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Ordinal sum of equality algebras

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Abstract

In this paper, by using the notion of ordinal sum of equality algebras which is defined in [4], we investigate some related properties and different kinds of filters of it such as prime and maximal. In addition, by focusing on chain equality algebra, we define the concept of a cut of an equality algebra and use it to introduce a new ordinal sum of them. By defining a relation relative to a cut, we show that the family of classes is made a cone algebra. Finally, we prove that every chain equality algebra can be represented as an ordinal sum of chain cone algebras.

Keywords: Equality algebra, ordinal sum, filter, cone algebra.

1 Introduction

The motivation for introducing equality algebras came from EQ-algebras which are defined by Novák in [22] and later elaborated in more detail in [23]. We have to emphasize that the operation of mlutiplication in EQ-algebras is important since it interprets a non-commutative conjunction in logic and, moreover, makes it possible to introduce only one implication unlike non-commutative conjuntion in logic based on residuated algebras. This new logical algebra, the equality algebra, has two connectives, a meet operation and an equivalence, and a constant. Equality algebra is introduced by Jenei [14] and it should also be mentioned that most of the properties of the equality algebras are inherited from EQ-algebras. The study of equality algebra is highly valuable for exploring possible algebraic semantics in fuzzy type theory. Recently, many mathematicians have studied this algebraic structure in various fields. For instance, the relation between equality algebra with other algebraic structures is investigated, the notions of annihilators and co-annihilators, commutativity are defined on equality algebras. It continued in [1, 2, 3, 6, 7, 12, 19, 20, 25]. In recent times, various aspects of equality algebras have been studied. These include equality logic, the category of equality algebras, tense operators on algebraic structures, the concept of involution, etc.

The definition of the ordinal sum in algebra is motivated by its utility in constructing new ordered structures from existing ones while preserving certain desirable properties, such as order relations and algebraic operations. The concept is particularly useful in areas like lattice theory, order theory, and algebraic structures with an inherent ordering (e.g., ordered groups or semigroups). As **Key Motivations**, we have to notice that ordinal sum allows the creation of a larger ordered structure by "stacking" smaller ordered structures in a way that preserves their internal order. For example, given two ordered sets A and B, the ordinal sum creates a new ordered set $A + B$ where every element of A is less than every element of B (**Construction of Larger Structures**).

In addition, the ordinal sum respects the order of the components. If A and B are totally ordered sets, the ordinal sum ensures the resulting structure is also totally ordered.

Moreover, in algebraic systems with order (e.g., lattices or semigroups), the ordinal sum is used to construct new systems while preserving algebraic and order-theoretic properties. For instance:

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Lattices: The ordinal sum of lattices preserves lattice properties while combining the underlying order structures.
Semigroups: Ordinal sum helps in defining ordered semigroups or other similar structures.
Finally, since many real-world phenomena and mathematical problems involve hierarchical or layered structures, the ordinal sum is a natural way to model such systems, where one part is "below" another in a strict sense.

Ordinal sums are one of the most important constructions used in the theory of t-norms on the unit interval. For example, based on the famous Mostert-Shields theorem [24], it can be proved that each continuous t-norm on the unit interval can be represented as an ordinal sum of continuous Archimedean t-norms. Ordinal sums of t-norms on bounded lattices have been studied intensively in recent years. In [26], the definition of an ordinal sum of t-norms defined on a set of mutually disjoint subintervals of a bounded lattice is provided, and it is shown that this ordinal sum does not necessarily have to be a t-norm, in general. Also, under which conditions the ordinal sum of any family of t-norms defined in this way is always a t-norm, are provided. In [18], a new characterization is given, concentrating on conditions that make possible to decide whether a specific ordinal sum, with respect to a particular family of t-norms, is a t-norm. In [27], it is investigated under which conditions the ordinal sum defined analogously as in [26], but this time on a sublattice of a lattice (this sublattice need not be an interval), can be extended to this lattice, and minimal and maximal such extensions are studied.

In this article, by employing the concept of ordinal sum of equality algebras which is defined in [4], we investigate some properties of it and different kinds of filters in ordinal sum of equality algebras. In addition, by focusing on chain equality algebra we show that every chain of equality algebra by some conditions is represented as ordinal sum of cone algebras.

2 Background information

In this section, we have some basic concepts about equality algebra that will be used in later sections. We only remind some definitions and results.

Definition 2.1. [14] Algebraic structure $(E; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is defined as an *equality algebra* if it meets the following conditions, for all $x, y, z \in E$,
$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1,
$(E2)$ $x \sim y = y \sim x$,
$(E3)$ $x \sim x = 1$,
$(E4)$ $x \sim 1 = x$,
$(E5)$ $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
$(E6)$ $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
$(E7)$ $x \sim y \leq (x \sim z) \sim (y \sim z)$.
On the equality algebra, we write $x \leq y$ if and only if $x \wedge y = x$. Then the relation "$\leq$" is a partial order relation on E. We note that a *lattice equality algebra* is an equality algebra which is a lattice. Also, we define the operations "$\rightarrow$" and "$\leftrightarrow$" on E as:

$$x \rightarrow y = x \sim (x \wedge y), \quad \text{and} \quad x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The set $\emptyset \neq X \subseteq E$ is called *subequality algebra* if it is closed under all operations on E. It means that for any $x, y \in X$, $x \wedge y, x \sim y \in X$.

For more information about type of algebras see [10].
Note: An equality algebra $(E; \wedge, \sim, 1)$ is denoted by E unless otherwise stated.

If there exists an element $0 \in E$ such that $0 \leq x$, for every $x \in E$, then E is called *bounded*. In a bounded equality algebra E, we define the negation operation " $'$ " by: $x' = x \rightarrow 0 = x \sim 0$, for all $x \in E$.

Definition 2.2. [3, 29] If for any $x, y, z \in E$,
(i) $x'' = x$, then E is called an *involutive equality algebra*.
(ii) 1 is the unique upper bound of the set $\{x \rightarrow y,\ y \rightarrow x\}$, then E is called *prelinear*.
(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, then E is called a *commutative equality algebra*.
(iv) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$, then E is called a *positive implicative* or *p-equality algebra*, for short.

Proposition 2.3. [14, 29] *The following conditions hold, for any $x, y, z \in E$:*
(i) $x \rightarrow y = 1$ *if and only if* $x \leq y$,
(ii) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$, *and* $x \rightarrow x = 1$,
(iii) $x \leq y \rightarrow x$,

(iv) $x \leq (x \to y) \to y$,
(v) $x \to (y \to z) = y \to (x \to z)$,
(vi) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,
(vii) $x \to y = x \to (x \wedge y)$,
$(viii)$ *If* E *is a lattice, then* $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
(ix) $((x \to y) \to y) \to y = x \to y$,
(x) $x \sim y \leq x \to y$,
(xi) $x \to y \leq (x \wedge z) \to (y \wedge z)$,
(xii) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \to y \leq (y \to z) \to (x \to z)$.

Definition 2.4. [15] Let F be a non-empty subset of E. Then F is called a *filter* of E if for all $x, y \in E$, we have
(i) $x \in F$ and $x \leq y$ imply $y \in F$,
(ii) $x \in F$ and $x \sim y \in F$ imply $y \in F$.
The set of all filters of E is denoted by $\mathcal{F}(E)$.

Proposition 2.5. [12, 15] *Let* $\emptyset \neq F \subseteq E$. *Then* $F \in \mathcal{F}(E)$ *if and only if, for all* $x, y \in E$, $1 \in F$, *and if* $x \in F$ *and* $x \to y \in F$, *then* $y \in F$.

Clearly, for any $F \in \mathcal{F}(E)$, $1 \in F$ and F is called a *proper filter* of E if $F \neq E$. Clearly, if E is bounded, then $F \in \mathcal{F}(E)$ is proper if and only if it is not containing 0. A non-trivial equality algebra E is called *simple* if $\mathcal{F}(E) = \{\{1\}, E\}$.

Let $F \in \mathcal{F}(E)$. Define the relation θ_F on E by

$$(x, y) \in \theta_F \quad \text{if and only if} \quad \{x \to y, y \to x\} \subseteq F. \tag{2.1}$$

Let $\dfrac{E}{F} = \{[x] |\ x \in E\}$ where $[x] = \{y \in E \mid\ (x, y) \in \theta_F\}$. Then define the binary relation $\leq_F$ on $\dfrac{E}{F}$ by:

$$[x] \leq_F [y] \quad \text{if and only if} \quad x \to y \in F, \tag{2.2}$$

which is an order relation on $\dfrac{E}{F}$. For any $x, y \in E$, define

$$[x] \sim_F [y] = [x \sim y] \quad \text{and} \quad [x] \wedge_F [y] = [x \wedge y]. \tag{2.3}$$

Then $\left(\dfrac{E}{F}, \sim_F, \wedge_F, 1_{\frac{E}{F}}\right)$ is called a *quotient equality algebra* and denoted by $\dfrac{E}{F}$, where $1_{\frac{E}{F}} = [1]_{\theta_F} = F$.

Definition 2.6. [20] Let $\emptyset \neq A \subseteq E$. The smallest filter of E containing A is called *the generated filter by* A *in* E which is denoted by $\langle A \rangle$. Indeed, $\langle A \rangle = \bigcap_{A \subseteq F \in \mathcal{F}(E)} F$.

Proposition 2.7. [20] *Let* $\emptyset \neq A \subseteq E$. *Then*

$$\langle A \rangle = \{x \in E \mid a_1 \to (a_2 \to (... \to (a_n \to x)...)) = 1,\ \textit{for some } n \in \mathbb{N} \textit{ and } a_1, ..., a_n \in A\}.$$

In particular, for any element $a \in E$, *we have* $\langle a \rangle = \{x \in E \mid a \to^n x = 1,\ \textit{for some } n \in \mathbb{N}\}$, *where* $x \to^0 y = y$ *and* $x \to^n y = x \to (x \to^{n-1} y)$, *for* $n \in \mathbb{N}$. *If* $F \in \mathcal{F}(E)$ *and* $a \in E \setminus F$, *then*

$$\langle F \cup \{a\} \rangle = \{x \in E \mid a \to^n x \in F,\ \textit{for some } n \in \mathbb{N}\}.$$

If $F, G \in \mathcal{F}(E)$, *then*

$$\langle F \cup G \rangle = \{x \in E \mid\ g \to x \in F,\ \textit{for some } g \in G\} = \{x \in E \mid\ f \to x \in G,\ \textit{for some } f \in F\}.$$

Definition 2.8. [7, 19, 20] Let F be a non-empty subset of E such that $1 \in F$. Then
(i) F is a *prime filter* of E if for any $x, y \in E$, $x \to y \in F$ or $y \to x \in F$. The set of all prime filters of E is denoted by $\mathcal{P}rime(E)$.
(ii) F is a *maximal filter* of E if it is not included in any other proper filter of E. The set of all maximal filters of E is denoted by $\mathcal{M}ax(E)$.

Note: We set

$$Rad(E) = \bigcap_{M \in \mathcal{M}ax(E)} M. \tag{2.4}$$

If E does not have any maximal filter we set $Rad(E) = E$. For more detail see [5].

3 Ordinal sum of equality algebras

In this section, we introduce the concept of ordinal sum of a family of equality algebras. Then, we focus on the ordinal sum of two equality algebras and some properties of them are investigated. Moreover, different kinds of ordinal sum and filters of equality algebra are studied.

Definition 3.1. [28] Let I be a linearly ordered set (simply chain) and $\{(E_i; \sim_i, \wedge_i, 1) \mid i \in I\}$ be a family of equality algebras such that $E_i \cap E_j = \{1\}$ for all distinct elements $i, j \in I$. Define the binary operations $\sim$ and $\wedge$ on $\bigcup_{i\in I} E_i$ as follows:

$$x \wedge y = \begin{cases} x \wedge_{E_i} y & x, y \in E_i \\ y & x \in E_j,\ y \in E_i \setminus \{1\}, i < j \\ x & x \in E_i \setminus \{1\},\ y \in E_j,\ i < j \end{cases} \qquad x \sim y = \begin{cases} x \sim_{E_i} y & x, y \in E_i \\ y & x \in E_j,\ y \in E_i \setminus \{1\},\ i < j \\ x & x \in E_i \setminus \{1\},\ y \in E_j,\ i < j \end{cases}$$

Then $(E, \wedge, \sim, 1^E)$ is called *the ordinal sum* of $\{(E_i, \sim_i, \wedge_i, 1^{E_i})\}_{i\in I}$ denoted by $E = \bigoplus_{i\in I} E_i$, where $1^E = 1^{E_i}$

Remark 3.2. (i) As $E = \bigcup_{i\in I} E_i$ is an equality algebra, $x \leq y$ if and only if $x \wedge y = x$, for any $x, y \in E$. Also, the order "$\leq$" on E is a partial order.
(ii) It is clear that for any $x \in E$, $x \leq 1$.
(iii) For any $i, j \in I$ such that $i \leq j$, if $x \in E_i \setminus \{1\}$ and $y \in E_j$, we have $x \leq y$. Since by definition of "$\wedge$", $x \wedge y = x$. Particularly, if I is bounded from below with minimal element i_0 and E_{i_0} is bounded, then E is bounded, too.

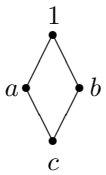
Throughout this paper, by the ordinal sum $\bigoplus_{i=1}^{n} E_i$, we mean that equality algebras $\{E_i \mid i \in I\}$ and an index set I satisfy the conditions in above definition. It is easy to see that the ordinal sum of finitely many equality algebras is associative. Hence, we feel free to write $E_1 \oplus E_2 \oplus \cdots \oplus E_n$ without parenthesis.

Example 3.3. Let $E_1 = \{0, q, r, 1\}$ be a chain where $0 \leq q \leq r \leq 1$. Define the operation $\sim$ on E_1 by:

$\sim_{E_1}$	0	q	r	1
0	1	0	0	0
q	0	1	r	q
r	0	r	1	r
1	0	q	r	1

$\to_{E_1}$	0	q	r	1
0	1	1	1	1
q	0	1	1	1
r	0	r	1	1
1	0	q	r	1

Then $E_1 = (E_1, \wedge, \sim, 1)$ is a bounded equality algebra. Now, let $E_2 = \{c, a, b, 1\}$ be a poset where $c \leq a, b \leq 1$. Define the operation $\sim$ on E_2 as follows:



$\sim_{E_2}$	c	a	b	1
c	1	b	a	c
a	b	1	c	a
b	a	c	1	b
1	c	a	b	1

$\to_{E_2}$	c	a	b	1
c	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	c	a	b	1

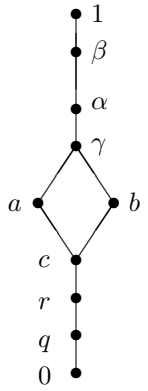
Then $(E_2, \wedge, \vee, \sim, 0, 1)$ is an equality algebra.
Also, let $(E_3 = \{\gamma, \alpha, \beta, 1\}, \leq)$ be a chain where $\gamma < \alpha < \beta < 1$. Define an operation $\sim$ on E_3 as follows:

$\sim_{E_3}$	γ	α	β	1
γ	1	γ	γ	γ
α	γ	1	α	α
β	γ	α	1	β
1	γ	α	β	1

$\to_{E_3}$	γ	α	β	1
γ	1	1	1	1
α	γ	1	1	1
β	γ	α	1	1
1	γ	α	β	1

Then $(E_3, \wedge, \sim, 0, 1)$ is an equality algebra.

Thus, the ordinal sum of them is $(E = E_1 \cup E_2 \cup E_3, \wedge, \sim, 1)$ with the following Hasse diagram:



and operation $\sim$ define on E as follows:

$\sim$	0	q	r	c	a	b	γ	α	β	1
0	1	0	0	0	0	0	0	0	0	0
q	0	1	r	q	q	q	q	q	q	q
r	0	r	1	r	r	r	r	r	r	r
c	0	q	r	1	b	a	c	c	c	c
a	0	q	r	b	1	c	a	a	a	a
b	0	q	r	a	c	1	b	b	b	b
γ	0	q	r	c	a	b	1	γ	γ	γ
α	0	q	r	c	a	b	γ	1	α	α
β	0	q	r	c	a	b	γ	α	1	β
1	0	q	r	c	a	b	γ	α	β	1

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra.

Remark 3.4. Let $\{E_i\}_{i\in I}$ be a family of equality algebras satisfying $E_i \cap E_j = \{1\}$, for any $i, j \in I$ such that $i \neq j$. Then the sub-equality of $\oplus_{i\in I} E_i$ is exactly the equality algebra of the form $\oplus_{i\in I} B_i$, where any B_i is (possibly trivial) a sub-equality algebra of E_i.

In the following, we focus on the ordinal sum on two equality algebras and so we have the definition of operations on this special case as:

$$x \wedge y = \begin{cases} x \wedge_{E_1} y & x, y \in E_1 \\ x \wedge_{E_2} y & x, y \in E_2 \\ y & x \in E_2,\ y \in E_1 \setminus \{1\} \\ x & x \in E_1 \setminus \{1\},\ y \in E_2 \end{cases} \tag{3.1}$$

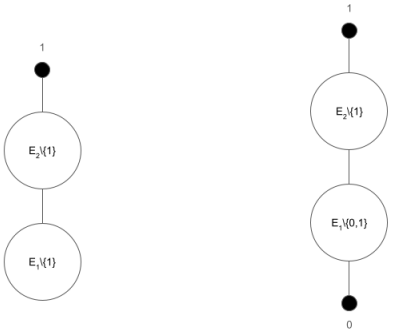
$$x \sim y = \begin{cases} x \sim_{E_1} y & x, y \in E_1 \\ x \sim_{E_2} y & x, y \in E_2 \\ y & x \in E_2,\ y \in E_1 \setminus \{1\} \\ x & x \in E_1 \setminus \{1\},\ y \in E_2 \end{cases} \tag{3.2}$$

In addition, we have the operation "$\to$" on E as: $x \to y = x \sim (x \wedge y)$. So, for ordinal sum of equality algebras, the implication operation is as follows:

$$x \to y = \begin{cases} x \to_{E_1} y & x, y \in E_1 \\ x \to_{E_2} y & x, y \in E_2 \\ y & x \in E_2,\ y \in E_1 \setminus \{1\} \\ 1 & x \in E_1 \setminus \{1\},\ y \in E_2 \end{cases} \tag{3.3}$$

It is straightforward to see that $E := E_1 \oplus E_2$ is an equality algebra and E_1 and E_2 are subalgebras of it (see [4]). Also, $E_1 \oplus E_2 \neq E_2 \oplus E_1$, where $E_1 \neq E_2$ and both are non-trivial. If $E_1 \cap E_2 \neq \{1\}$, then E_1 and E_2 are replaced

by isomorphic copies with the sole intersection 1 such as $\{1\} \times E_1$ and $E_2 \times \{1\}$, where intersection is $\mathbf{1} = (1, 1)$ and define their ordinal sum as above.



Definition 3.5. An equality algebra which can not be obtained as the ordinal sum of two non-trivial equality algebras is said to be *sum irreducible* and is denoted by $\oplus$-irreducible.

It is obvious that every equality algebra E is the ordinal sum of E with the trivial equality algebra $\{1\}$, i. e., $\{1\} \oplus E = E = E \oplus \{1\}$.

Definition 3.6. [4] Recall that an algebraic structure E is *subdirectly irreducible* if and only if E has a congruence $\mu \neq \Delta$ that is contained in every non-trivial congruence of E.

Definition 3.7. [4] An equality algebra E is a subdirect product of a family $\{E_i\}_{i \in I}$ of equality algebras if and only if there exists a one-to-one homomorphism $\phi : E \longrightarrow \prod_{i \in I} E_i$ such that the map $\pi_j \circ \phi$ is an onto homomorphism, for all $j \in I$.

Note: Recall that an algebraic structure E is subdirectly irreducible if and only if E has a congruence $\mu \neq \Delta$ that is contained in every non-trivial congruence of E. Assume E is a subdirectly irreducible equality algebra with μ. Then $U = [1]_\mu$ is the unique minimal filter of E distinct from $\{1\}$. It means that U is contained in every non-trivial filter of E.

Definition 3.8. [4] Let E be a subdirectly irreducible equality algebra with minimal filter U. The set of all fixed elements of E is denoted by $\mathcal{F}ix_U(E)$ and defined by

$$\mathcal{F}ix_U(E) = \{a \in E \mid \text{ for all } u \in U;\ u \to a = a\}.$$

Let $\mathcal{S} = (E \setminus \mathcal{F}ix_U(E)) \cup \{1\}$.

Proposition 3.9. [4] *Let E be a subdirectly irreducible p-equality and $U \neq \{1\}$ is the minimal filter of E. Consider $a \in \mathcal{F}ix_U(E)$ and $a \neq 1$. Then*
(i) For any $u \in U$, $a \leq u$.
(ii) For any $b \in E$, if $b \leq a$, then $b \in \mathcal{F}ix_U(E)$, which means $\mathcal{F}ix_U(E)$ is a down set.
(iii) For all $b \in E$, $a \to b, b \to a \in \mathcal{F}ix_U(E)$.
(iv) For any $b \in \mathcal{S}$, $a \leq b$ and $b \to a = a$.
(v) $\mathcal{F}ix_U(E) \in \mathcal{F}(E)$.

Theorem 3.10. [4] *Let E be a subdirectly irreducible p-equality algebra, U be it's minimal filter. Then*
(i) $\mathcal{F}ix_U(E)$ and $\mathcal{S}$ are subalgebras of E. In addition $\mathcal{S} \in \mathcal{F}(E)$.
(ii) $U \subseteq \mathcal{S}$ and $\langle \mathcal{S}, \wedge, \sim, 1 \rangle$ is a commutative subdirectly irreducible p-equality algebra. In particular, $\mathcal{S}$ is totally ordered.
(iii) $E = \mathcal{F}ix_U(E) \oplus \mathcal{S}$.

Corollary 3.11. *If E is a subdirectly irreducible p-equality algebra, then it is not $\oplus$-irreducible.*

Proof. It is clear by Theorem 3.10(iii). □

Note: In the following, by $E_1 \oplus E_2$, we mean the ordinal sum of non-trivial equality algebras E_1 and E_2, unless otherwise stated.

Proposition 3.12. *Let E_1 be bounded. Then $E = E_1 \oplus E_2$ is not involutive.*

Proof. Let E_1 be bounded. By Remark 3.2(iii), $E = E_1 \oplus E_2$ is bounded, too. If $x \in E_2 \setminus \{1\}$, then by (3.3), we have

$$x'' = (x \to 0) \to 0 = 0 \to 0 = 0 \to_{E_1} 0 = 1 \neq x.$$

□

Proposition 3.13. *If E_1 and E_2 are*
(i) lattices such that E_2 is bounded, then $E = E_1 \oplus E_2$ is a lattice;
(ii) chains, then $E = E_1 \oplus E_2$ is a chain;
(iii) prelinears, then $E = E_1 \oplus E_2$ is prelinear;
(iv) p-equalities, then $E = E_1 \oplus E_2$ is a p-equality.

Proof. By Remark 3.2 the proofs of $(i) - (iii)$ are clear.
(iv) Let E_1 and E_2 be p-equalities and $x, y, z \in E$. Then, we have the following cases:
Cases 1 and 2: If $x, y, z \in E_1$ or $x, y, z \in E_2$, then the proof is clear.
Suppose $x, y, z \in E_i \setminus \{1\}$. Then, we have the following cases:
Case 3: If $x, y \in E_1$ and $z \in E_2$, then by (3.3), we have

$$x \to (y \to z) = x \to 1 = 1 = (x \to_{E_1} y) \to 1 = (x \to y) \to (x \to z).$$

Case 4: If $x \in E_1$ and $y, z \in E_2$, then

$$x \to (y \to z) = 1 = 1 \to 1 = (x \to y) \to (x \to z).$$

Case 5: If $y \in E_1$ and $x, z \in E_2$, then

$$x \to (y \to z) = x \to 1 = 1 = y \to (x \to_{E_2} z) = (x \to y) \to (x \to z).$$

Case 6: If $y, z \in E_1$ and $x \in E_2$, then

$$x \to (y \to z) = x \to (y \to_{E_1} z) = y \to_{E_1} z = y \to z = (x \to y) \to (x \to z).$$

Case 7: If $z \in E_1$ and $x, y \in E_2$, then

$$x \to (y \to z) = x \to z = z = (x \to_{E_2} y) \to z = (x \to y) \to (x \to z).$$

Case 8: If $x, z \in E_1$ and $y \in E_2$, then

$$x \to (y \to z) = x \to z = x \to_{E_1} z = 1 \to (x \to_{E_1} z) = (x \to y) \to (x \to z).$$

Case 9: If at least one of x or y or z equal 1, then

$$y \to z = x \to (y \to z) = (x \to y) \to (x \to z) = (1 \to y) \to (1 \to z) = y \to z.$$
$$x \to z = x \to (1 \to z) = x \to (y \to z) = (x \to 1) \to (x \to z) = 1 \to (x \to z) = x \to z.$$
$$1 = x \to (y \to 1) = x \to (y \to z) = (x \to y) \to (x \to 1) = (x \to y) \to 1 = 1.$$

Thus, in all above cases, the equation $x \to (y \to z) = (x \to y) \to (x \to z)$ holds. Therefore, $E = E_1 \oplus E_2$ is a p-equality algebra. □

Note: Clearly, if E_1 or E_2 is not a lattice, then $E = E_1 \oplus E_2$ can not be a lattice. Moreover, the condition "bounded" Proposition 3.13(i) for E_2 is a necessary condition, as you can see in the following example.

Example 3.14. Let $E_1 = \{0, a, b, 1\}$ where $0 < a, b < 1$. Define an operation $\sim$ on E_1 as follows,

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

Then $(E_1, \wedge, \sim, 0, 1)$ is a bounded equality algebra. Also, suppose $E_2 = (0, 1]$ and define the operation $\wedge$ and $\sim$ on E_2 as follows:

$$x \wedge y = \min\{x, y\} \qquad x \sim y = (x \to y) \wedge (y \to x), \text{ where } x \to y = \begin{cases} 1 & x \leq y \\ (1-x) \vee y & x > y \end{cases}$$

Then $(E_2, \wedge, \sim, 1)$ is an equality algebra which is not bounded. Now, if we made $E = E_1 \oplus E_2$, then according to definition of E, we know that any element of E_1 is less than all elements of E_2, so the question is this: What is the $\sup\{a, b\}$? Since E_2 is not bounded, we can not determine the $\sup\{a, b\}$ so, E is not a lattice. Because of this, we said that E_2 has to be a bounded lattice. So, the $\sup\{a, b\}$ will be 0_{E_2}.

Proposition 3.15. *If E_1 and E_2 are non-trivial, then in any condition, $E = E_1 \oplus E_2$ is not a commutative equality algebra.*

Proof. Consider $x, y \in E$ such that $x \in E_1 \setminus \{1\}$ and $y \in E_2 \setminus \{1\}$. Then, we have

$$(x \to y) \to y = 1 \to y = y \neq 1 = x \to x = (y \to x) \to x.$$

□

Proposition 3.16. *Let $E = E_1 \oplus E_2$. Then the following statements hold:*
(i) $F \in \mathcal{F}(E_2)$, then $F \in \mathcal{F}(E)$.
(ii) $F \in \mathcal{F}(E)$ implies $F_1 = F \cap E_1 \in \mathcal{F}(E_1)$.
(iii) If $F_1 \in \mathcal{F}(E_1)$, then $F = F_1 \cup E_2 \in \mathcal{F}(E)$.

Proof. (i) Clearly, $1 \in F \subseteq E_2$. Suppose $x, x \to y \in F$, for $x, y \in E$. Since $E_1 \cap E_2 = \{1\}$, we get $x, x \to y \in E_2$. Also, by (3.3) and $x \to y \in E_2$, we get $y \in E_2$. From $F \in \mathcal{F}(E_2)$, we get $y \in F$. Hence, $F \in \mathcal{F}(E)$.
(ii) Let $F \in \mathcal{F}(E)$. Take $F_1 := F \cap E_1$. Clearly, $1 \in F_1$. In order to check that F_1 is a filter of E_1, we take $x, y \in E_1$ such that $x, x \to y \in F_1$, then we have $x, x \to y \in F$, by $F \in \mathcal{F}(E)$ we get $y \in F$. Thus $y \in F \cap E_1$, and so $y \in F_1$. Hence, $F_1 \in \mathcal{F}(E_1)$.
(iii) Clearly, $1 \in F$. Suppose $x, x \to y \in F$. Then, we have the following cases:
Case 1: If $x, x \to y \in F_1$, then since $F_1 \in \mathcal{F}(E_1)$, we have $y \in F_1$, and so $y \in F$.
Case 2: If $x, x \to y \in E_2$, then since $E_2 \in \mathcal{F}(E_2)$, we get $y \in E_2$, and so $y \in F$.
Case 3: If $x \in F_1 \setminus \{1\}$ and $x \to y \in E_2 \setminus \{1\}$, then $y \notin E_2$. because of $y \in E_2$ implies that $x \to y = 1$, a contradiction. Thus $y \in E_1$ and so $x \to y = x \to_{E_1} y \in E_1$. Hence, $x \to y \in E_1 \cap E_2 = \{1\}$, which is a contradiction. So, this case does not happen.
Case 4: If $x \in E_2 \setminus \{1\}$ and $x \to y \in F_1 \setminus \{1\}$, then if $y \in E_2 \setminus \{1\}$, we get $y \in F$ and so the proof is complete. If $y \in E_1$, then $x \to y = y \in F_1$, and so $y \in F$. Thus $F \in \mathcal{F}(E)$.
Case 5: If $x = 1$ and $x \to y \in F$, then by (3.3), $y \in F$ and so $F \in \mathcal{F}(E)$.
Case 6: If $x \in F \setminus \{1\}$ and $x \to y = 1$, then we have two cases. If $x \in F_1$, then since $x \to y = 1 \in F_1$, we get $y \in F_1$, and so $y \in F$. If $x \notin F_1$, we get $x \in E_2 \setminus \{1\}$, and since $x \to y = 1 \in E_2$, we get $y \in E_2$ and so $y \in F$. Hence, $F \in \mathcal{F}(E)$. Therefore, in all above cases $F \in \mathcal{F}(E)$. □

Now, we give general form of filters in the ordinal sum of equality algebra.

Theorem 3.17. *In the ordinal sum $\bigoplus_{i=1}^{n} E_i$ we have the following statements:*

(i) *If F_j is a filter of E_j, for some $j \in \{1, 2, \cdots, n\}$, then $F = F_j \cup \left(\bigcup_{k>j} E_k \right)$ is a filter of $\bigoplus_{i=1}^{n} E_i$.*

(ii) *If F is a non-trivial filter of $\bigoplus_{i=1}^{n} E_i$ and $k_j = \min\{i \in \{1, 2, \cdots, n\} \mid (F \setminus \{1\}) \cap E_i \neq \emptyset\}$, then $F_{k_j} = F \cap E_{k_j} \in \mathcal{F}(E_{k_j})$ and $F = F_{k_j} \cup \left(\bigcup_{k>k_j} E_k \right)$.*

Proof. (i) Similar to the proof of Proposition 3.16, for $x, x \to y \in F$, if $x, x \to y \in F_j$, since F_j is a filter of E_j, clearly $y \in F_j$. In other cases, since F_j is upset and clearly $\bigcup_{k>j} E_k$ is a filter of $\bigoplus E_i$, we get that $y \in \bigcup_{k>j} E_k \subseteq F$. Hence, F is a filter of $\bigoplus E_i$.
(ii) Since $i : E_i \hookrightarrow \bigoplus E_i$, it is straightforward that if F is a filter of $\bigoplus E_i$, then $i^{-1}(F) = F \cap E_k$ is a filter of E_k. Similar to the proof of Proposition 3.16, the proof of other case is trivial. □

Remark 3.18. From the above theorem, we can see that any non-trivial filter F of the ordinal sum $\bigoplus_{i=1}^{n} E_i$ of equality algebras is uniquely determined by filter F_i of E_i, where $i = \min\{k \mid F \cap E_k \neq \emptyset\}$.

Proposition 3.19. *Let $E = E_1 \oplus E_2$. Then, we have the following statements:*

(i) *Suppose $P \in \mathcal{P}rime(E)$ and $P_1 = P \cap E_1$. If P_1 is proper, then $P_1 \in \mathcal{P}rime(E_1)$.*

(ii) *If $P_1 \in \mathcal{P}rime(E_1)$, then $P = P_1 \cup E_2 \in \mathcal{P}rime(E)$.*

Proof. (i) Suppose $P_1 = P \cap E_1$. By assumption and Proposition 3.16, P_1 is a proper filter of E_1. Consider $x, y \in E_1$. Since $E_1 \subseteq E$, we get $x, y \in E$. In addition, $P \in \mathcal{P}rime(E)$, we get $x \to y \in P$ or $y \to x \in P$. On the other side, $x \to y = x \to_{E_1} y \in E_1$ and $y \to x = y \to_{E_1} x \in E_1$. Hence, $x \to y \in P \cap E_1$ or $y \to x \in P \cap E_1$. Hence, $P_1 \in \mathcal{P}rime(E_1)$.
(ii) Suppose $P_1 \in \mathcal{P}rime(E_1)$ and $P = P_1 \cup E_2$. Consider $x, y \in E$. If $x = 1$ or $y = 1$, then $x \to y = 1$ or $y \to x = 1$ and the proof is complete. Let $x \neq 1 \neq y$. Then, we have the following cases:
Case 1: If $x, y \in E_1$, then $x \to y = x \to_{E_1} y$ and $y \to x = y \to_{E_1} x$. Since $P_1 \in \mathcal{P}rime(E_1)$ and $x, y \in E_1$, we get $x \to y \in P_1$ or $y \to x \in P_1$ and from $P_1 \subseteq P$, we obtain $x \to y \in P$ or $y \to x \in P$.
Case 2: If $x, y \in E_2$, then $x \to y = x \to_{E_2} y$ and $y \to x = y \to_{E_2} x \in E_2$. Since $E_2 \subseteq P$, we get $x \to y \in P$ or $y \to x \in P$.
Case 3: Consider $x \in E_1$ and $y \in E_2$. According to definition of "$\leq$", we know that $x \leq y$ and so $x \to y = 1 \in P_1 \subseteq P$. Hence, $x \to y \in P$.
Case 4: Consider $x \in E_2$ and $y \in E_1$. In this case $y \to x = 1 \in P_1 \subseteq P$. Hence, $y \to x \in P$, and so $P \in \mathcal{P}rime(E)$.
Therefore, in all above cases $P \in \mathcal{P}rime(E)$. □

In the following, we give an example of unbounded equality algebra which has not any maximal filter.

Let us consider the two elements Boolean algebra $L = \{0, 1\}$. Then L is an equality algebra and $1 \neq 0$. From now on, we use **2** to denote L, for simplicity. Indeed, $\mathbf{2} := (\{0, 1\}, \wedge, \sim, 1)$ is the unique equality algebra with two elements, where

$$x \sim y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Example 3.20. Consider $E_i = \{-i, 1\}$, where $i \geq 2$, $1 := -1$ and $E_i \cong \mathbf{2}$. Clearly, $E_i \cap E_j = \{1\}$. Consider $E = \oplus_{i \geq 2} E_i$. Thus by Definition 3.1, $E = \{-n : n \in \mathbb{N}\}$ is an unbounded equality algebra. Easy calculation shows that $\mathcal{F}(E) = \{F_i\}_{i \geq 1}$, where $F_i = \{-i, \cdots, -2, 1\}$. We claim that E has no maximal filter. If $M \in \mathcal{M}ax(E)$, then there exists $n \in \mathbb{N}$ such that $M = F_n = \{-n, -n+1, \cdots, -2, 1\}$ and so $-n-1 \notin M$. Thus $F_{n+1} := \{-n-1, -n, \cdots, -2, 1\} \in \mathcal{F}(E)$ such that $M \subsetneq F_{n+1} \subsetneq E$, which is a contradiction. Therefore, E has no maximal filter.

Note: According to the above example, every equality algebra does not have maximal filters, in general. In the following, we have some results about equality algebras and maximal filters of them.

Proposition 3.21. *Let $E = E_1 \oplus E_2$ such that $\mathcal{M}ax(E) \neq \emptyset$. Then, we have the following statements:*

(i) *Suppose $M \in \mathcal{M}ax(E)$ and $M_1 = M \cap E_1$. Then $M_1 \in \mathcal{M}ax(E_1)$.*

(ii) *If $M_1 \in \mathcal{M}ax(E_1)$, then $M = M_1 \cup E_2 \in \mathcal{M}ax(E)$.*

Proof. (i) Since $i : E_1 \hookrightarrow E_1 \oplus E_2$ is an embedding and $i^{-1}(M) = M \cap E_1$ we get $M_1 \in \mathcal{M}ax(E_1)$.
(ii) Let $M_1 \in \mathcal{M}ax(E_1)$. Then M_1 is a proper filter of E_1. Now, take $M := M_1 \cup E_2$. Then by Proposition 3.16, M is a proper filter of E. Now, show that $M \in \mathcal{M}ax(E)$. For this, suppose $M \subseteq H \subsetneq E$. Then $M_1 = M \cap E_1 \subseteq H \cap E_1 \subsetneq E \cap E_1 = E_1$. Since $M_1 \in \mathcal{M}ax(E_1)$, we get $M_1 = H \cap E_1$. Thus,

$$M = M_1 \cup E_2 = (H \cap E_1) \cup E_2 = H \cup E_2.$$

Hence, $M = H \cup E_2$. By assumption, $M \subseteq H$. Since $M = H \cup E_2$, we get $H \cup E_2 \subseteq H \subseteq H \cup E_2$. Hence, $H = H \cup E_2 = M$. Therefore, $M \in \mathcal{M}ax(E)$. □

Note: Proposition 3.21(ii) is logically correct, if premise is false, i.e., $\mathcal{M}ax(E_1)$ is empty, then it trivially holds.

Theorem 3.22. *Let $E = E_1 \oplus E_2$, $\mathcal{M}ax(E) \neq \emptyset$ and E_1 be simple. Then $M \in \mathcal{M}ax(E)$ if and only if $M = E_2$.*

Proof. Let $M \in \mathcal{M}ax(E)$. We show that $M = E_2$. By Proposition 3.21, $M \cap E_1 \in \mathcal{M}ax(E_1)$. Since E_1 is simple, we get $M \cap E_1 = \{1\}$. Also, $E_1 \cap E_2 = \{1\}$, thus $M \subseteq E_2$. On the other hand, let $x \in E_2 \setminus M$, we get $M \subsetneq \langle M \cup \{x\}\rangle \subseteq E$. From $M \in \mathcal{M}ax(E)$, we have $\langle M \cup \{x\}\rangle = E$. Since $M \subseteq E_2$ and $x \in E_2$, we obtain $\langle M \cup \{x\}\rangle \subseteq E_2$, thus $E \subseteq E_2$ and so $E_1 \subseteq E_2$, which is a contradiction. Hence, $M = E_2$.

Conversely, consider $M = E_2$. Then we show that $E_2 \in \mathcal{M}ax(E)$. Suppose $H \in \mathcal{F}(E)$ such that $E_2 \subseteq H \subseteq E$. Then by Proposition 3.16, we get $H \cap E_1 \in \mathcal{F}(E_1)$. By assumption, E_1 is simple, so $H \cap E_1 = \{1\}$ or $H \cap E_1 = E_1$. If $H \cap E_1 = \{1\}$, since $E_1 \cap E_2 = \{1\}$ we get $H \subseteq E_2$. Also, we know, $E_2 \subseteq H$. Hence, $H = E_2$. If $H \cap E_1 = E_1$, then $E_1 \subseteq H$. Also, we know, $E_2 \subseteq H$. Thus, $E = E_1 \cup E_2 \subseteq H \subseteq E$, and so, $H = E$. Therefore, $E_2 \in \mathcal{M}ax(E)$. □

Corollary 3.23. *Let $E = E_1 \oplus E_2$ such that $\mathcal{M}ax(E) \neq \emptyset$. If E_1 is simple, then by* (2.4)*, $Rad(E) = E_2$.*

Proposition 3.24. *Let $E = E_1 \oplus E_2$ and E_1 be simple. Then*
(i) $\mathcal{F}(E) \setminus \{E\} = \mathcal{F}(E_2)$.
(ii) E is subdirectly irreducible if and only if E_2 is subdirectly irreducible.

Proof. (i) By Proposition 3.16(i), $\mathcal{F}(E_2) \subseteq \mathcal{F}(E)$. For the other side of inclusion, let $F \in \mathcal{F}(E)$, then we have two cases:
Case 1: if $F \cap (E_1 \setminus \{1\}) = \emptyset$, then $F \subseteq E_2$ and so $F \in \mathcal{F}(E_2)$.
Case 2: if $F \cap (E_1 \setminus \{1\}) \neq \emptyset$, then by Proposition 3.16(ii), $F \cap E_1 \in \mathcal{F}(E_1)$. Since E_1 is simple, we get $F \cap E_1 = \{1\}$ or $F \cap E_1 = E_1$. If $F \cap E_1 = E_1$, then $E_1 \subseteq F$. Since F is an upset and Remark 3.2, we get $E_2 \subseteq F$ and so $E = E_1 \cup E_2 \subseteq F$, i.e. $F = E$, which is a contradiction. Hence, $F \cap E_1 = \{1\}$ and so $F \subseteq E_2$. Therefore, $F \in \mathcal{F}(E_2)$.
(ii) It is the result of (i). □

Definition 3.25. An equality algebra E such that $\mathcal{M}ax(E) \neq \emptyset$ is called

(i) *semisimple* if $Rad(E) = \{1\}$.

(ii) *local* if E has only one maximal filter.

(iii) *perfect* if $E = Rad(E) \cup Rad'(E)$, where E is bounded and $Rad'(E) = \{a' \in E \mid a \in Rad(E)\}$.

Theorem 3.26. *Let $E = E_1 \oplus E_2$ such that $\mathcal{M}ax(E) \neq \emptyset$. Then, we have the following statements:*

(i) *If E is semisimple, then E_1 is semisimple.*

(ii) *E is simple, then E_1 is local.*

Proof. (i) According to (2.4), we claim that $Rad(E_1) \subseteq Rad(E)$; Let $a \in Rad(E_1)$. Then for any $M_1 \in \mathcal{M}ax(E_1)$, $a \in M_1$. Also, if $M \in \mathcal{M}ax(E)$ is arbitrary, then by Proposition 3.21(i), $M \cap E_1 \in \mathcal{M}ax(E_1)$. Hence, $a \in M \cap E_1$ and so $a \in M$. Thus, for any $M \in \mathcal{M}ax(E)$, $a \in M$, and so $a \in Rad(E)$.

Since E is semisimple, we get that $\{1\} \subseteq Rad(E_1) \subseteq \{1\}$. Hence, $Rad(E_1) = \{1\}$, and so E_1 is semisimple.
(ii) If E_1 is simple, then by Theorem 3.22, $\mathcal{M}ax(E) = E_2$. Thus, E has just one maximal filter, and so E is local. □

Note: We have to notice that if E_1 is semisimple, E may not be semisimple, where $\mathcal{M}ax(E) \neq \emptyset$. Since for any $M \in \mathcal{M}ax(E_1)$, by Proposition 3.21, $M \cup E_2 \in \mathcal{M}ax(E)$, and so the set of all maximal filters of E_1 is a subset of the set of all maximal filters of E. Thus, by (2.4), $Rad(E) \subseteq Rad(E_1) \cup E_2 = \{1\} \cup E_2 = E_2$.

Theorem 3.27. *Let $E = E_1 \oplus E_2$ be bonded and $\mathcal{M}ax(E) \neq \emptyset$. Then E is perfect and E_1 is simple if and only if $E_1 = \{0, 1\}$.*

Proof. Since E_1 is simple, by Corollary 3.23 and (2.4), $Rad(E) = E_2$. Also, by assumption, since E is perfect, we have $E = Rad(E) \cup Rad'(E)$, thus, $E = E_2 \cup Rad'(E)$, since clearly $Rad(E) \cap Rad'(E) = \emptyset$, we get $Rad'(E) = E_1 \setminus \{1\}$. Now, we prove $Rad'(E) = \{0\}$. Since $1' = 0$, we have $\{0\} \subseteq Rad'(E)$. If $x \in Rad'(E)$, then there exists $y \in Rad(E)$ such that $x = y'$. By (3.3), we have

$$x = y' = y \to 0 = 0.$$

Hence, $x = 0$, and so $Rad'(E) \subseteq \{0\}$. Therefore, $Rad'(E) = \{0\}$ and so $E_1 = \{0, 1\}$.

Conversely, consider $E_1 = \{0, 1\}$. Clearly, E_1 is simple, $E = \{0, 1\} \cup E_2 = \{0\} \cup E_2$ and $Rad(E) \cup Rad'(E) \subseteq E$. If $x \in E$, then $x = 0$ or $x \in E_2$. If $x = 0$, then $0' = 1 \in Rad(E)$, thus $0 \in Rad'(E)$. Also, since E_1 is simple, by Corollary 3.23, $x \in E_2 = Rad(E)$. Thus, $E \subseteq Rad(E) \cup Rad'(E)$. Hence, $E = Rad(E) \cup Rad'(E)$. Therefore, E is perfect. □

4 Ordinal sum of chain equality algebras

In this section, we investigate ordinal sum of chain equality algebras and study some properties of them and by using this notion we prove that every chain commutative equality algebra is represented as an ordinal sum of chain cone algebras.

Definition 4.1. Let $\emptyset \neq X \subseteq E \setminus \{1\}$. Then X is called a *cut* of E if
(c1) $x < y$, for all $x \in X$ and $y \in E \setminus X$.
(c2) $E \setminus X$ is closed under "$\to$".
(c3) $y \to x = x$, for all $x \in X$ and $y \in E \setminus X$.
We say that a cut X is trivial, if $X = \emptyset$ or $X = E \setminus \{1\}$.

Example 4.2. Let $E = E_1 \oplus E_2$. Consider $X := E_1 \setminus \{1\}$. Then by (3.3), X is a cut of E.

Note: From now on, let an equality algebra $(E; \wedge, \sim, 1)$ be a chain, unless otherwise stated.

Proposition 4.3. *Let X be a a non-trivial cut of an equality algebra E, and denote $E_1 = X \cup \{1\}$ and $E_2 = E \setminus X$. Then*

(i) *X is downset of $(E, \leq)$.*

(ii) *$E \setminus X$ is upset.*

(iii) *$(E_1, \wedge, \sim, 1)$ and $(E_2, \wedge, \sim, 1)$ are subalgebras of E.*

(iv) *$E = E_1 \oplus E_2$. In addition, E_1 and E_2 are non-trivial.*

Proof. (i) Let $x < y$ and $y \in X$. If $x \in X$, then the proof is complete. Consider $x \in E \setminus X$. Then by Definition 4.1(c3), we have $x \to y = y$. On the other hand, by $x < y$, we get $x \to y = 1$, and so $y = 1$ and $y \in X \subseteq E \setminus \{1\}$, which is a contradiction. Hence, $x \in X$ and so X is downset.
(ii) Let $x \in E \setminus X$ and $x \leq y$, for $y \in E$. If $y \notin E \setminus X$, then $y \in X$. By (i), X is downset and so $x \in X$, which is a contradiction. Thus, $y \in E \setminus X$ and so $E \setminus X$ is upset.
(iii) Let $x, y \in E_1 = X \cup \{1\}$. Then, we have the following cases:
Case 1: If $x = y = 1$, or $x = 1$ and $y \in X$ or $x \in X$ and $y = 1$ or $x = y$, clearly we get $x \wedge y \in E_1$ and $x \sim y \in E_1$.
Case 2: Consider $x \neq y \in X$. Since $X \subseteq E$ and E is a chain, we have $x < y$ or $y < x$. If $x < y$ and $y \to x \notin X$, then $y \to x \in E \setminus X$. By Definition 4.1(c3), since $x \in X$ and $y \to x \in E \setminus X$, we get $(y \to x) \to x = x$. On the other hand, by Proposition 2.3(iv), $y \leq (y \to x) \to x = x$, which is a contradiction. Also, by Proposition 2.3(x), $x \sim y \leq y \to x$ and by (i), $x \sim y \in X$. Clearly, $x \wedge y \leq x, y$ and by (i), $x \wedge y \in X$. Hence, $(E_1, \wedge, \sim, 1)$ is a subalgebra of E.

Now, consider $x, y \in E_2$. Then, we have the following cases:
Case 1: If $x = 1$ or $y = 1$ or $x = y$, then it is clear that E_2 is closed under the operations $\wedge$ and $\sim$.
Case 2: If $x \neq y \in E_2 \setminus \{1\}$. Since E is chain, we have $x < y$ or $y < x$. Without lost of generality, suppose $x < y$. Obviously, by Definition 4.1(c2), E_2 is closed under the operation $\to$. So, $y \to x \in E_2$. Now, consider $y \to x = y \sim (y \wedge x) = y \sim x$. Thus, $y \sim x \in E_2$. The proof of other case is similar. Hence, $(E_2, \wedge, \sim, 1)$ is a subalgebra of E.
(iv) Clearly, $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \{1\}$. By Definition 4.1, (3.1) and (3.2) the proof is straightforward. □

Note: Proposition 4.3 holds also for trivial cuts, except in point (iv), decomposition is no-longer non-trivial, if X is a trivial cut, then $E_1 = \{1\}$ and $E_2 = E$, or $E_1 = E$ and $E_2 = \{1\}$.

Let us consider the condition (L) if for any $x, y, z \in E$, we have

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \qquad (L)$$

Example 4.4. (i) Consider the equality algebra E_1 as in Example 3.3. Clearly, E_1 satisfies the condition (L);
(ii) [16] Let $(E = \{0, a, b, 1\}, \leq)$ be a chain, where $0 < a < b < 1$. Define the operation $\sim$ on E as follows:

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	a	a
b	a	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra and the condition (L) does not hold in this equality algebra, since

$$b = 1 \to b = (a \to b) \to (a \to 0) \neq (b \to a) \to (b \to 0) = a \to a = 1.$$

Definition 4.5. Let $a \in E \setminus \{1\}$. Define

$$E_a = \{x \in E \setminus \{1\} \mid a \to x = x\}.$$

Clearly, $1, a \notin E_a$.

Lemma 4.6. *Consider $a, b \in E \setminus \{1\}$. Then*
(i) If $x \in E_a$, then $x < a$ and if $a \leq y$, then $y \notin E_a$. Indeed, $E_a \subseteq \{x \in E |\ x < a\}$.
(ii) If $a \leq b$, then $E_a \subseteq E_b$.
If E satisfies the condition (L), then
(iii) E_a is downset.
(iv) $E \setminus E_a$ is upset and is closed under operation "$\to$".
(v) $x < y$, for any $x \in E_a$ and $y \in E \setminus E_a$.
(vi) $y \to x = x$, for any $x \in E_a$ and $y \in E \setminus E_a$.

Proof. (i) Since E is a chain, we have $x < a$ or $a < x$. Let $x \in E_a$ and $a < x$, then $a \to x = x$ and $a \to x = 1$. So, $x = 1$, which is a contradiction. Hence, $x < a$. In addition, if $a \leq y$ and $y \in E_a$, then $1 = a \to y = y$, and so $y = 1$, which is a contradiction. Therefore, if $a \leq y$, then $y \notin E_a$.
(ii) Let $x \in E_a$ and $a \leq b$. Then $a \to x = x$. By Proposition 2.3(vi), and (iii), $x \leq b \to x \leq a \to x = x$. Thus, $b \to x = x$, and so $x \in E_b$. Hence, $E_a \subseteq E_b$.
(iii) Consider $y \in E_a$ and $x < y$. Then $a \to y = y$. By Proposition 2.3(vi), we have $a \to x \leq a \to y = y$. Then $(a \to x) \to y = 1$. Hence, we have

$$\begin{aligned}
(a \to x) \to x &= 1 \to ((a \to x) \to x) && \text{by Proposition 2.3(ii)} \\
&= ((a \to x) \to y) \to ((a \to x) \to x) \\
&= (y \to (a \to x)) \to (y \to x) && \text{by condition (L)} \\
&= ((a \to y) \to (a \to x)) \to (y \to x) && \text{since } y \in E_a \\
&= ((y \to a) \to (y \to x)) \to (y \to x) && \text{by condition (L)} \\
&= (1 \to (y \to x)) \to (y \to x) && \text{since } y \in E_a, \text{ by (i)} \\
&= (y \to x) \to (y \to x) && \text{by Proposition 2.3(ii)} \\
&= 1. && \text{by Proposition 2.3(ii)}
\end{aligned}$$

So, $a \to x \leq x$. Also, by Proposition 2.3(iii), $x \leq a \to x$. Hence, $a \to x = x$, and so $x \in E_a$. Therefore, E_a is downset.
(iv) Let $x \in E \setminus E_a$ and $x \leq y$. If $y \in E_a$, then by (iii), E_a is downset and so $x \in E_a$, which is a contradiction. Hence, $y \in E \setminus E_a$ and $E \setminus E_a$ is upset. Moreover, suppose $x, y \in E \setminus E_a$. If $x \to y \in E_a$, then by Proposition 2.3(iii), $y \leq x \to y$. Thus by (iii), E_a is downset, so $y \in E_a$, which is a contradiction. Hence, $x \to y \in E \setminus E_a$. Similarly, we can see that $y \to x \in E \setminus E_a$.
(v) Let $x \in E_a$ and $y \in E \setminus E_a$. Since E is a chain, we get $x < y$ or $y < x$. If $y < x$, then from $x \in E_a$ and (i), we get $y \in E_a$, which is a contradiction. Hence, $x < y$.
(vi) Assume $x \in E_a$ and $y \in E \setminus E_a$. By Proposition 2.3(iii) and (iv), $x \leq y \to x$ and $y \leq (y \to x) \to x$, respectively. Since by (iv), $E \setminus E_a$ is upset, we get $(y \to x) \to x \in E \setminus E_a$. By (i), $a \leq (y \to x) \to x$. Since $x \in E_a$, by Proposition 2.3(v) we have

$$1 = a \to ((y \to x) \to x) = (y \to x) \to (a \to x) = (y \to x) \to x.$$

Thus, $y \to x \leq x$. Hence, $y \to x = x$. □

Theorem 4.7. *If E satisfies the condition (L), then E is an ordinal sum of $E_a \cup \{1\}$ and $E \setminus E_a$, for any $a \in E$. Indeed, $E = E_a \cup \{1\} \oplus (E \setminus E_a)$.*

Proof. By Proposition 4.3, the proof is clear. □

Definition 4.8. [**?**] The structure $(B, \to, 1)$ of the type $(2, 0)$ is called a *BCK-algebra* if the following conditions hold, for any $x, y, z \in B$:
$(BCK1)$ $(x \to y) \to ((y \to z) \to (x \to z)) = 1$.
$(BCK2)$ $1 \to x = x$.
$(BCK3)$ $x \to 1 = 1$.
$(BCK4)$ $x \to y = 1$ and $y \to x = 1$ imply $x = y$.

The algebraic structure $(B, \leq)$ is a poset, where $x \leq y$ if and only if $x \to y = 1$, for any $x, y \in B$. Also, B is called *bounded* if there exists an element $0 \in B$ such that $0 \leq x$, for all $x \in B$. If $(B, \leq)$ is a meet-semilattice, then B is called a BCK-meet-semilattice and is denoted by $(B, \wedge, \to, 1)$, where $x \leq y$ if and only if $x \wedge y = x$. The equivalence operation $\leftrightarrow$ of B is defined by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. A BCK-meet-semilattice $(B, \wedge, \to, 1)$ is called a *BCK(D)-meet-semilattice*, if it satisfies in the condition (D): $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, for any $x, y, z \in B$.

Theorem 4.9. [11] *The following statements are equivalent:*
(i) For any equality algebra $E = (E, \wedge, \sim, 1)$, $\psi(E) = (E, \wedge, \to, 1)$ is a BCK-meet-semilattice, where $\to$ denotes the implication of E.
(ii) For any BCK(D)-meet-semilattice $B = (E, \wedge, \to, 1)$, $\varphi(B) = (E, \wedge, \leftrightarrow, 1)$ is an equality algebra, where $\leftrightarrow$ denotes the equivalence operation of B. Moreover, the implication of $\varphi(B)$ coincides with $\to$, that is, we have, $x \to y = x \leftrightarrow (x \wedge y)$

Proposition 4.10. *Consider E satisfying the condition (L); then the following statements are equivalent:*
(i) E is sum irreducible.
(ii) For all $x, y \in E$ if $x \to y = y$, then $x = 1$ or $y = 1$.
(iii) E is a commutative BCK-algebra.

Proof. $(i \Rightarrow ii)$ If $x \to y = y$ and $x, y \neq 1$, then E_x is a non-trivial cut of E; thus by Proposition 4.3, E is not sum irreducible, which is a contradiction. Hence, $x = 1$ or $y = 1$.
$(ii \Rightarrow iii)$ By Theorem 4.9, E is a BCK-meet semilattice algebra. So, it is enough to show that it is commutative, it means that for any $x, y \in E$, we have $(x \to y) \to y = (y \to x) \to x$. Since by assumption, E is a chain, thus $x < y$ or $y < x$. Without lose of generality, we suppose $x \leq y$. Then, we have to prove $y = (y \to x) \to x$. For this, we have

$$\begin{aligned}
[((y \to x) \to x) \to y] \to (y \to x) &= [((y \to x) \to x) \to y] \to [[((y \to x) \to x) \to x] \quad \text{by Proposition 2.3(ix)} \\
&= [y \to ((y \to x) \to x)] \to (y \to x) \qquad \text{by condition } (L) \\
&= [(y \to x) \to (y \to x)] \to (y \to x) \qquad \text{by Proposition 2.3(v)} \\
&= 1 \to (y \to x) \qquad \text{by Proposition 2.3(ii)} \\
&= y \to x. \qquad \text{by Proposition 2.3(ii)}
\end{aligned}$$

Hence, $[((y \to x) \to x) \to y] \to (y \to x) = y \to x$, and so by (ii), we have $((y \to x) \to x) \to y = 1$ or $y \to x = 1$. If $y \to x = 1$, then $y \leq x$, which is a contradiction. Thus, $((y \to x) \to x) \to y = 1$ and so $(y \to x) \to x \leq y$. By Proposition 2.3(iv), we get $(y \to x) \to x = y$. Therefore, E is a commutative BCK-algebra.
$(iii \Rightarrow i)$ Suppose E is not sum irreducible. Then there exist two non-trivial subalgebras E_1 and E_2 such that $E = E_1 \oplus E_2$. Suppose $x \in E_1 \setminus \{1\}$ and $y \in E_2 \setminus \{1\}$. Then by Remark 3.2, $x \leq y$, thus

$$(x \to y) \to y = 1 \to y = y \neq 1 = x \to x = (y \to x) \to x.$$

Hence, E can not be commutative, which is a contradiction. Therefore, E is sum irreducible. □

Let define the following relation on chain equality algebras. For any $a, b \in E \setminus \{1\}$, define

$$a \equiv b \;\Leftrightarrow\; E_a = E_b.$$

Obviously, this relation is an equivalence relation on E. Set $[a]$ denote the $\equiv$-class of $a \in E \setminus \{1\}$. This equivalence relation is a congruence relation on $(E \setminus \{1\}, \leq)$ regarded as a lattice, i.e., $a \equiv b$ if and only if $a \wedge c \equiv b \wedge c$ and $a \vee c \equiv b \vee c$, for all $c \in E \setminus \{1\}$.

Indeed, skipping the trivialities, if $a \equiv b$ and $a < c < b$, then by Lemma 4.6(ii), $E_a \subseteq E_c \subseteq E_b = E_a$, thus $E_a = E_c = E_b$, and so $a \equiv b \equiv c$. The quotient set $\dfrac{E \setminus \{1\}}{\equiv}$ have a natural linear order as follows:

$$[a] \leq [b] \;\Leftrightarrow\; E_a \subseteq E_b \;\Leftrightarrow a \leq b.$$

Note that if $(E, \leq)$ has bottom element 0, then $[0]$ is the bottom element of $\dfrac{E \setminus \{1\}}{\equiv}$ with respect to this ordering.

Proposition 4.11. *Consider E satisfying the condition (L) and $a, b \in E \setminus \{1\}$. We have the following statements:*
(i) If $a \not\equiv b$, then $a \in E_b$ if $a < b$ and $b \in E_a$ if $a > b$.
(ii) $a \equiv b$ and $a \neq b$ imply $a \equiv b \to a$.

Proof. (i) Let $a, b \in E$. By assumption, without loss of generality suppose $a < b < 1$. Since if $a = b$, then $a \equiv a$, which is a contradiction. Since $a < b < 1$, by Lemma 4.6(ii), $E_a \subseteq E_b$. By assumption, $a \not\equiv b$, then $E_b \nsubseteq E_a$, because if $E_b \subseteq E_a$, then $E_b = E_a$, and so $a \equiv b$, which is a contradiction. So, there exists $x \in E_b \setminus E_a$. Since $x \in E_b$, then $b \to x = x$ and $x \notin E_a$. Thus, $a \to x \neq x$, and so by Proposition 2.3(iii), $x < a \to x$. Also, we have

$$\begin{aligned} b \to ((a \to x) \to x) &= (a \to x) \to (b \to x) && \text{by Proposition 2.3(v)} \\ &= (a \to x) \to x. && \text{since } x \in E_b \end{aligned}$$

Thus, $(a \to x) \to x \in E_b$ or $(a \to x) \to x = 1$. The case $(a \to x) \to x = 1$ does not happen since $x < a \to x$.

Thus, $(a \to x) \to x \in E_b$. In addition, by Proposition 2.3(iv), $a \leq (a \to x) \to x$ and E_b is down set, so $a \in E_b$. The proof of other case is similar.
(ii) Let $a, b \in E \setminus \{1\}$ such that $a < b < 1$. By Proposition 2.3(iii), $a \leq b \to a \leq 1$. If $b \to a = 1$, then $b \leq a$ and by assumptions $a = b$, which is a contradiction. If $a = b \to a$, then $a \in E_b$, since $a \equiv b$, we have $E_a = E_b$, and so $a \in E_a$, which is a contradiction. Hence, $a < b \to a < 1$. Now, suppose $a \not\equiv b \to a$. By (i), $a \in E_{b \to a}$, i.e., $(b \to a) \to a = a$. Thus, by Proposition 2.3(v) we have

$$b \to a = b \to ((b \to a) \to a) = (b \to a) \to (b \to a) = 1.$$

So $b \leq a$, which is a contradiction. Hence, $a \equiv b \to a$. □

Proposition 4.12. *Consider E satisfying the condition (L). Then $[a] \cup \{1\}$ is closed under the operations $\wedge$ and $\to$, for any $a \in E \setminus \{1\}$.*

Proof. Since E is a chain, clearly $[a] \cup \{1\}$ is a chain and is closed under the operation $\wedge$. For proving $[a] \cup \{1\}$ is closed under the operation $\to$, suppose $x, y \in [a] \cup \{1\}$, then we have the following cases:
Case 1: If $x = y = 1$, then $x \to y = y \to x = 1 \in [a] \cup \{1\}$.
Case 2: If $x \in [a]$ and $y = 1$, then $x \to y = 1$ and $y \to x = x$. Thus $x \to y, y \to x \in [a] \cup \{1\}$.
Case 3: If $y \in [a]$ and $x = 1$, then the proof is similar to Case 2.
Case 4: If $x, y \in [a]$, then $x \equiv a \equiv y$. Thus, $E_a = E_x = E_y$. Also, by Proposition 4.11(ii), $x \equiv y$, and so $x \equiv y \to x$ and $y \equiv x \to y$. Hence, $E_x = E_{y \to x}$ and $E_y = E_{x \to y}$, and so $E_a = E_{x \to y} = E_{y \to x}$. Thus, $x \to y, y \to x \in [a] \subseteq [a] \cup \{1\}$. Therefore, in all above cases, $[a] \cup \{1\}$ is closed under the operation $\to$. □

Definition 4.13. [13] A *cone algebra* is an algebra $(A, \to, 1)$ of type $(2, 2, 0)$ satisfying the identities:

(i) $1 \to x = x$,

(ii) $x \to x = 1$,

(iii) $x \to (y \to z) = y \to (x \to z)$,

(iv) $(x \to y) \to y = (y \to x) \to x$,

(v) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$.

Theorem 4.14. *Let E be a commutative chain equality algebra satisfying the condition (L). Then $(\bigcup_{a \in E \setminus \{1\}} [a] \cup \{1\}, \to, 1)$ makes a cone algebra.*

Proof. Suppose I is a linearly ordered set where $i < j$ if and only if $[a_i] \leq [a_j]$ if and only if $E_{a_i} \subseteq E_{a_j}$. Clearly, by Proposition 4.12, if $x, y \in [a_i] \cup \{1\}$, then since $[a_i] \cup \{1\}$ is closed under the operation $\to$, for all $a_i \in E \setminus \{1\}$, we have $x \to y, y \to x \in [a_i] \cup \{1\}$. If $x \in [a_i]$ and $y \in [a_j]$, where $i < j$, then since $[a_i] \leq [a_j]$, we get $x \to y = 1$. Also, if $x \in [a_i]$ and $y \in [a_j]$, where $i > j$, then by Proposition 4.11(i), $x \in [a_j]$ and by Proposition 4.12, $x \to y, y \to x \in [a_j]$. Now, it is straightforward to see that the set of all $([a_i] \cup \{1\}, \to, 1)$ where $a_i \in E \setminus \{1\}$ makes a cone algebra. □

Theorem 4.15. [13] *Every chain commutative BCK-algebra satisfying the condition (L) can uniquely be represented as an ordinal sum of chain cone algebras.*

Theorem 4.16. *Every chain commutative equality algebra satisfying the condition* (L) *can uniquely be represented as an ordinal sum of chain cone algebras.*

Proof. By Theorems 4.9 and 4.15, the proof is clear. □

Definition 4.17. [8, 9] A *hoop* is an algebraic structure $(H, \odot, \to, 1)$ with two binary operations $\odot$, $\to$ and one constant 1 such that $(H, \odot, 1)$ is a commutative monoid and for any $x, y, z \in H$, we have:
(H1) $x \to x = 1$,
(H2) $x \odot (x \to y) = y \odot (y \to x)$,
(H3) $x \to (y \to z) = (x \odot y) \to z$.

In a hoop H, if we define the order $\leq$ by "$x \leq y$ if and only if $x \to y = 1$", then $(H, \leq)$ is a poset.

Definition 4.18. [17] A *BL-algebra* is an algebraic structure $(B, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that for any $x, y, z \in B$ the following conditions hold:
(BL1) $(B, \wedge, \vee, 0, 1)$ is a bounded lattice.
(BL2) $(B, \odot, 1)$ is a commutative monoid.
(BL3) $x \odot y \leq z$ if and only if $x \leq y \to z$.
(BL4) $(x \to y) \vee (y \to x) = 1$.
(BL5) $x \wedge y = x \odot (x \to y)$.

Theorem 4.19. *For every chain equality algebra, the following statements are equivalent:*

(i) *E is a commutative equality algebra satisfies in condition (L).*

(ii) *E is an ordinal sum of chain cone algebras.*

(iii) *E is a $\{\to, 1\}$-subreduct of a chain hoop.*

(iv) *E is a $\{\to, 1\}$-subreduct of a chain BL-algebra.*

Proof. By Theorems 4.9 and [13, Theorem 2.9], the proof is clear. □

5 Conclusions and future works

This paper explores the concept of the ordinal sum of equality algebras, examining its related properties and various types of filters, including prime and maximal filters. Furthermore, the study introduces the concept of a cut within a chain equality algebra, which is then used to define a new type of ordinal sum for such algebras. By employing a relation associated with a cut, it is demonstrated that the resulting family of classes forms a cone algebra. Finally, it is proven that any chain equality algebra can be expressed as an ordinal sum of chain cone algebras.

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