Iranian Journal of Fuzzy Systems

Volume 22, Number 1, (2025), pp. 185-207

Original paper

University of Sistan and Baluchestan

New types of approximations via ⊛-β-soft fuzzy complement neighborhood and ⊙-β-soft fuzzy complement neighborhood and their applications in multiple attribute decision-making (MADM)

A. M. Khalil [1], S. G. Li [2] and H. L. Yang [3]

[1,2,3] *School of Mathematics and Statistics, Shaanxi Normal University, Xi′an 710062, China*
[1] *Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt*
[2] *School of Mathematics and Physics, Guangxi Minzu University, Nanning 530006, China*

a.khalil@azhar.edu.eg, shengganglinew@126.com, yanghailong@snnu.edu.cn

Abstract

A β-soft fuzzy complement neighborhood is the initial concept proposed by Zhang and Zhan (International Journal of Machine Learning and Cybernetics 10(2019) 1487-1502), which we will refer to in this study as ⊛-β-soft fuzzy complement neighborhood. In the present paper, we first introduce three new approximation types based on β-soft fuzzy covering via ⊛-β-soft fuzzy complement neighborhood, along with several essential features and examples. The new approximation types are significant because they satisfy the inclusion property (i.e., the upper approximation contains the lower approximation, which is one of the essential features of rough set models). In addition, we update some algorithms to a very easy-to-understand state. As a result, improved decision-making procedures will be observable and trustworthy in order to arrive at the optimal choice. Second, we offer a novel idea of ⊙-β-soft fuzzy complement neighborhood and then present three other new approximation types based on β-soft fuzzy covering via ⊙-β soft fuzzy complement neighborhood, besides outlining some of its key characteristics and giving some examples. On the theoretical side, by using three other new approximations via ⊙-β-FCN$\mathscr{A}$, we also study some basic fuzzy topology properties. Third, we use ⊙-β-FCN$\mathscr{A}$ to construct a new method that can be applied to the MADM field. We illustrate this method using a real-world problem: a candidate seeking employment in a company. Finally, to demonstrate the advantages of the proposed work, we will compare our proposed method with published β-soft fuzzy MADM methods.

Keywords: ⊛-β-soft fuzzy complement neighborhood, ⊙-β-soft fuzzy complement neighborhood, new types approximations, inclusion property, MADM, numerical example, comparison.

1 Introduction

One useful mathematical technique to address the ambiguity of data processing and information systems are the rough set theory, as proposed by Pawlak [28, 29]. When given an equivalence relation $\mathcal{R}$ on a set X (and thus an approximation space $(X, \mathcal{R})$), a rough set theory's main contribution is to offer a lower approximation $\mathcal{R}^-(A)$ and an upper approximation $\mathcal{R}^+(A)$ for characterizing an arbitrary given subset A of X; this is much like the situation of a lower sum, an upper sum, and the definite integral $\int_a^b f(t)dt$ of a function f on a finite closed interval $[a, b]$. Numerous studies (e.g., [4, 8, 10, 12, 20, 21, 36, 56]) rely on the idea of rough set theory and its nine fundamental properties (see Table 1 in [7]), particularly the inclusion property which is essential for resolving issues in the real world, are given. In the context of covering approximation space, Pawlak's rough set models are relevant to the existing covering-based rough set models were introduced in [19, 30, 31, 42]. Conversely, Dubois and Prade [9] presented the concept of fuzzy rough sets by combining rough sets with fuzzy sets [45]. Following this line of reasoning, Li et al. [16] offered two

Corresponding Author: S. G. Li
Received: October 2024; Revised: December 2024; Accepted: February 2025.
https://doi.org/10.22111/ijfs.2025.50036.8834

broad fuzzy covering-based rough set models, and Zhou et al. [55] defined three different types of them, as well as the definition, traits, and relationships of each model, which were described in depth by D'eer and Cornelis [5]. Numerous scholars studied the concepts of the aforementioned principles, in addition to β-fuzzy covering-based rough sets in [2, 6, 22, 32, 34, 36, 37, 38, 39, 51].

Soft set theory is a novel approach proposed by renowned mathematician Molodtsov [26]. According to some interpretations, it modifies the crisp set notion for parametrically describing uncertainties and imprecise data. The soft set theory is a parameterized family of subsets in the universal set, as opposed to the fuzzy set theory (i.e., which is a tool for giving components membership degrees). This approach provides a dependable way to handle ambiguity, whose numerous computer science, data analysis, and decision-making situations can benefit from its application. In addition to being universal and able to handle a wide range of uncertainties and imprecisions, soft set theory does not necessitate any additional quantitative measurements, unlike fuzzy theory and probability theory. It is the process of handling and displaying undefined and ambiguous data, which makes it suitable for a variety of applications such as pattern identification, data mining, and decision-making in imprecise circumstances. Since its original formulation, the theory of soft sets has undergone significant improvement, leading to several extensions and generalizations. Maji et al. [24] presented the idea of soft fuzzy sets by combining soft sets with fuzzy sets. Following that, some soft fuzzy set extension models (e.g., [1, 13, 14, 15, 23, 25, 35, 40, 41]) underwent rapid development. Additionally, Zhan et al. [48, 52] developed the idea of soft rough sets, which blend rough and soft sets. Zhan et al. [47] offered a unique uncertain soft set model over soft fuzzy rough sets. Li et al. [17] presented the idea of soft coverings and provided an algorithm for parameter reductions by fusing soft sets and coverings. Recent publications by authors in [43, 44, 46, 49, 50] presented the idea of soft covering-based rough sets, their key characteristics, and decision-making use.

Based on the discussion above, it should be noted that there are application-specific constraints to the notion of soft covering-based rough sets. Take the following simplified scenario. For instance, let $X = \{x_1, x_2, x_3\}$ be a set of three airlines (which represent: Sichuan, Turkish, and Emirates, respectively) and $I = \{i_1, i_2, i_3\}$ be a set of three parameters (which represent: Price, Quality of meals, and Time discipline, respectively). For these three parameters, we have $\mathscr{A} = \{\mathscr{A}(i_1), \mathscr{A}(i_2), \mathscr{A}(i_3)\}$, where $\mathscr{A}(i_1), \mathscr{A}(i_2), \mathscr{A}(i_3)$ are given as $\mathscr{A}(i_1) = \dfrac{0.77}{x_1} + \dfrac{0.92}{x_2} + \dfrac{0.83}{x_3}$, $\mathscr{A}(i_2) = \dfrac{0.8}{x_1} + \dfrac{0.63}{x_2} + \dfrac{0.95}{x_3}$, and $\mathscr{A}(i_3) = \dfrac{0.84}{x_1} + \dfrac{0.78}{x_2} + \dfrac{0.67}{x_3}$. There is a dilemma, which is the critical value provided by the passengers if they wish to select just one airline out of X for evaluation to determine which one is the best for them. In the case of $\mathscr{A}(i_1)(x_1) = 0.77$ and $\mathscr{A}(i_1)(x_2) = 0.92$, for instance, we are unable to select the optimal airline using the soft covering-based rough sets whose critical value is 1. This example represents a common evaluation issue, where passengers find it challenging to select an airline that meets their needs while adopting soft-covering techniques. To overcome these limitations, Zhang and Zhan [53] generalized the soft covering to β-soft fuzzy covering based on the ideas of Ma [21] and Yang and Hu [38]. In addition, they presented the concepts of β-soft fuzzy covering, β-soft fuzzy neighborhood, $\circledast$-β-soft fuzzy complement neighborhood, types of approximations based on β-soft fuzzy covering, and described a new algorithm in the MADM field. This demonstrates the significance of the use of approximation types in β-soft fuzzy covering due to its reliance on soft fuzzy sets. Some recent studies also used β-soft fuzzy covering (e.g., [27, 54]).

2 Preliminaries

The key concepts needed for this article are explained in this section, as indicated below.

2.1 Fuzzy sets, soft sets, and soft fuzzy sets

Definition 2.1. [24, 26, 45] *For a given set X, let 2^X be the set of all subsets of X, $[0,1]^X$ (resp., $([0,1]^X)^I = [0,1]^{XI}$) be the set of all mappings from X (resp., I) to $[0,1]$ (resp., $[0,1]^X$). Then,*

(1) *Each element $\mathcal{A} \in [0,1]^X$ is called a fuzzy set (briefly, FS) on X.*

(2) *Each element $\Phi \in 2^{X\times I} = (2^X)^I$ is called a soft set (briefly, SS) on X indexed by I (it can also be looked as a subset of $X \times I$).*

(3) *Each element $\mathscr{A} \in ([0,1]^X)^I = [0,1]^{X\times I}$ is called a soft fuzzy set (briefly, SFS) on X indexed by I (it can also be looked as a fuzzy set on $X \times I$).*

(4) *The point-wise order and the point-wise order-reversing involution on $[0,1]^X$ (resp., $[0,1]^{X\times I}$) induced by the ordinary order $\leq$ and the ordinary order-reversing involution $'$ on $[0,1]$ is denoted by $\leqslant$ and $'$, respectively.*

(5) *For a subset* $A \subseteq [0,1]$ *(resp.,* $\mathcal{A} \subseteq [0,1]^X$*), we use* $\bigwedge A$ *and* $\bigvee A$ *(resp.,* $\bigwedge \mathcal{A}$ *and* $\bigvee \mathcal{A}$*) to denote the infimum and the supremum of* A *(resp., of* $\mathcal{A}$*) in* $([0,1], \leq, ')$ *(resp., in* $([0,1]^X, ')$ *respectively.*

(6) *An appointment that* $\bigvee \emptyset = 0$ *(or* 0_X*) and* $\bigwedge \emptyset = 1$ *(or* 1_X*) is made, where* b_Y *is an FS on* X *taking constant value* b *on* Y *and* 0 *on* $X - Y$ $(\emptyset \neq Y \subseteq X)$*. In addition, we identify* 1_Y *with* Y*.*

2.2 β-Soft fuzzy covering, β-soft fuzzy neighborhood, ⊛-β-soft fuzzy complement neighborhood, and types of approximations

Definition 2.2. [53] *A SF* $\mathscr{A} \in [0,1]^{X \times I}$ *satisfying* $\bigvee \mathscr{A} = \bigvee \{\mathscr{A}(i) \mid i \in I\} \geq \beta_X$ *is called a* β*-fuzzy covering (briefly,* β*-FC); a* 1*-FC is also called FC.*

Definition 2.3. [53] *Let* $\mathscr{A} \in [0,1]^{X \times I}$ *be a* β*-FC,* $\beta \in (0,1]$*, and* $x \in X$*. Then, a* β*-fuzzy neighborhood of* x *induced by the SFS* $\mathscr{A}$ *(briefly,* β*-FN*$\mathscr{A}$*), is defined by* $\mathcal{N}^{\beta}_{\mathscr{A}x} = \bigwedge \{\mathscr{A}(i) \mid i \in I, \mathscr{A}(i)(x) \geq \beta\} \in [0,1]^X$*.*

Definition 2.4. [53] *Let* $\mathscr{A} \in [0,1]^{X \times I}$ *be a* β*-FC,* $\beta \in (0,1]$*, and* $x \in X$*. Then, a* $\circledast$*-*β*-fuzzy complement neighborhood of* x *induced by the SFS* $\mathscr{A}$ *(briefly,* $\circledast$*-*β*-FCN*$\mathscr{A}$*), is defined by* $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) = \mathcal{N}^{\beta}_{\mathscr{A}y}(x)$ $(\forall y \in X)$*.*

The following definition was put out by Zhang and Zhan [53] based on $\circledast$-β-FCN$\mathscr{A}$:

Definition 2.5. [53] *Let* $\mathcal{A} \in [0,1]^X$*,* $\beta \in (0,1]$*,* $\mathscr{A} \in [0,1]^{X \times I}$ *be a* β*-FC, and* $x, y \in X$*.*

(1) *The first type approximation of* $\mathcal{A}$ *based on* β*-FC*$\mathscr{A}$ *is a pair* $\left(\mathscr{A}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}), \mathscr{A}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})\right)$*, defined by*

$$\mathscr{A}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \bigwedge_{y \in X} ([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \vee \mathcal{A}(y)) \text{ and } \mathscr{A}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \bigvee_{y \in X} [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \mathcal{A}(y)].$$

(2) *The second type approximation of* $\mathcal{A}$ *based on* β*-FC*$\mathscr{A}$ *is a pair* $\left(\mathscr{A}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A}), \mathscr{A}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})\right)$*, defined by*

$$\mathscr{A}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x) = \bigwedge_{y \in X} \left([\mathcal{N}^{\beta}_{\mathscr{A}x}(y)]' \vee [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \vee \mathcal{A}(y)\right) \text{ and } \mathscr{A}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x) = \bigvee_{y \in X} [\mathcal{N}^{\beta}_{\mathscr{A}x}(y) \wedge \mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \mathcal{A}(y)].$$

Nawar et al. [27] *redefined the* $\mathscr{A}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A})$ *and* $\mathscr{A}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})$*, respectively, as follows:*

(2)′ *The second type approximation of* $\mathcal{A}$ *based on* β*-FC*$\mathscr{A}$ *is a pair* $\left(\tilde{\mathscr{A}}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A}), \tilde{\mathscr{A}}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})\right)$*, defined by*

$$\tilde{\mathscr{A}}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x) = \bigvee_{y \in X} \left([\mathcal{N}^{\beta}_{\mathscr{A}x}(y)]' \wedge [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \wedge \mathcal{A}(y)\right) \text{ and } \tilde{\mathscr{A}}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x) = \bigwedge_{y \in X} [\mathcal{N}^{\beta}_{\mathscr{A}x}(y) \vee \mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \vee \mathcal{A}(y)].$$

(3) *The third type approximation of* $\mathcal{A}$ *based on* β*-FC*$\mathscr{A}$ *is a pair* $\left(\mathscr{A}^{-\circledast}_{\beta\mathcal{C}3}(\mathcal{A}), \mathscr{A}^{+\circledast}_{\beta\mathcal{C}3}(\mathcal{A})\right)$*, defined by*

$$\mathscr{A}^{-\circledast}_{\beta\mathcal{C}3}(\mathcal{A})(x) = \bigwedge_{y \in X} \left(([\mathcal{N}^{\beta}_{\mathscr{A}x}(y)]' \wedge [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]') \vee \mathcal{A}(y)\right) \text{ and } \mathscr{A}^{+\circledast}_{\beta\mathcal{C}3}(\mathcal{A})(x) = \bigvee_{y \in X} [(\mathcal{N}^{\beta}_{\mathscr{A}x}(y) \vee \mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)) \wedge \mathcal{A}(y)].$$

2.3 Fuzzy topology

Definition 2.6. [11, 18] *A subset* $\mathcal{P} \subseteq [0,1]^X$ *is called a fuzzy topology on a set* X *if it meets the three conditions listed below:*

$(\mathcal{L}1)$ $0_X,\ 1_X \in \mathcal{P}$,

$(\mathcal{L}2)$ $\bigvee\limits_{j \in J} \mathcal{A}_j \in \mathcal{P}$ $(\forall \{\mathcal{A}_j\}_{j \in J} \subseteq \mathcal{P})$*, where* J *is any index set,*

$(\mathcal{L}3)$ $\mathcal{A} \wedge \mathcal{B} \in \mathcal{P}$ $(\forall \mathcal{A}, \mathcal{B} \in \mathcal{P})$.

Definition 2.7. [11, 18] *A subset* $\mathcal{P}_{co} \subseteq [0,1]^X$ *is called a fuzzy co-topology on a set* X *if it meets the three conditions listed below:*

$(\mathcal{CO}1)$ $0_X,\ 1_X \in \mathcal{P}_{co}$,

$(\mathcal{CO}2)$ $\bigwedge\limits_{j \in J} \mathcal{A}_j \in \mathcal{P}_{co}$ $(\forall \{\mathcal{A}_j\}_{j \in J} \subseteq \mathcal{P}_{co})$*, where* J *is any index set,*

$(\mathcal{CO}3)$ $\mathcal{A} \vee \mathcal{B} \in \mathcal{P}_{co}$ $(\forall \mathcal{A}, \mathcal{B} \in \mathcal{P}_{co})$.

Definition 2.8. [11, 18] $(\mathcal{LCO}1)$ *A fuzzy topological space* $(X, \mathcal{P})$ *(resp., a fuzzy co-topological space* $(X, \mathcal{P}_{co})$*) is called an Alexandroff space if it satisfies* $\bigwedge\limits_{j \in J} \mathcal{A}_j \in \mathcal{P}$ *(resp.,* $\bigvee\limits_{j \in J} \mathcal{A}_j \in \mathcal{P}_{co}$*) for each* $\{\mathcal{A}_j\}_{j \in J} \subseteq \mathcal{P}_{co})$*), where* J *is any index set.*

$(\mathcal{LCO}2)$ *A subset* $\mathcal{T} \subseteq [0,1]^X$ *(resp.,* $\mathcal{W} \subseteq [0,1]^X$*) is called a fuzzy semi-topology (resp., a fuzzy semi-co-topology) on a set* X*, if it satisfies* $(\mathcal{L}1)$ *and* $(\mathcal{L}3)$ *(resp., it satisfies* $(\mathcal{CO}1)$ *and* $(\mathcal{CO}3)$*) in Definition 2.6 (resp., Definition 2.7).*

$(\mathcal{LCO}3)$ *Clearly, a fuzzy topological space* $(X, \mathcal{P})$ *is Alexandroff if and only if* $(X, \mathcal{P}_{co})$ *is, at the same time, a fuzzy co-topological space.*

2.4 The motivation for this study

(1) Zhang and Zhan [53] provided the three notions of approximations based on β-FC$\mathscr{A}$ via $\circledast$-β-FCN$\mathscr{A}$, as mentioned in Definition 2.5. However, the following example shows that these notions do not satisfy the inclusion property.

Example 2.9. *Let $X = \{x, y, z\}$ be a 3-element set, $I = \{1, 2\}$ be a 2-element parameter set, and $\mathscr{A} \in [0,1]^{X\times I}$ is defined by $\mathscr{A}(1) = \frac{0.8}{x} + \frac{0.3}{y} + \frac{0.4}{z}$ and $\mathscr{A}(2) = \frac{0.7}{x} + \frac{0.9}{y} + \frac{0.7}{z}$. Then $\mathscr{A} \in [0,1]^{X\times I}$ is a β-fuzzy covering induced by the soft fuzzy sets $(0 < \beta \leq 0.6)$ by Definition 2.2. Take a $\beta = 0.5$. By Definitions 2.3 and 2.4, we obtain: $\mathcal{N}_{\mathscr{A}x}^{0.5} = \frac{0.7}{x} + \frac{0.3}{y} + \frac{0.4}{z}, \mathcal{N}_{\mathscr{A}y}^{0.5} = \mathcal{N}_{\mathscr{A}z}^{0.5} = \frac{0.7}{x} + \frac{0.9}{y} + \frac{0.7}{z}, \mathcal{C}_{\mathscr{A}x}^{0.5\circledast} = \frac{0.7}{x} + \frac{0.7}{y} + \frac{0.7}{z}, \mathcal{C}_{\mathscr{A}y}^{0.5\circledast} = \frac{0.3}{x} + \frac{0.9}{y} + \frac{0.9}{z}$, and $\mathcal{C}_{\mathscr{A}z}^{0.5\circledast} = \frac{0.4}{x} + \frac{0.7}{y} + \frac{0.7}{z}$. Again take a $\mathcal{A} = \frac{1}{x} + \frac{0}{y} + \frac{0}{z} \in [0,1]^X$. By Definition 2.5, $\mathscr{A}_{\beta\mathcal{C}1}^{-\circledast}(\mathcal{A}) = \dfrac{0.3}{x} + \dfrac{0.1}{y} + \dfrac{0.3}{z}$, $\mathscr{A}_{\beta\mathcal{C}k}^{+\circledast}(\mathcal{A}) = \dfrac{0.7}{x} + \dfrac{0.3}{y} + \dfrac{0.4}{z}$ $(k = 1, 2)$, $\mathscr{A}_{\beta\mathcal{C}2}^{-\circledast}(\mathcal{A}) = \dfrac{0.6}{x} + \dfrac{0.1}{y} + \dfrac{0.3}{z}$, $\mathscr{A}_{\beta\mathcal{C}3}^{-\circledast}(\mathcal{A}) = \dfrac{0.3}{x} + \dfrac{0.1}{y} + \dfrac{0.1}{z}$, $\mathscr{A}_{\beta\mathcal{C}3}^{+\circledast}(\mathcal{A}) = \frac{0.7}{x} + \frac{0.7}{y} + \frac{0.7}{z}$, and thus, $\mathscr{A}_{\beta\mathcal{C}k}^{-\circledast}(\mathcal{A}) \not\subseteq \mathcal{A} \not\supseteq \mathscr{A}_{\beta\mathcal{C}k}^{+\circledast}(\mathcal{A})$ $(k = 1, 2, 3)$.*

(2) *From the aforementioned reason in (1) and in order to meet the inclusion property, we will propose three new approximation types based on β-FC$\mathscr{A}$ via $\circledast$-β-FCN$\mathscr{A}$. As a result, the methods in* [2, 27, 53] *will be updated to determine the best decision.*

(3) *We will present a new concept of $\circledcirc$-β-soft fuzzy complement neighborhoods, abbreviated as $\circledcirc$-β-FCN$\mathscr{A}$, which is a generalization of Definition 4 in* [22]. *The usefulness of this new concept is to define three other types of approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$, as well as list some of its key characteristics and give some examples.*

(4) *We will investigate some fundamental fuzzy topology properties as a theoretical application of three new approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$.*

(5) *We will propose a unique technique via $\mathbb{C}$-$\circledcirc$-approximation (in the sense of defining three new approximations) of $\mathcal{A}$, based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$ in the MADM field as a practical application.*

(6) *We will provide a comparison with the various β-soft fuzzy covering MADM models in* [27, 53] *and Algorithm 1 in Section 3 to illustrate the benefits of the proposed model.*

To make the sections of the paper (from Section 1 to Section 6) easier for readers to understand, the following organizational style will be used:

Section 1 (Introduction).
Section 2 (Preliminaries) divided into
2.1. Fuzzy sets, soft sets, and soft fuzzy sets.
2.2. β-Soft fuzzy covering, β-soft fuzzy neighborhood, $\circledast$-β-soft fuzzy complement neighborhood, and types of approximations.
2.3. Fuzzy topology.
2.4. The motivation for this study.
Section 3: (Three new types of approximations based on β-FC$\mathscr{A}$ via $\circledast$-β-FCN$\mathscr{A}$) divided into
3.1. Three new types of $\circledast$-lower and $\circledast$-upper approximations of $\mathcal{A}$.
3.2. Revise and update some of the algorithms found in [2, 27, 53].
Section 4: ($\circledcirc$-β-FCN$\mathscr{A}$ and three new types of approximations based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$) divided into
4.1. Fuzzy topology with three new approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$.
Section 5: (An innovative method by $\mathbb{C}$-$\circledcirc$-approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$ in the MADM field) divided into
5.1. Comparison with other MADM models based on β-FC$\mathscr{A}$.
5.2. Limitations of the proposed model in this work.
Section 6: (Conclusions and future work).

3 Three new types of approximations based on β-FC$\mathscr{A}$ via $\circledast$-β-FCN$\mathscr{A}$

As demonstrated in Section 2, we conclude that the Definition 2.5 (1)-(3) proposed by Zhang and Zhan [53] does not satisfy the inclusion property based on β-FC$\mathscr{A}$ via on $\circledast$-β-FCN$\mathscr{A}$, as illustrated in Example 2.9. Thus, the following will be presented in this section:

(1) Present three new types of approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledast$-β-FCN$\mathscr{A}$ which satisfies the inclusion property, together with an explanation of some fundamental characteristics and examples.

(2) Revise and update some of the algorithms found in [2, 27, 53] and provide several examples.

Now, we will go over this section in the manner described below:

3.1 Three new types of ⊛-lower and ⊛-upper approximations of $\mathcal{A}$

Definition 3.1. *Let $\mathcal{A} \in [0,1]^X$, $\beta \in (0,1]$, $\mathscr{A} \in [0,1]^{X\times I}$ be a β-FC, $x, y \in X$, and $k = 1,2,3$. The three new types of approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via ⊛-β-FCN$\mathscr{A}$ is a pair $\left(\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}), \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})\right)$, defined by*

$$\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})(x) = \begin{cases} \mathscr{A}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})(x), & \mathcal{A}(x) \geq 1-\beta, \\ 0, & \mathcal{A}(x) < 1-\beta \end{cases}$$

(called the new type k-⊛-lower approximation of $\mathcal{A}$) and

$$\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})(x) = \begin{cases} \mathscr{A}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})(x), & \mathcal{A}(x) \leq \beta, \\ 1, & \mathcal{A}(x) > \beta \end{cases}$$

(called the new type k-⊛-upper approximation of $\mathcal{A}$).

If $k = 1$ (resp., $k = 2$ and $k = 3$), then the a pair $\left(\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}), \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})\right)$ (resp., $\left(\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A}), \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})\right)$ and $\left(\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}3}(\mathcal{A}), \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}3}(\mathcal{A})\right)$) is called first (resp., second and third) new type approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via ⊛-β-FCN$\mathscr{A}$.

An illustration of Definition 3.1 is the following two examples:

Example 3.2. *Assume that $X = \{x, y, z\}$ be a 3-element set, $I = \{1, 2\}$ be a 2-element parameter set, and $\mathscr{A} \in [0,1]^{X\times I}$ is defined by $\mathscr{A}(1) = \frac{0.8}{x} + \frac{0.6}{y} + \frac{0.4}{z}$ and $\mathscr{A}(2) = \frac{0.3}{x} + \frac{0.4}{y} + \frac{0.7}{z}$. Then $\mathscr{A} \in [0,1]^{X\times I}$ is a β-FC$\mathscr{A}$ $(0 < \beta \leq 0.6)$ by Definition 2.2. Take a $\beta = 0.5$. By Definitions 2.3 and 2.4, we obtain: $\mathcal{N}^{0.5}_{\mathscr{A}x} = \mathcal{N}^{0.5}_{\mathscr{A}y} = \frac{0.8}{x} + \frac{0.6}{y} + \frac{0.4}{z}$, $\mathcal{N}^{0.5}_{\mathscr{A}z} = \frac{0.3}{x} + \frac{0.4}{y} + \frac{0.7}{z}$, $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x} = \frac{0.8}{x} + \frac{0.8}{y} + \frac{0.3}{z}$, $\mathcal{C}^{0.5\circledast}_{\mathscr{A}y} = \frac{0.6}{x} + \frac{0.6}{y} + \frac{0.4}{z}$, and $\mathcal{C}^{0.5\circledast}_{\mathscr{A}z} = \frac{0.4}{x} + \frac{0.4}{y} + \frac{0.7}{z}$. Again take a $\mathcal{A} = \frac{0.9}{x} + \frac{0.1}{y} + \frac{0.3}{z} \in [0,1]^X$. By Definition 3.1, $\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}k}(\mathcal{A}) = \frac{0.2}{x} + \frac{0}{y} + \frac{0}{z}$ $(k = 1, 3)$, $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}1}(\mathcal{A}) = \frac{1}{x} + \frac{0.6}{y} + \frac{0.4}{z}$, $\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}2}(\mathcal{A}) = \frac{0.4}{x} + \frac{0}{y} + \frac{0}{z}$, $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}2}(\mathcal{A}) = \frac{1}{x} + \frac{0.6}{y} + \frac{0.3}{z}$, $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}3}(\mathcal{A}) = \frac{1}{x} + \frac{0.8}{y} + \frac{0.4}{z}$, and thus, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}) \leq \mathcal{A} \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})$ $(k = 1, 2, 3)$.*

Example 3.3. *Consider Examples 4.10, 4.16, and 4.20 in [53], where $X = \{x_1, x_2, ..., x_6\}$, $\mathcal{A} = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.6}{x_6}$. By Definition 3.1, $\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}1}(\mathcal{A}) = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} + \frac{0.6}{x_6}$, $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}1}(\mathcal{A}) = \frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{1}{x_6}$, $\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}2}(\mathcal{A}) = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} + \frac{0.6}{x_6}$, $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}2}(\mathcal{A}) = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{1}{x_6}$, $\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}3}(\mathcal{A}) = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} + \frac{0.4}{x_6}$, and $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}3}(\mathcal{A}) = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{1}{x_6}$. Thus, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}) \leq \mathcal{A} \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})$ $(k = 1, 2, 3)$.*

The above inclusions (a fundamental characteristic of the innovative lower and upper approximation operators) actually hold in general.

Theorem 3.4. *Let $\mathcal{A} \in [0,1]^X$, $\beta \in (0,1]$, and $\mathscr{A} \in [0,1]^{X\times I}$ be a β-FC. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}) \leq \mathcal{A} \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})$ $(k = 1, 2, 3)$.*

Proof. Let $x \in X$. Then, we only prove $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}) \leq \mathcal{A} \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})$ when $k = 1$ by considering the following 6 cases:

Case 1. $\mathcal{A}(x) < 1 - \beta \leq \beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) = 0$ by Definition 3.1. Consequently, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(x) \leq \mathcal{A}(x)$. Again, by Definition 3.1, we get $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \bigvee\limits_{y\in X} [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \mathcal{A}(y)] \geq \mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x) \wedge \mathcal{A}(x) = \mathcal{A}(x)$ (since $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x) \geq \beta$). That is, $\mathcal{A}(x) \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$, and hence $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) \leq \mathcal{A}(x) \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$.

Case 2. $1 - \beta \leq \mathcal{A}(x) \leq \beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \bigwedge\limits_{y\in X} ([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \vee \mathcal{A}(y)) \leq [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)]' \vee \mathcal{A}(x) = \mathcal{A}(x)$ (because $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x) \geq \beta$ implies $[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)]' \leq \mathcal{A}(x)$). Also, $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \bigvee\limits_{y\in X} [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \mathcal{A}(y)] \geq \mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x) \wedge \mathcal{A}(x) = \mathcal{A}(x)$. Hence, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x) \leq \mathcal{A}(x) \leq \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$.

Case 3. $1-\beta \le \beta < \mathcal{A}(x)$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=1$ by Definition 3.1. Consequently, $\mathcal{A}(x) \le \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$. Again, by Definition 3.1, we get $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=\bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]'\vee\mathcal{A}(y)) \le [\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)]'\vee\mathcal{A}(x)=\mathcal{A}(x)$ (because $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)\ge\beta$ implies $[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)]'\le 1-\beta$). That is, $\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$, and hence $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)\le\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$.

Case 4. $\mathcal{A}(x)<\beta<1-\beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=0$ by Definition 3.1. Consequently, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)\le\mathcal{A}(x)$. Again, by Definition 3.1, we get $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=\bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)\wedge\mathcal{A}(y)]\ge\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)\wedge\mathcal{A}(x)=\mathcal{A}(x)$ (since $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)\ge\beta$). That is, $\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$, and hence $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)\le\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$.

Case 5. $\beta<\mathcal{A}(x)<1-\beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=0$ and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=1$ by Definition 3.1. Hence, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)\le\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$.

Case 6. $\beta<1-\beta\le\mathcal{A}(x)$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=1$ by Definition 3.1. Consequently, $\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$. Again, by Definition 3.1, we get $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=\bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]'\vee\mathcal{A}(y))\le[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)]'\vee\mathcal{A}(x)=\mathcal{A}(x)$ (because $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)\ge\beta$ implies $[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(x)]'\le 1-\beta$). That is, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)\le\mathcal{A}(x)$, and hence $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)\le\mathcal{A}(x)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)$. □

Furthermore, the three new types approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledast$-β-FCN$\mathscr{A}$ have the following properties (where $\mathcal{A}\in[0,1]^X$, $\beta\in(0,1]$, $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC, $\{\mathcal{A}_j\}_{j\in J}\subseteq[0,1]^X$, $J\ne\emptyset$, and $k=1,2,3$):

Theorem 3.5. (1) $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}')=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]'$ *and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A}')=[\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]'$.

(2) $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(b_X)=\begin{cases}b_X, & b\ge 1-\beta,\\ 0_X, & \text{otherwise},\end{cases}$ *and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(b_X)=\begin{cases}b_X, & b\le\beta,\\ 1_X, & \text{otherwise}.\end{cases}$ *Especially,* $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(b_X)=\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(b_X)=b_X$ *if* $b\in[1-\beta,\beta]$, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(0_X)=\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(0_X)=0_X$, *and* $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(1_X)=\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(1_X)=1_X$.

(3) $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}\Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big)=\bigwedge\limits_{j\in J}\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_j)$ *(i.e.,* $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}$ *is* $\wedge$*-preserving) and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big)=\bigvee\limits_{j\in J}\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_j)$ *(i.e.,* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}$ *is* $\vee$*-preserving).*

(4) $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_1)\le\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_2)$ *and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_1)\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_2)$ *if* $\mathcal{A}_1\le\mathcal{A}_2$.

(5) $\bigvee\limits_{j\in J}\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_j)\le\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big)$ *and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}\Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big)\le\bigwedge\limits_{j\in J}\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_j)$.

(6) $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}[\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]\le\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})$ *and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]$.

(7) $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})\le\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}[\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]$ *and* $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]\le\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})$.

Proof. (1) Let $x\in X$. We only prove $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}')(x)=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A})]'(x)$ when $k=1$ by considering the following 6 cases:

Case 1. $\mathcal{A}(x)\le\beta\le 1-\beta$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=\bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)\wedge\mathcal{A}(y)]$ by Definition 3.1. As $1-\beta\le 1-\mathcal{A}(x)$, it follows that $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=\bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]'\vee\mathcal{A}'(y))=\bigwedge\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)\wedge\mathcal{A}(y)]'=\Big(\bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)\wedge\mathcal{A}(y)]\Big)'=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)]'=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})]'(x)$.

Case 2. $\beta<\mathcal{A}(x)<1-\beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=0$ (since $\beta<1-\mathcal{A}(x)<1-\beta$) and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=1$ by Definition 3.1. Thus, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})]'(x)$.

Case 3. $\beta<1-\beta<\mathcal{A}(x)$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=1$ by Definition 3.1. Again, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=0$ (since $1-\mathcal{A}(x)<1-\beta$). Thus, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})]'(x)$.

Case 4. $\mathcal{A}(x)\le 1-\beta\le\beta$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=\bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)\wedge\mathcal{A}(y)]$ by Definition 3.1. Again, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=\bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]'\vee\mathcal{A}'(y))$ (since $1-\beta\le 1-\mathcal{A}(x)$) and $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})]'(x)$ by Case 1.

Case 5. $1-\beta\le\mathcal{A}(x)\le\beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=\bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]'\vee\mathcal{A}'(y))$ (since $1-\beta\le 1-\mathcal{A}(x)$) and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=\bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)\wedge\mathcal{A}(y)]$ by Definition 3.1. Thus $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})]'(x)$ by Case 1.

Case 6. If $1-\beta\le\beta<\mathcal{A}(x)$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})(x)=1$ by Definition 3.1. Again, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=0$ (since $1-\mathcal{A}(x)<1-\beta$). Thus, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}')(x)=[\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A})]'(x)$.

(2) Let $x\in X$, $k=1$, and consider the following 6 cases:

Case 1. $b \leq \beta < 1-\beta$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(b_X)(x) = \bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(y) \wedge b_X(y)] = b$ (since $\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(x) \geq \beta$). As $b < 1-\beta$, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 0$ by Definition 3.1.

Case 2. $\beta < b < 1-\beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 0$ and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 1$ by Definition 3.1.

Case 3. $\beta < 1-\beta \leq b$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(b_X)(x) = \bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{b_X x}(y)]' \vee b_X(y)) = b$ (since $\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(x) \geq \beta$) and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 1$ (since $\beta < b$) by Definition 3.1.

Case 4. $1-\beta \leq b \leq \beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(b_X)(x) = \bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(y)]' \vee b_X(y)) = b$ (since $\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(x) \geq \beta$) and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 1$ (since $\beta < b$) by Definition 3.1.

Case 5. $b < 1-\beta \leq \beta$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 0$ (since $b < 1-\beta$) by Definition 3.1 and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(b_X)(x) = \bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(y) \wedge b_X(y)] = b$ (since $\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(x) \geq \beta \geq b$).

Case 6. $1-\beta \leq \beta < b$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(b_X)(x) = \bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(y)]' \vee b_X(y)) = b$ (since $\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(x) \geq \beta$ which implies $[\mathcal{C}^{\beta\circledast}_{\{b_X\}x}(y)]' \leq 1-\beta \leq b$) and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(b_X)(x) = 1$ (since $\beta < b$) by Definition 3.1.

(3) Let $x \in X$. We first prove $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}\Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big) = \bigwedge\limits_{j\in J}\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_j)$ when $k = 1$ by considering the following 2 cases:

Case 1. $1-\beta \leq \Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big)(x)$. Then $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}\Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big)(x) = \bigwedge\limits_{y\in X}\Big([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \vee \bigwedge\limits_{j\in J}\mathcal{A}_j(y)\Big) = \bigwedge\limits_{y\in X}\bigwedge\limits_{j\in J}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \vee \mathcal{A}_j(y)) = \bigwedge\limits_{j\in J}\bigwedge\limits_{y\in X}([\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y)]' \vee \mathcal{A}_j(y)) = \bigwedge\limits_{j\in J}\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_j)(x)$ by Definition 3.1.

Case 2. $(\bigwedge\limits_{j\in J}\mathcal{A}_j)(x) < 1-\beta$. Then $(\mathcal{A}_i)(x) < 1-\beta$ (thus $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_i)(x) = 0$) for some $i \in J$, which implies $\bigwedge\limits_{j\in J}\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_j)(x) = 0$. Therefore, $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}\Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big)(x) = \bigwedge\limits_{j\in J}\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_j)(x)$.

Now, we prove $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big) = \bigvee\limits_{j\in J}\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}k}(\mathcal{A}_j)$ when $k = 1$ by considering the following 2 cases:

Case 1. $\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big)(x) \leq \beta$. Then $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big)(x) = \bigvee\limits_{y\in X}\Big[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \bigvee\limits_{j\in J}\mathcal{A}_j(y)\Big] = \bigvee\limits_{y\in X}\bigvee\limits_{j\in J}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \mathcal{A}_j(y)] = \bigvee\limits_{j\in J}\bigvee\limits_{y\in X}[\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}(y) \wedge \mathcal{A}_j(y)] = \bigvee\limits_{j\in J}\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_j)(x)$ by Definition 3.1.

Case 2. $\beta < (\bigvee\limits_{j\in J}\mathcal{A}_j)(x)$. Then $\beta < (\mathcal{A}_i)(x)$ (thus $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_i)(x) = 1$) for some $i \in J$, which implies $\bigvee\limits_{j\in J}\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_j)(x) = 1$. Therefore, $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big)(x) = \bigvee\limits_{j\in J}\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}1}(\mathcal{A}_j)(x)$.

(4) and (5) follows from (3), and (6) and (7) follow from Theorem 3.4. □

3.2 Revise and update some of the algorithms found in [2, 27, 53]

This subsection will be covered in the following manner:

(1) Update algorithms in [2, 27, 53] (i.e., these algorithms are based on rough approximations without the inclusion property) based on Definition 3.1, in addition to some examples.

(2) Obtaining the best decision by updating the algorithms in [2, 27, 53]. This is different from the procedures in [2, 27, 53] that failed to achieve the desired outcome because of the inclusion property, which generally prevents the right choice from being made. Stated differently, the outcomes of our updated methodologies for decision-making will be observable and trustworthy.

Algorithm 1 (Update of algorithms in [27, 53]).

Step 1. It is the same as Step I in [53].

Step 2. Compute the values $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x)$ and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x)$, as per Definition 3.1 $(\forall x \in X)$.

Step 3. Compute the ranking function value $\mathscr{I}(x)$ $(x \in X)$:

$$\mathscr{I}(x) = \mathcal{A}(x) + \mathscr{Q}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x) + \mathscr{Q}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x).$$

Step 4. Getting the final optimal decision-making according to the values $\mathscr{I}(x)$ $(x \in X)$ of the ranking function $\mathscr{I}$.

Further, we will use Algorithm 1 as an updated version by replacing the value lower and upper approximations in Definition 3.12 in [27] by the values $\mathscr{Q}^{-\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x)$ and $\mathscr{Q}^{+\circledast}_{\beta\mathcal{C}2}(\mathcal{A})(x)$, as per Definition 3.1, respectively, in the second step, the first step is the same as Steps 1 and 2 in [27], and the third and fourth steps are in Algorithm 1.

Next is an illustration of two examples.

Example 3.6. [27, 53] *$X = \{x_1, x_2, ..., x_6\}$, $E = \{\epsilon_1, \epsilon_2, ..., \epsilon_5\}$, $\mathscr{A} \in [0,1]^{X\times E}$, $\mathcal{N}^{0.5}_{\mathscr{A}x}$, and $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x}$ are given as the following (see also Tables 10–12 in Example 6.1 [53] and Tables 1–3 in Example 4.1 [27]):*

Step 1. (1) *$\mathscr{A} \in [0,1]^{X\times E}$ are* $\mathscr{A}(\epsilon_1) = \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.3}{x_5} + \frac{0.1}{x_6}$, $\mathscr{A}(\epsilon_2) = \frac{0.5}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.8}{x_5} + \frac{0.7}{x_6}$, $\mathscr{A}(\epsilon_3) = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.2}{x_6}$, $\mathscr{A}(\epsilon_4) = \frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.7}{x_5} + \frac{0.8}{x_6}$, *and* $\mathscr{A}(\epsilon_5) = \frac{0.4}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.3}{x_6}$.

(2) *The β-FN$\mathscr{A}$ are* $\mathcal{N}^{0.5}_{\mathscr{A}x_1} = \frac{0.5}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.5}_{\mathscr{A}x_2} = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.5}_{\mathscr{A}x_3} = \frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.2}{x_6}$, $\mathcal{N}^{0.5}_{\mathscr{A}x_4} = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.5}_{\mathscr{A}x_5} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5} + \frac{0.2}{x_6}$, *and* $\mathcal{N}^{0.5}_{\mathscr{A}x_6} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.7}{x_5} + \frac{0.7}{x_6}$.

(3) *The $\circledast$-β-FCN$\mathscr{A}$ are* $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x_1} = \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.1}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x_2} = \frac{0.2}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.2}{x_6}$, $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x_3} = \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6}$, $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x_4} = \frac{0.1}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x_5} = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}$, *and* $\mathcal{C}^{0.5\circledast}_{\mathscr{A}x_6} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4} + \frac{0.2}{x_5} + \frac{0.7}{x_6}$.

Step 2. *$\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}2}(\mathcal{A})(x)$ and $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}2}(\mathcal{A})(x)$ for $\mathcal{A} = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.6}{x_6}$ are computed by Definition 3.1 as follows:* $\mathscr{Q}^{-\circledast}_{0.5\mathcal{C}2}(\mathcal{A}) = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} + \frac{0.6}{x_6}$ *and* $\mathscr{Q}^{+\circledast}_{0.5\mathcal{C}2}(\mathcal{A}) = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{1}{x_6}$.

Step 3. *Compute the ranking function $\mathscr{I}(x)$:* $\mathscr{I}(x_1) = 0.6$, $\mathscr{I}(x_2) = 1.5$, $\mathscr{I}(x_3) = 2.3$, $\mathscr{I}(x_4) = 0.8$, $\mathscr{I}(x_5) = 0.8$, *and* $\mathscr{I}(x_6) = 2.2$.

Step 4. *Ranking results:* $x_3 \succ x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$.

The comparison of the Algorithm 1 based on Definition 3.1, Zhang and Zhan's approach [53], *and Nawar et al.'s approach* [27] *are listed in the following Table 1:*

Table 1: Comparison of the two approaches for Example 3.6

	Ranking	Best Decision
Algorithm 1	$x_3 \succ x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$	x_3
Zhang and Zhan's approach [53]	$x_3 \approx x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$	x_3 and x_6
Nawar et al.'s approach [27]	$x_3 \approx x_6 \succ x_2 \succ x_1 \approx x_4 \approx x_5$	x_3 and x_6

Example 3.7. [53] *X, E, $\mathscr{A} \in [0,1]^{X\times I}$ are as in Example 3.6, and $\mathcal{N}^{0.4}_{\mathscr{A}x}$ and $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x}$ are given as the following (see also Tables 13–14 in Example 6.3 [53]):*

Step 1. (1) *The β-FN$\mathscr{A}$ are* $\mathcal{N}^{0.4}_{\mathscr{A}x_1} = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.4}_{\mathscr{A}x_2} = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.4}_{\mathscr{A}x_3} = \frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.4}_{\mathscr{A}x_4} = \frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{N}^{0.4}_{\mathscr{A}x_5} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5} + \frac{0.2}{x_6}$, *and* $\mathcal{N}^{0.4}_{\mathscr{A}x_6} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.7}{x_5} + \frac{0.7}{x_6}$.

(2) *The $\circledast$-β-FCN$\mathscr{A}$ are* $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x_1} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x_2} = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4} + \frac{0.2}{x_5} + \frac{0.2}{x_6}$, $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x_3} = \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6}$, $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x_4} = \frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.1}{x_5} + \frac{0.1}{x_6}$, $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x_5} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}$,

and $\mathcal{C}^{0.4\circledast}_{\mathscr{A}x_6} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} + \frac{0.2}{x_5} + \frac{0.7}{x_6}$.

Step 2. $\mathscr{Q}^{-\circledast}_{0.4\mathcal{C}2}(\mathcal{A})(x)$ *and* $\mathscr{Q}^{+\circledast}_{0.4\mathcal{C}2}(\mathcal{A})(x)$ *for* $\mathcal{A} = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.6}{x_6}$ *are computed by Definition 3.1 as follows:* $\mathscr{Q}^{-\circledast}_{0.4\mathcal{C}2}(\mathcal{A}) = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} + \frac{0.6}{x_6}$ *and* $\mathscr{Q}^{-\circledast}_{0.4\mathcal{C}2}(\mathcal{A}) = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{1}{x_6}$.

Step 3. *Compute the ranking function* $\mathscr{I}(x)$*:* $\mathscr{I}(x_1) = 0.6,\ \mathscr{I}(x_2) = 1.5,\ \mathscr{I}(x_3) = 2.3,\ \mathscr{I}(x_4) = 0.8,\ \mathscr{I}(x_5) = 0.8$, *and* $\mathscr{I}(x_6) = 2.2$.

Step 4. *Ranking results:* $x_3 \succ x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$.

The comparison of the Algorithm 2 based on Definition 3.1 and Zhang and Zhan's approach [53] *are listed in the following Table 2:*

Table 2: Comparison of the two approaches for Example 3.7

	Ranking	Best Decision
Algorithm 1	$x_3 \succ x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$	x_3
Zhang and Zhan's approach [53]	$x_6 \succ x_3 \succ x_2 \succ x_4 \succ x_1 \approx x_5$	x_6

Algorithm 2 (Update of algorithm in [2]).

Step 1. It is the same as Steps 1-3 in [2].

Step 2. Define and compute the new OMGCFRLAO $\mathscr{D}^{-\circledast}_{\beta\sum_{i=1}^n \mathbb{C}_i}(\mathcal{A})(x)$ and the new OMGCFRUAO $\mathscr{D}^{+\circledast}_{\beta\sum_{i=1}^n \mathbb{C}_i}(\mathcal{A})(x)$ in the sense of Definition 3.1 as follows ($\mathcal{A} \in [0,1]^X$, $\beta \in (0,1]$, and $x, y \in X$):

$$\mathscr{D}^{-\circledast}_{\beta\sum_{i=1}^n \mathbb{C}_i}(\mathcal{A})(x) = \begin{cases} \bigvee\limits_{i\in I}^{n} \bigwedge\limits_{y\in X} ([\mathcal{C}^{\beta\circledast}_{\mathbb{C}_i x}(y)]' \vee \mathcal{A}(y)), & \mathcal{A}(x) \geq 1-\beta, \\ 0, & \mathcal{A}(x) < 1-\beta \end{cases}$$

and

$$\mathscr{D}^{+\circledast}_{\beta\sum_{i=1}^n \mathbb{C}_i}(\mathcal{A})(x) = \begin{cases} \bigwedge\limits_{i\in I}^{n} \bigvee\limits_{y\in X} [\mathcal{C}^{\beta\circledast}_{\mathbb{C}_i x}(y) \wedge \mathcal{A}(y)], & \mathcal{A}(x) \leq \beta, \\ 1, & \mathcal{A}(x) > \beta. \end{cases}$$

Step 3. Compute the ranking function value $\mathscr{I}(x)$ $(x \in X)$:

$$\mathscr{I}(x) = \mathcal{A}(x) + \mathscr{D}^{-\circledast}_{\beta\sum_{i=1}^n \mathbb{C}_i}(\mathcal{A})(x) + \mathscr{D}^{+\circledast}_{\beta\sum_{i=1}^n \mathbb{C}_i}(\mathcal{A})(x).$$

Step 4. Getting the final optimal decision-making according to the values $\mathscr{I}(x)$ $(x \in X)$ of the ranking function $\mathscr{I}$.

Next is an illustration example.

Example 3.8. [2] $X = \{x_1, x_2, ..., x_6\}$, $\mathbb{C}_1 = \{C_{11}, C_{12}, ..., C_{15}\} \subseteq [0,1]^X$, $\mathbb{C}_2 = \{C_{21}, C_{22}, ..., C_{25}\} \subseteq [0,1]^X$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x}$, *and* $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2 x}$ *are given as the following (see also Tables 5–6 in Example 15* [2]*):*

Step 1. (1) $\mathbb{C}_1, \mathbb{C}_2 \in [0,1]^X$ *are* $C_{11} = \frac{0.8}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6}$, $C_{12} = \frac{0.4}{x_1} + \frac{0.1}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{0.5}{x_5} + \frac{0.3}{x_6}$, $C_{13} = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.8}{x_4} + \frac{0.7}{x_5} + \frac{0.2}{x_6}$, $C_{14} = \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.8}{x_5} + \frac{0.1}{x_6}$, $C_{15} = \frac{0.7}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5} + \frac{0.8}{x_6}$, $C_{21} = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.2}{x_4} + \frac{0.7}{x_5} + \frac{0.3}{x_6}$, $C_{22} = \frac{0.3}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.6}{x_4} + \frac{0.3}{x_5} + \frac{0.6}{x_6}$, $C_{23} = \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.1}{x_6}$, $C_{24} = \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.5}{x_5} + \frac{0.6}{x_6}$, *and* $C_{25} = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.9}{x_3} + \frac{0.4}{x_4} + \frac{0.7}{x_5} + \frac{0.7}{x_6}$.

(2) *The* ⊛*-β-FCN*$\mathscr{A}$ *of* $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x}$ *and* $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2 x}$ *are* $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x_1} = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.7}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x_2} = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x_3} = \frac{0.4}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.3}{x_5} + \frac{0.6}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x_4} = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x_5} = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_1 x_6} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.7}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2 x_1} = \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4} +$

$\frac{0.3}{x_5}+\frac{0.3}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2x_2}=\frac{0.6}{x_1}+\frac{0.6}{x_2}+\frac{0.1}{x_3}+\frac{0.2}{x_4}+\frac{0.1}{x_5}+\frac{0.1}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2x_3}=\frac{0.3}{x_1}+\frac{0.3}{x_2}+\frac{0.6}{x_3}+\frac{0.3}{x_4}+\frac{0.3}{x_5}+\frac{0.6}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2x_4}=\frac{0.2}{x_1}+\frac{0.2}{x_2}+\frac{0.2}{x_3}+\frac{0.6}{x_4}+\frac{0.2}{x_5}+\frac{0.3}{x_6}$, $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2x_5}=\frac{0.5}{x_1}+\frac{0.5}{x_2}+\frac{0.3}{x_3}+\frac{0.3}{x_4}+\frac{0.6}{x_5}+\frac{0.3}{x_6}$, *and* $\mathcal{C}^{0.6\circledast}_{\mathbb{C}_2x_6}=\frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.3}{x_3}+\frac{0.1}{x_4}+\frac{0.1}{x_5}+\frac{0.6}{x_6}$.

Step 2. *$\mathscr{D}^{-\circledast}_{0.6\mathbb{C}_1+\mathbb{C}_2}(\mathcal{A})(x)$ and $\mathscr{D}^{+\circledast}_{0.6\mathbb{C}_1+\mathbb{C}_2}(\mathcal{A})(x)$ for $\mathcal{A}=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.1}{x_4}+\frac{0.5}{x_5}+\frac{0.8}{x_6}$ are computed as follows: $\mathscr{D}^{-\circledast}_{0.6\mathbb{C}_1+\mathbb{C}_2}(\mathcal{A})=\frac{0.5}{x_1}+\frac{0}{x_2}+\frac{0.7}{x_3}+\frac{0}{x_4}+\frac{0.5}{x_5}+\frac{0.7}{x_6}$ and $\mathscr{D}^{+\circledast}_{0.6\mathbb{C}_1+\mathbb{C}_2}(\mathcal{A})=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{1}{x_3}+\frac{0.3}{x_4}+\frac{0.5}{x_5}+\frac{1}{x_6}$.*

Step 3. *Compute the ranking function $\mathscr{I}(x)$: $\mathscr{I}(x_1)=1.7$, $\mathscr{I}(x_2)=0.6$, $\mathscr{I}(x_3)=2.4$, $\mathscr{I}(x_4)=0.4$, $\mathscr{I}(x_5)=1.5$, and $\mathscr{I}(x_6)=2.5$.*

Step 4. *Ranking results: $x_6\succ x_3\succ x_1\succ x_5\succ x_2\succ x_4$.*

The comparison of the Algorithm 2 based on new OMGCFRLAO and the new OMGCFRUAO in the sense of Definition 3.1 and Atef and El Atik's approach [2] are listed in the following Table 3[1] *:*

Table 3: Comparison of the two approaches for Example 3.8

	Ranking	Best Decision
Algorithm 2	$x_6\succ x_3\succ x_1\succ x_5\succ x_2\succ x_4$	x_6
Atef and El Atik's approach [2]	$x_3\approx x_6\succ x_5\succ x_1\succ x_2\approx x_4$	x_3 and x_6

4 ⊙-β-FCN$\mathscr{A}$ and three new types of approximations based on β-FC$\mathscr{A}$ via ⊙-β-FCN$\mathscr{A}$

This section's goal is as follows:

(1) Introduce a new notion of ⊙-β-FCN$\mathscr{A}$, whose idea is taken from Definition 4 in Ma [22] and present three new approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via ⊙-β-FCN$\mathscr{A}$, along with using examples and an explanation of certain basic properties.

(2) Study the fundamental characteristics of the fuzzy topology related to three new approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via ⊙-β-FCN$\mathscr{A}$.

As seen below, we now start with the idea of ⊙-β-FCN$\mathscr{A}$.

Definition 4.1. *Let $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC, $\beta\in(0,1]$, and $x\in X$. Then $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}=\bigwedge\{[\mathscr{A}(i)]'\mid i\in I,[\mathscr{A}(i)]'(x)>1-\beta\}=\bigwedge\{[\mathscr{A}(i)]'\mid i\in I,\mathscr{A}(i)(x)<\beta\}=[\bigvee\{\mathscr{A}(i)\mid i\in I,\mathscr{A}(i)(x)<\beta\}]'$ on X is called a ⊙-β-fuzzy complement neighborhood of element x induced by the SFS $\mathscr{A}$.*

Example 4.2. *(Continued from Example 3.2). As $[\mathscr{A}(1)]'=\frac{0.2}{x}+\frac{0.4}{y}+\frac{0.6}{z}$ and $[\mathscr{A}(2)]'=\frac{0.7}{x}+\frac{0.6}{y}+\frac{0.3}{z}$. Then, $\mathcal{C}^{0.5\odot}_{\mathscr{A}x}=\mathcal{C}^{0.5\odot}_{\mathscr{A}y}=\frac{0.7}{x}+\frac{0.6}{y}+\frac{0.3}{z}$ and $\mathcal{C}^{0.5\odot}_{\mathscr{A}z}=\frac{0.2}{x}+\frac{0.4}{y}+\frac{0.6}{z}$ by Definition 4.1.*

Remark 4.3. *The notions of $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}$ and $\mathcal{C}^{\beta\circledast}_{\mathscr{A}x}$ are different. For example, $\mathcal{C}^{0.5\odot}_{\mathscr{A}x}\neq\mathcal{C}^{0.5\circledast}_{\mathscr{A}x}$, $\mathcal{C}^{0.5\odot}_{\mathscr{A}y}\neq\mathcal{C}^{0.5\circledast}_{\mathscr{A}y}$, and $\mathcal{C}^{0.5\odot}_{\mathscr{A}y}\neq\mathcal{C}^{0.5\circledast}_{\mathscr{A}y}$ by Examples 3.2 and 4.2.*

Theorem 4.4. *Let $\mathcal{A}\in[0,1]^X$, $\beta\in(0,1]$, and $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC. Then*

(1) $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(x)\geq 1-\beta$ $(\forall x\in X)$ [2].

(2) *If $1-\beta<\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y)$ and $1-\beta<\mathcal{C}^{\beta\odot}_{\mathscr{A}y}(z)$, then $1-\beta<\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(z)$ $(\forall\{x,y,z\}\subseteq X)$.*

(3) *For each $i\in I$, $\bigvee\{\mathcal{C}^{\beta\odot}_{\mathscr{A}x}\mid x\in X,\mathscr{A}(i)(x)<\beta\}\leq[\mathscr{A}(i)]'$.*

(4) $\mathcal{C}^{\beta_2\odot}_{\mathscr{A}x}\leq\mathcal{C}^{\beta_1\odot}_{\mathscr{A}x}$ *if $\beta_1\leq\beta_2$ $(\beta_1,\beta_2\in(0,1])$.*

Proof. (1) and (4) are obvious.

(2) Notice that $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y)>1-\beta$ iff $[\mathscr{A}(i)]'(y)>a$ (whenever $[\mathscr{A}(i)]'(x)>1-\beta$) for some $a>1-\beta$ and that $\mathcal{C}^{\beta\odot}_{\mathscr{A}y}(z)>1-\beta$ iff $[\mathscr{A}(i)]'(z)>b$ (whenever $[\mathscr{A}(i)]'(y)>1-\beta$) for some $b>1-\beta$, we have $[\mathscr{A}(i)]'(z)>\min\{a,b\}$ (whenever $[\mathscr{A}(i)]'(x)>1-\beta$), which implies $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(z)>1-\beta$.

(3) Assume $\mathscr{A}(i)(x)<\beta$, then $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}\leq[\mathscr{A}(i)]'$ by Definition 4.1, which implies $\bigvee\{\mathcal{C}^{\beta\odot}_{\mathscr{A}x}\mid\mathscr{A}(i)(x)<\beta\}\leq[\mathscr{A}(i)]'$. □

[1] The ranking functions $\mathscr{I}(x)$ of Atef and El Atik's approach [2] are $\mathscr{I}(x_1)=0.42$, $\mathscr{I}(x_2)=\mathscr{I}(x_2)=0.36$, $\mathscr{I}(x_3)=\mathscr{I}(x_6)=0.66$, and $\mathscr{I}(x_5)=0.5$ by Step 4 in Subsection 7.3 [51]

[2] $\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(x)>1-\beta$ holds if $\{1-\mathscr{A}(i)(x)\mid i\in I,1-\mathscr{A}(i)(x)>1-\beta\}$ a finite set.

Three new approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ are what we propose in the following:

Definition 4.5. *Let $\mathcal{A} \in [0,1]^X$, $\beta \in (0,1]$, $\mathscr{A} \in [0,1]^{X\times I}$ be a β-FC, and $x, y \in X$.*

(1) *The first new type of approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ is a pair $\big(\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A}), \mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A})\big)$, defined by*

$$\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \begin{cases} \bigwedge\limits_{y\in X} ([\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y)]' \vee \mathcal{A}(y)), & \mathcal{A}(x) \geq 1-\beta, \\ 0, & \mathcal{A}(x) < 1-\beta \end{cases}$$

(called the new type 1-$\odot$-lower approximation of $\mathcal{A}$) and

$$\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A})(x) = \begin{cases} \bigvee\limits_{y\in X} [\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y) \wedge \mathcal{A}(y)], & \mathcal{A}(x) \leq \beta, \\ 1, & \mathcal{A}(x) > \beta \end{cases}$$

(called the new type 1-$\odot$-upper approximation of $\mathcal{A}$).

(2) *The second new type of approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ is a pair $\big(\mathscr{Q}^{-\odot}_{\beta\mathcal{C}2}(\mathcal{A}), \mathscr{Q}^{+\odot}_{\beta\mathcal{C}2}(\mathcal{A})\big)$, defined by*

$$\mathscr{Q}^{-\odot}_{\beta\mathcal{C}2}(\mathcal{A})(x) = \begin{cases} \bigwedge\limits_{y\in X} ([\mathcal{N}^{\beta}_{\mathscr{A}x}(y)]' \vee [\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y)]' \vee \mathcal{A}(y)), & \mathcal{A}(x) \geq 1-\beta \\ 0, & \mathcal{A}(x) < 1-\beta \end{cases}$$

(called the new type 2-$\odot$-lower approximation of $\mathcal{A}$) and

$$\mathscr{Q}^{+\odot}_{\beta\mathcal{C}2}(\mathcal{A})(x) = \begin{cases} \bigvee\limits_{y\in X} [\mathcal{N}^{\beta}_{\mathscr{A}x}(y) \wedge \mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y) \wedge \mathcal{A}(y)], & \mathcal{A}(x) \leq \beta \\ 1, & \mathcal{A}(x) > \beta \end{cases}$$

(called the new type 2-$\odot$-upper approximation of $\mathcal{A}$).

(3) *The third new type of approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ is a pair $\big(\mathscr{Q}^{-\odot}_{\beta\mathcal{C}3}(\mathcal{A}), \mathscr{Q}^{+\odot}_{\beta\mathcal{C}3}(\mathcal{A})\big)$, defined by*

$$\mathscr{Q}^{-\odot}_{\beta\mathcal{C}3}(\mathcal{A})(x) = \begin{cases} \bigwedge\limits_{y\in X} \big(([\mathcal{N}^{\beta}_{\mathscr{A}x}(y)]' \wedge [\mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y)]') \vee \mathcal{A}(y)\big), & \mathcal{A}(x) \geq 1-\beta \\ 0, & \mathcal{A}(x) < 1-\beta \end{cases}$$

(called the new type 3-$\odot$-lower approximation of $\mathcal{A}$) and

$$\mathscr{Q}^{+\odot}_{\beta\mathcal{C}3}(\mathcal{A})(x) = \begin{cases} \bigvee\limits_{y\in X} [(\mathcal{N}^{\beta}_{\mathscr{A}x}(y) \vee \mathcal{C}^{\beta\odot}_{\mathscr{A}x}(y)) \wedge \mathcal{A}(y)], & \mathcal{A}(x) \leq \beta \\ 1, & \mathcal{A}(x) > \beta \end{cases}$$

(called the new type 3-$\odot$-upper approximation of $\mathcal{A}$).

The next numerical example is provided to serve as an illustration of Definition 4.5.

Example 4.6. *(Continued from Examples 3.2 and 4.2). By Definition 4.5, we obtain $\mathscr{Q}^{-\odot}_{0.5\mathcal{C}k}(\mathcal{A}) = \frac{0.4}{x} + \frac{0}{y} + \frac{0}{z}$ $(k = 1,2,3)$, $\mathscr{Q}^{+\odot}_{0.5\mathcal{C}k}(\mathcal{A}) = \frac{1}{x} + \frac{0.7}{y} + \frac{0.3}{z}$ $(k = 1,2)$, and $\mathscr{Q}^{+\odot}_{0.5\mathcal{C}3}(\mathcal{A}) = \frac{1}{x} + \frac{0.8}{y} + \frac{0.3}{z}$.*

The three new types approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ have the following properties (where $\mathcal{A} \in [0,1]^X$, $\beta \in (0,1]$, $\mathscr{A} \in [0,1]^{X\times I}$ be a β-FC, $\{\mathcal{A}_j\}_{j\in J} \subseteq [0,1]^X$, $J \neq \emptyset$, and $k = 1,2,3$):

Theorem 4.7. (1) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}) \leq \mathcal{A} \leq \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})$.

(2) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}') = [\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})]'$ *and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A}') = [\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})]'$.

(3) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(b_X) = \begin{cases} b_X, & b \geq 1-\beta, \\ 0_X, & \text{otherwise}, \end{cases}$ *and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(b_X) = \begin{cases} b_X, & b \leq \beta, \\ 1_X, & \text{otherwise}. \end{cases}$ *Especially,* $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(b_X) = \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(b_X) = b_X$ *if* $b \in [1-\beta, \beta]$, $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(0_X) = \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(0_X) = 0_X$, *and* $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(1_X) = \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(1_X) = 1_X$.

(4) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}\Big(\bigwedge\limits_{j\in J} \mathcal{A}_j\Big) = \bigwedge\limits_{j\in J} \mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}_j)$ *(i.e., $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}$ is $\wedge$-preserving) and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}\Big(\bigvee\limits_{j\in J} \mathcal{A}_j\Big) = \bigvee\limits_{j\in J} \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A}_j)$ *(i.e., $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}$ is $\vee$-preserving).*

(5) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}_1) \leq \mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}_2)$ *and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A}_1) \leq \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A}_2)$ *if* $\mathcal{A}_1 \leq \mathcal{A}_2$.

(6) $\bigvee\limits_{j\in J}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}_j)\leq\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}\Big(\bigvee\limits_{j\in J}\mathcal{A}_j\Big)$ *and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}\Big(\bigwedge\limits_{j\in J}\mathcal{A}_j\Big)\leq\bigwedge\limits_{j\in J}\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A}_j)$.
(7) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}[\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})]\leq\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})$ *and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}[\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})]$.
(8) $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})\leq\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}[\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})]$ *and* $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}[\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})]\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})$.

Proof. By Definition 4.5, the proof of (1) and (2)-(8) are similar to those of Theorem 3.4 and Theorem 3.5, respectively. □

4.1 Fuzzy topology with three new approximations of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$

Definition 4.8. *Let $\mathcal{A}\in[0,1]^X$, $\beta\in(0,1]$, $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC, and $k=1,2,3$. If $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})=\mathcal{A}$ (resp., $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})=\mathcal{A}$), then $\mathcal{A}$ is called a $\odot$-lower (resp. $\odot$-upper) $[0,1]$-definable with respect to $\mathscr{A}$. The set of all $\odot$-lower and $\odot$-upper definable fuzzy sets is denoted by, respectively,*

$$\mathcal{H}^{-\odot}=\{\mathcal{A}\in[0,1]^X:\ \mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})=\mathcal{A}\}\ \textit{and}\ \mathcal{H}^{+\odot}=\{\mathcal{A}\in[0,1]^X:\ \mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})=\mathcal{A}\}.$$

We offer some fundamental features on $\mathcal{H}^{-\odot}$ and $\mathcal{H}^{-\odot}$, as shown below.

Theorem 4.9. *Let $\mathcal{A}\in[0,1]^X$, $\beta\in(0,1]$, $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC, and $k=1,2,3$. Then*
(1) *$\mathcal{H}^{-\odot}$ and $\mathcal{H}^{+\odot}$ are Alexandroff fuzzy topologies.*
(2) $\mathcal{H}^{\odot}=\{\mathcal{A}\in[0,1]^X:\ \mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A})=\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A})\}$.

Proof. (1) Firstly, we only prove $\mathcal{H}^{-\odot}$ is an Alexandroff fuzzy topology when $k=1$ by considering the following $(\mathcal{L}1)$ and $(\mathcal{L}2)$ of Definition 2.6 and $(\mathcal{LCO}1)$ of Definition 2.8:

$(\mathcal{L}1)$ $0_X,1_X\in\mathcal{H}^{-\odot}$ can be obtained from Theorem 4.7(3).

$(\mathcal{L}2)$ $\forall\{\mathcal{A}_j\}_{j\in J}\subseteq\mathcal{H}^{-\odot}$ and J is any index set, one gets that

$$\begin{aligned}\bigvee_{j\in J}\mathcal{A}_j&=\bigvee_{j\in J}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\qquad\text{(by the definition of }\mathcal{H}^{-\odot})\\&\leq\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}\Big(\bigvee_{j\in J}\mathcal{A}_j\Big)\qquad\text{(by Theorem 4.7(6))}\\&\leq\bigvee_{j\in J}\mathcal{A}_j\quad\text{(by Theorem 4.7(1)).}\end{aligned}$$

Consequently, it may be concluded that $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\bigvee\limits_{j\in J}\mathcal{A}_j)=\bigvee\limits_{j\in J}\mathcal{A}_j$, and thus, $\bigvee\limits_{j\in J}\mathcal{A}_j\in\mathcal{H}^{-\odot}$.

$(\mathcal{LCO}1)$ $\forall\{\mathcal{A}_j\}_{j\in J}\subseteq\mathcal{H}^{-\odot}$ and J is any index set, one gets that

$$\begin{aligned}\bigwedge_{j\in J}\mathcal{A}_j&=\bigwedge_{j\in J}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\qquad\text{(by the definition of }\mathcal{H}^{-\odot})\\&=\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}\Big(\bigwedge_{j\in J}\mathcal{A}_j\Big)\qquad\text{(by Theorem 4.7(4)).}\end{aligned}$$

Consequently, it may be concluded that $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\bigwedge\limits_{j\in J}\mathcal{A}_j)=\bigwedge\limits_{j\in J}\mathcal{A}_j$, and thus, $\bigwedge\limits_{j\in J}\mathcal{A}_j\in\mathcal{H}^{-\odot}$. Hence, $\mathcal{H}^{-\odot}$ is an Alexandroff fuzzy topology.

Secondly, we only prove $\mathcal{H}^{+\odot}$ is an Alexandroff fuzzy topology when $k=1$ by considering the following $(\mathcal{L}1)$ and $(\mathcal{L}2)$ of Definition 2.6 and $(\mathcal{LCO}1)$ of Definition 2.8:

$(\mathcal{L}1)$ $0_X,1_X\in\mathcal{H}^{+\odot}$ can be obtained from Theorem 4.7(3).

$(\mathcal{L}2)$ $\forall\{\mathcal{A}_j\}_{j\in J}\subseteq\mathcal{H}^{+\odot}$ and J is any index set, one gets that

$$\begin{aligned}\bigvee_{j\in J}\mathcal{A}_j&\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}\Big(\bigvee_{j\in J}\mathcal{A}_j\Big)\qquad\text{(by Theorem 4.7(1))}\\&=\bigvee_{j\in J}\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\qquad\text{(by Theorem 4.7(4))}\\&=\bigvee_{j\in J}\mathcal{A}_j\qquad\text{(by the definition of }\mathcal{H}^{+\odot}).\end{aligned}$$

Consequently, it may be concluded that $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\bigvee\limits_{j\in J}\mathcal{A}_j)=\bigvee\limits_{j\in J}\mathcal{A}_j$, and thus $\bigvee\limits_{j\in J}\mathcal{A}_j\in\mathcal{H}^{+\odot}$.

($\mathcal{LCO}1$) $\forall\{\mathcal{A}_j\}_{j\in J}\subseteq\mathcal{H}^{-\odot}$ and J is any index set, one gets that

$$\begin{aligned}\bigwedge_{j\in J}\mathcal{A}_j&\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}\Big(\bigwedge_{j\in J}\mathcal{A}_j\Big)\quad\text{(by Theorem 4.7(1))}\\&\leq\bigwedge_{j\in J}\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\quad\text{(by Theorem 4.7(6))}\\&=\bigwedge_{j\in J}\mathcal{A}_j\quad\text{(by the definition of }\mathcal{H}^{+\odot}).\end{aligned}$$

Consequently, it may be concluded that $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\bigwedge\limits_{j\in J}\mathcal{A}_j)=\bigwedge\limits_{j\in J}\mathcal{A}_j$, and thus, $\bigwedge\limits_{j\in J}\mathcal{A}_j\in\mathcal{H}^{+\odot}$. Hence, $\mathcal{H}^{+\odot}$ is an Alexandroff fuzzy topology.

(2) We only prove $\mathcal{H}$ is an Alexandroff fuzzy topology when $k=1$ by considering the following ($\mathcal{L}1$) and ($\mathcal{L}2$) of Definition 2.6 and ($\mathcal{LCO}1$) of Definition 2.8:

($\mathcal{L}1$) $0_X,1_X\in\mathcal{H}^{\odot}$ can be obtained from Theorem 4.7(3).

($\mathcal{L}2$) $\forall\{\mathcal{A}_j\}_{j\in J}\subseteq\mathcal{H}^{\odot}$ and J is any index set, one gets that

$$\begin{aligned}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}\Big(\bigvee_{j\in J}\mathcal{A}_j\Big)&\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}\Big(\bigvee_{j\in J}\mathcal{A}_j\Big)\quad\text{(by Theorem 4.7(1))}\\&=\bigvee_{j\in J}\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\quad\text{(by Theorem 4.7(4))}\\&=\bigvee_{j\in J}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\quad\text{(by the definition of }\mathcal{H}^{\odot})\\&\leq\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}\Big(\bigvee_{j\in J}\mathcal{A}_j\Big)\quad\text{(by Theorem 4.7(6)).}\end{aligned}$$

Consequently, it may be concluded that $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\bigvee\limits_{j\in J}\mathcal{A}_j)=\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\bigvee\limits_{j\in J}\mathcal{A}_j)$, and thus, $\bigvee\limits_{j\in J}\mathcal{A}_j\in\mathcal{H}^{\odot}$.

($\mathcal{LCO}1$) $\forall\{\mathcal{A}_j\}_{j\in J}\subseteq\mathcal{H}^{\odot}$ and J is any index set, one gets that

$$\begin{aligned}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}\Big(\bigwedge_{j\in J}\mathcal{A}_j\Big)&\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}\Big(\bigwedge_{j\in J}\mathcal{A}_j\Big)\quad\text{(by Theorem 4.7(1))}\\&\leq\bigwedge_{j\in J}\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\quad\text{(by Theorem 4.7(6))}\\&=\bigwedge_{j\in J}\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A}_j)\quad\text{(by the definition of }\mathcal{H}^{\odot})\\&=\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}\Big(\bigwedge_{j\in J}\mathcal{A}_j\Big)\quad\text{(by Theorem 4.7(4)).}\end{aligned}$$

Consequently, it may be concluded that $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\bigwedge\limits_{j\in J}\mathcal{A}_j)=\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\bigwedge\limits_{j\in J}\mathcal{A}_j)$, and thus, $\bigwedge\limits_{j\in J}\mathcal{A}_j\in\mathcal{H}^{\odot}$. Hence, $\mathcal{H}^{\odot}$ is an Alexandroff fuzzy topology. □

Theorem 4.10. *Let $\mathcal{A}\in[0,1]^X$, $\beta\in(0,1]$, $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC, and $k=1,2,3$. Then, $\mathcal{H}^{\odot}=\mathcal{H}^{-\odot}\wedge\mathcal{H}^{+\odot}$.*

Proof. We will prove $\mathcal{H}^{\odot}=\mathcal{H}^{-\odot}\wedge\mathcal{H}^{+\odot}$ and take $k=1$. Firstly, by Definition 4.8, for each $\mathcal{A}\in\mathcal{H}^{-\odot}\wedge\mathcal{H}^{+\odot}$, we obtain $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A})=\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A})=\mathcal{A}$ and thus, $\mathcal{A}\in\mathcal{H}^{\odot}$. Secondly, for each $\mathcal{A}\in\mathcal{H}^{\odot}$ and by Theorem 4.7(1), we obtain $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A})\leq\mathcal{A}\leq\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A})$. Also, by $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A})=\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A})$ for each $\mathcal{A}\in\mathcal{H}^{\odot}$, we obtain $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}1}(\mathcal{A})=\mathcal{A}=\mathscr{Q}^{+\odot}_{\beta\mathcal{C}1}(\mathcal{A})$, and thus $\mathcal{A}\in\mathcal{H}^{-\odot}\wedge\mathcal{H}^{+\odot}$. Hence, $\mathcal{H}^{\odot}=\mathcal{H}^{-\odot}\wedge\mathcal{H}^{+\odot}$. □

In the follow-up, we indicate $\mathcal{Z}^{-\odot}=\{\mathscr{Q}^{-\odot}_{\beta\mathcal{C}k}(\mathcal{A}):\mathcal{A}\in[0,1]^X\}$ and $\mathcal{Z}^{+\odot}=\{\mathscr{Q}^{+\odot}_{\beta\mathcal{C}k}(\mathcal{A}):\mathcal{A}\in[0,1]^X\}$ $(k=1,2,3)$.

Theorem 4.11. *Let $\mathcal{A}\in[0,1]^X$, $\beta\in(0,1]$, $\mathscr{A}\in[0,1]^{X\times I}$ be a β-FC, and $k=1,2,3$. Then,*

(1) *$\mathcal{Z}^{-\odot}$ is a fuzzy semi-topology.*

(2) *$\mathcal{Z}^{+\odot}$ is a fuzzy semi-co-topology.*

Proof. (1) We only prove $\mathcal{Z}^{-\circledcirc}$ is a fuzzy semi-topology when $k=1$ by considering the following $(\mathcal{L}1)$ and $(\mathcal{L}3)$ of Definition 2.6 (see, Definition 2.8 $(\mathcal{LCO}2)$):

$(\mathcal{L}1)$ $0_X, 1_X \in \mathcal{H}^{-\circledcirc}$ can be obtained from Theorem 4.7(3).

$(\mathcal{L}3)$ By Theorem 4.7(4) and by definition of $\mathcal{Z}^{-\circledcirc}$, that is available for any $\mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A})$, $\mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) \in \mathcal{Z}^{-\circledcirc}$ $(\forall \mathcal{A},\ \mathcal{B} \in [0,1]^X)$ and $\mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A}) \wedge \mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) = \mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A} \wedge \mathcal{B})$. Then, $\mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A}) \wedge \mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) \in \mathcal{Z}^{-\circledcirc}$ $(\forall \mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A}),\ \mathscr{Q}^{-\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) \in \mathcal{Z}^{-\circledcirc})$ and thus, $\mathcal{Z}^{-\circledcirc}$ is a fuzzy semi-topology.

(2) We only prove $\mathcal{Z}^{+\circledcirc}$ is a fuzzy semi-co-topology when $k=1$ by considering the following $(\mathcal{CO}1)$ and $(\mathcal{CO}3)$ of Definition 2.7 (see, Definition 2.8 $(\mathcal{LCO}2)$):

$(\mathcal{CO}1)$ $0_X, 1_X \in \mathcal{H}^{+\circledcirc}$ can be obtained from Theorem 4.7(3).

$(\mathcal{CO}3)$ By Theorem 4.7(4) and by definition of $\mathcal{Z}^{+\circledcirc}$, that is available for any $\mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A})$, $\mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) \in \mathcal{Z}^{-\circledcirc}$ $(\forall \mathcal{A},\ \mathcal{B} \in [0,1]^X)$ and $\mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A}) \vee \mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) = \mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A} \wedge \mathcal{B})$. Then, $\mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A}) \wedge \mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) \in \mathcal{Z}^{-\circledcirc}$ $(\forall \mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{A}),\ \mathscr{Q}^{+\circledcirc}_{\beta\mathcal{C}1}(\mathcal{B}) \in \mathcal{Z}^{+\circledcirc})$ and thus, $\mathcal{Z}^{+\circledcirc}$ is a fuzzy semi-co-topology. □

5 An innovative method by $\mathbb{C}$-$\circledcirc$-approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$ in the MADM field

The aim of this section is to:

(1) Use the $\mathbb{C}$-$\circledcirc$-approximation of $\mathcal{A}$ (in the sense of Definition 4.5) based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$ to propose a novel method to solve the decision-making problem (i.e., an application of β-FC$\mathscr{A}$ in MADM field).

(2) An innovative method aims to select the best scheme or rank all schemes based on the assessment data of the attribute set supplied by decision-makers.

(3) To illustrate the efficacy and benefits of our suggested methodology, we will present a comparative analysis and discussions with the various methods, such as the Algorithm 1, Zhang and Zhan's method [53], and Nawar et al.'s method [27].

(4) The proposed method has certain limitations, which will be discussed in this study.

Regarding our proposed method (i.e., Algorithm 3), we will outline the fundamentals of our methodology in this section, covering Steps 1 through 8 as indicated below.

Assume that X be an element set of option schemes, I be an element parameter set of schemes, $\mathscr{S} = \{\mathscr{A}_k(i) \mid i \in I, k \in N\} \subseteq [0,1]^{X\times I}$ be a family of β-FC$\mathscr{A}$, and $\beta \in (0,1]$.

Algorithm 3. Description of the decision-making method based on β-FC$\mathscr{A}$ via $\circledcirc$-β-FCN$\mathscr{A}$.

Step 1. Input the $\mathscr{S} = \{\mathscr{A}_k(i) \mid i \in I, k \in N\} \subseteq [0,1]^{X\times I}$ be a collection of β-FC.

Step 2. Define and compute $\mathscr{P}^{\vee}_k$ (the $\vee-$ideal solution) and $\mathscr{P}^{\wedge}_k$ (the $\wedge-$ideal solution) of $\{\mathscr{A}_k(i) \mid i \in I, k \in N\}$, respectively, as

$$\mathscr{P}^{\vee}_k(\mathscr{A}_k(i)) = \bigvee \Big(\mathscr{A}_k(i)(x)\Big)\ \ (\forall\ x \in X)\ \text{ and }\ \mathscr{P}^{\wedge}_k(\mathscr{A}_k(i)) = \bigwedge \Big(\mathscr{A}_k(i)(x)\Big)\ \ (\forall\ x \in X).$$

Step 3. Define and compute $\mathscr{D}^{\vee}_k(x)$ (the $\vee-$similarity between the $x \in X$ and $\mathscr{P}^{\vee}_k(\mathscr{A}_k(i))$) and $\mathscr{D}^{\wedge}_k(x)$ (the $\wedge-$similarity between the $x \in X$ and $\mathscr{P}^{\wedge}_k(\mathscr{A}_k(i))$) for each $x \in X$, respectively, as

$$\mathscr{D}^{\vee}_k(x) = \frac{\sum \Big[\mathscr{A}_k(i)(x) \times \mathscr{P}^{\vee}_k(\mathscr{A}_k(i))\Big]}{\sum \mathscr{P}^{\vee}_k(\mathscr{A}_k(i)) \vee \sum \mathscr{A}_k(i)(x)}\ (\forall i \in I)\ \text{ and }\ \mathscr{D}^{\wedge}_k(x) = \frac{\sum \Big[\mathscr{A}_k(i)(x) \times \mathscr{P}^{\wedge}_k(\mathscr{A}_k(i))\Big]}{\sum \mathscr{P}^{\wedge}_k(\mathscr{A}_k(i)) \vee \sum \mathscr{A}_k(i)(x)}\ (\forall i \in I).$$

Step 4. Define and compute $\mathcal{G}^{\vee}(x)$ (the $\vee-$group similarity) and $\mathcal{G}^{\wedge}(x)$ (the $\wedge-$group similarity) for each $x \in X$, respectively, as

$$\mathcal{G}^{\vee}(x) = w_1\mathscr{D}^{\vee}_1(x) + w_2\mathscr{D}^{\vee}_2(x) + ... + w_k\mathscr{D}^{\vee}_k(x)$$

and

$$\mathcal{G}^{\wedge}(x) = w_1\mathscr{D}^{\wedge}_1(x) + w_2\mathscr{D}^{\wedge}_2(x) + ... + w_k\mathscr{D}^{\wedge}_k(x),$$

where $w = (w_1, w_2, ..., w_k)$ is the weight vector, entered by the decision-makers.

Step 5. Define and compute the $\mathbb{C}$-$\odot$-approximation of $\mathcal{G}^{\vee},\ \mathcal{G}^{\wedge} \in [0,1]^X$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ with respect to $\mathscr{S}$ are separately for each $x \in X$, respectively, as

$$\mathscr{Q}^{-\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\vee})(x) = \begin{cases} \bigwedge\limits_{y\in X}([\mathbb{C}^{\beta\odot}_{\mathscr{S}x}(y)]' \vee \mathcal{G}^{\vee}(y)), & \mathcal{G}^{\vee}(x) \geq 1-\beta, \\ 0, & \mathcal{G}^{\vee}(x) < 1-\beta \end{cases}$$

(called the $\mathbb{C}$-$\odot$-lower approximation of $\mathcal{G}^{\vee}$),

$$\mathscr{Q}^{+\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\vee})(x) = \begin{cases} \bigvee\limits_{y\in X}[\mathbb{C}^{\beta\odot}_{\mathscr{S}x}(y) \wedge \mathcal{G}^{\vee}(y)], & \mathcal{G}^{\vee}(x) \leq \beta, \\ 1, & \mathcal{G}^{\vee}(x) > \beta \end{cases}$$

(called the $\mathbb{C}$-$\odot$-upper approximation of $\mathcal{G}^{\vee}$),

$$\mathscr{Q}^{-\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\wedge})(x) = \begin{cases} \bigwedge\limits_{y\in X}([\mathbb{C}^{\beta\odot}_{\mathscr{S}x}(y)]' \vee \mathcal{G}^{\wedge}(y)), & \mathcal{G}^{\wedge}(x) \geq 1-\beta, \\ 0, & \mathcal{G}^{\wedge}(x) < 1-\beta \end{cases}$$

(called the $\mathbb{C}$-$\odot$-lower approximation of $\mathcal{G}^{\wedge}$),

$$\mathscr{Q}^{+\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\wedge})(x) = \begin{cases} \bigvee\limits_{y\in X}[\mathbb{C}^{\beta\odot}_{\mathscr{S}x}(y) \wedge \mathcal{G}^{\wedge}(y)], & \mathcal{G}^{\wedge}(x) \leq \beta, \\ 1, & \mathcal{G}^{\wedge}(x) > \beta \end{cases}$$

(called the $\mathbb{C}$-$\odot$-upper approximation of $\mathcal{G}^{\wedge}$), and $\mathbb{C}^{\beta}_{\mathscr{S}x} = \bigwedge\limits_{\mathscr{A}_k\in\mathscr{S}} \mathcal{C}^{\beta\odot}_{\mathscr{A}_k x}$ $(\forall x \in X)$, where $\mathcal{C}^{\beta\odot}_{\mathscr{A}_k x}$ is given in Definition 4.1.

Step 6. Define and compute $\mathscr{U}^{\vee}, \mathscr{U}^{\wedge} \in [0,1]^X$ (the aggregation evaluation value) of the $\mathbb{C}$-$\odot$-approximation of $\mathcal{G}^{\vee},\ \mathcal{G}^{\wedge} \in [0,1]^X$ based on β-FC $\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ with respect to $\mathscr{S}$ for each $x \in X$, respectively, as[3]

$$\mathscr{U}^{\vee}(x) = \lambda\mathscr{Q}^{-\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\vee})(x) + (1-\lambda)\mathscr{Q}^{+\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\vee})(x)$$

and

$$\mathscr{U}^{\wedge}(x) = \lambda\mathscr{Q}^{-\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\wedge})(x) + (1-\lambda)\mathscr{Q}^{+\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\wedge})(x),$$

where the decision-makers preference is represented by $\lambda \in [0,1]$; that is, the higher the value of λ, the more risk-preferring the decision-maker is, and the lower the value of λ, the risk-averse.

Step 7. Define and compute the relative closeness $\Re_c$ for each $x \in X$ with respect to $\mathscr{U}^{\vee}, \mathscr{U}^{\wedge} \in [0,1]^X$ as

$$\Re_c(x) = \frac{\lambda\mathscr{U}^{\vee}(x)}{\lambda\mathscr{U}^{\vee}(x) + (1-\lambda)\mathscr{U}^{\wedge}(x)}.$$

Step 8. Rank the schemes based on the values $\Re_c(x)$ $(x \in X)$. The larger the value of $\Re_c(x)$, the better the ranking order of scheme x.

The following can be depicted to illustrate the steps of the aforementioned algorithm:

Algorithm 3. An algorithm that uses $\mathbb{C}$-$\odot$-approximation to determine the ranking of all schemes.

Input: A $\mathscr{S} = \{\mathscr{A}_k(i) \mid i \in I, k \in N\} \subseteq [0,1]^{X\times I}$ be a family of β-FC $\mathscr{A}$ (i.e., a set of option schemes) and $x \in X$.
Output: Decision Making.
1: Compute $\mathscr{P}^{\vee}_k(\mathscr{A}_k(i))$ and $\mathscr{P}^{\wedge}_k(\mathscr{A}_k(i))$;
2: Compute $\mathscr{D}^{\vee}_k(x)$ and $\mathscr{D}^{\wedge}_k(x)$;
3: Compute $\mathcal{G}^{\vee}(x)$ and $\mathcal{G}^{\wedge}(x)$;
4: Compute $\mathscr{Q}^{-\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\vee})(x)$, $\mathscr{Q}^{+\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\vee})(x)$, $\mathscr{Q}^{-\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\wedge})(x)$, and $\mathscr{Q}^{+\odot}_{\beta\mathbb{C}}(\mathcal{G}^{\wedge})(x)$;
5: Compute $\mathscr{U}^{\vee}(x)$ and $\mathscr{U}^{\wedge}(x)$;
6: Compute $\Re_c(x)$;
7: Obtain the decision.

The following numerical example of a candidate wanting to get a job in a company, as referred to in [53], and we will demonstrate the applicability of the present $\mathbb{C}$-$\odot$-approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ in the MADM field through the aforementioned steps.

[3] Motivated from the fuzzy set on decision set found in Subsection 5.2 [33].

Example 5.1. *A company wants to select the best candidates from a list of six applicants. Let us say* $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *are six applicants. Three expert groups (decision-makers) of the company want to help the manager choose the best candidate depending on evaluating the four parameters* $I = \{i_1, i_2, i_3, i_4\}$*, where* i_1 *stands for the 'language expression skills ',* i_2 *stands for the 'comprehensive analysis capabilities ',* i_3 *stands for 'self-control ability ',* i_4 *stands for 'interpersonal skills', and three different weights (i.e.,* $w_1 = 0.5,\ w_2 = 0.2,\ w_3 = 0.3$*) are described by decision-makers. The three expert groups give a critical value of* $\beta = 0.5\ (0 < \beta \leq 0.6)$ *to judge whether the candidate reaches the skill requirements. We prefer to select candidates who possess a strong grasp of the abilities rather than those who are only familiar with them. Therefore,* $\mathscr{S} = \{\mathscr{A}_k(i) \mid i \in I, k = 1, 2, 3\} \subseteq [0,1]^{X\times I}$ *denotes the value for each candidate has skill requirements as declared by experts according to their experience. By Algorithm 3, we obtain the following:*

Step 1. *The evaluation result can be described by* $\mathscr{S} \subseteq [0,1]^{X\times I}$ *is a collection of* β*-FC*$\mathscr{A}$ $(0 < \beta \leq 0.6)$ *which is given as*

$$\mathscr{A}_1(i_1) = \frac{0.8}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.1}{x_5} + \frac{0.3}{x_6},\ \mathscr{A}_1(i_2) = \frac{0.7}{x_1} + \frac{0.4}{x_2} + \frac{0.1}{x_3} + \frac{0.2}{x_4} + \frac{0.6}{x_5} + \frac{0.8}{x_6},$$
$$\mathscr{A}_1(i_3) = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6},\ \mathscr{A}_1(i_4) = \frac{0.5}{x_1} + \frac{0.9}{x_2} + \frac{0.7}{x_3} + \frac{0.9}{x_4} + \frac{0.9}{x_5} + \frac{0.1}{x_6},$$
$$\mathscr{A}_2(i_1) = \frac{0.7}{x_1} + \frac{0.4}{x_2} + \frac{0.2}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} + \frac{0.3}{x_6},\ \mathscr{A}_2(i_2) = \frac{0.9}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.7}{x_6},$$
$$\mathscr{A}_2(i_3) = \frac{0.2}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} + \frac{0.4}{x_6},\ \mathscr{A}_2(i_4) = \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.9}{x_4} + \frac{0.5}{x_5} + \frac{0.6}{x_6},$$
$$\mathscr{A}_3(i_1) = \frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.6}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.3}{x_6},\ \mathscr{A}_3(i_2) = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.2}{x_3} + \frac{0.3}{x_4} + \frac{0.8}{x_5} + \frac{0.6}{x_6},$$
$$\mathscr{A}_3(i_3) = \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.7}{x_6},\ \mathscr{A}_3(i_4) = \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.9}{x_4} + \frac{0.7}{x_5} + \frac{0.2}{x_6}.$$

Step 2. *The* $\vee-$*ideal solution* $\mathscr{P}_k^{\vee}$ *and* $\wedge-$*ideal solution* $\mathscr{P}_k^{\wedge}$ *from Step 1 are computed, respectively, as follows* $(k = 1, 2, 3)$:

$\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_1)) = \mathscr{A}_1(i_1)(x_1) \vee \mathscr{A}_1(i_1)(x_2) \vee ... \vee \mathscr{A}_1(i_1)(x_6) = 0.8 \vee 0.3 \vee ... \vee 0.3 = 0.8$. *Similarly,*

$$\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_2)) = \mathscr{P}_2^{\vee}(\mathscr{A}_2(i_3)) = \mathscr{P}_3^{\vee}(\mathscr{A}_3(i_2)) = \mathscr{P}_3^{\vee}(\mathscr{A}_3(i_3)) = 0.8,$$
$$\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_3)) = \mathscr{P}_2^{\vee}(\mathscr{A}_2(i_1)) = \mathscr{P}_3^{\vee}(\mathscr{A}_3(i_1)) = 0.7,$$
$$\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_4)) = \mathscr{P}_2^{\vee}(\mathscr{A}_2(i_2)) = \mathscr{P}_2^{\vee}(\mathscr{A}_2(i_4)) = \mathscr{P}_3^{\vee}(\mathscr{A}_3(i_4)) = 0.9,$$
$$\mathscr{P}_1^{\wedge}(\mathscr{A}_1(i_1)) = \mathscr{P}_1^{\wedge}(\mathscr{A}_1(i_2)) = \mathscr{P}_1^{\wedge}(\mathscr{A}_1(i_4)) = \mathscr{P}_2^{\wedge}(\mathscr{A}_2(i_2)) = \mathscr{P}_3^{\wedge}(\mathscr{A}_3(i_1)) = \mathscr{P}_3^{\wedge}(\mathscr{A}_3(i_4)) = 0.1,$$
$$\mathscr{P}_1^{\wedge}(\mathscr{A}_1(i_3)) = \mathscr{P}_3^{\wedge}(\mathscr{A}_3(i_3)) = 0.3,\ \mathscr{P}_2^{\wedge}(\mathscr{A}_2(i_1)) = \mathscr{P}_2^{\wedge}(\mathscr{A}_2(i_3)) = \mathscr{P}_3^{\wedge}(\mathscr{A}_3(i_2)) = 0.2,$$

and $\mathscr{P}_2^{\wedge}(\mathscr{A}_2(i_4)) = 0.4$.

Step 3. *The* $\vee-$*similarity* $\mathscr{D}_k^{\vee}(x)$ *and* $\wedge-$*similarity* $\mathscr{D}_k^{\wedge}(x)$ *from Step 2 are computed, respectively, as follows* $(k = 1, 2, 3,\ \forall x \in X)$:

$$\mathscr{D}_1^{\vee}(x_1) = \frac{\sum\left[\mathscr{A}_1(i)(x_1) \times \mathscr{P}_1^{\vee}(\mathscr{A}_1(i))\right]}{\sum \mathscr{P}_1^{\vee}(\mathscr{A}_1(i)) \vee \sum \mathscr{A}_1(i)(x_1)} = \frac{1.86}{3.2} = 0.5812,\ \textit{where}\ \sum\left[\mathscr{A}_1(i)(x_1) \times \mathscr{P}_1^{\vee}(\mathscr{A}_1(i))\right] = [(\mathscr{A}_1(i_1)(x_1) \times$$
$$\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_1))) + (\mathscr{A}_1(i_2)(x_1) \times \mathscr{P}_1^{\vee}(\mathscr{A}_1(i_2))) + (\mathscr{A}_1(i_3)(x_1) \times \mathscr{P}_1^{\vee}(\mathscr{A}_1(i_3))) + (\mathscr{A}_1(i_3)(x_1) \times \mathscr{P}_1^{\vee}(\mathscr{A}_1(i_4)))] = [(0.8 \times 0.8) +$$
$$(0.7 \times 0.8) + (0.3 \times 0.7) + (0.5 \times 0.9)] = 1.86;\ \sum \mathscr{P}_1^{\vee}(\mathscr{A}_1(i)) \vee \sum \mathscr{A}_1(i)(x_1) = (\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_1)) + \mathscr{P}_1^{\vee}(\mathscr{A}_1(i_2)) + \mathscr{P}_1^{\vee}(\mathscr{A}_1(i_3)) +$$
$$\mathscr{P}_1^{\vee}(\mathscr{A}_1(i_4))) \vee (\mathscr{A}_1(i_1)(x_1) + \mathscr{A}_1(i_2)(x_1) + \mathscr{A}_1(i_3)(x_1) + \mathscr{A}_1(i_4)(x_1)) = (0.8 + 0.8 + 0.7 + 0.9) \vee (0.8 + 0.7 + 0.3 + 0.5) = 3.2.$$

Similarly, we can arrange it as following list:

$$\mathscr{D}_1^{\vee} = \frac{0.5812}{x_1} + \frac{0.5594}{x_2} + \frac{0.4312}{x_3} + \frac{0.5406}{x_4} + \frac{0.5375}{x_5} + \frac{0.4562}{x_6},$$
$$\mathscr{D}_2^{\vee} = \frac{0.5788}{x_1} + \frac{0.4697}{x_2} + \frac{0.5182}{x_3} + \frac{0.4273}{x_4} + \frac{0.4424}{x_5} + \frac{0.5152}{x_6},$$
$$\mathscr{D}_3^{\vee} = \frac{0.4062}{x_1} + \frac{0.5062}{x_2} + \frac{0.5031}{x_3} + \frac{0.45}{x_4} + \frac{0.4937}{x_5} + \frac{0.4469}{x_6},$$

$$\mathscr{D}_1^{\wedge} = \frac{0.1261}{x_1} + \frac{0.1545}{x_2} + \frac{0.1588}{x_3} + \frac{0.1381}{x_4} + \frac{0.1476}{x_5} + \frac{0.1737}{x_6},$$
$$\mathscr{D}_2^{\wedge} = \frac{0.2043}{x_1} + \frac{0.2265}{x_2} + \frac{0.225}{x_3} + \frac{0.3}{x_4} + \frac{0.25}{x_5} + \frac{0.225}{x_6},$$
$$\mathscr{D}_3^{\wedge} = \frac{0.1823}{x_1} + \frac{0.21}{x_2} + \frac{0.16}{x_3} + \frac{0.1647}{x_4} + \frac{0.1737}{x_5} + \frac{0.2111}{x_6}.$$

Step 4. *The $\vee-$group similarity $\mathcal{G}^{\vee}(x)$ and the $\wedge-$group similarity $\mathcal{G}^{\wedge}(x)$ from Step 3 are computed, respectively, as follows* $(\forall x \in X)$:

$\mathcal{G}^{\vee}(x_1) = w_1\mathscr{D}_1^{\vee}(x_1) + w_2\mathscr{D}_2^{\vee}(x_1) + w_3\mathscr{D}_3^{\vee}(x_1) = 0.5 \times 0.5812 + 0.2 \times 0.5788 + 0.3 \times 0.4062 = 0.5282.$

Similarly, we obtain:

$$\mathcal{G}^{\vee} = \frac{0.5282}{x_1} + \frac{0.5255}{x_2} + \frac{0.4702}{x_3} + \frac{0.4907}{x_4} + \frac{0.5053}{x_5} + \frac{0.4652}{x_6}$$

and

$$\mathcal{G}^{\wedge} = \frac{0.1586}{x_1} + \frac{0.1855}{x_2} + \frac{0.1724}{x_3} + \frac{0.1785}{x_4} + \frac{0.1759}{x_5} + \frac{0.1952}{x_6}.$$

Step 5. *The $\mathbb{C}$-$\odot$-approximation of $\mathcal{G}^{\vee},\ \mathcal{G}^{\wedge} \in [0,1]^X$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ with respect to $\mathscr{S}$ from Step 1 and Step 4, is computed, respectively, as follows:*

First: Take a $\beta = 0.5$ and by Definition 4.1, we obtain:

$$\mathcal{C}_{\mathscr{A}1x_1}^{0.5\odot} = \frac{0.7}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4} + \frac{0.5}{x_5} + \frac{0.3}{x_6},\ \mathcal{C}_{\mathscr{A}1x_2}^{0.5\odot} = \mathcal{C}_{\mathscr{A}1x_3}^{0.5\odot} = \frac{0.2}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.2}{x_6},$$
$$\mathcal{C}_{\mathscr{A}1x_4}^{0.5\odot} = \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4} + \frac{0.4}{x_5} + \frac{0.2}{x_6},\ \mathcal{C}_{\mathscr{A}1x_5}^{0.5\odot} = \frac{0.2}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.9}{x_5} + \frac{0.7}{x_6},$$
$$\mathcal{C}_{\mathscr{A}1x_6}^{0.5\odot} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.7}{x_6},\ \mathcal{C}_{\mathscr{A}2x_1}^{0.5\odot} = \frac{0.8}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} + \frac{0.6}{x_6},$$
$$\mathcal{C}_{\mathscr{A}2x_2}^{0.5\odot} = \frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5} + \frac{0.3}{x_6},\ \mathcal{C}_{\mathscr{A}2x_3}^{0.5\odot} = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.8}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.7}{x_6},$$
$$\mathcal{C}_{\mathscr{A}2x_4}^{0.5\odot} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} + \frac{0.3}{x_6},\ \mathcal{C}_{\mathscr{A}2x_5}^{0.5\odot} = \frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.3}{x_6},$$
$$\mathcal{C}_{\mathscr{A}2x_6}^{0.5\odot} = \frac{0.3}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.2}{x_5} + \frac{0.6}{x_6},\ \mathcal{C}_{\mathscr{A}3x_1}^{0.5\odot} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.2}{x_5} + \frac{0.4}{x_6},$$
$$\mathcal{C}_{\mathscr{A}3x_2}^{0.5\odot} = \mathcal{C}_{\mathscr{A}3x_6}^{0.5\odot} = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5} + \frac{0.7}{x_6},\ \mathcal{C}_{\mathscr{A}3x_3}^{0.5\odot} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3} + \frac{0.7}{x_4} + \frac{0.2}{x_5} + \frac{0.4}{x_6},$$
$$\mathcal{C}_{\mathscr{A}3x_4}^{0.5\odot} = \frac{0.3}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.3}{x_6},\ \mathcal{C}_{\mathscr{A}3x_5}^{0.5\odot} = \frac{0.3}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.7}{x_5} + \frac{0.3}{x_6}.$$

Second: As $\mathbb{C}_{\mathscr{S}x}^{\beta} = \bigwedge\limits_{\mathscr{A}_k \in \mathscr{S}} \mathcal{C}_{\mathscr{A}_k x}^{\beta\odot}$ $(\forall x \in X)$, then we obtain:

$$\mathbb{C}_{\mathscr{S}x_1}^{0.5\odot} = \frac{0.6}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.2}{x_5} + \frac{0.3}{x_6},\ \mathbb{C}_{\mathscr{S}x_2}^{0.5\odot} = \frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5} + \frac{0.2}{x_6},$$
$$\mathbb{C}_{\mathscr{S}x_3}^{0.5\odot} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.2}{x_5} + \frac{0.2}{x_6},\ \mathbb{C}_{\mathscr{S}x_4}^{0.5\odot} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.2}{x_6},$$
$$\mathbb{C}_{\mathscr{S}x_5}^{0.5\odot} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.4}{x_4} + \frac{0.6}{x_5} + \frac{0.3}{x_6},\ \mathbb{C}_{\mathscr{S}x_6}^{0.5\odot} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.6}{x_6}.$$

Third: $\mathscr{Q}_{0.5\mathbb{C}}^{-\odot}(\mathcal{G}^{\vee}),\ \mathscr{Q}_{0.5\mathbb{C}}^{+\odot}(\mathcal{G}^{\vee}),\ \mathscr{Q}_{0.5\mathbb{C}}^{-\odot}(\mathcal{G}^{\wedge})$ and $\mathscr{Q}_{0.5\mathbb{C}}^{+\odot}(\mathcal{G}^{\wedge})$, for each $x \in X$, are computed, respectively, as follows:

$$\mathscr{Q}_{0.5\mathbb{C}}^{-\odot}(\mathcal{G}^{\vee}) = \frac{0.5282}{x_1} + \frac{0.5255}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{0.5053}{x_5} + \frac{0}{x_6},$$
$$\mathscr{Q}_{0.5\mathbb{C}}^{+\odot}(\mathcal{G}^{\vee}) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{0.4702}{x_3} + \frac{0.4907}{x_4} + \frac{1}{x_5} + \frac{0.4652}{x_6},$$

$\mathscr{Q}^{-\odot}_{0.5\mathbb{C}}(\mathcal{G}^{\wedge}) = 0_X(\textit{each element of } X \textit{ equals } 0)$, *and* $\mathscr{Q}^{+\odot}_{0.5\mathbb{C}}(\mathcal{G}^{\wedge}) = 0.1952_X(\textit{each element of } X \textit{ equals } 0.1952)$.

Step 6. *Let* $\lambda = 0.6$ *and by Step 5, the aggregation evaluation values* $\mathscr{U}^{\vee}(x)$ *and* $\mathscr{U}^{\wedge}(x)$ *are computed, respectively, as follows* $(\forall x \in X)$:

$$\mathscr{U}^{\vee}(x_1) = (0.6)(0.5282) + (1-0.6)(1) = 0.7169.$$

Similarly, we obtain:

$$\mathscr{U}^{\vee} = \frac{0.7169}{x_1} + \frac{0.7153}{x_2} + \frac{0.1881}{x_3} + \frac{0.1963}{x_4} + \frac{0.7032}{x_5} + \frac{0.1861}{x_6}$$

and

$$\mathscr{U}^{\wedge} = 0.0781_X(\textit{each element of } X \textit{ equals } 0.0781).$$

Step 7. *The relative closeness* $\Re_c(x)$ *from Step 6 is computed as follows* $(\forall x \in X)$:

$$\Re_c(x_1) = \frac{(0.6)(0.7169)}{(0.6)(0.7169) + (0.4)(0.0781)} = 0.9322.$$

Similarly, $\Re_c(x_2) = 0.9321, \Re_c(x_3) = 0.7832,\ \Re_c(x_4) = 0.7904,\ \Re_c(x_5) = 0.9311,$ *and* $\Re_c(x_6) = 0.7814.$

Step 8. *By the result of Step 7, we obtain the ranking of six candidates* $x_1 \succ x_2 \succ x_5 \succ x_4 \succ x_3 \succ x_6$. *In our decision method, the best candidate is* x_1.

5.1 Comparison with other MADM models based on β-FC$\mathscr{A}$

The following comparisons and discussions are conducted to illustrate the efficacy and benefits of our proposed $\mathbb{C}$-$\odot$-approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ in the MADM field. Regarding the illustrative Example 5.1, we will compare the proposed approach (i.e., Algorithm 3) with the approaches of Algorithm 1, Zhang and Zhan [53], and Nawar et al. [27], respectively, where Table 4 displays all rankings and best decision that were determined using the aforementioned methods.

Table 4: Comparison and ranking results

	Method	Ranking	Best Decision
Algorithm 3	$\mathbb{C}$-$\odot$-approximation depend on $\odot$-β-FCN$\mathscr{A}$	$x_1 \succ x_2 \succ x_5 \succ x_4 \succ x_3 \succ x_6$	x_1
Algorithm 1	$\mathscr{Q}^{-\circledast}_{\beta C2}(\mathcal{A})$ and $\mathscr{Q}^{+\circledast}_{\beta C2}(\mathcal{A})$ depend on $\circledast$-β-FCN$\mathscr{A}$	$x_3 \succ x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$	x_3
Zhang and Zhan's approach [53]	$\mathscr{A}^{-\circledast}_{\beta C2}(\mathcal{A})$ and $\mathscr{A}^{+\circledast}_{\beta C2}(\mathcal{A})$ depend on $\circledast$-β-FCN$\mathscr{A}$	$x_3 \approx x_6 \succ x_2 \succ x_4 \approx x_5 \succ x_1$	x_3 and x_6
Nawar et al.'s approach [27]	$\mathscr{A}^{-\circledast}_{\beta C2}(\mathcal{A})$ and $\mathscr{A}^{+\circledast}_{\beta C2}(\mathcal{A})$ depend on $\circledast$-β-FCN$\mathscr{A}$	$x_3 \approx x_6 \succ x_2 \succ x_1 \approx x_4 \approx x_5$	x_3 and x_6

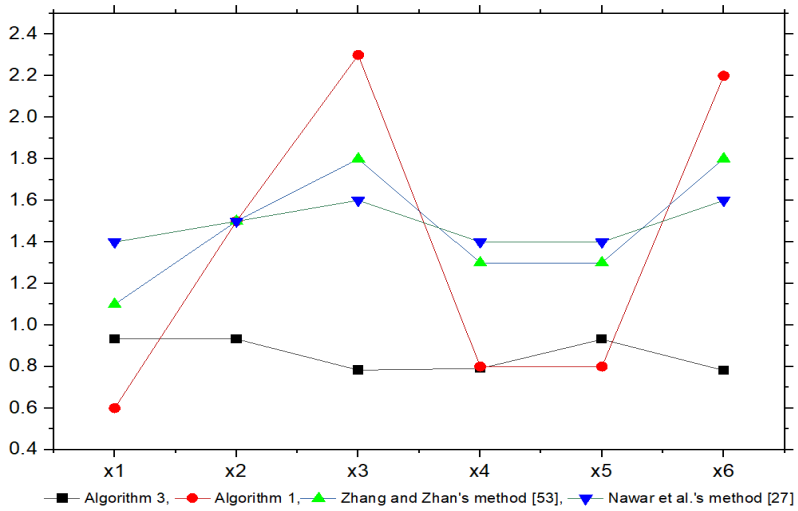

Figure 1: The ranking results of MADM models based on β-FC$\mathscr{A}$.

As indicated in Table 4 and Figure 1, the final results and the ranking position of candidates of Algorithm 3 of Example 5.1 differ from those of Algorithm 1, Zhang and Zhan's method [53], and Nawar et al.'s method [27]. Below, we discuss the distinctions.

(1) Our proposed approach (i.e., Algorithm 3) is based on $\mathbb{C}$-$\odot$-approximation via $\odot$-β-FCN$\mathscr{A}$. It also accomplishes the inclusion property, regarded as one of the most crucial characteristics of rough sets. As a result, our approach succeeds in obtaining the strict order via $\odot$-β-FCN$\mathscr{A}$ and the best decision is a candidate x_1.

(2) Algorithm 1 depends on $\mathscr{Q}^{-\odot}_{\beta\mathcal{C}2}(\mathcal{A})$ and $\mathscr{Q}^{+\odot}_{\beta\mathcal{C}2}(\mathcal{A})$ via $\circledast$-β-FCN$\mathscr{A}$ and was able to satisfy the inclusion property but failed to obtain the strict order (which is $x_4 \approx x_5$), and the ranking results are not fully ranked. Consequently, the outcome is untrustworthy.

(3) Zhang and Zhan [53] (resp., Nawar et al. [27]) depend on $\mathscr{A}^{-\odot}_{\beta\mathcal{C}2}(\mathcal{A})$ and $\mathscr{A}^{+\odot}_{\beta\mathcal{C}2}(\mathcal{A})$ (resp., $\tilde{\mathscr{A}}^{-\odot}_{\beta\mathcal{C}2}(\mathcal{A})$ and $\tilde{\mathscr{A}}^{+\odot}_{\beta\mathcal{C}2}(\mathcal{A})$) via $\circledast$-β-FCN$\mathscr{A}$ can not satisfy the inclusion property and also failed to obtain the strict order which are $x_3 \approx x_6$ and $x_4 \approx x_5$ (resp., $x_3 \approx x_6$ and $x_1 \approx x_4 \approx x_5$). Furthermore, the company will need to invest more time and energy in comparing the two candidates (x_3 and x_6, which are the best decisions in [27, 53]) again. In contrast, the company needs to hire a single individual. Consequently, we discover that the methods of [27, 53] have numerous shortcomings, which leads to inaccurate and untrustworthy conclusions.

We find that our developed decision-making strategies by $\mathbb{C}$-$\odot$-approximation of $\mathcal{A}$ based on β-FC$\mathscr{A}$ via $\odot$-β-FCN$\mathscr{A}$ produce better results than Algorithm 1 and approaches of [27, 53] and have observable and reliable results when compared to the two distinctions discussed above.

5.2 Limitations of the proposed model in this work

During our analysis of the proposed method, we found the following limitations:

(1) Examining the proposed method reveals that Steps 2, 3, and 5 require more calculation time and have a more complex procedure. Therefore, it is challenging to compute further examples when taking into account the complexity of these stages. So, we only computed one example using Algorithm 3 in Section 5.

(2) The existence of large sets of alternatives and parameters in several MADM issues is another limitation of the proposed method, which is difficult to calculate manually. So it requires appropriate code to be built using various mathematical software programs, including MAPLE and MATLAB.

(3) Steps 6 and 7 in the proposed method require rephrasing with more precision in how to reach challenges like sensitivity to parameter selection (β) or the impact of soft fuzzy set configurations on outcomes.

6 Conclusions

This paper uses $\odot$-β-FCN$\mathscr{A}$ to solve MADM problems based on β-soft fuzzy covering through the introduction of new types of approximations. For this, a number of clear benefits and advantages are studied, such as defining new types of approximations via $\circledast$-β-FCN$\mathscr{A}$, meeting the inclusion property, understandably updating some algorithms, introducing a novel idea of $\odot$-β-FCN$\mathscr{A}$, researching some fundamental fuzzy topology properties, and developing a novel approach that can be used in the MADM field. Regarding the directions of future research:

(1) Biswas et al. [3] presented a spherical fuzzy set based on multi-criteria group decision-making (MAGDM). In the future, we define a spherical fuzzy based on β-soft fuzzy covering via $\odot$-β-FCN$\mathscr{A}$ to get further results that should be applied.

(2) Using a family of β-fuzzy complement neighborhoods, Atef and ElAtik [2] constructed covering-based optimistic (pessimistic) multigranulation fuzzy rough sets. Therefore, in the future, we will expand our work to include optimistic (pessimistic) multigranulation fuzzy rough sets by using a family of $\odot$-β-FCN$\mathscr{A}$.

(3) Redefine the notion of soft fuzzy covering-based rough model (see, Definition 4.4 in [49]), which satisfy the inclusion property.

(4) Generalization of the new types of approximations based on β-fuzzy covering via $\odot$-β-FCN$\mathscr{A}$ produced two different universes.

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

Compliance with ethical standards

Conflict of Interest: A. M. Khalil, S. G. Li and H. L. Yang declare that they have no conflict of interest.
Ethical approval: This article does not contain any studies with human participants performed by any of the authors.
Informed consent: Informed consent was obtained from all individual participants included in the study.
Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 11771263) and the Fundamental Research Funds for the Central Universities (Grant No. GK202105007, GK202304053).

References

[1] D. Ahmed, B. Dai, A. M. Khalil, *Picture m-polar fuzzy soft sets and their application in decision-making problems*, Iranian Journal of Fuzzy Systems, **19**(9) (2022), 161-173. https://doi.org/10.22111/IJFS.2022.7218

[2] M. Atef, A. E. F. El Atik, *Some extensions of covering-based multigranulation fuzzy rough sets from new perspectives*, Soft Computing, **25** (2021), 6633-6651. https://doi.org/10.1007/s00500-021-05659-8

[3] S. Biswas, D. Božanić, D. Pamučar, D. Marinković, *A spherical fuzzy based decision making framework with einstein aggregation for comparing preparedness of smes in quality 4.0*, Facta Universitatis, Series: Mechanical Engineering, **21**(3) (2023), 453-478. https://doi.org/10.22190/FUME230831037B

[4] J. H. Dai, S. C. Gao, G. J. Zheng, *Generalized rough set models determined by multiple neighborhoods generated from a similarity relation,* Soft Computing, **12** (2017), 1-14. https://doi.org/10.1007/s00500-017-2672-x

[5] L. D'eer, C. Cornelis, *A comprehensive study of fuzzy covering based rough set models: Definitions, properties and interrelationships,* Fuzzy Sets and Systems, **336** (2018), 1-26. https://doi.org/10.1016/j.fss.2017.06.010

[6] L. D'eer, C. Cornelis, L. Godo, *Fuzzy neighborhood operators based on fuzzy coverings*, Fuzzy Sets and Systems, **312** (2017), 17-35. https://doi.org/10.1016/j.fss.2016.04.003

[7] L. D'eer, N. Verbiest, C. Cornelis, L. Godo, *A comprehensive study of implicator-conjunctor-based and noise-tolerant fuzzy rough sets: Definitions, properties and robustness analysis*, Fuzzy Sets and Systems, **275** (2015), 1-38. https://doi.org/10.1016/j.fss.2014.11.018

[8] W. S. Du, B. Q. Hu, *Dominance-based rough set approach to incomplete ordered information systems*, Information Sciences, **346-347** (2016), 106-129. https://doi.org/10.1016/j.ins.2016.01.098

[9] D. Dubois, H. Prade, *Rough fuzzy sets and fuzzy rough sets*, International Journal of General Systems, **17** (1990), 191-201. https://doi.org/10.1080/03081079008935107

[10] S. Greco, B. Matarazzo, R. Slowinski, *Rough approximation by dominance relations*, International Journal of Intelligent Systems, **17** (2002), 153-171. https://doi.org/10.1002/int.10014

[11] U. Höhle, A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, Kluwer Academic Publishers, Boston, (1999), 123-272. https://doi.org/10.1007/978-1-4615-5079-2_5

[12] Q. H. Hu, D. Yu, J. Liu, C. Wu, *Neighborhood rough set based heterogeneous feature subset selection*, Information Sciences, **178** (2008), 3577-3594. https://doi.org/10.1016/j.ins.2008.05.024

[13] A. M. Khalil, C. Dunqian, A. Abdelfatah, S. Florentin, R. A. Wedad, *Combination of the single-valued neutrosophic fuzzy set and the soft set with applications in decision-making*, Symmetry, **12** (2020), 1361. https://doi.org/10.3390/sym12081361

[14] A. M. Khalil, S. G. Li, H. Garg, H. Li, S. Ma, *New operations on interval-valued picture fuzzy set, interval-valued picture fuzzy soft set and their applications*, IEEE Access, **7** (2019), 51236-51253. https://doi.org/10.1109/ACCESS.2019.2910844

[15] A. M. Khalil, A. M. Zahran, R. Basheer, *A novel diagnosis system for detection of kidney disease by a fuzzy soft decision-making problem*, Mathematics and Computers in Simulation, **203** (2023), 271-305. https://doi.org/10.1016/j.matcom.2022.06.014

[16] T. J. Li, Y. Leung, W. X. Zhang, *Generalized fuzzy rough approximation operators based on fuzzy covering*, International Journal of Approximate Reasoning, **48** (2008), 836-856. https://doi.org/10.1016/j.ijar.2008.01.006

[17] Z. Li, N. Xie, G. Wen, *Soft coverings and their parameter reductions*, Applied Soft Computing, **31** (2015), 48-60. https://doi.org/10.1016/j.asoc.2015.02.027

[18] Y. M. Liu, M. K. Luo, *Fuzzy topology*, World Scientific Publication, Singapore, 1998. https://doi.org/10.1142/3281

[19] L. W. Ma, *On some types of neighborhood-related covering rough sets*, International Journal of Approximate Reasoning, **53** (2012), 901-911. https://doi.org/10.1016/j.ijar.2012.03.004

[20] L. W. Ma, *Some twin approximation operators on covering approximation spaces*, International Journal of Approximate Reasoning, **56** (2015), 59-70. https://doi.org/10.1016/j.ijar.2014.08.003

[21] L. W. Ma, *Two fuzzy covering rough set models and their generalizations over fuzzy lattices*, Fuzzy Sets and Systems, **294** (2016), 1-17. https://doi.org/10.1016/j.fss.2015.05.002

[22] L. W. Ma, *Couple fuzzy covering rough set models and their generalizations to CCD lattices*, International Journal of Approximate Reasoning, **126** (2020), 48-69. https://doi.org/10.1016/j.ijar.2020.08.003

[23] T. Mahmood, M. Asif, U. Rehman, J. Ahmmad, *T-bipolar soft semigroups and related results*, Spectrum of Mechanical Engineering and Operational Research, **1**(1) (2024), 258-271. https://doi.org/10.31181/smeor11202421

[24] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy soft sets*, Journal of Fuzzy Mathematics, **203** (2001), 589-602.

[25] P. K. Maji, A. R. Roy, R. Biswas, *On intuitionistic fuzzy soft sets*, Journal of Fuzzy Mathematics, **12** (2004), 669-683.

[26] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, **37** (1999), 19-31. https://doi.org/10.1016/S0898-1221(99)00056-5

[27] A. S. Nawar, M. Atef, A. M. Khalil, *Certain types of fuzzy soft β-covering based fuzzy rough sets with application to decision-making*, Journal of Intelligent and Fuzzy Systems, **40** (2021), 10825-10836. https://doi.org/10.3233/JIFS-201822

[28] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences, **11** (1982), 341-356. https://doi.org/10.1007/BF01001956

[29] Z. Pawlak, *Rough sets and intelligent data analysis*, Information Sciences, **147** (2002), 1-12. https://doi.org/10.1016/S0020-0255(02)00197-4

[30] J. A. Pomykala, *Approximation operations in approximation space*, Bulletin of the Polish Academy of Sciences Mathematics, **35** (1987), 653-662. https://mathscinet.ams.org/mathscinet/relay-station?mr=937744

[31] J. A. Pomykala, *On definability in the nondeterministic information system*, Bulletin of the Polish Academy of Sciences Mathematics, **36** (1988), 193-210. https://zbmath.org/pdf/04110174.pdf

[32] J. Su, Y. Wang, J. Li, *A novel fuzzy covering rough set model based on generalized overlap functions and its application in MCDM*, Symmetry, **15** (2023), 647. https://doi.org/10.3390/sym15030647

[33] B. Sun, W. Ma, Y. Qian, *Multigranulation fuzzy rough set over two universes and its application to decision making*, Knowledge-Based Systems, **123** (2017), 61-74. https://doi.org/10.1016/j.knosys.2017.01.036

[34] C. Wang, D. Chen, B. Sun, Q. Hu, *Communication between information systems with covering based rough sets*, Information Sciences, **216** (2012), 17-33. https://doi.org/10.1016/j.ins.2012.06.010

[35] X. Wang, A. M. Khalil, *A new kind of generalized Pythagorean fuzzy soft set and its application in decision-making*, Computer Modeling in Engineering and Sciences, **136**(3) (2023), 2861-2871. https://doi.org/10.32604/cmes.2023.026021

[36] C. Z. Wang, Y. P. Shi, X. D. Fan, M. W. Shao, *Attribute reduction based on k-nearest neighborhood rough sets*, International Journal of Approximate Reasoning, **106** (2019), 18-31. https://doi.org/10.1016/j.ijar.2018.12.013

[37] B. Yang, B. Hu, *A fuzzy covering-based rough set model and its generalization over fuzzy lattice*, Information Sciences, **367** (2016), 463-486. https://doi.org/10.1016/j.ins.2016.05.053

[38] B. Yang, B. Hu, *On some types of fuzzy covering-based rough sets*, Fuzzy Sets and Systems, **312** (2017), 36-65. https://doi.org/10.1016/j.fss.2016.10.009

[39] B. Yang, B. Hu, *Fuzzy neighborhood operators and derived fuzzy coverings*, Fuzzy Sets and Systems, **370** (2019), 1-13. https://doi.org/10.1016/j.fss.2016.04.003

[40] Y. Yang, C. Liang, S. Ji, T. Liu, *Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making*, Journal of Intelligent and Fuzzy Systems, **29** (2015), 1711-1722. https://doi.org/10.3233/IFS-15164

[41] X. Yang, T. Y. Lin, J. Yang, Y. L. A. Dongjun, *Combination of interval-valued fuzzy set and soft set*, Computers and Mathematics with Applications, **58** (2009), 521-527. https://doi.org/10.1016/j.camwa.2009.04.019

[42] Y. Y. Yao, *Relational interpretations of neighborhood operators and rough set approximation operators*, Information Sciences, **111** (1998), 239-259. https://doi.org/10.1016/S0020-0255(98)10006-3

[43] Ş. Yüksel, Z. G. Ergül, N. Tozlu, *Soft covering based rough sets and their application*, The Scientific World Journal, **2014** (2014), 1-9. https://doi.org/10.1155/2014/970893

[44] Ş. Yüksel, N. Tozlu, T. H. Dizman, *An application of multicriteria group decision making by soft covering based rough sets*, Filomat, **29** (2015), 209-219. https://doi.org/10.2298/FIL1501209Y

[45] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[46] J. Zhan, J. C. R. Alcantud, *A novel type of soft rough covering and its application to multicriteria group decision making*, Artificial Intelligence Review, **52** (2019), 2381-2410. https://doi.org/10.1007/s10462-018-9617-3

[47] J. Zhan, M. I. Ali, N. Mehmood, *On a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods*, Applied Soft Computing, **56** (2017), 446-457. https://doi.org/10.1016/j.asoc.2017.03.038

[48] J. Zhan, Q. Liu, T. Herawan, *A novel soft rough set: Soft rough hemirings and corresponding multicriteria group decision making*, Applied Soft Computing, **54** (2017), 393-402. https://doi.org/10.1016/j.asoc.2016.09.012

[49] J. Zhan, B. Sun, *Covering-based soft fuzzy rough theory and its application to multiple criteria decision making*, Computational and Applied Mathematics, **38** (2019), 149. https://doi.org/10.1007/s40314-019-0931-4

[50] J. Zhan, Q. Wang, *Certain types of soft coverings based rough sets with applications*, International Journal of Machine Learning and Cybernetics, **10** (2019), 1065-1076. https://doi.org/10.1007/s13042-018-0785-x

[51] J. Zhan, X. Zhang, Y. Yao, *Covering based multigranulation fuzzy rough sets and corresponding applications*, Artificial Intelligence Review, **53** (2020), 1093-1126. https://doi.org/10.1007/s10462-019-09690-y

[52] J. Zhan, K. Zhu, *A novel soft rough fuzzy set: Z-soft rough fuzzy ideals of hemirings and corresponding decision making*, Soft Computing, **21** (2017), 1923-1936. https://doi.org/10.1007/s00500-016-2119-9

[53] L. Zhang, J. Zhan, *Fuzzy soft β-covering based fuzzy rough sets and corresponding decision-making applications*, International Journal of Machine Learning and Cybernetics, **10** (2019), 1487-1502. https://doi.org/10.1007/s13042-018-0828-3

[54] L. Zhang, J. Zhan, J. C. R. Alcantud, *Novel classes of fuzzy soft β-coverings-based fuzzy rough sets with applications to multi-criteria fuzzy group decision making*, Soft Computing, **23** (2019), 5327-5351. https://doi.org/10.1007/s00500-018-3470-9

[55] J. L. Zhou, F. S. Xu, Y. Y. Guan, H. K. Wang, *Three types of fuzzy covering-based rough set models*, Fuzzy Sets and Systems, **423** (2021), 122-148. https://doi.org/10.1016/j.fss.2020.11.014

[56] W. Zhu, *Topological approaches to covering rough sets*, Information Sciences, **177** (2007), 1499-1508. https://doi.org/10.1016/j.ins.2006.06.009