq u''$, $v' = v''$ and $u' = u''$, it follows for every $v \leq v'$ and $u \leq u'$ in $[0, 1]$ that

$$[v - C(u, v)][v' - C(u', v')] \leq [v - C(u', v)][v' - C(u, v')].$$

Conversely, if $[v - C(u, v)][v' - C(u', v')] \leq [v - C(u', v)][v' - C(u, v')]$ for all $u, v, u', v' \in [0, 1]$ such that $u \leq u'$ and $v \leq v'$, then, given $u, u' \in [0, 1]$ and $w, w' \in [-1, 0]$, in order to prove that the random variables are $\mathrm{I}(1, -1)$, we consider three possible cases:

1. If $u > u', w > w'$ we have that

$$\mathbb{P}[U > u, -V > w | U > u', -V > w'] = \frac{\mathbb{P}[U > u, -V > w]}{\mathbb{P}[U > u', -V > w']},$$

   is nondecreasing in u', w'.

2. If $u \leq u'$ and $w \leq w'$ we have that $\mathbb{P}[U > u, -V > w | U > u', -V > w'] = 1$, and therefore it is nondecreasing in u', w'.

3. Let us consider, without loss of generality, $u \leq u'$ and $w > w'$. Then we have

$$\mathbb{P}[U > u, -V > w | U > u', -V > w'] = \frac{\mathbb{P}[U > u', -V > w]}{\mathbb{P}[U > u', -V > w']}.$$

   Since $\mathbb{P}[U > u', -V > w'] \geq \mathbb{P}[U > u', -V > w'']$ for all u', w', w'' such that $w' \leq w''$, we have that the probability above is nondecreasing in w'. So, in order to prove that this equation is nondecreasing in u', considering u'' such that $u' \leq u''$, we need to verify that

$$P[-V > w | U > u', -V > w'] \leq \mathbb{P}[-V > w | U > u'', -V > w'],$$

   which, taking $v = -w$ and $v' = -w'$, is equivalent to

$$P[V < v | U > u', V < v'] \leq \mathbb{P}[V < v | U > u'', V < v'].$$

   But, this is equivalent to

$$\frac{v - C(u', v)}{v' - C(u', v')} \leq \frac{v - C(u'', v)}{v' - C(u'', v')},$$

   for $u' \leq u''$ and $v \leq v'$, which is the statement's condition.

In all the cases we have proved that C is $\mathrm{I}(1, -1)$, and the proof is completed. □

Example 3.2. *Let $\{C_\delta\}_{\delta \in [-1,1]}$ be the Ali-Mikhail-Haq one-parameter family of 2-copulas given by*

$$C_\delta(u, v) = \frac{uv}{1 + \delta(1 - u)(1 - v)},$$

for all $(u, v) \in [0, 1]^2$ (see [1]). It is easy to prove that C_δ is $I(-1, 1)$ and $I(1, -1)$ for $\delta \in [0, 1]$, and C_δ is $I(1, 1)$ and $I(-1, -1)$ for $\delta \in [-1, 0]$.

3.2 The trivariate case

In this subsection we study the I(α) dependence concept for 3-copulas. Despite the difficulty of working with three random variables and the complexity of the expressions obtained, drawn results take us to similar conclusions to the bivariate case.

Theorem 3.3. *Let (U,V,W) be three random variables with associated 3-copula C and uniform marginals on $[0,1]$. Then C is:*

i. *$I(1,1,1)$ if, and only if, $\widehat{C}(u,v,w)\widehat{C}(u',v',w') \geq \widehat{C}(u,v',w')\widehat{C}(u',v,w)$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$.*

ii. *$I(1,1,-1)$ if, and only if, $[w - C_{23}(v,w) - C_{13}(u,w) + C(u,v,w)] \cdot [w' - C_{23}(v',w') - C_{13}(u',w') + C(u',v',w')] \leq [w - C_{23}(v',w) - C_{13}(u',w) + C(u',v',w)] \cdot [w' - C_{23}(v,w') - C_{13}(u,w') + C(u,v,w')]$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$;*

iii. *$I(1,-1,1)$ if, and only if, $[v - C_{23}(v,w) - C_{12}(u,v) + C(u,v,w)] \cdot [v' - C_{23}(v',w') - C_{12}(u',v') + C(u',v',w')] \leq [v - C_{23}(v,w') - C_{12}(u',v) + C(u',v,w')] \cdot [v' - C_{23}(v',w) - C_{12}(u,v') + C(u,v',w)]$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$;*

iv. *$I(-1,1,1)$ if, and only if, $[u - C_{13}(u,w) - C_{12}(u,v) + C(u,v,w)] \cdot [u' - C_{13}(u',w') - C_{12}(u',v') + C(u',v',w')] \leq [u - C_{13}(u,w') - C_{12}(u,v') + C(u,v',w')] \cdot [u' - C_{13}(u',w) - C_{12}(u',v) + C(u',v,w)]$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$;*

v. *$I(1,-1,-1)$ if, and only if, $[C_{23}(v,w) - C(u,v,w)] \cdot [C_{23}(v',w') - C(u',v',w')] \leq [C_{23}(v,w) - C(u',v,w)] \cdot [C_{23}(v',w') - C(u,v',w')]$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$;*

vi. *$I(-1,1,-1)$ if, and only if, $[C_{13}(u,w) - C(u,v,w)] \cdot [C_{13}(u',w') - C(u',v',w')] \leq [C_{13}(u,w) - C(u,v',w)] \cdot [C_{13}(u',w') - C(u',v,w')]$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$;*

vii. *$I(-1,-1,1)$ if, and only if, $[C_{12}(u,v) - C(u,v,w)] \cdot [C_{12}(u',v') - C(u',v',w')] \leq [C_{12}(u,v) - C(u,v,w')] \cdot [C_{12}(u',v') - C(u',v',w)]$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$;*

viii. *$I(-1,-1,-1)$ if, and only if, $C(u,v,w)C(u',v',w') \geq C(u,v',w')C(u',v,w)$ for all u,v,w,u',v',w' in $[0,1]$ such that $u \leq u'$, $v \leq v'$ and $w \leq w'$.*

Proof. We prove part ii. The rest of the parts can be proved in a similar way, so we omit their proofs.

Let (U,V,W) be three random variables with associated 3-copula C. Assume C is I$(1,1,-1)$, then we have that

$$\mathbb{P}[U > u, V > v, -W > t | U > u', V > v', -W > t'],$$

is nondecreasing in (u',v',t') for all (u,v,t), i.e.,

$$\mathbb{P}[U > u, V > v, -W > t | U > u', V > v', -W > t'] \leq \mathbb{P}[U > u, V > v, -W > t | U > u'', V > v'', -W > t''], \quad (2)$$

for all $u,v,u',v',u'',v'' \in [0,1]$ and $t,t',t'' \in [-1,0]$ such that $u' \leq u''$, $v' \leq v''$ and $t' \leq t''$. Condition (2) is equivalent to

$$\frac{\mathbb{P}[U > u, V > v, -W > t, U > u', V > v', -W > t']}{\mathbb{P}[U > u', V > v', -W > t']} \leq \frac{\mathbb{P}[U > u, V > v, -W > t, U > u'', V > v'', -W > t'']}{\mathbb{P}[U > u'', V > v'', -W > t'']},$$

i.e.,

$$\frac{\mathbb{P}[U > u, V > v, W < -t, U > u', V > v', W < -t']}{\mathbb{P}[U > u', V > v', W < -t']} \leq \frac{\mathbb{P}[U > u, V > v, W < -t, U > u'', V > v'', W < -t'']}{\mathbb{P}[U > u'', V > v'', W < -t'']},$$

hence,

$$\frac{\mathbb{P}[U > \max\{u,u'\}, V > \max\{v,v'\}, W < \min\{-t,-t'\}]}{\mathbb{P}[U > u', V > v', W < -t']}$$
$$\leq \frac{\mathbb{P}[U > \max\{u,u''\}, V > \max\{v,v''\}, W < \min\{-t,-t''\}]}{\mathbb{P}[U > u'', V > v'', W < -t'']},$$

that is,

$$\begin{aligned}
&\frac{\min\{-t,-t'\}-C_{23}(\max\{v,v'\},\min\{-t,-t'\})-C_{13}(\max\{u,u'\},\min\{-t,-t'\})}{-t'-C_{23}(v',-t')-C_{13}(u',-t')+C(u',v',-t')}\\
&+\frac{C(\max\{u,u'\},\max\{v,v'\},\min\{-t,-t'\})}{-t'-C_{23}(v',-t')-C_{13}(u',-t')+C(u',v',-t')}\\
&\le\frac{\min\{-t,-t''\}-C_{23}(\max\{v,v''\},\min\{-t,-t''\})-C_{13}(\max\{u,u''\},\min\{-t,-t''\})}{-t''-C_{23}(v'',-t'')-C_{13}(u'',-t'')+C(u'',v'',-t'')}\\
&+\frac{C(\max\{u,u''\},\max\{v,v''\},\min\{-t,-t''\})}{-t''-C_{23}(v'',-t'')-C_{13}(u'',-t'')+C(u'',v'',-t'')}.
\end{aligned}$$

Taking $w=-t$, $w'=-t'$ and $w''=-t''$ we obtain

$$\begin{aligned}
[\min\{w,w'\}\ &-\ C_{13}(\max\{u,u'\},\min\{w,w'\})-C_{23}(\max\{v,v'\},\min\{w,w'\})\\
&+\ C(\max\{u,u'\},\max\{v,v'\},\min\{w,w'\})]\cdot[w''-C_{13}(u'',w'')-C_{23}(v'',w'')+C(u'',v'',w'')]\\
&\le\ [\min\{w,w''\}-C_{13}(\max\{u,u''\},\min\{w,w''\})-C_{23}(\max\{v,v''\},\min\{w,w''\})\\
&+\ C(\max\{u,u''\},\max\{v,v''\},\min\{w,w''\})]\cdot[w'-C_{13}(u',w')-C_{23}(v',w')+C(u',v',w')],
\end{aligned}\tag{3}$$

for all $u,v,w,u',v',w',u'',v'',w''$ in $[0,1]$ such that $u'\le u''$, $v'\le v''$ and $w''\le w'$. By setting $u=u'\le u''$, $v=v'\le v''$ and $w'=w''$ in (3), for every $w\le w'$ in $[0,1]$ it follows (renaming $u'=u''$ and $v'=v''$)

$$\begin{aligned}
&[w-C_{23}(v,w)-C_{13}(u,w)+C(u,v,w)]\cdot[w'-C_{23}(v',w')-C_{13}(u',w')+C(u',v',w')]\\
&\le[w-C_{23}(v',w)-C_{13}(u',w)+C(u',v',w)]\cdot[w'-C_{23}(v,w')-C_{13}(u,w')+C(u,v,w')].
\end{aligned}\tag{4}$$

Conversely, suppose that (4) holds for all $u,v,u',v',w,w'\in[0,1]$ such that $u\le u'$, $v\le v'$ and $w\le w'$. Then, given $u,u',v,v'\in[0,1]$ and $t,t'\in[-1,0]$, we consider three possible cases:

1. If $u>u',v>v'$ and $t>t'$ we have that

$$\mathbb{P}[U>u,V>v,-W>t|U>u',V>v',-W>t']=\frac{\mathbb{P}[U>u,V>v,-W>t]}{\mathbb{P}[U>u',V>v',-W>t']},$$

   is nondecreasing in u',v',t'.

2. If $u\le u',v\le v'$ and $t\le t'$ we have that $\mathbb{P}[U>u,V>v,-W>t|U>u',V>v',-W>t']=1$, and therefore it is nondecreasing in u',v',t'.

3. Let us consider, without loss of generality, $u\le u',v\le v'$ and $t>t'$. Then we have

$$\begin{aligned}
\mathbb{P}[U>u,V>v,-W>t|U>u',V>v',-W>t']&=\frac{\mathbb{P}[U>u,V>v,-W>t,U>u',V>v',-W>t']}{\mathbb{P}[U>u',V>v',-W>t']}\\
&=\frac{\mathbb{P}[U>u',V>v',-W>t]}{\mathbb{P}[U>u',V>v',-W>t']}.
\end{aligned}$$

   Since $\mathbb{P}[U>u',V>v',-W>t']\ge\mathbb{P}[U>u',V>v',-W>t'']$ for all u',v',t',t'' such that $t\le t''$, we have that the expression above is nondecreasing in t'. In order to prove that this expression is nondecreasing in u' and v', considering u'' and v'' such that $u'\le u''$ and $v'\le v''$, we need to verify that

$$P[-W>t|U>u',V>v',-W>t']\le\mathbb{P}[-W>t|U>u'',V>v'',-W>t'],$$

   which is equivalent to

$$P[W<w|U>u',V>v',W<w']\le\mathbb{P}[W<w|U>u'',V>v'',W<w'],$$

   taking $w=-t$ and $w'=-t'$. But, this is equivalent to

$$\frac{w-C_{23}(v',w)-C_{13}(u',w)+C(u',v',w)}{w'-C_{23}(v',w')-C_{13}(u',w')+C(u',v',w')}\le\frac{w-C_{23}(v'',w)-C_{13}(u'',w)+C(u'',v'',w)}{w'-C_{23}(v'',w')-C_{13}(u'',w')+C(u'',v'',w')},$$

   for $u'\le u'',v'\le v''$ and $w\le w'$, which is the statement's condition.

Hence, we obtain that C is $I(1,1,-1)$ in all the cases, and the proof is complete. □

3.3 The multivariate case

Now, we will explore the n-dimensional case. To achieve our goals, we need the following preliminary lemma, which will allow us to relate the probability of the event $\bigcap_{i=1}^{n}(\alpha_i X_i > x_i)$ to the associated n-copula of the random vector $(X_1, X_2, \dots, X_n)$ and its margins. In the sequel, $|A|$ will denote the cardinality of a set A.

Lemma 3.4. *Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be an n-dimensional random vector and let $\alpha \in \mathbb{R}^n$ such that $|\alpha_i| = 1$ for all $i = 1, 2, ..., n$. Let $I \subseteq \{1, ..., n\}$, such that $\alpha_k = -1$, if $k \in I$, and let $J = \{1, ..., n\} \setminus I$ $(I, J \neq \emptyset)$. For all $\mathbf{x} \in \overline{\mathbb{R}}^n = [-\infty, \infty]^n$, we have*

$$\mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i X_i > x_i)\right] = \mathbb{P}\left[\bigcap_{i\in I}(X_i < \overline{x}_i)\right] - \sum_{j\in J}\mathbb{P}\left[\bigcap_{i\in I\cup\{j\}}(X_i < \overline{x}_i)\right] + \sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j_1,j_2\}}(X_i < \overline{x}_i)\right] - \cdots$$

$$+ (-1)^{|J|-1}\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}\cdots\sum_{\substack{j_{|J|-1}\in J\\ j_{|J|-1}>j_{|J|-2}}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j_1,\dots,j_{|J|-1}\}}(X_i < \overline{x}_i)\right] + (-1)^{|J|}\mathbb{P}\left[\bigcap_{i=1}^{n}(X_i < \overline{x}_i)\right],$$

where

$$\overline{x}_i = \begin{cases} x_i, & \text{if } i \in J, \\ -x_i, & \text{if } i \in I. \end{cases}$$

Proof. We will apply the induction method over the number of components of α that are equal to 1; i.e., over the cardinal of J. We will consider the following change of variable throughout this proof: if $-X_i > x_i$, then $X_i < -x_i$ so we consider $\overline{x}_i = -x_i$ if $i \in I$ and $\overline{x}_i = x_i$ if $i \in J$. Let us start proving the case $|J| = 1$: If we suppose that $\alpha_l = 1$ and the remaining coordinates are all equal to -1, then

$$\mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i X_i > x_i)\right] = \mathbb{P}\left[x_l < X_l, \bigcap_{\substack{i=1\\ i\neq l}}^{n}(X_i < \overline{x}_i)\right]$$
$$= \mathbb{P}\left[\bigcap_{\substack{i=1\\ i\neq l}}^{n}(X_i < \overline{x}_i)\right] - \mathbb{P}\left[\bigcap_{i=1}^{n}(X_i < \overline{x}_i)\right].$$

Now we consider the statement true for $|J| = m$ and our goal is claiming the same for $|J| = m+1$. So, if $l_{m+1} \in J$ we have

$$\mathbb{P}\left[\bigcap_{i=1}^{n}(\alpha_i X_i > x_i)\right] = \mathbb{P}\left[\bigcap_{i\in J}(X_i > x_i), \bigcap_{i\in I}(X_i < \overline{x}_i)\right]$$
$$= \mathbb{P}\left[x_{l_{m+1}} < X_{l_{m+1}}, \bigcap_{\substack{i\in J\\ i\neq l_{m+1}}}(X_i > x_i), \bigcap_{i\in I}(X_i < \overline{x}_i)\right]$$
$$= \mathbb{P}\left[\bigcap_{\substack{i\in J\\ i\neq l_{m+1}}}(X_i > x_i), \bigcap_{i\in I}(X_i < \overline{x}_i)\right] - \mathbb{P}\left[\bigcap_{\substack{i\in J\\ i\neq l_{m+1}}}(X_i > x_i), \bigcap_{i\in I\cup\{l_{m+1}\}}(X_i < \overline{x}_i)\right].$$

Applying the induction hypothesis we obtain that the first expression is equal to

$$\mathbb{P}\left[\bigcap_{\substack{i\in J\\ i\neq l_{m+1}}}(X_i > x_i), \bigcap_{i\in I}(X_i < \overline{x}_i)\right] = \mathbb{P}\left[\bigcap_{i\in I}(X_i < \overline{x}_i)\right] - \sum_{\substack{j\in J\\ j\neq l_{m+1}}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j\}}(X_i < \overline{x}_i)\right]$$
$$+ \sum_{\substack{j_1\in J\\ j_1\neq l_{m+1}}}\sum_{\substack{j_2\in J\\ j_2\neq l_{m+1}\\ j_2>j_1}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j_1,j_2\}}(X_i < \overline{x}_i)\right] - \cdots + (-1)^m\mathbb{P}\left[\bigcap_{\substack{i=1\\ i\neq l_{m+1}}}^{n}(X_i < \overline{x}_i)\right].$$

Whereas the second one is equal to the following expression

$$\mathbb{P}\left[\bigcap_{\substack{i\in J\\ i\neq l_{m+1}}}(X_i>x_i),\bigcap_{i\in I\cup\{l_{m+1}\}}(X_i<\overline{x}_i)\right]=\mathbb{P}\left[\bigcap_{i\in I\cup\{l_{m+1}\}}(X_i<\overline{x}_i)\right]-\sum_{\substack{j\in J\\ j\neq l_{m+1}}}\mathbb{P}\left[\bigcap_{i\in I\cup\{l_{m+1},j\}}(X_i<\overline{x}_i)\right]$$
$$+\sum_{\substack{j_1\in J\\ j_1\neq l_{m+1}}}\sum_{\substack{j_2\in J\\ j_2\neq l_{m+1}\\ j_2>j_1}}\mathbb{P}\left[\bigcap_{i\in I\cup\{l_{m+1},j_1,j_2\}}(X_i<\overline{x}_i)\right]-\cdots+(-1)^m\mathbb{P}\left[\bigcap_{i=1}^{n}(X_i<\overline{x}_i)\right].$$

Now, we subtract both expressions. Note that we can add the first term of the second expression to the second term of the first one; ie.,

$$-\sum_{\substack{j\in J\\ j\neq l_{m+1}}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j\}}(X_i<\overline{x}_i)\right]-\mathbb{P}\left[\bigcap_{i\in I\cup\{l_{m+1}\}}(X_i<\overline{x}_i)\right]=-\sum_{j\in J}\mathbb{P}\left[\bigcap_{i\in I\cup\{j\}}(X_i<\overline{x}_i)\right].$$

It follows similarly with the third term of the first expression and the second term of the second one

$$\sum_{\substack{j_1\in J\\ j_1\neq l_{m+1}}}\sum_{\substack{j_2\in J\\ j_2\neq l_{m+1}\\ j_2>j_1}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j_1,j_2\}}(X_i<\overline{x}_i)\right]+\sum_{\substack{j\in J\\ j\neq l_{m+1}}}\mathbb{P}\left[\bigcap_{i\in I\cup\{l_{m+1},j\}}(X_i<\overline{x}_i)\right]$$
$$=\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}\mathbb{P}\left[\bigcap_{i\in I\cup\{j_1,j_2\}}(X_i<\overline{x}_i)\right],$$

and so on with the remaining terms. Finally, our first and last terms are, respectively,

$$\mathbb{P}\left[\bigcap_{i\in I}(X_i<\overline{x}_i)\right]\qquad\text{and}\qquad(-1)^{m+1}\mathbb{P}\left[\bigcap_{i=1}^{n}(X_i<\overline{x}_i)\right].$$

Thus, grouping all these terms together we reach the desired result. ☐

Remark 3.5. *Note that if the n-dimensional random vector $\mathbf{X}$ has associated n-copula C, then the expression of Lemma 3.4 can be written in terms of the copula and its margins via Sklar's theorem. Thus, let $F_1, F_2, ..., F_n$ be the marginal distribution functions of $\mathbf{X}$, and $\alpha\in\mathbb{R}^n$ such that $|\alpha_i|=1$ for all $i=1,2,...,n$. Let $I\subseteq\{1,2,...,n\}$, such that $\alpha_i=-1$ if $i\in I$, and $\alpha_i=1$ if $i\in J=\{1,2,...,n\}\setminus I$. For every $\mathbf{x}\in\mathbb{R}^n$, denote $\overline{x}_i=x_i$ if $i\in J$, and $\overline{x}_i=-x_i$ if $i\in I$, and let $\mathbf{u}=(F_1(\overline{x}_1),F_2(\overline{x}_2),...,F_n(\overline{x}_n))\in[0,1]^n$. Given any subset $K=\{i_1,i_2,...,i_p\}\subseteq\{1,2,...,n\}$, with $i_1<i_2<\cdots<i_p$, denote by C_K the p-marginal of C for the random vector $(X_{i_1},X_{i_2},...,X_{i_p})$, as defined before. Obviously, $C_{\{1,2,...,n\}}=C$. Similarly, denote $\mathbf{u}_K=(u_{i_1},u_{i_2},...,u_{i_p})$. Then, the right-hand side of the expression in Lemma 3.4 can be written as follows*

$$C_I(\mathbf{u}_I)-\sum_{j\in J}C_{I\cup\{j\}}(\mathbf{u}_{I\cup\{j\}})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C_{I\cup\{j_1,j_2\}}(\mathbf{u}_{I\cup\{j_1,j_2\}})-\cdots$$
$$+(-1)^{|J|-1}\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}\cdots\sum_{\substack{j_{|J|-1}\in J\\ j_{|J|-1}>j_{|J|-2}}}C_{I\cup\{j_1,...,j_{|J|-1}\}}(\mathbf{u}_{I\cup\{j_1,...,j_{|J|-1}\}})$$
$$+(-1)^{|J|}C(\mathbf{u}).$$

Moreover, given any subset $K=\{i_1,i_2,...,i_p\}\subseteq\{1,2,...,n\}$, with $i_1<i_2<\cdots<i_p$, we denote by $(\mathbf{u}_K,\mathbf{1})$ the point in $[0,1]^n$, such that $u_i=F_i(\overline{x}_i)$ if $i\in K$ and $u_i=1$ if $i\in\{1,2,...,n\}\setminus K$. Then, the above expression can be written as follows

$$C(\mathbf{u}_I,\mathbf{1})-\sum_{j\in J}C(\mathbf{u}_{I\cup\{j\}},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\mathbf{u}_{I\cup\{j_1,j_2\}},\mathbf{1})-\cdots$$
$$+(-1)^{|J|-1}\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}\cdots\sum_{\substack{j_{|J|-1}\in J\\ j_{|J|-1}>j_{|J|-2}}}C(\mathbf{u}_{I\cup\{j_1,\dots,j_{|J|-1}\}},\mathbf{1})$$
$$+(-1)^{|J|}C(\mathbf{u}).$$

Now, on the basis of Remark 3.5, we are able to state and prove the following characterization of the multivariate I(α) dependence concept.

Theorem 3.6. *Let $\mathbf{U}=(U_1,U_2,\dots,U_n)$ be an n-dimensional random vector with associated n-copula C, and uniform marginals on $[0,1]$. Let $\alpha\in\mathbb{R}^n$ such that $|\alpha_i|=1$ for all $i=1,2,...,n$. Let $I\subseteq\{1,2,...,n\}$, such that $\alpha_i=-1$ if $i\in I$, and $\alpha_i=1$ if $i\in J=\{1,2,...,n\}\setminus I$, with $I,J\neq\emptyset$. Then C is $I(\alpha)$ if, and only if,*

$$\frac{C(\mathbf{u}^{(1)}_I,\mathbf{1})-\sum_{j\in J}C(\mathbf{u}^{(1)}_{I\cup\{j\}},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\mathbf{u}^{(1)}_{I\cup\{j_1,j_2\}},\mathbf{1})-\cdots+(-1)^{|J|}C(\mathbf{u}^{(1)})}{C(\overline{\mathbf{u}}'_I,\mathbf{1})-\sum_{j\in J}C(\overline{\mathbf{u}}'_{I\cup\{j\}},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\overline{\mathbf{u}}'_{I\cup\{j_1,j_2\}},\mathbf{1})-\cdots+(-1)^{|J|}C(\overline{\mathbf{u}}')}$$
$$\le\frac{C(\mathbf{u}^{(2)}_I,\mathbf{1})-\sum_{j\in J}C(\mathbf{u}^{(2)}_{I\cup\{j\}},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\mathbf{u}^{(2)}_{I\cup\{j_1,j_2\}},\mathbf{1})-\cdots+(-1)^{|J|}C(\mathbf{u}^{(2)})}{C(\overline{\mathbf{u}}''_I,\mathbf{1})-\sum_{j\in J}C(\overline{\mathbf{u}}''_{I\cup\{j\}},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\overline{\mathbf{u}}''_{I\cup\{j_1,j_2\}},\mathbf{1})-\cdots+(-1)^{|J|}C(\overline{\mathbf{u}}'')},\tag{5}$$

for all $\mathbf{u},\mathbf{u}',\mathbf{u}''$, such that $u'_i\le u''_i$ for all $i=1,2,...,n$, where the components of $\mathbf{u}^{(1)},\mathbf{u}^{(2)},\overline{\mathbf{u}}'$ and $\overline{\mathbf{u}}''$ are given by

$$u^{(1)}_i=\begin{cases}\max\{u_i,u'_i\}, & \text{if } i\in J\\ -\max\{u_i,u'_i\}, & \text{if } i\in I\end{cases},\qquad u^{(2)}_i=\begin{cases}\max\{u_i,u''_i\}, & \text{if } i\in J\\ -\max\{u_i,u''_i\}, & \text{if } i\in I\end{cases},$$
$$\overline{u}'_i=\begin{cases}u'_i, & \text{if } i\in J\\ -u'_i, & \text{if } i\in I\end{cases}\quad\text{and}\quad\overline{u}''_i=\begin{cases}u''_i, & \text{if } i\in J\\ -u''_i, & \text{if } i\in I\end{cases}.$$

Proof. Let $\mathbf{U}=(U_1,U_2,\dots,U_n)$ be an n-dimensional random vector with associated n-copula C. Since C is I(α), we have that

$$\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(-U_i>u_i)|\bigcap_{i\in J}(U_i>u'_i),\bigcap_{i\in I}(-U_i>u'_i)\right],$$

is nondecreasing in $\mathbf{u}'$ for all $\mathbf{u}$, i.e.,

$$\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(-U_i>u_i)|\bigcap_{i\in J}(U_i>u'_i),\bigcap_{i\in I}(-U_i>u'_i)\right]$$
$$\le\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(-U_i>u_i)|\bigcap_{i\in J}(U_i>u''_i),\bigcap_{i\in I}(-U_i>u''_i)\right],$$

for all $u_i,u'_i,u''_i\in[0,1]$ such that $i\in J$, and $u_i,u'_i,u''_i\in[-1,0]$ such that $i\in I$, verifying that $u'_i\le u''_i$ for all $1\le i\le n$. The inequality above is equivalent to the following expression

$$\frac{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(-U_i>u_i),\bigcap_{i\in J}(U_i>u'_i),\bigcap_{i\in I}(-U_i>u'_i)\right]}{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u'_i),\bigcap_{i\in I}(-U_i>u'_i)\right]}$$
$$\le\frac{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(-U_i>u_i),\bigcap_{i\in J}(U_i>u''_i),\bigcap_{i\in I}(-U_i>u''_i)\right]}{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u''_i),\bigcap_{i\in I}(-U_i>u''_i)\right]},$$

i.e.,

$$\frac{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(U_i<-u_i),\bigcap_{i\in J}(U_i>u'_i),\bigcap_{i\in I}(U_i<-u'_i)\right]}{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u'_i),\bigcap_{i\in I}(U_i<-u'_i)\right]}$$
$$\le\frac{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i),\bigcap_{i\in I}(U_i<-u_i),\bigcap_{i\in J}(U_i>u''_i),\bigcap_{i\in I}(U_i<-u''_i)\right]}{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u''_i),\bigcap_{i\in I}(U_i<-u''_i)\right]},$$

and therefore,

$$\begin{aligned}&\frac{\mathbb{P}\left[\bigcap_{i\in J}(U_i>\max\{u_i,u_i'\}),\bigcap_{i\in I}(U_i<\min\{-u_i,-u_i'\}),\right]}{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i'),\bigcap_{i\in I}(U_i<-u_i')\right]}\\ \leq\ &\frac{\mathbb{P}\left[\bigcap_{i\in J}(U_i>\max\{u_i,u_i''\}),\bigcap_{i\in I}(U_i<\min\{-u_i,-u_i''\})\right]}{\mathbb{P}\left[\bigcap_{i\in J}(U_i>u_i''),\bigcap_{i\in I}(U_i<-u_i'')\right]}.\end{aligned}$$

Using Lemma 3.4 and Remark 3.5, it is possible to express this inequality in terms of the copula C as it follows

$$\begin{aligned}&\frac{C(\mathbf{u}_I^{(1)},\mathbf{1})-\sum_{j\in J}C(\mathbf{u}_{I\cup\{j\}}^{(1)},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\mathbf{u}_{I\cup\{j_1,j_2\}}^{(1)},\mathbf{1})-\cdots+(-1)^{|J|}C(\mathbf{u}^{(1)})}{C(\overline{\mathbf{u}}_I',\mathbf{1})-\sum_{j\in J}C(\overline{\mathbf{u}}_{I\cup\{j\}}',\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\overline{\mathbf{u}}_{I\cup\{j_1,j_2\}}',\mathbf{1})-\cdots+(-1)^{|J|}C(\overline{\mathbf{u}}')}\\ \leq&\frac{C(\mathbf{u}_I^{(2)},\mathbf{1})-\sum_{j\in J}C(\mathbf{u}_{I\cup\{j\}}^{(2)},\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\mathbf{u}_{I\cup\{j_1,j_2\}}^{(2)},\mathbf{1})-\cdots+(-1)^{|J|}C(\mathbf{u}^{(2)})}{C(\overline{\mathbf{u}}_I'',\mathbf{1})-\sum_{j\in J}C(\overline{\mathbf{u}}_{I\cup\{j\}}'',\mathbf{1})+\sum_{j_1\in J}\sum_{\substack{j_2\in J\\ j_2>j_1}}C(\overline{\mathbf{u}}_{I\cup\{j_1,j_2\}}'',\mathbf{1})-\cdots+(-1)^{|J|}C(\overline{\mathbf{u}}'')},\end{aligned}$$

where the components of $\mathbf{u}^{(1)},\mathbf{u}^{(2)},\overline{\mathbf{u}}'$ and $\overline{\mathbf{u}}''$ are given by

$$u_i^{(1)}=\begin{cases}\max\{u_i,u_i'\}, & \text{if } i\in J\\ -\max\{u_i,u_i'\}, & \text{if } i\in I\end{cases},\qquad u_i^{(2)}=\begin{cases}\max\{u_i,u_i''\}, & \text{if } i\in J\\ -\max\{u_i,u_i''\}, & \text{if } i\in I\end{cases},$$

$$\overline{u}_i'=\begin{cases}u_i', & \text{if } i\in J\\ -u_i', & \text{if } i\in I\end{cases}\quad\text{and}\quad \overline{u}_i''=\begin{cases}u_i'', & \text{if } i\in J\\ -u_i'', & \text{if } i\in I\end{cases}.$$

Thus, we obtain inequality (5).

Conversely, it is obvious that from inequality (5), and following back the equivalences used before, we get that C is I(α).

□

Next, we will provide several examples to illustrate the I(α) concept within the context of the multivariate case.

Example 3.7. *For all $n\geq 2$, the n-copula M^n is I($\mathbf{1}$) and I($-\mathbf{1}$).*

Example 3.8. *For all $n\geq 2$, the n-copula Π^n is I(α) for any direction α.*

Example 3.9. *Let C be the n-copula given by the convex linear combination of the n-copulas Π^n and M^n, i.e.,*

$$C(\mathbf{u})=\theta\Pi^n(\mathbf{u})+(1-\theta)M^n(\mathbf{u}),$$

for all $\mathbf{u}\in[0,1]^n$, and $\theta\in[0,1]$. Then, C is I($\mathbf{1}$) and I($-\mathbf{1}$) for all $\theta\in[0,1]$.

Example 3.10. *Let $\{C_\lambda\}_{\lambda\in[-1,1]}$ be a generalization of the Farlie-Gumbel-Morgenstern one-parameter family of 2-copulas, given by*

$$C_\lambda(\mathbf{u})=\prod_{i=1}^n u_i\left[1+\lambda\prod_{i=1}^n(1-u_i)\right],$$

for all $\mathbf{u}\in[0,1]^n$ (see [13]). Let $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)\in\mathbb{R}^n$ such that $|\alpha_i|=1$ for all $i=1,2,...,n$, and $J=\{i\in\{1,2,...,n\}:\alpha_i=1\}$. It can be proved, after some tedious computations, that C_λ is I(α) for $\lambda\in[0,1]$ if $|J|$ is even, and C_λ is I(α) for $\lambda\in[-1,0]$ if $|J|$ is odd.

4 Conclusions

In this paper we have characterized the monotonicity according to a direction of a random vector, a new concept of positive dependence, —particularly the I(α) notion— by using copulas. We started with the bivariate and trivariate cases, due to their simplicity and straightforwardness, aiming to illustrate the procedure to be followed in the multivariate case. Initially, we obtained in both cases a characterization in terms of an inequality involving the associated copula and its margins evaluated at maximum and minimum values, in order to subsequently simplify this characterization to be applied to specific cases. Throughout this paper, examples of copulas with I(α) dependence have been provided for different vectors α.

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