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Generalized Schur-constant bivariate distribution and its associated Archimedean copula

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Abstract

This paper explores an extension of the bivariate Schur-constant model, introducing an additional parameter to its associated Archimedean copula for greater flexibility. We analyze the dependence properties of the proposed model and illustrate our findings with several examples. Furthermore, provide a likelihood ratio test to compare the performance of the extended Archimedean copula with that of the traditional Archimedean subfamily. Two real-data analysis are also included.

Keywords: Archimedean copula, Schur-constant model, stochastic dependence.

1 Introduction

A pair (X, Y) of continuous, non-negative random variables is Schur-constant if its joint survival function is of the particular form

$$P(X > x, Y > y) = S(x + y), \quad x, y > 0, \tag{1}$$

for some appropriate function $S : R_+ \to [0, 1]$, called generator. Obviously, the marginal distributions of X and Y are specified by the survival functions $S(x) = P(X > x)$ and $S(y) = P(Y > y)$, respectively. Thus, $S(.)$ is continuous and non-increasing, with $S(0) = 1$ and $S(\infty) = 0$. Schur-constant models represent a no-aging property among a finite set of exchangeable lifetimes, as discussed by Barlow and Mendel [3] and Caramellino and Spizzichino [6]. In reliability, survival analysis, and actuarial sciences, the focus is typically on absolutely continuous random variables valued in R_+. Chi et al. [9] and Castaner et al. [8] have explored applications of Schur-constant models in insurance mathematics. Nelsen [33] demonstrated that survival copulas generated from bivariate Schur-constant models possess several useful properties. Nair and Sankaran [31, 32] characterized bivariate Schur-constant distributions through properties of random variable functions and reliability concepts. Ta and Van [36, 37] examined the Laplace transform of bivariate Schur-constant distributions, showing that the resulting function is a copula. Lefèvre and Simon [27] highlighted connections with L_1-symmetric Dirichlet vectors and Archimedean copulas, while Lefèvre and Simon [25] linked continuous partially Schur-constant vectors to partially Archimedean copulas, following Ressel's work [35] on multivariate function monotonicity. Unnikrishnan Nair et al. [38] discussed properties of Laplace transforms of bivariate Schur-constant distributions from renewal theory, emphasizing their application in copula construction. Recent investigations into Schur-constancy for discrete lifetimes can be found in [7, 23, 26].

The primary characteristic of the bivariate Schur-constant model is its Archimedean survival copula, which confers numerous advantageous properties typical of this copula class [33]. Among copulas, strict Archimedean copulas are particularly notable [18, 34]. Various generalizations of Archimedean copulas have been explored. Dolati and Karbasian [12] and Durante et al. [14] studied bivariate copulas defined by two univariate functions that generalize the Archimedean family. Xie et al. [40] provided a multidimensional extension of this model. Xie et al. [41] introduced a family of

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copulas that generalizes both Archimedean and Marshall-Olkin copulas, including well-known copulas as special cases. Ressel [35] developed a new class of generalized Archimedean copulas using a Williamson-type theorem, overcoming the exchangeability limitation of standard Archimedean copulas, which require no additional compatibility conditions for their generators. Doodman et al. [13] proposed an extension of Archimedean copulas with concave multiplicative generators in the style of FGM family of copulas, enabling the modeling of higher positive dependence. Hua and Joe [21] used the comonotonic latent variables to extend exchangeable Archimedean copulas. Morillas [30] provided a method to derive new copulas from a given strictly increasing continuous function, generalizing the construction of Archimedean copulas from a product copula. Valdez and Xiao [39] extended Archimedean copulas using distortion functions.

In this paper, we propose an extension of the Schur-constant model. The associated Archimedean copula of the proposed model has an additional parameter compared to the original Schur-constant model, enhancing its flexibility. The paper is organized as follows: Section 2 introduces the proposed generalized Schur-constant model; Section 3 derives dependence properties and Section 4 discusses the convexity properties of the associated copula of the proposed model; Sections 5 and 6 present a simulation algorithm for the associated copula and propose a likelihood ratio test (LRT) for certain subfamilies of the model. Section 7 illustrates the results using two real datasets, and Section 8 concludes the paper.

2 A generalized Schur-constant model

Let $S : [0, \infty] \to [0, 1]$ be a continuous survival function (i.e, S is continuous and strictly decreasing, with $S(0) = 1$ and $S(\infty) = 0$). With the idea of increasing the flexibility of Schur-constant model, consider the extension $R : [0, \infty]^2 \to [0, 1]$, defined by

$$R(x, y) = S(x + y + \alpha xy), \quad x, y > 0, \tag{2}$$

where α is a suitable constant. The case $\alpha \to 0$, the bivariate survival function defined by (2) leads to the survival function of a bivariate Schur-constant distribution. Thus, (2) could be considered as an extension of Schur-constant model. The following result provides for which conditions the structure defined by (2) is a bivariate survival function.

Proposition 2.1. *Let $S : [0, \infty] \to [0, 1]$ be a continuous and twice differentiable survival function. If $S(.)$ is convex, then for $\alpha \in (0, r]$, where $r = \inf_{t\geq 0} \left|\frac{S''(t)}{S'(t)}\right|$, the function $R(x, y) = S(x+y+\alpha xy)$ is a valid absolutely continuous bivariate survival function.*

Proof. Its enough to show that $\frac{\partial}{\partial x \partial y} R(x, y) \geq 0$. It follows that

$$\frac{\partial}{\partial x \partial y} R(x, y) = \alpha S'(x + y + \alpha xy) + (1 + \alpha x)(1 + \alpha y) S''(x + y + \alpha xy).$$

Put $x + y + \alpha xy = t$. Then, $\frac{\partial}{\partial x \partial y} R(x, y) := \alpha S'(t) + (1 + \alpha t) S''(t)$. Since $S(.)$ is convex, for $\alpha > 0$, we have that $\alpha S'(t)+(1+\alpha t)S''(t) \geq \alpha S'(t)+S''(t)$. Since, $S(.)$ is non-increasing, then, for $\alpha \leq \frac{S''(t)}{-S'(t)}$ we have that $\alpha S'(t)+S''(t) \geq 0$, which gives the required result. □

We now provide several examples.

Example 2.2. *Let $S(t) = e^{-t}$, $t \geq 0$ be the survival function of the ordinary exponential random variable. Since $\inf_{t\geq 0} \left|\frac{S''(t)}{S'(t)}\right| = 1$, then for $\alpha \in (0, 1]$*

$$R(x, y) = e^{-(x+y+\alpha xy)}, \quad x, y \geq 0, \tag{3}$$

is a survival function. Note that it is the survival function of the bivariate Gumbel-Barnett distribution [4, 20] *and has received applications in many areas* [1, 10, 15, 24].

We note that a large class of convex survival functions could be defined by

$$S(t) = E(e^{-tZ}), \quad t > 0, \tag{4}$$

which is the Laplace-Stieltjes transform of a non-negative random variable Z. In the following we provide several examples.

Example 2.3. *Let Z be a geometric random variable with the probability density function $P(Z = k) = (1-\theta)\theta^{k-1}$, $k \in \{1, 2, 3, ...\}$, $\theta \in (0, 1)$. Then (4) gives the survival function $S(t) = \frac{(1-\theta)e^{-t}}{1-\theta e^{-t}}$, $t \geq 0$. From Proposition 2.1 the extended bivariate survival function (2), with $\alpha \in (0, 1]$ is given by*

$$R(x, y) = \frac{(1-\theta)e^{-(x+y+\alpha xy)}}{1-\theta e^{-(x+y+\alpha xy)}}, \quad x, y \geq 0,$$

which is the bivariate Gumbel-Barnett survival function (3) when $\theta \to 0$.

Example 2.4. *Let Z be a two-valued random variable with the probability density function $P(Z = k) = \theta^{k-1}(1-\theta)^{2-k}$, $k = 1, 2$, $\theta \in (0, 1]$. Then (4) gives the survival function $S(t) = e^{-t}[1 - \theta(1 - e^{-t})]$. From Proposition 2.1 the extended bivariate survival function (2), with $\alpha \in (0, 1]$ is given by*

$$R(x, y) = e^{-(x+y+\alpha xy)}[1 - \theta(1 - e^{-(x+y+\alpha xy)})], \quad x, y > 0,$$

which is the bivariate Gumbel-Barnett survival function (3) when $\theta \to 0$.

Example 2.5. *Let Z be a zero-truncated Poison random variable with the probability density function $P(Z = k) = \frac{\theta^{k-1}e^{-\theta}}{(k-1)!}$, $k = 1, 2, 3, ...$, $\theta > 0$. Then (4) gives the survival function $S(t) = \exp\{-t - \theta(1 - e^{-t})\}$. From Proposition 2.1 the extended bivariate survival function (2), with $\alpha \in (0, 1]$ is given by*

$$R(x, y) = \exp\{-x - y - \alpha xy - \theta(1 - \exp[-x - y - \alpha xy])\}, \quad x, y > 0,$$

which is the survival function of the extended bivariate Gompertz-Makeham distribution studied in [16].

3 Generated copula

In this section we discuss the copula of the extended Schur-constant model. We show that the corresponding copula is Archimedean. For more convenience, let us recall here the definitions of Archimedean copula. Before presenting the main results of this article, it is worth recalling the precise definition of a bivariate copula. It is important to note that, here, the notion of copula will be understood in the absolutely continuous bivariate case. In the absolutely continuous bivariate case, a copula is a differentiable function $C : [0, 1]^2 \to [0, 1$, such that the following conditions are fulfilled [34]:

(C1) Boundary condition: For any $t \in [0, 1]$, we have $C(t, 0) = C(0, t) = 0$ and $C(t, 1) = t, C(1, t) = t$.
(C2) Positive derivative condition: For any $(u, v) \in [0, 1]^2$, we have $\frac{\partial}{\partial u \partial v} C(u, v) \geq 0$.

The classical definition of a strict generator is recalled below (see [18, 34]).

Definition 3.1. *A function $\phi : [0, 1] \to [0, \infty]$, is said to be a strict generator, if and only if it satisfies the following conditions:*
(1) $\phi(1) = 0$,
(2) $\lim_{u \to 0^+}(u) = +\infty$,
(3) $\phi'(u) < 0$ *for any* $u \in [0, 1)$ *and* $\phi'(1) < 0$,
(4) $\phi''(u) \geq 0$, *for any* $u \in [0, 1]$.

Based on a specific strict generator, a copula is Archimedean, as described in the next definition.

Definition 3.2. *The strict Archimedean copula associated to a strict generator function $\phi(.)$ is defined by*

$$C_\phi(u, v) = \phi^{-1}\{\phi(u) + \phi(v)\},$$

for $(u, v) \in [0, 1]^2$.

Archimedean copulas have received increasing attention for their flexibility to model various dependence structures of random variables. Most of the Archimedean copulas have few parameters, which make the copulas less flexible. Thus it is interesting to consider some generalization of this family of copulas. Several generalizations of the Archimedean copulas were studied in [12, 14, 40, 41]. In the following, we discuss the copula structure of the proposed model. Let (X, Y) be a vector of continuous random variables with the survival function defined by (2). From Proposition 2.1, since for $r = \inf_{t \geq 0} \left| \frac{S''(t)}{S'(t)} \right|$, the function $R(x, y) = S(x + y + \alpha xy)$ is a valid absolutely continuous bivariate survival function,

with the same univariate marginal survival function $S(.)$, then in view of Sklar's Theorem [34] there exists a copula C such that

$$R(x,y) = C(S(x), S(y)),$$

so that

$$S(x+y+\alpha xy) = C(S(x), S(y)), \quad x,y \geq 0. \tag{5}$$

Since $S(t) > 0$ for all $t \geq 0$ and $S(.)$ is decreasing on $[0,\infty)$, then the ordinary inverse of S exists, i.e., for $u \in [0,1]$ and $t \in [0,\infty)$, $S(t) = u$ is equivalent to $t = S^{-1}(u)$. Putting $x = S^{-1}(u)$ and $y = S^{-1}(v)$ in (5), we get

$$S\{S^{-1}(u) + S^{-1}(v) + \alpha S^{-1}(u)S^{-1}(v)\} = C(u,v), \quad (u,v) \in [0,1]^2.$$

Define the function $\phi : [0,1] \to [0,\infty]$, with $\phi(t) = S^{-1}(t)$, which is decreasing and convex with $\phi(0) = \infty$ and $\phi(1) = 0$. The generated copula has the form

$$C_\phi^\alpha(u,v) = \phi^{-1}\{\phi(u) + \phi(v) + \alpha\phi(u)\phi(v)\}, \quad (u,v) \in [0,1]^2. \tag{6}$$

The copula (6) can be rewritten as

$$C_\phi^\alpha(u,v) = \phi^{-1}\left(\frac{(1+\alpha\phi(u))(1+\alpha\phi(v)) - 1}{\alpha}\right). \tag{7}$$

In the following we show that (6) is an Archimedean copula. The following result shows that (6) is also an Archimedean copula.

Proposition 3.3. *Let ϕ be generator of an Archimedean copula. Then, for $\alpha \in (0, r]$, where $r = \inf_{t\geq 0} \frac{\phi''(t)}{(\phi'(t))^2}$, the function C_ϕ^α, defined by (6) is an Archimedean copula with the generator $\psi(t) = \ln(1+\alpha\phi(t))$.*

Proof. We note that $\psi^{-1}\{\psi(u)+\psi(v)\} = \phi^{-1}\{\phi(u)+\phi(v)+\alpha\phi(u)\phi(v)\}$. Thus, to show (6) is an Archimedean copula, we need to check that $\psi(.)$, is a generator: (1) $\psi(1) = 0$, (2) $\psi(0) = \infty$, (3) since $\frac{d}{dt}\psi(t) = \frac{\alpha\phi'(t)}{1+\alpha\phi(t)} \leq 0$, then $\psi(.)$ is decreasing, to check the convexity condition (4) in Definition 3.1, for $\psi(.)$, note that the second derivative of $\psi(.)$ is given by

$$\frac{d^2}{dt^2}\psi(t) = \frac{\phi''(t)(1+\alpha\phi(t)) - \alpha(\phi'(t))^2}{(1+\alpha\phi(t))^2}.$$

Since $\phi(.)$ is convex, then $\phi''(t)(1+\alpha\phi(t)) - \alpha(\phi'(t))^2 \geq \phi''(t) - \alpha(\phi'(t))^2$. Thus, if $\alpha \leq \frac{\phi''(t)}{(\phi'(t))^2}$, for all t, or $\alpha \leq \inf_t \frac{\phi''(t)}{(\phi'(t))^2}$, then $\psi''(t) \geq 0$. □

Proposition 3.3 shows that a pair (X,Y) of continuous random variables having the bivariate survival function (2) possess an Archimedean survival copula. A converse of this result also holds.

Proposition 3.4. *If (U,V) is a pair of uniform (0,1) random variables whose joint distribution function is the Archimedean copula C_ϕ^α defined by (6), then the pair $(X,Y) = (\phi(U), \phi(V))$ has the survival function of the form (2).*

Remark 3.5. *We note that $\lim_{\alpha\to 0}\psi(t) = \phi(t)$, and thus. $\lim_{\alpha\to 0} C_\phi^\alpha(u,v) = C_\phi(u,v)$. We call the Archimedean copula C_ϕ^α, associated with the genealized Schur-constant model (2), as the extended version of the Archimedean copula C_ϕ. By the proposed approach the strict well-known one-parameter copulas could be extend by incorporating a second parameter, thus enriching the toolbox of available copula models for various applications.*

The following examples show some particular cases and new sub-families.

Example 3.6. *Consider $\phi(t) = -\ln(t)$, which is the generator of the product copula $\Pi(u,v) = uv$. Then for $\alpha \in (0,1]$, the extended Archimedean copula, with the generator $\psi(t) = \ln(1+\alpha\phi(t))$, is given by*

$$C^\alpha(u,v) = uve^{-\alpha\ln(u)\ln(v)}, \tag{8}$$

which is the Gumble-Barnett family of copulas associated with the survival function (3).

Example 3.7. *Consider* $\phi(t) = \ln(\frac{1-\theta(1-t)}{t})$, $\theta \in [-1,1]$, *which is the generator of the Ali-Mikhail-Haq (AMH) family of copulas* [34], *given by*

$$C^{\theta}(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}. \tag{9}$$

From Proposition 3.3, the admissible range of α *is* $(0, 1+\theta]$, *with* $\theta \in [-1,1]$. *The extended Archimedean copula with the generator* $\psi(t) = \ln\Big(1 + \alpha \ln(\frac{1-\theta(1-t)}{t})\Big)$, *is given by*

$$C^{\alpha,\theta}(u,v) = \frac{\frac{uv(1-\theta)}{(1-\theta(1-u))(1-\theta(1-v))} e^{\alpha \ln\left(\frac{u}{1-\theta(1-u)}\right)\ln\left(\frac{v}{1-\theta(1-v)}\right)}}{1 - \frac{uv\theta}{(1-\theta(1-u))(1-\theta(1-v))} e^{\alpha \ln\left(\frac{u}{1-\theta(1-u)}\right)\ln\left(\frac{v}{1-\theta(1-v)}\right)}}. \tag{10}$$

which is the copula of the bivariate distribution given in Example 2.3, for the case $\theta \in [0,1]$. *The case* $\alpha = 0$, *reduces to the AMH family of copulas. The case* $\theta = 0$, *reduces to the Gumble-Barnett copula (8). Since the parameter* α *and* θ *satisfy* $\alpha \in (0, 1+\theta]$, *we can construct a new one-parameter copula, by letting* $\alpha = 1+\theta$, *in (10).*

Example 3.8. *Consider the bivariate distribution given in Example 2.4. The associated Archimedean copula is then*

$$C^{\alpha,\theta}(u,v) = g(u)g(v)e^{-\alpha \ln(g(u))\ln(g(v))}\left(1 - \theta(1-g(u)g(v)e^{-\alpha \ln(g(u))\ln(g(v))}\right), \tag{11}$$

where $\alpha \in (0,1]$, $\theta \in (0,1]$ *and,* $g(t) = \frac{\theta - 1 + \sqrt{(1-\theta)^2 + 4\theta t}}{2\theta}$. *Note that the case* $\theta = 1$, *reduces to the Gumble-Barnett copula (8).*

Example 3.9. *Consider* $\phi(t) = -\ln(\frac{e^{-\theta t}-1}{e^{-\theta}-1})$, $\theta \in (-\infty, \infty)$, *which is the generator of the Frank copula* [34]. *From Proposition 3.3, the admissible range of* α *is* $(0,1]$, *and the two parameter extenstion of the Frank copula is given by*

$$C^{\alpha,\theta}(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1} e^{-\frac{\alpha}{\theta}\ln\left(\frac{e^{-\theta u}-1}{e^{-\theta}-1}\right)\ln\left(\frac{e^{-\theta v}-1}{e^{-\theta}-1}\right)}\right). \tag{12}$$

Example 3.10. *Consider* $\phi(t) = -\ln(1-(1-t)^{\theta})$, $\theta \in [1,\infty)$, *which is the generator of the Joe copula* [34]. *From Proposition 3.3, the admissible range of* α *is* $(0, \frac{1}{\theta}]$ *with* $\theta \geq 1$. *The two parameters extenstion of the Joe copula with the generator* $\psi(t) = \ln\big(1 - \alpha \ln(1-(1-t)^{\theta})\big)$, *is given by*

$$C^{\alpha,\theta}(u,v) = 1 - \left(1 - (1-\bar{u}^{\theta})(1-\bar{v}^{\theta})e^{-\alpha \ln(1-\bar{u}^{\theta})\ln(1-\bar{v}^{\theta})}\right)^{\frac{1}{\theta}}, \tag{13}$$

where $\bar{u} = 1-u$ *and* $\bar{v} = 1-v$. *Note that for* $\theta = 1$, *the copula (11) reduces to the Gumble-Barnett copula (8).*

Example 3.11. *Consider the Clayton copula with generator* $\phi(t) = (t^{-\theta}-1)/\theta$, *where* $\theta > 0$. *From Proposition 3.3, the admissible range of* α *is* $(0, 1+\theta]$, *and the two parameters extenstion of the Clayton's copula with the generator* $\psi(t) = \ln(1 + \frac{\alpha}{\theta}(t^{-\theta}-1))$, *is given by*

$$C^{\alpha,\theta}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1 + \frac{\alpha}{\theta}(u^{-\theta}-1)(v^{-\theta}-1)\right)^{-\frac{1}{\theta}}. \tag{14}$$

The case $\theta = 1$, *this copula reduces to the AMH family of copula (9) with the parameter* $1-\alpha$.

Example 3.12. *Consider the Gumbel-Hougaard family of copulas with the generator* $\phi(t) = (-\ln(t))^{\theta}$, *where* $\theta \in [1,\infty)$. *From Proposition 3.3, the admissible range of* α *is* $(0,1]$, *and the two parameters extenstion of the Gumbel-Hougaard copula with the generator* $\psi(t) = \ln(1 + \alpha(-\ln(t))^{\theta})$, *is given by*

$$C^{\alpha,\theta}(u,v) = e^{-[(-\ln(u))^{\theta} + (-\ln(v))^{\theta} + \alpha(-\ln(u))^{\theta}(-\ln(v))^{\theta}]^{\frac{1}{\theta}}}. \tag{15}$$

The case $\theta = 1$, *reduces to the Gumble-Barnett copula (8).*

Remark 3.13. *In the case where the generating function* $\phi(.)$ *is not strict, Proposition 3.3 does not hold. However, the general structure (7) can be used to add a parameter to a non-strict Archimedean copula, although the resulting extension may not be an Archimedean, as the following example shows.*

Example 3.14. *Consider the Fréchet-Hoeffding lower bound copula* $W(u,v) = \max(u+v-1, 0)$, *which is an Archimedean with the generator* $\phi(t) = 1-t$, *for* $t \in [0,1]$ *and* $\phi^{[-1]}(t) = \max(1-t, 0)$. *It is easy to see that the function* $\psi(t) = \ln(1+\alpha(1-t))$ *is concave for all* α *and therefore, is not a generator of an Archimedean copula. But the extension (7) leads to*

$$W^{\alpha}(u,v) = \max(u+v-1-\alpha(1-u)(1-v), 0), \tag{16}$$

which is 2-increasing for $\in [-1,0]$. *Note that for* $\alpha = -1$, W^{α} *reduces to* $\Pi(u,v) = uv$ *and,* $W^{0}(u,v) = W(u,v)$.

4 Some dependence properties

In the following, we study basic dependence properties of the associated Archimedean copula defined in (6) of the genealized Schur-constant model proposed in (2). Let $C_v(u) = \frac{\partial}{\partial v}C(u,v)$ be the partial derivative of a copula C. Let $\mathcal{C}$ be the class of all bivariate copulas and let $\mathcal{C}_i$, $i = 1,2,3,4$ be the subclasses defined by

$$\begin{aligned}
\mathcal{C}_1 &= \{C \in \mathcal{C} \colon C(u,v) \geq uv\}, \\
\mathcal{C}_2 &= \left\{C \in \mathcal{C} \colon \frac{C(u',v)}{u'} \leq \frac{C(u,v)}{u}, \forall\ 0 < u < u', \forall\ 0 \leq v \leq 1\right\}, \\
\mathcal{C}_3 &= \{C \in \mathcal{C} \colon C_{v'}(u) \leq C_v(u), \forall\ 0 < v < v', \forall\ 0 \leq u \leq 1\}, \\
\mathcal{C}_4 &= \{C \in \mathcal{C} \colon C \text{ is } TP_2\}.
\end{aligned}$$

A function $g \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ is TP_2 if $g(x',y')g(x'',y'') \geq g(x',y'')g(x'',y')$ with $x' < x''$ and $y' < y''$). Let (X,Y) be a pair of continuous random variables whose copula is C. Then (X,Y) is said to be *positive quadrant dependent* (PQD) if $C \in \mathcal{C}_1$. The random variable Y is said to be *left tail decreasing in* X (LTD$(Y|X)$) if $C \in \mathcal{C}_2$. The random vector (X,Y) is *left corner set decreasing* (LCSD) if $C \in \mathcal{C}_4$ (see [34] for a complete study). If C_1 and C_2 are two copulas, we say that C_2 is *more concordant than* C_1 (written $C_1 \prec_c C_2$) if $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in [0,1]^2$. A parametric family $\{C_\theta\}$ of copulas is positively ordered if $C_{\theta_1} \prec_c C_{\theta_2}$ whenever $\theta_1 \leq \theta_2$ [34]. For the family (6) of copulas, we have the following result.

Proposition 4.1. *The parametric Archimedean family* $\{C_\phi^\alpha\}$ *is negatively ordered; i.e.,* $C_\phi^{\alpha_2} \prec_c C_\phi^{\alpha_1}$ *whenever* $0 \leq \alpha_1 \leq \alpha_2$.

Proof. Let $0 \leq \alpha_1 \leq \alpha_2$. Then, for all $u,v \in [0,1]$, $\phi(u) + \phi(v) + \alpha_1\phi(u)\phi(v) \leq \phi(u) + \phi(v) + \alpha_2\phi(u)\phi(v)$, since ϕ^{-1} is decreasing on $[0,1]$, then $C_\phi^{\alpha_1}(u,v) \geq C_\phi^{\alpha_2}(u,v)$. □

As a result for an Archimedean copula C_ϕ and its associated extension C_ϕ^α, we have that $C_\phi^\alpha \prec_c C_\phi$, for every α.

Proposition 4.2. *Let* C_{ϕ_1} *and* C_{ϕ_2} *be two strict Archimedean copulas and* $C_{\phi_1}^\alpha$ *and* $C_{\phi_2}^\alpha$ *be their associated extensions. Then, for every* α,

i) $C_{\phi_1}^\alpha \underset{PQD}{\leq} C_{\phi_2}^\alpha$ *if and only if* $f(x+y+\alpha xy) < f(x) + f(y) + \alpha f(x)f(y)$, *where* $f(t) = \phi_1(\phi_2^{-1}(t))$;

ii) *If* $C_{\phi_1} \prec_c C_{\phi_2}$, *then* $C_{\phi_1}^\alpha \prec_c C_{\phi_2}^\alpha$.

Proof. Part (i) is straightforward. For part (ii), note that the function f is continuous, non-decreasing, and $f(0) = 0$. Since $C_{\phi_1} \prec_c C_{\phi_2}$, then in view of Corollary 4.4.4 in [34], $f(.)$ is concave. Let $g(t) = \ln(1+\alpha t)$. Since C_ϕ^α is Archimedean with the generator $t \to \ln(1+\alpha\phi(t))$, then $C_{\phi_1}^\alpha \prec_c C_{\phi_2}^\alpha$, if $g(f(\frac{e^t-1}{\alpha}))$ is concave. Because $f(.)$ concave and $g(.)$ is increasing, then $g(f(.))$ is cancave and the result follows. □

Proposition 4.3. *Let* C_ϕ *be an Archimedean copula with the strict generator* $\phi(.)$ *and let* C_ϕ^α *be its associated extension. If* $C_\phi \in \mathcal{C}_i$, *for each* $i = 1,2,3,4$, *then also* $C_\phi^\alpha \in \mathcal{C}_i$.

Proof. Since the Archimedean copula $C_\phi \in \mathcal{C}_i$, $i = 1,2,3,4$, then its generator $\phi(.)$ satisfies $-\ln(\phi^{-1}(t))$ is sub-additive (because of PQD property), $\ln(-\frac{d}{dt}\phi^{-1}(t))$ is convex (for SI property) and $\phi^{-1}(.)$ is completely monotone (for LCSD and LTD properties) (see, e.g., [34]). Now, since the generator of the Archimedean copula C_ϕ^α is given by $g(\phi(t))$, where $g(t) = \ln(1+\alpha t)$, then the result hold for C_ϕ^α from the non-decreasingness of g. □

The Kendall's distribution function of an Archimedean copula with the generator ϕ is given by $K_\phi(t) = t - \frac{\phi(t)}{\phi'(t)}$, for $0 < t \leq 1$ [34]. Then the Kendall's distribution of the associated extension C_ϕ^α is given by

$$K_\phi^\alpha(t) = t - \frac{1+\alpha\phi(t)}{\alpha\phi\prime(t)}\ln(1+\alpha\phi(t)), \quad 0 < t \leq 1.$$

A copula C is said to be Positive K-Dependent (PKD) if and only if $K(t) \leq t - t\ln(t)$, for all $0 < t \leq 1$. Given two copulas C_1 and C_2 with the corresponding Kendall's distribution functions K_1 and K_2, the copula C_1 is less than C_2 in PKD order (denoted by $C_1 \prec_{PKD} C_2$) if and only if $K_1(t) \leq K_2(t)$, for all $0 \leq t \leq 1$ [5]. So, for the associated extension C_ϕ^α, we have the following result by some calculation.

Proposition 4.4. *Let C_ϕ be an Archimedean copula with the strict generator $\phi(.)$ and let C_ϕ^α be its associated extension. Then the followings hold:*

i) *If C_ϕ is PKD then C_ϕ^α is PKD.*

ii) *C_ϕ^α is PKD if and only if $\frac{\ln(\phi^{-1}(t))}{\ln(1+\alpha t)}$ is nondecreasing in t.*

iii) *$C_\phi^\alpha \prec_{PKD} C_\phi$.*

Proposition 4.5. *Let C_{ϕ_1} and C_{ϕ_2} be two Archimedean copulas and $C_{\phi_1}^\alpha$ and $C_{\phi_2}^\alpha$ be their associated extensions. Then for for every α:*

i) *$C_{\phi_1}^\alpha \prec_{PKD} C_{\phi_2}^\alpha$, if and only if $\frac{\ln(1+\alpha\phi_1(t))}{\ln(1+\alpha\phi_2(t))}$ is non-decreasing in t;*

ii) *If $C_{\phi_1} \prec_{PKD} C_{\phi_2}$, then $C_{\phi_1}^\alpha \prec_{PKD} C_{\phi_2}^\alpha$.*

4.1 Measures of association

The population version of the most common nonparametric measures of association between the components of a continuous random pair (X, Y) are Kendall's tau (τ), Spearman's rho (ρ), Gini's gamma (γ), and the Blomqvist's beta (β). These measures depend only on the copula C and are given by:

$$\tau(C) = 4\int_0^1\int_0^1 C(x,y)\,dC(x,y) - 1, \tag{17}$$

$$\rho(C) = 12\int_0^1\int_0^1 (C(x,y) - xy)\,dxdy, \tag{18}$$

$$\gamma(C) = 4\left(\int_0^1 C(u,1-u)\,du - \int_0^1 (u - C(u,u))\,du\right), \tag{19}$$

$$\beta(C) = 4C\left(\frac{1}{2},\frac{1}{2}\right) - 1, \tag{20}$$

respectively [34]. The Kendall's tau for a copula C could be written in terms of K_C (its Kendall's distribution function) as $\tau(C) = 3 - 4\int_0^1 K_C(t)dt$ and $\rho(C) = 3 - 6\int\int_{[0,1]^2}\left(u\frac{\partial C(u,v)}{\partial u} + v\frac{\partial C(u,v)}{\partial v}\right)du\,dv$ (see, [34]). Therefore, by some elementary calculations for the associated extension C_ϕ^α, these measures are given by

$$\tau(C_\phi^\alpha) = 1 + 4\int_0^1 \frac{(1+\alpha\phi(t))\ln(1+\alpha\phi(t))}{\alpha\phi'(t)}dt,$$

$$\rho(C_\phi^\alpha) = 3 - 6\int\int_{[0,1]^2} \frac{u(1+\alpha\phi(v))\phi\prime(u) + v(1+\alpha\phi(u))\phi\prime(v)}{\phi\prime(C_\phi^\alpha(u,v))}\,du\,dv,$$

$$\gamma(C_\phi^\alpha) = 4\left[\int_0^1\left(\phi^{-1}\left(\frac{(1+\alpha\phi(u))(1+\alpha\phi(1-u))-1}{\alpha}\right) - u + \phi^{-1}\left(\frac{(1+\alpha\phi(u))^2-1}{\alpha}\right)\right)du\right],$$

and

$$\beta(C_\phi^\alpha) = 4\phi^{-1}\left(\frac{(1+\alpha\phi(\frac{1}{2}))^2-1}{\alpha}\right) - 1,$$

respectively. Recall that given two copulas C_1 and C_2 such that $C_1 \prec_c C_2$, then we have $\tau(C_1) \le \tau(C_2)$, $\rho(C_1) \le \rho(C_2)$, $\gamma(C_1) \le \gamma(C_2)$ and $\beta(C_1) \le \beta(C_2)$ (see [34]). Since $C_\phi^\alpha \prec_c C_\phi$, for every $\alpha > 0$, then from Proposition 4.1, we have $\tau(C_\phi^\alpha) \le \tau(C_\phi)$, $\rho(C_\phi^\alpha) \le \rho(C_\phi)$, $\gamma(C_\phi^\alpha) \le \gamma(C_\phi)$ and $\beta(C_\phi^\alpha) \le \tau(C_\phi)$. Thus, the upper bounds for these measures for the copula C_ϕ^α are equal to those of the C_ϕ.

Example 4.6. *For the non-strict Archimedean copula $W^\alpha(u,v) = \max(u+v-1-\alpha(1-u)(1-v), 0)$, with $\alpha \in [-1,0]$, given by (16), we have tha*

$$\tau(W^\alpha) = \frac{-2(1+\alpha)[(1+\alpha)\ln(1+\alpha) - \alpha]}{\alpha},$$

$$\rho(W^\alpha) = \frac{-3(1+\alpha)[\alpha(2+\alpha) - 2(1+\alpha)\ln(1+\alpha)]}{\alpha^3},$$

$$\gamma(W^\alpha) = \frac{(\alpha+1)(\alpha-2)[\sqrt{(\alpha+1)^3} - \frac{3}{2}\alpha - 1]}{3\alpha^2},$$

and

$$\beta(W^\alpha) = -(\alpha+1).$$

Note that the Archimedean copula W^α satisfies $W \prec_c W^\alpha \prec_c \Pi$, for $\alpha \in [-1, 0]$, and

$$\lim_{\alpha\to 0} \tau(W^\alpha) = \lim_{\alpha\to 0} \rho(W^\alpha) = \lim_{\alpha\to 0} \gamma(W^\alpha) = \lim_{\alpha\to 0} \beta(W^\alpha) = -1,$$

and

$$\lim_{\alpha\to -1} \tau(W^\alpha) = \lim_{\alpha\to -1} \rho(W^\alpha) = \lim_{\alpha\to -1} \gamma(W^\alpha) = \lim_{\alpha\to -1} \beta(W^\alpha) = 0.$$

For a given copula C, the lower tail dependence coefficient is defined by [34]

$$\lambda_L(C) = \lim_{u\to 0^+} \frac{C(u,u)}{u}, \tag{21}$$

and the *upper tail dependence coefficient* as

$$\lambda_U(C) = 2 - \lim_{u\to 1^-} \frac{1 - C(u,u)}{1-u}. \tag{22}$$

For the proposed model (6), the lower and upper tail dependence coefficients are given by

$$\lambda_L(C^\alpha_\phi) = \lim_{x\to\infty} \frac{\phi^{-1}(x(2+\alpha x))}{\phi^{-1}(x)},$$

and

$$\lambda_U(C^\alpha_\phi) = 2 - \lim_{x\to 0^+} \frac{1 - \phi^{-1}(x(2+\alpha x))}{1 - \phi^{-1}(x)},$$

respectively. In the following, we compute the values of dependence measures for associated extesions of some Archimedean copulas.

Example 4.7. *Consider the associated extension of the Ali-Mikhail-Haq (AMH) family of copulas given by (10). Figure 1 shows the values of dependence measures for this copula for $\alpha \in \{0.5, 1\}$. We see that all of dependence measures increase as θ increases. For the associated extension with $\alpha = 0.5$ and $\alpha = 1$, the positive values of dependence measures are smaller than those of AMH copula, but negative values of dependence measures are bigger than those of AMH. The lower and upper tail dependence coefficients of this family are $\lambda_L = \lambda_U = 0$.*

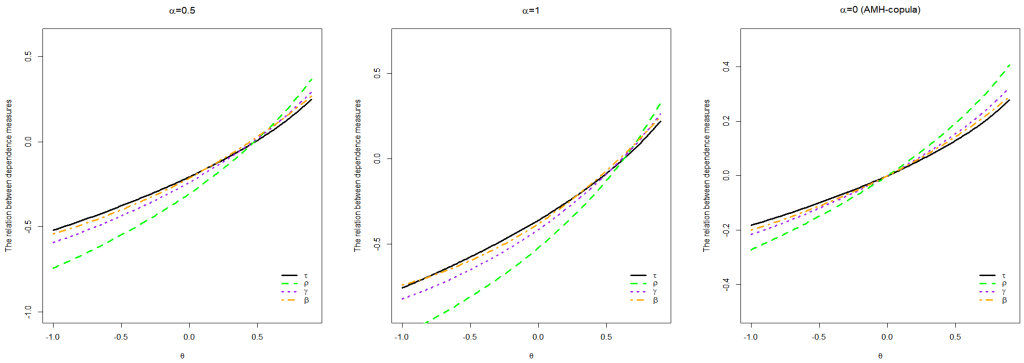

Figure 1: The values of Kendall's τ, Spearman's ρ, Gini's γ, and Blomqvist's β of associated extension of AMH copula for $\alpha \in \{0.5, 1\}$, $\theta > 0$ and the case $\alpha \to 0$, corresponding to the AMH copula.

Example 4.8. *Consider the extension of the Gumbel-Barnett copula given by (11). Figure 2 shows values of dependence measures for $\alpha \in \{0.5, 1\}$. The lower and upper tail dependence coefficients of this family are $\lambda_L = \lambda_U = 0$.*

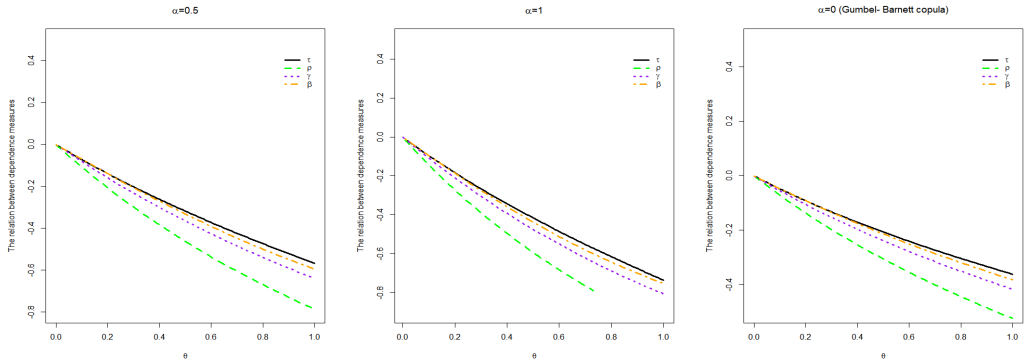

Figure 2: The values of Kendall's τ, Spearman's ρ, Gini's γ, and Blomqvist's β for associated extension of the Gumbel-Barnett copula for $\alpha \in \{0.5, 1\}$, $\theta > 0$, and the case $\alpha \to 0$, corresponding to the Gumbel-Barnett copula.

5 Convexity properties

In this section, we study some convexity (concavity) properties for the associated extension of an Archimedean copula. A copula C is *Schur-concave* if

$$C(u, v) \leq C(\lambda u + (1-\lambda)v, \lambda v + (1-\lambda)u), \tag{23}$$

for all $u, v \in (0, 1)$ and $\lambda \in [0, 1]$ [34]. A copula C is said to be *quasi-concave* if for all [11] $u, v, u', v' \in [0, 1]$ and all $\lambda \in [0, 1]$,

$$C(\lambda u + (1-\lambda)v, \lambda u' + (1-\lambda)v') \geq \min\{C(u, u'), C(v, v')\}.$$

Proposition 5.1. *The associated extension C_ϕ^α of an Archimedean copula, is Schur-concave and quasi-concave.*

Proof. Since ϕ is non-increasing, we have

$$u < \lambda v + (1-\lambda)u \Rightarrow \phi(u) > \phi(\lambda v + (1-\lambda)u),$$

and

$$v < \lambda u + (1-\lambda)v \Rightarrow \phi(v) > \phi(\lambda u + (1-\lambda)v).$$

Since ϕ^{-1} is non-increasing,

$$\phi^{-1}\big(\phi(u) + \phi(v) + \alpha\phi(u)\phi(v)\big) < \phi^{-1}\big(\phi(\lambda v + (1-\lambda)u) + \phi(\lambda u + (1-\lambda)v) + \alpha\phi(\lambda u + (1-\lambda)v)\phi(\lambda v + (1-\lambda)u),$$

and thus C_ϕ^α is Schur-concave. For quasi-concavity of C_ϕ^α, we note that if $c \leq a$ and $d \leq b$ then

$$c < \lambda a + (1-\lambda)c \quad \textit{and} \quad d < \lambda b + (1-\lambda)d,$$

and thus

$$C_\phi^\alpha(c, d) \leq C_\phi^\alpha\big(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\big).$$

Similarly, if $a \leq c$ and $b \leq d$, then $C_\phi^\alpha(a, b) \leq C_\phi^\alpha\big(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\big)$, which completes the proof. □

6 Random data generation

In this section, an Algorithm is proposed based on the method introduced in [18] for generating random data from the extended Archimedean copula (6).

Algorithm 1 Random vector generation from associated Archimedean extension C_ϕ^α

For $i = 1$ to n
Let $\alpha = \alpha_0$.
Generate u and t from $U(0, 1)$ distribution, independently.
Calculate w using $w = \phi^{-1}\big[\frac{1}{\alpha}\{(1 + \alpha\phi(u))^{\frac{1}{t}} - 1\}\big]$.
Calculate v through $v = \phi^{-1}\big[\frac{\phi(w)-\phi(u)}{(1+\alpha\phi(u)}\big]$. So, $(u_i, v_i),\ i = 1, \ldots, n$ follows the copula C_ϕ^α.

Figure 3 shows scatterplot of 1000 observations from the copula $C^{\alpha,\theta}$ and the corresponding Archimedean copula C^{θ}, for Frank (12) and Joe (13) copulas. The right panel is for associated extensions with $\alpha = 1$ and the left one is for the corresponding Archimedean copulas (i.e. when $\alpha = 0$).

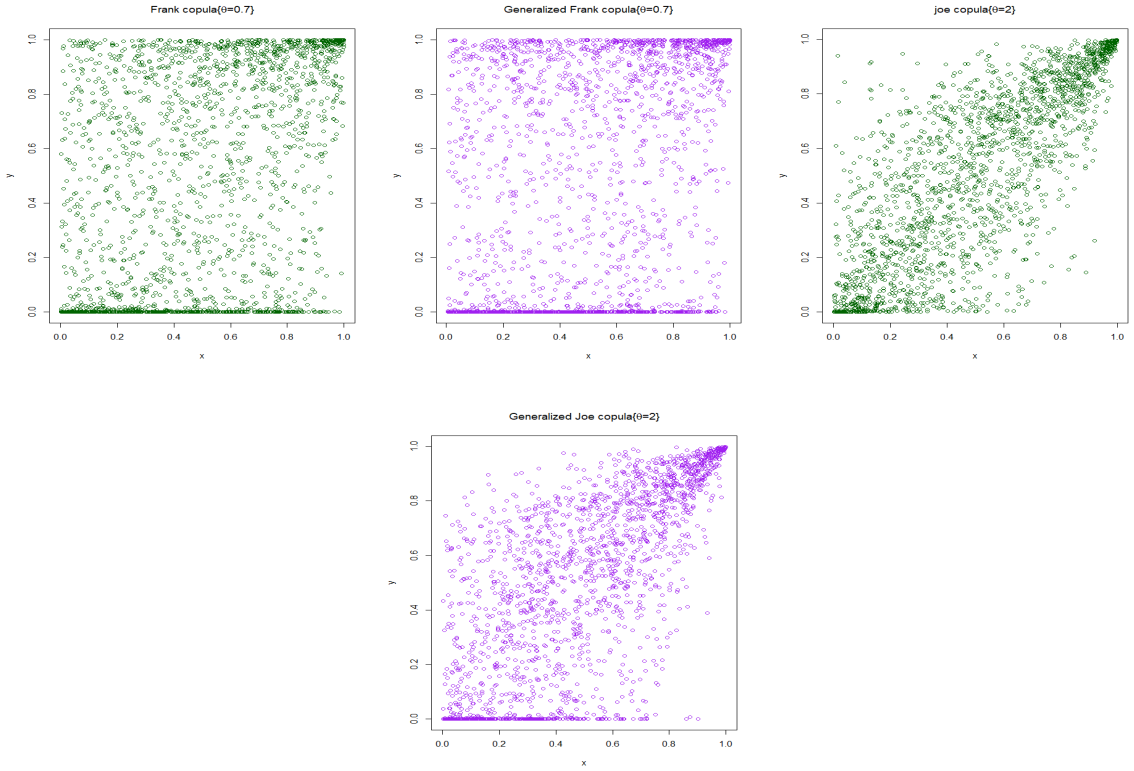

Figure 3: Scatter plots of data generated from Frank and Joe copulas ($\alpha \to 0$) (left panel) and their corresponding associated extensions for $\alpha = 1$ (right panel) for specific values of θ.

7 Testing $H_0 : \alpha = 0$

Note that the Archimedean copula $C_\phi(u, v) = \phi^{-1}\{\phi(u) + \phi(v)\}$, is a special case of its associated extension $C^\alpha_\phi(u, v) = \phi^{-1}\{\phi(u) + \phi(v) + \alpha\phi(u)\phi(v)\}$, obtained by $\alpha \to 0$. To choose between copulas C_ϕ and C^α_ϕ in model fitting, we may perform the following test of hypothesis

$$H_0 : \alpha = 0 \quad versus \quad H_1 : \alpha \neq 0, \tag{24}$$

When analyzing the data, a two-parameter copula is often simplified to one of its one-parameter subfamilies if it provides the best fit. Thus, using a two-parameter copula effectively compares all its one-parameter subfamilies, eliminating the need to fit each subfamily separately. Below, we present a likelihood ratio statistic (LR) for testing this hypothesis problem (24). Let $(U_1, V_1), \ldots, (U_n, V_n)$ be a random sample from (U, V), having the associated Archimedean extension C^α_ϕ and let,

$$L(\alpha, \theta) = \prod_{i=1}^n c^{\alpha,\theta}(u_i, v_i), \tag{25}$$

be the likelihood function based on the sample $(u_1, v_1), \ldots, (u_n, v_n)$, where $c^{\alpha,\theta}(u, v) = \frac{\partial^2}{\partial u \partial v} C^{\alpha,\theta}(u, v)$, is the copula density function. Using the likelihood ratio statistic $\Lambda = \frac{L(0,\hat{\theta})}{L(\hat{\alpha},\hat{\theta})}$ whose approximate distribution $G = -2 \ln \Lambda \sim \chi^2(1)$, we can test the hypothesis (24). The test is, therefore, to reject the null hypothesis when G is higher than $\chi^2(1)$ at specified level of significance.

8 Application to data

Analyzing data with a two-parameter copula simplifies to comparing its one-parameter subfamilies. In the following application, we fit two real datasets using the two-parameter Archimedean extension and its special case where $\alpha = 0$. The first dataset pertains to kidney infection data [28, 29], collected from an experiment involving 23 individuals, where the times between recurrences of a specific event were recorded. Each patient has two recurrence times: the first recurrence time (FRT) and the second recurrence time (SRT).

Figure 4 shows the scatter plot of observations (x_i, y_i) and normalized ranks $(u_i, v_i) = (\frac{rank(x_i)}{n+1}, \frac{rank(y_i)}{n+1})$. The Kendall's tau and the Spearman's rho are given by $\tau_n = 0.18$ and $\rho_n = 0.27$, respectively. We observe positive relationship between FRT and SRT variables. We analyzed the data using the Gumbel-Hougaard (15), Clayton (14),

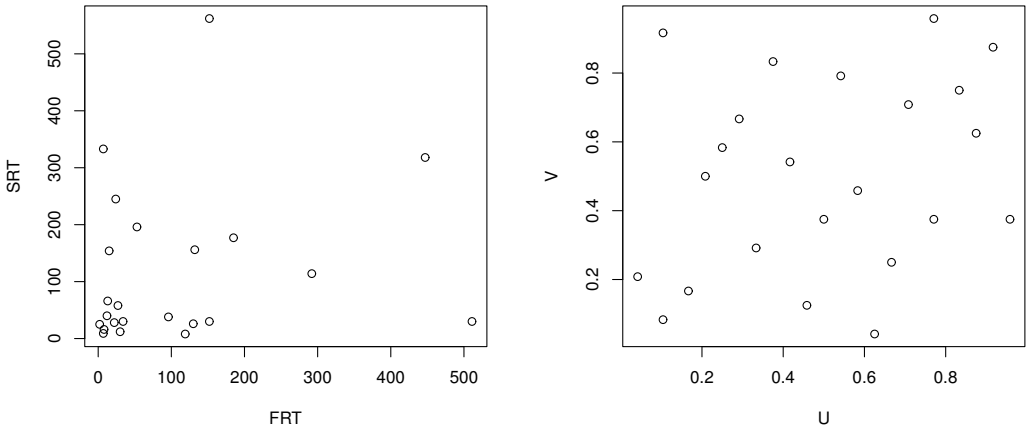

Figure 4: Pairs of observations (left panel) and pairs of normalised ranks (right panel) for variables FRT and SRT of kidney data

Frank (12), Joe (13) and AMH (10) copulas. Tables 1 and 2 display the maximum likelihood estimates (MLEs) for the parameters, along with Loglikelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). Parameter estimation was conducted using a maximum likelihood procedure implemented with the `optim` function in R. The likelihood ratio test statistics G and p-values for testing $H_0 : \alpha = 0$ are presented in Table 3. The null hypothesis is rejected at the significance level of 0.05 for all five copulas, indicating that the two-parameter model offers a better fit. According to Table 1, the Clayton copula, with parameters $\theta = 0.928$ and $\alpha = 0.507$, is the best fit among them. The second dataset used is "daxreturns," which contains transformed standardized residuals of daily log

Table 1: Fitted two-parameters copulas $C^{\alpha,\theta}$ to kidney data

Copula	$\widehat{\theta}$ (standard error)	$\widehat{\alpha}$(standard eror)	Loglike	AIC	BIC
Gumbel-Hougard	1.157 (0.196)	0.045 (0.176)	21.22	-40.44	-39.30
Clayton	**0.928 (0.0.646)**	**0.507 (0.456)**	**21.28**	**-40.56**	**-39.43**
Frank	0.186 (0.197)	2.080 (2.312)	21.01	-40.02	-38.88
Joe	0.657 (0.214)	0.316 (0.436)	21.23	-40.46	-39.33
AMH	1.213 (0.034)	0.0079 (0.187)	21.23	-40.46	-39.33

Table 2: Fitted one-parameters copulas C^{θ} to kidney data

Copula	$\widehat{\theta}$ (standard error)	Loglike	AIC	BIC
Gumbel-Hougard	1.26 (0.228)	0.851	0.296	1.43
Clayton	**0.53 (0.398)**	**0.969**	**0.062**	**1.197**
Frank	1.88 (1.4)	0.880	0.230	1.370
Joe	0.73 (0.346)	0.933	0.134	1.269
AMH	1.36 (0.34)	0.651	0.696	1.830

returns for 15 major German stocks in the DAX index, spanning from January 2005 to August 2009. It includes 1,158

Table 3: LRT statistic for testing $H_0 : \alpha = 0$ for kidney data

Test	Gumbel-Hougard	Clayton	Frank	Joe	AMH
G-Statistic	6.5	6.17	6.34	6.24	6.97
p-value	0.010	0.012	0.011	0.012	0.008

observations across 15 variables, with column names representing the stock ticker symbols. For our analysis, we focused on the columns $DTE.DE$ and $VOW3.DE$. The dataset is available in the R package $VineCopula$. We evaluated the

Table 4: Fitted two-parameters copulas $C^{\alpha,\theta}$ to daxreturns data

Copula	$\widehat{\theta}$ (standard error)	$\widehat{\alpha}$(standard eror)	Loglike	AIC	BIC
Gumbel-Hougard	1.012 (0.021)	0.085 (0.014)	1139.05	-2274.1.	-2264.9
Clayton	0.282 (0.037)	0.101 (0.025)	1137.2	-2270.4	-2269.7
Frank	**0.562 (0.124)**	**0.272 (0.100)**	**1141.9**	**-2279.8**	**-2269.7**
Joe	0.998 (0.042)	0.918 (0.014)	1139.1	-2274.3	-2262.7
AMH	0.171 (0.100)	0.077 (0.016)	1138.4	-2272.8	-2262.7

Table 5: Fitted one-parameters copulas C^{θ} to daxreturns data

Copula	$\widehat{\theta}$ (standard error)	Loglike	AIC	BIC
Gumbel-Hougard	1.17 (0.026)	29.12	-40.2	-35.2
Clayton	0.45 (0.046)	40.09	-78.8	-73.1
Frank	1.73 (0.182)	45.43	-88.8	-83.8
Joe	1.17 (0.037)	14.63	-27.3	-22.2
AMH	0.68 (0.046)	49.12	-96.2	-91.2

Table 6: LRT statistic for testing $H_0 : \alpha = 0$ for daxreturns data

Test	Gumbel-Hougard	Clayton	Frank	Joe	AMH
G-Statistic	7.33	6.69	6.44	8.7	6.28
p-value	0.007	0.010	0.011	0.003	0.012

Gumbel-Hougaard (15), Clayton (14), Frank (12), Joe (13) and AMH (10) copulas for fitting the data. Tables 4 and 5 present the maximum likelihood estimates (MLEs) for the parameters alongside the log-likelihood, AIC, and BIC. The likelihood ratio test statistics G and p-values for testing $H_0 : \alpha = 0$ are reported in Table 6. The null hypothesis is rejected at a significance level of 0.05 for all five copulas, indicating that the two-parameter model provides a better fit. Among these, the Frank copula with parameters $\theta = 0.562$ and $\alpha = 0.272$ is identified as the best fit, as shown in Table 4.

9 Conclusions

In this research, we extend the bivariate Schur-constant model by introducing an additional parameter to its Archimedean copula for enhanced flexibility. We analyze the dependence properties of this new structure and illustrate our findings with several examples. Additionally, we perform a likelihood ratio test to compare the performance of the extended Archimedean copula against the traditional Archimedean subfamily. Two real datasets are analyzed, demonstrating the model's versatility in capturing various dependencies. A multivariate extension of the model proposed in (6) can be defined as

$$C_{\phi}^{\alpha}(u_1, ..., u_n) = \phi^{-1}\left(\sum_{i=1}^{n}\phi(u_i) + \alpha_1\sum_{i<j}\phi(u_i)\phi(u_j) + \alpha_2\sum_{i<j<k}\phi(u_i)\phi(u_j)\phi(u_k) + ... + \alpha_n\prod_{i=1}^{n}\phi(u_i)\right), \qquad (26)$$

where for $i = 1, ..., n-2$, $\alpha_i = \alpha^i$. The bivariate marginal copulas are given by $C_{\phi}^{\alpha}(u_i, u_j) = \phi^{-1}\{\phi(u_i) + \phi(u_j) + \alpha\phi(u_i)\phi(u_j)\}$, $i, j \in \{1, 2, ..., n\}$, and for $n = 3$, it is

$$C_{\phi}^{\alpha}(u_1, u_2, u_3) = \phi^{-1}\left(\phi(u_1) + \phi(u_2) + \phi(u_3) + \alpha\phi(u_1)\phi(u_2) + \alpha\phi(u_1)\phi(u_3) + \alpha\phi(u_2)\phi(u_3) + \alpha^2\phi(u_1)\phi(u_2)\phi(u_3)\right).$$

Future work will focus on exploring the properties of the copulas defined by (26).

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