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# A novel approach to time-fractional equations using fuzzy beta Laplace transform iterative technique and its applications in fluid dynamics

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**Abstract**

This work proposes a novel approach for modeling complex systems with uncertain data using fuzzy calculus and linear time-fractional differential equations. We provide the fuzzy beta generalized Hukuhara derivative, which maintains inherent uncertainties without transforming fuzzy issues into precise formulations, improving both accuracy and interpretability. We present a fuzzy beta Laplace transform iterative approach to solve fuzzy linear time-fractional equations in fluid dynamics effectively. Case studies, including the fuzzy time-fractional diffusion and advection-dispersion equations, demonstrate the effectiveness of our strategy in capturing system dynamics under uncertainty. This study enhances the amalgamation of fuzzy calculus and fractional modeling, offering a solid foundation for the analysis of uncertain complex systems.

*Keywords:* Fuzzy time-fractional oxygen diffusion equation, the fuzzy beta generalized Hukuhara time-fractional derivative, the fuzzy beta-Laplace transform iterative method, fuzzy time-fractional diffusion equation.

## 1 Introduction

The modeling of complex environmental processes-such as pollutant dispersion, sediment transport, and groundwater flow-has increasingly adopted time-fractional differential equations due to their capacity to incorporate memory effects and hereditary characteristics in dynamic systems [15, 16, 34]. These equations offer a more realistic depiction of temporal behavior than classical integer-order models. However, their practical applicability remains limited by uncertainties inherent in environmental parameters, boundary conditions, and the modeling of spatial heterogeneity.

One of the central challenges in employing time-fractional models for environmental systems lies in addressing these pervasive uncertainties. Traditional formulations often assume precise inputs, which contrasts sharply with the ambiguous and imprecise nature of real-world environmental data. This discrepancy necessitates modeling frameworks that allow for a systematic treatment of epistemic uncertainties [9, 12, 13, 14, 19, 21, 22, 23, 24, 27, 28, 29, 30] To address these limitations, fuzzy set theory and fuzzy differential equations emerged as promising tools for modeling uncertainty in complex systems [4, 20]. These approaches became more robust with the development of non-local fuzzy fractional operators, enabling more flexible formulations of uncertain dynamical systems [25, 31]. Significant contributions in this direction include the work of Agarwal et al. [1], who demonstrated the potential of fuzzy fractional models in uncertain settings.

Recent developments have greatly expanded fuzzy fractional calculus' theoretical foundation. Van Hoa [32] used Caputo generalised Hukuhara differentiability to study the existence and uniqueness of solutions for fuzzy fractional functional differential equations. At the same time, Long et al. [33] introduced new techniques based on Schauder-type fuzzy-valued continuous functions and Banach fixed point theory to show that generalised Hukuhara-weak solutions exist. Numerous non-local fuzzy fractional differential operators have been developed as a result of the widespread

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application of fuzzy fractional calculus in various scientific domains. Allahviranloo et al. [5] modified Riemann-Liouville differentiability to the fuzzy context using Mittag-Leffler functions to derive explicit solutions for uncertain fractional differential equations. Al-Smadi et al. [6] developed fuzzy fractional differential equations based on the Atangana-Baleanu-Caputo operator, proving their existence and uniqueness as well as a novel computer algorithm. Friedman et al. [17] developed numerical procedures for fuzzy differential and integral equations using the embedding method, while Viet Long et al. [33] defined fuzzy fractional integrals and Caputo gH-partial derivatives for multivariable functions, proving solution existence via fixed point theorems. Additionally, Allahviranloo et al. [3] investigated fuzzy Caputo fractional differential equations under generalised Hukuhara differentiability, comprehensively proving their existence and uniqueness with convincing instances.

Despite these advances, considerable challenges persist- particularly in obtaining analytical or approximate solutions for fuzzy time-fractional differential equations in applied domains like fluid dynamics. The main difficulties stem from the non-locality of fractional operators and the complexity of preserving fuzzy uncertainty throughout the solution process. To overcome these constraints, this paper presents a unique method based on the fuzzy beta generalised Hukuhara time-fractional derivative. By working directly in the fuzzy domain, the suggested approach maintains the intrinsic uncertainties that come with environmental modelling. In order to efficiently approximate solutions for fuzzy linear time-fractional equations in fluid dynamics, this research proposes an iterative fuzzy beta Laplace transform method. This technique combines fuzzy arithmetic with series expansion strategies to improve both stability and approximation accuracy. Notably, the iterative structure supports efficient numerical implementation and avoids common pitfalls of indirect or overly rigid transformation-based techniques.

The main contributions of this study are threefold: (1) the introduction of a novel fuzzy fractional derivative tailored to uncertain systems; (2) the development of a direct fuzzy numerical method based on Laplace transforms; and (3) the demonstration of the method's effectiveness through case studies involving fuzzy time-fractional diffusion and advection-dispersion equations. These advancements provide a more reliable and accurate framework for analyzing real-world systems governed by fractional dynamics and subject to uncertain data.

The following portions of this document are structured as outlined below. In Section 2, we introduce essential principles of fuzzy mathematics. Section 3 introduces the fuzzy beta-integral and the fuzzy beta generalised Hukuhara time-fractional derivative, outlining their fundamental features. Section 4 analyses the fuzzy beta-Laplace transform and its essential characteristics. In Section 5, we examine a fuzzy linear time-fractional equation defined by the fuzzy $[beta]_g$-time derivative and outline the fuzzy beta Laplace transform iterative method as a numerical approach for estimating fuzzy solutions to this problem. Section 6 demonstrates the practical applicability of the proposed method through specific examples of fluid dynamics problems, including the fuzzy time-fractional Advection-Dispersion equation and the fuzzy time-fractional diffusion equation. These examples distinctly demonstrate the solution phases and emphasise the efficacy of the proposed technique in addressing real-world problems. Final conclusions are presented in Section 7.

## 2 Preliminaries

In this section, we review specific definitions related to fuzzy partial differential equations and introduce the notations and fundamental concepts that will be used throughout the paper. For a deeper understanding of fuzzy theory, we suggest that readers refer to the articles and books cited in [2, 4, 10, 18].

A fuzzy set on the real line $\mathbb{R}$ is called a fuzzy number if each of its $r$-cuts is non-empty, bounded, and closed. In this paper, we denote the set of all fuzzy numbers by $\mathcal{F}$.

Let $\check{\psi} : \Omega \to \mathcal{F}$ be a fuzzy function, where $\Omega \subseteq \mathbb{R} \times [0, \infty)$, and $\mathcal{F}$ denotes the set of all fuzzy numbers. The parametric representation of this fuzzy function is given by

$$[\check{\psi}(\zeta, \tau)]^r = [\psi^-(\zeta, \tau; \mathfrak{r}), \psi^+(\zeta, \tau; \mathfrak{r})],$$

for all $r$ belongs to $[0, 1]$.

**Definition 2.1.** [10, 11] *Let $\check{\psi}(\zeta, \tau) : \Omega \to \mathbb{F}$ be a mapping. We say that $\check{\psi}$ is* fuzzy generalized Hukuhara partially differentiable *(or* $[gH - p]$-differentiable*) with respect to $\tau$ at the point $(\zeta_0, \tau_0)$, which belongs to $\Omega$ if the generalized Hukuhara difference $\check{\psi}(\zeta_0, \tau_0 + \hbar) \ominus_{gH} \check{\psi}(\zeta_0, \tau_0)$ exists for any sufficiently small $\hbar > 0$, and the following limit holds:*

$$\frac{\partial \check{\psi}(\zeta, \tau)}{\partial \tau}\Big|_{(\zeta_0, \tau_0)} = \lim_{\hbar \to 0} \frac{\check{\psi}(\zeta_0, \tau_0 + \hbar) \ominus_{gH} \check{\psi}(\zeta_0, \tau_0)}{\hbar}. \tag{1}$$

*Moreover, this definition remains valid provided that the quantity $\frac{\partial \check{\psi}(\zeta, \tau)}{\partial \tau}\Big|_{(\zeta_0, \tau_0)}$ is a well-defined fuzzy function.*

**Remark 2.2.** [20] *Consider a triangular fuzzy function* $\check{\psi}(\zeta,\tau)$ *that is* $[gH-p]$*-differentiable with respect to* $\zeta$ *and has no switching points in its domain* $\Omega$*. Suppose that* $\check{\psi}(\zeta,\tau)$ *is expressed as*

$$\check{\psi}(\zeta,\tau)=\left(\psi_1(\zeta,\tau),\psi_2(\zeta,\tau),\psi_3(\zeta,\tau)\right). \tag{2}$$

*The* $[gH-p]$*-differentiability of* $\check{\psi}(\zeta,\tau)$ *with respect to* $\zeta$ *can be classified as follows:*

1. *The function* $\check{\psi}(\zeta,\tau)$ *is* $[i-p]$-differentiable *if and only if*

$$\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}=\left(\frac{\partial\psi_1(\zeta,\tau)}{\partial\zeta},\frac{\partial\psi_2(\zeta,\tau)}{\partial\zeta},\frac{\partial\psi_3(\zeta,\tau)}{\partial\zeta}\right).$$

2. *The function* $\check{\psi}(\zeta,\tau)$ *is* $[ii-p]$-differentiable *if and only if*

$$\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}=\left(\frac{\partial\psi_3(\zeta,\tau)}{\partial\zeta},\frac{\partial\psi_2(\zeta,\tau)}{\partial\zeta},\frac{\partial\psi_1(\zeta,\tau)}{\partial\zeta}\right).$$

*Furthermore, suppose that* $\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}$ *is* $[gH-p]$*-differentiable at* $(\zeta,\tau)$*, which belongs to* $\Omega$ *and has no switching points in* $\Omega$*. If both* $\check{\psi}(\zeta,\tau)$ *and its derivative* $\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}$ *share the same type of* $[gH-p]$*-differentiability, then* $\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}$ *is* $[i-p]$*-differentiable and satisfies*

$$\frac{\partial^2\check{\psi}(\zeta,\tau)}{\partial\zeta^2}=\left(\frac{\partial^2\psi_1(\zeta,\tau)}{\partial\zeta^2},\frac{\partial^2\psi_2(\zeta,\tau)}{\partial\zeta^2},\frac{\partial^2\psi_3(\zeta,\tau)}{\partial\zeta^2}\right).$$

*However, if* $\check{\psi}(\zeta,\tau)$ *and its derivative* $\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}$ *exhibit different types of differentiability, then* $\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}$ *is* $[ii-p]$*-differentiable, and its second-order derivative is given by*

$$\frac{\partial^2\check{\psi}(\zeta,\tau)}{\partial\zeta^2}=\left(\frac{\partial^2\psi_3(\zeta,\tau)}{\partial\zeta^2},\frac{\partial^2\psi_2(\zeta,\tau)}{\partial\zeta^2},\frac{\partial^2\psi_1(\zeta,\tau)}{\partial\zeta^2}\right).$$

# 3 Exploring the fuzzy beta-integral and fuzzy beta generalized Hukuhara time-fractional derivative

This section presents the notions of the beta-integral and the time-fractional beta generalized Hukuhara derivative for a fuzzy function represented by $\check{\psi}(\zeta,\tau)$. We also analyze several essential properties related to these derivatives.

We further examine the fuzzy function $\check{\psi}$ and assume that its first generalized Hukuhara derivative is Riemann integrable and fuzzy continuous over the domain. Furthermore, we presume that no switching points are present within the function's domain.

## 3.1 Fuzzy beta-integral

This section examines the fundamental concept of the fuzzy beta-integral. Additionally, we present multiple characterizations that clarify the attributes of this specific fuzzy fractional integral.

**Definition 3.1.** *The fuzzy beta-integral of a function* $\check{\psi}$ *of order* $\beta$ *with respect to the parameter* $\tau$*, denoted by* ${}^{*}\mathbb{I}^{\beta}_{\mathfrak{a},\tau}\left(\check{\psi}(\zeta,\tau)\right)$*, is defined as follows:*

$${}^{*}\mathbb{I}^{\beta}_{\mathfrak{a},\tau}\left(\check{\psi}(\zeta,\tau)\right)=\int_{\mathfrak{a}}^{\tau}\left((\xi-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\check{\psi}(\zeta,\xi)\,\mathrm{d}\xi,$$

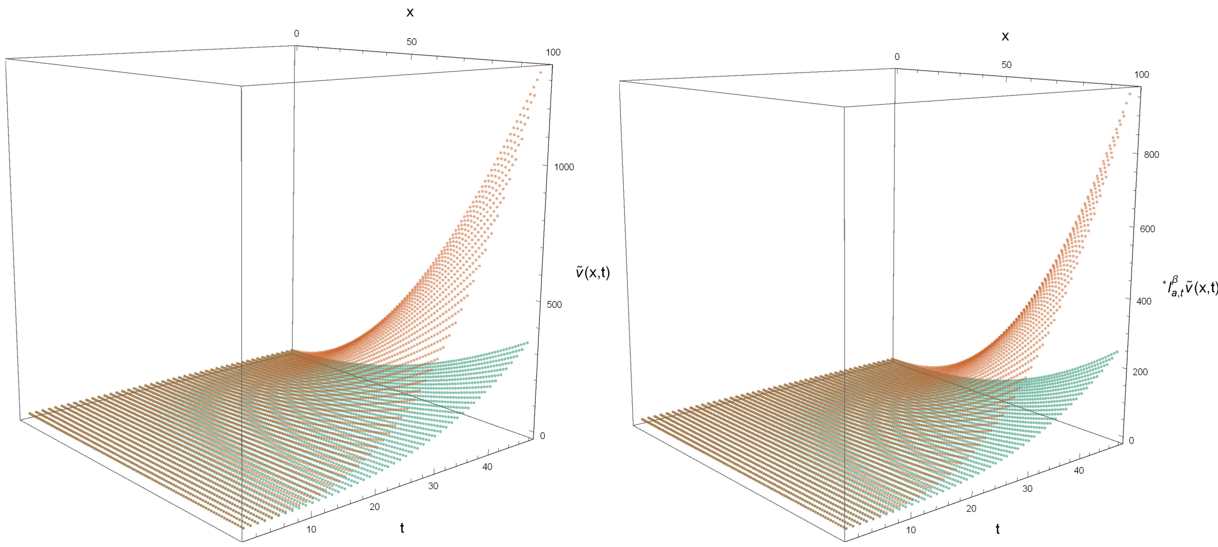
*where* $\Gamma$ *denotes the gamma function. This definition is applicable if the integral exists.*

**Theorem 3.2.** *Let* $\check{\psi}:\Omega\to\mathcal{F}$ *and* $\beta$*, which belongs to* $(0,1)$*. Under these conditions, the following properties hold:*

i. *The function* ${}^{*}\mathbb{I}^{\beta}_{\mathfrak{a},\tau}\left(\check{\psi}(\zeta,\tau)\right)$ *belongs to the set* $\mathcal{F}$*.*

ii. *For any* $\mathfrak{r}$ *belongs to* $[0,1]$*, the function* $\left[{}^{*}\mathbb{I}^{\beta}_{\mathfrak{a},\tau}\left(\check{\psi}(\zeta,\tau)\right)\right]^{\mathfrak{r}}$ *can be expressed as a pair:*

$$\left[{}^{*}\mathbb{I}^{\beta}_{\mathfrak{a},\tau}\left(\check{\psi}(\zeta,\tau)\right)\right]^{\mathfrak{r}}=\left[I^{\beta}_{\mathfrak{a},\tau}\left(\psi^{-}(\zeta,\tau;\mathfrak{r})\right),I^{\beta}_{\mathfrak{a},\tau}\left(\psi^{+}(\zeta,\tau;\mathfrak{r})\right)\right].$$




(a) (b)

Figure 1: Visual representation of fuzzy beta-integral in Example 3.3. (a). $\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$ .(b). $\left[{}^{*}\mathbb{I}^{\frac{1}{2}}_{0,\tau}\left(\check{\psi}(\zeta,\tau)\right)\right]^{\mathfrak{r}}$

*In this context, $I^{\beta}_{\mathfrak{a},\tau}$ denotes the classical beta-integral operator acting on real-valued functions, as defined in* [8]*, and extended here to fuzzy-valued functions via level-wise application.*

*Proof.* The proof follows directly from Definition 3.1 and the properties of the fuzzy integral. $\square$

**Example 3.3.** *Consider the triangular fuzzy function $\check{\psi} : [0,2] \times [0,3] \longrightarrow \mathcal{F}$ defined as follows:*

$$\check{\psi}(\zeta,\tau) = \left(2\tau^2\zeta^3, 6\tau^2\zeta^3, 12\tau^2\zeta^3\right).$$

*We compute the fuzzy beta-integral of order $\frac{1}{2}$ for the function $\check{\psi}(\zeta,\tau)$, obtaining the following result:*

$$\begin{aligned}
{}^{*}\mathbb{I}^{\frac{1}{2}}_{0,\tau}\left(\check{\psi}(\zeta,\tau)\right) &= \left(I^{\frac{1}{2}}_{0,\tau}\left(\psi_1(\zeta,\tau)\right), I^{\frac{1}{2}}_{0,\tau}\left(\psi_2(\zeta,\tau)\right), I^{\frac{1}{2}}_{0,\tau}\left(\psi_3(\zeta,\tau)\right)\right) \\
&= \left(\frac{4\left(\sqrt{1+\sqrt{\pi}\tau}\left(8-4\sqrt{\pi}\tau+3\pi\tau^2\right)-8\right)\zeta^3}{15\pi^{\frac{5}{4}}},\right. \\
&\qquad \frac{4\left(\sqrt{1+\sqrt{\pi}\tau}\left(8-4\sqrt{\pi}\tau+3\pi\tau^2\right)-8\right)\zeta^3}{5\pi^{\frac{5}{4}}}, \\
&\qquad \left.\frac{8\left(\sqrt{1+\sqrt{\pi}\tau}\left(8-4\sqrt{\pi}\tau+3\pi\tau^2\right)-8\right)\zeta^3}{5\pi^{\frac{5}{4}}}\right).
\end{aligned}$$

*Figure 1 illustrates the original function $\check{\psi}(\zeta,\tau)$ alongside the computed fuzzy beta-integral ${}^{*}\mathbb{I}^{\frac{1}{2}}_{0,\tau}\left(\check{\psi}(\zeta,\tau)\right)$.*

## 3.2 Fuzzy beta generalized Hukuhara time-fractional derivative

This section focuses on the beta fractional derivative as defined by the generalized Hukuhara derivative. Furthermore, certain characteristics pertaining to this form of differentiability will be demonstrated.

**Definition 3.4.** *The fuzzy beta generalized Hukuhara time-fractional derivative of order $0 < \beta < 1$ for a function $\check{\psi} : \Omega \to \mathcal{F}$, with respect to $\tau$ at the point $(\zeta,\tau)$, which belongs to $\Omega$, is defined as follows:*

$${}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau) = \lim_{\epsilon\to 0}\frac{\check{\psi}\left(\zeta,\tau+\epsilon\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau)}{\epsilon},$$

*for all* $\tau \geq \mathfrak{a}$ *and* $0 < \beta \leq 1$. *This definition is valid under the condition that* ${}_{g}^{*}\mathbb{D}_{\mathfrak{a},\tau}^{\beta}\check{\psi}(\zeta,\tau)$ *is a fuzzy function. In simpler terms, we refer to* $\check{\psi}$ *as a* $[beta]_g$*-time differentiable fuzzy function.*

**Theorem 3.5.** *Let us consider a fuzzy function* $\check{\psi} : \Omega \to \mathcal{F}$ *that is fuzzy* $[beta]_g$*-time differentiable at a point* $\tau_0 \geq \mathfrak{a}$. *It follows that* $\check{\psi}$ *exhibits fuzzy continuity at the point* $\tau_0$.

*Proof.* We have

$$\check{\psi}\left(\zeta,\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau_0)=\frac{\check{\psi}\left(\zeta,\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau_0)}{\epsilon}\times\epsilon.$$

As a result, the application of Theorem 3.2 in [7] is warranted.

$$\begin{aligned}\lim_{\epsilon\to0}\left(\check{\psi}\left(\zeta,\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau_0)\right) &= \lim_{\epsilon\to0}\frac{\check{\psi}\left(\zeta,\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau_0)}{\epsilon}\\ &\times \lim_{\epsilon\to0}\epsilon\\ &= {}_{g}^{*}\mathbb{D}_{\mathfrak{a},\tau}^{\beta}\check{\psi}(\zeta,\tau_0)\times 0\\ &= 0,\end{aligned}$$

therefore,

$$\begin{aligned}\lim_{\epsilon\to0}\check{\psi}\left(\zeta,\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) &= \lim_{\epsilon\to0}\left(\check{\psi}\left(\zeta,\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau_0)\right)\oplus\check{\psi}(\zeta,\tau_0)\\ &= 0\oplus\check{\psi}(\zeta,\tau_0)=\check{\psi}(\zeta,\tau_0).\end{aligned}$$

Assuming that $\tau=\tau_0+\epsilon\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}$, we can express $\epsilon$ as $\epsilon=\frac{(\tau-\tau_0)}{\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}}$.

The outcome will be represented by

$$\lim_{\frac{(\tau-\tau_0)}{\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}}\to0}\check{\psi}(\zeta,\tau)=\check{\psi}(\zeta,\tau_0).$$

Given that $\left(\tau_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\neq 0$, the preceding equation can be expressed as

$$\lim_{\tau-\tau_0\to0}\check{\psi}(\zeta,\tau)=\check{\psi}(\zeta,\tau_0),$$

therefore $\lim_{\tau\to\tau_0}\check{\psi}(\zeta,\tau)=\check{\psi}(\zeta,\tau_0)$, and the proof has been finalized. □

**Theorem 3.6.** *Consider a fuzzy function* $\check{\psi} : \Omega \to \mathcal{F}$ *that is fuzzy* $[beta]_g$*-time differentiable on* $\Omega$, *where* $0 \leq \beta < 1$. *Then*

$${}_{g}^{*}\mathbb{D}_{\mathfrak{a},\tau}^{\beta}\check{\psi}(\zeta,\tau)=\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{\partial_g\check{\psi}(\zeta,\tau)}{\partial\tau}.$$

*Proof.* By assuming $\mathtt{h}=\epsilon\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}$ in Definition 3.4, we derive that $\epsilon=\mathtt{h}\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}$. Therefore

$$\begin{aligned}{}_{g}^{*}\mathbb{D}_{\mathfrak{a},\tau}^{\beta}\check{\psi}(\zeta,\tau) &= \lim_{\epsilon\to0}\frac{\check{\psi}\left(\zeta,\tau+\epsilon\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau)}{\epsilon}\\ &= \lim_{\mathtt{h}\to0}\frac{\check{\psi}(\zeta,\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}}\\ &= \left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\lim_{\mathtt{h}\to0}\frac{\check{\psi}(\zeta,\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}}\\ &= \left((\tau-\mathfrak{a})+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{\partial_g\check{\psi}(\zeta,\tau)}{\partial\tau}.\end{aligned}$$

Consequently, the desired result was achieved. □

**Theorem 3.7.** *Assume that a fuzzy function $\check{\psi} : \Omega \to \mathcal{F}$ is a generalized Hukuhara differentiable function; thus, it is also $[beta]_g$-time differentiable.*

*Proof.* Let $\check{\psi}$ be a function that is generalized Hukuhara differentiable. The application of Eq.(1) ensures the existence of the limit $\lim_{\mathtt{h}\to 0} \frac{\check{\psi}(\zeta,\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}}$. Consequently, the subsequent limit also exists.

$$\lim_{\mathtt{h}\to 0} \frac{\check{\psi}(\zeta,\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}} \left((\tau-\mathfrak{a})+\frac{1}{\Gamma(ß)}\right)^{1-ß}. \tag{3}$$

Letting $\epsilon = \mathtt{h}\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(ß)}\right)^{ß-1}$ implies $\mathtt{h} = \epsilon\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(ß)}\right)^{1-ß}$. Consequently, Eq.(3) can be reformulated as

$$\lim_{\epsilon\to 0} \frac{\check{\psi}\left(\zeta,\tau+\epsilon\left((\tau-\mathfrak{a})+\frac{1}{\Gamma(ß)}\right)^{1-ß}\right)\ominus_{gH}\check{\psi}(\zeta,\tau)}{\epsilon} = {}^{*}_{g}\mathbb{D}^{ß}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau).$$

The evidence is therefore definitive. □

**Definition 3.8.** *Let $\check{\psi} : \Omega \to \mathcal{F}$ be a fuzzy-valued function that is $[\beta]_g$-time differentiable of order $0 < ß < 1$. We say that $\check{\psi}$ is $[\beta]_{i.g}$-time differentiable at a point $(\zeta,\tau)\in\Omega$, denoted by ${}^{*}_{i.g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)$, if for all $\mathfrak{r}\in[0,1]$ and $\tau\in\Omega$, the following condition holds:*

$$\left[{}^{*}_{i.g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}} = \left[\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi^{-}(\zeta,\tau;\mathfrak{r}),\ \mathbb{D}^{\beta}_{0,\tau}\psi^{+}(\zeta,\tau;\mathfrak{r})\right]. \tag{4}$$

*Similarly, we say that $\check{\psi}$ is $[\beta]_{ii.g}$-time differentiable at $(\zeta,\tau)\in\Omega$, denoted by ${}^{*}_{ii.g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)$, if for all $\mathfrak{r}\in[0,1]$ and $\tau\in\Omega$, the following condition holds:*

$$\left[{}^{*}_{ii.g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}} = \left[\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi^{+}(\zeta,\tau;\mathfrak{r}),\ \mathbb{D}^{\beta}_{0,\tau}\psi^{-}(\zeta,\tau;\mathfrak{r})\right]. \tag{5}$$

**Remark 3.9.** *Let $\check{\psi} : \Omega \to \mathcal{F}$ be a function that is $[beta]_g$-time differentiable with an order of $0 < ß < 1$. Moreover, $\check{\psi}$ constitutes a triangular fuzzy function, specifically defined as $\check{\psi}(\zeta,\tau) = \Big(\psi_1(\zeta,\tau),\psi_2(\zeta,\tau),\psi_3(\zeta,\tau)\Big)$.*

i. $\check{\psi}$ *is regarded as $[beta]_{i.g}$-time differentiable at the point $(\zeta,\tau)$ within the set $\Omega$ if the subsequent equation is satisfied:*

$${}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau) = \left(\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi_1(\zeta,\tau),\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi_2(\zeta,\tau),\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi_3(\zeta,\tau)\right).$$

ii. $\check{\psi}$ *is deemed $[beta]_{ii.g}$-time differentiable at the point $(\zeta,\tau)$ within the collection $\Omega$ if the subsequent equation is satisfied:*

$${}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau) = \left(\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi_3(\zeta,\tau),\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi_2(\zeta,\tau),\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\psi_1(\zeta,\tau)\right).$$

**Example 3.10.** *We commence by analyzing the fuzzy function $\check{\psi}$, which is defined over the domain $[0,2]\times[0,\frac{\pi}{4}]$.Monetary amounts, as detailed below*

$$\check{\psi}(\zeta,\tau) = \left(2\zeta^3\sin 2\tau, 5\zeta^3\sin 2\tau, 12\zeta^3\sin 2\tau\right).$$

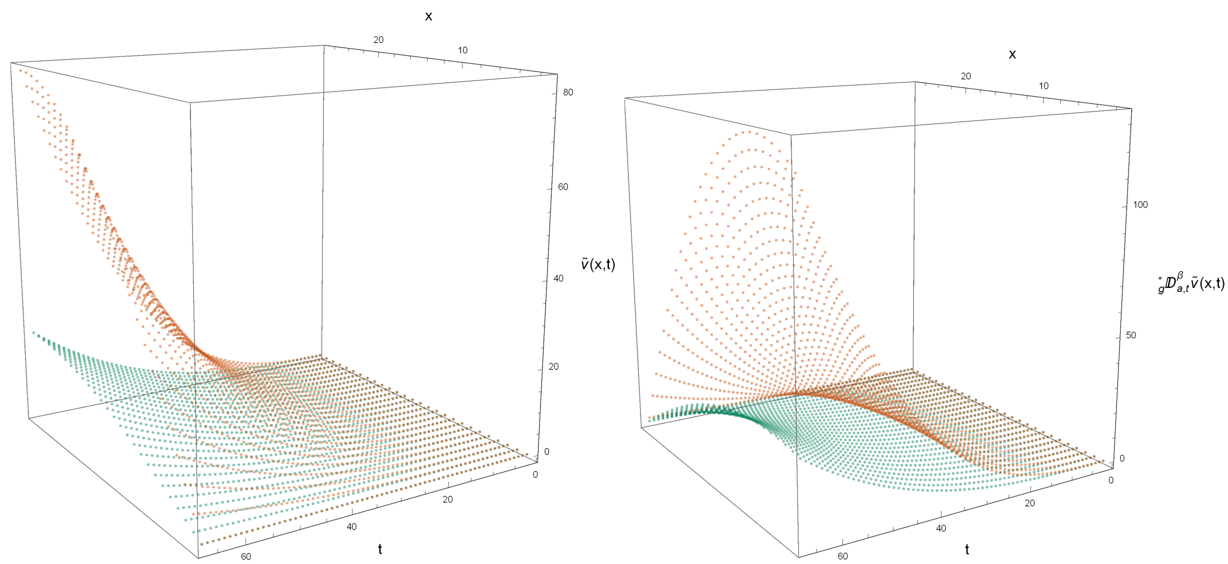
*Subsequently, we investigate the ${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)$ is expressed as:*

$${}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{0,\tau}\check{\psi}(\zeta,\tau) = \left(4\sqrt{\frac{1}{\sqrt{\pi}}+\tau}\zeta^3\cos 2\tau, 10\sqrt{\frac{1}{\sqrt{\pi}}+\tau}\zeta^3\cos 2\tau, 24\sqrt{\frac{1}{\sqrt{\pi}}+\tau}\zeta^3\cos 2\tau\right).$$

*The fuzzy function $\check{\psi}(\zeta,\tau)$ is classified as $[beta]_{i.g}-$time differentiable in the defined domain of $[0,2]\times[0,\frac{\pi}{4}]$.*

*To graphically illustrate the notion of $[beta]_g-$ time differentiability for this solution, we have produced a figure depicting the behavior of both $\check{\psi}(\zeta,\tau)$ and ${}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{0,\tau}\check{\psi}(\zeta,\tau)$ in Figure 2.*

*The lower cut (in green) and the higher cut (in orange) for both $\check{\psi}(\zeta,\tau)$ and ${}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{0,\tau}\check{\psi}(\zeta,\tau)$ are invariant, underscoring that $\check{\psi}(\zeta,\tau)$ is a $[beta]_{i.g}-$time differentiable function.*




(a) (b)

Figure 2: Demonstrating the behavior of $\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$ and its $[beta]_g$ derivative for the specified parameters in Example 3.10: $\beta = \frac{1}{2}$, $\mathfrak{r} = 0.2$. (a).$\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$. (b).$\left[{}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$.

**Example 3.11.** *Examine the function $\check{\psi}(\zeta,\tau)$, which is defined as follows:*

$$\check{\psi}(\zeta,\tau) = \left(\zeta^4 e^{-\tau^2}, 8\zeta^4 e^{-\tau^2}, 16\zeta^4 e^{-\tau^2}\right).$$

*We now explore the notion of a fuzzy $[beta]_g-$derivative of order $\beta = \frac{1}{3}$. It may be expressed as:*

$${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau) = \left(-32e^{-\tau^2}\tau\zeta^4\left(\tau + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\right)^{\frac{2}{3}}, -16e^{-\tau^2}\tau\zeta^4\left(\tau + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\right)^{\frac{2}{3}}, -2e^{-\tau^2}\tau\zeta^4\left(\tau + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\right)^{\frac{2}{3}}\right).$$

*Thus, the fuzzy function $\check{\psi}(\zeta,\tau)$ is regarded as $[beta]_{ii.g}-$time differentiable.*

*To visually comprehend the concept of $[beta]_g-$differentiability for this solution, we have generated a plot that depicts the behavior of both $\check{\psi}(\zeta,\tau)$ and ${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)$ with an order of $\beta = \frac{1}{3}$ in Figure 3.*

*The positions of the lower cut (in Green) and the upper cut (in Orange) for ${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)$ has been adjusted. As a result, $\check{\psi}(\zeta,\tau)$ is a $[beta]_{ii.g}-$time differentiable function.*

**Theorem 3.12.** *Let $\check{\psi} : \Omega \to \mathcal{F}$ be a fuzzy-valued function. Suppose that the point $(\zeta_0,\tau_0)$ lies in $\Omega$, and that the order satisfies $0 < \beta \leq 1$. Consider the following two cases:*

i. *$\check{\psi}$ is differentiable at $(\zeta_0,\tau_0)$ in the sense of $[i - gH]$ if and only if $\check{\psi}$ is $[beta]_{i.g}$ differentiable at $(\zeta_0,\tau_0)$.*

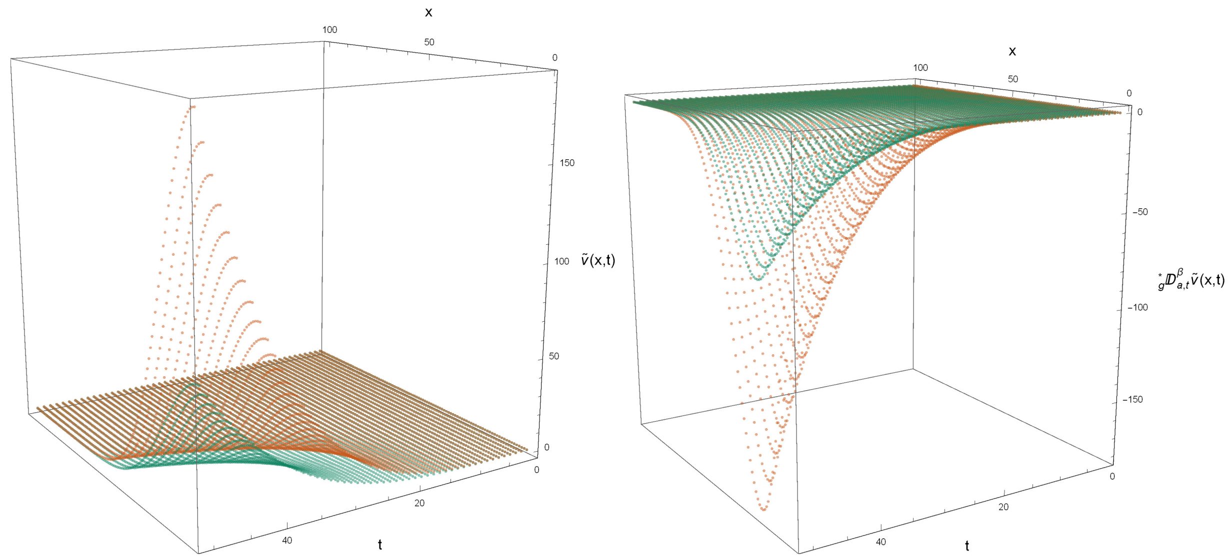
ii. *$\check{\psi}$ is $[ii - gH]-$ differentiable at $(\zeta_0,\tau_0)$ if and only if $\check{\psi}$ is $[beta]_{ii.g}$-time differentiable at $(\zeta_0,\tau_0)$.*

*Proof.* By employing Theorems 3.6 and 3.7, the intended result can be readily achieved. □

**Remark 3.13.** *Define $\check{\psi}(\zeta,\tau) = \left(\psi_1(\zeta,\tau), \psi_2(\zeta,\tau), \psi_3(\zeta,\tau)\right)$ represents a $[beta]_g$-time differentiable triangular fuzzy function. According to Theorem 3.6, if $\check{\psi}(\zeta,\tau)$ is a $[bata]_{i.g}$-differentiable function, then*

$${}^{*}_{i.g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau) = \left(\left((\tau-\mathfrak{a}) + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta}\psi_1'(\zeta,\tau), \left((\tau-\mathfrak{a}) + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta}\psi_2'(\zeta,\tau), \left((\tau-\mathfrak{a}) + \tfrac{1}{\Gamma(\beta)}\right)^{1-\beta}\psi_3'(\zeta,\tau)\right).$$

*and if $\check{\psi}(\zeta,\tau)$ is a $[beta]_{ii.g}$-differentiable function,*




(a) (b)

Figure 3: Visualizing the dynamics of $\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$ and its $[beta]_g$-derivative under specific parameters outlined in Example 3.11: $\beta = \frac{1}{3}$ and $\mathfrak{r} = 0.5$. (a). $\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$. (b). $\left[{}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$.

$$
{}^{*}_{ii.g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau) = \left( \left((\tau-\mathfrak{a}) + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\psi_3'(\zeta,\tau),\ \left((\tau-\mathfrak{a}) + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\psi_2'(\zeta,\tau),\ \left((\tau-\mathfrak{a}) + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\psi_1'(\zeta,\tau)\right).
$$

**Definition 3.14.** *Let $\check{\psi} : \Omega \to \mathcal{F}$ be fuzzy fractional $[beta]_g$-time differentiable in $\Omega$, where $0 < \beta \leq 1$ and $\xi_0 \in \Omega$.*

- *A point $\xi_0$ is designated as a switching point **type I** for the fuzzy $[beta]_g$-time differentiability of $\check{\psi}$ if there exist points $\xi_1$ and $\xi_2$ ($\xi_1 < \xi_0 < \xi_2$) such that at $\xi_1$, Eq.(4) is satisfied while Eq.(5) is not, and at $\xi_2$, Eq.(5) is satisfied while Eq.(4) is not.*
- *A specific location Point $\xi_0$ is classified as a switching point **type II** for the fuzzy $[beta]_g$-time differentiability of $\check{\psi}$ if there exist points $\xi_1$ and $\xi_2$ ($\xi_1 < \xi_0 < \xi_2$) such that at $\xi_1$, Eq.(5) is satisfied while Eq.(4) is not, and at $\xi_2$, Eq.(4) is satisfied while Eq.(5) is not.*

**Example 3.15.** *Let $\check{\psi} : [0,2]\times[0,4] \to \mathcal{F}$ be defined by $\check{\psi}(\zeta,\tau) = \left(16\zeta^2(\tau^2-6\tau), 8\zeta^2(\tau^2-6\tau), \zeta^2(\tau^2-6\tau)\right)$. To compute the $[beta]_g-$time derivative of $\check{\psi}(\zeta,\tau)$, we apply Definition 3.4 and Remark 3.13. In this instance, given $(\zeta,\tau) \in [0,2]\times[0,3)$ and $\beta = \frac{1}{4}$, we obtain*

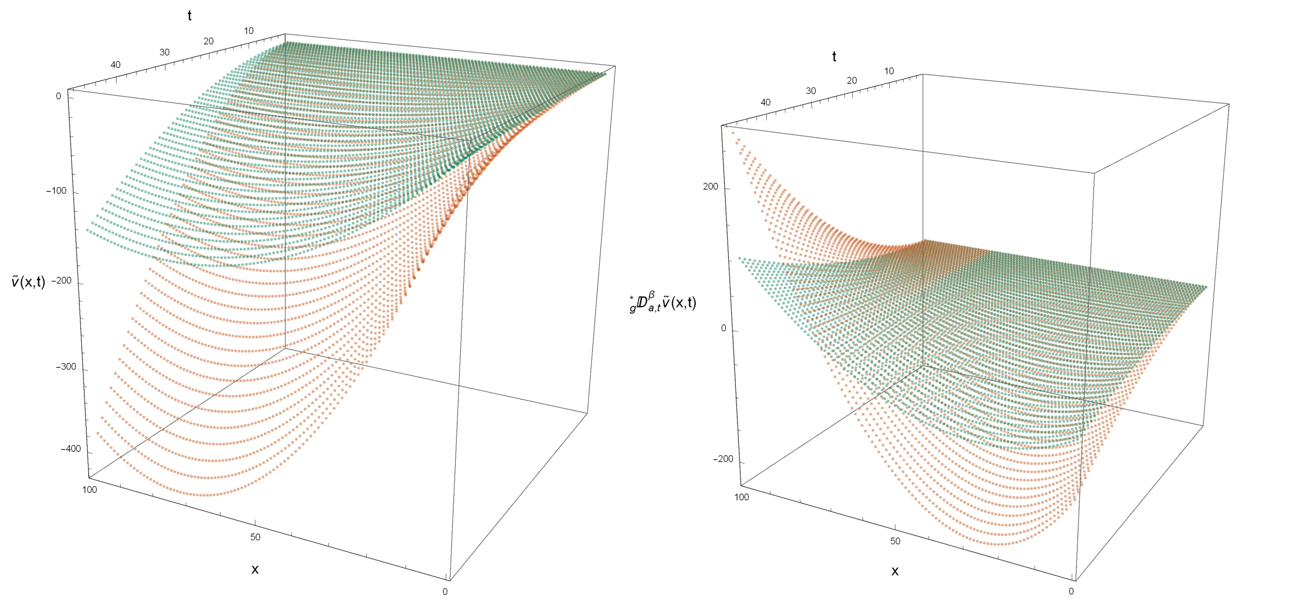
$$
{}^{*}_{i.g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau) = \left(16(2\tau-6)\zeta^2\left(\tau+\frac{1}{\Gamma(\frac{1}{4})}\right)^{\frac{3}{4}}, 8(2\tau-6)\zeta^2\left(\tau+\frac{1}{\Gamma(\frac{1}{4})}\right)^{\frac{3}{4}}, (2\tau-6)\zeta^2\left(\tau+\frac{1}{\Gamma(\frac{1}{4})}\right)^{\frac{3}{4}}\right),
$$

*and for $(\zeta,\tau) \in [0,2]\times[3,4]$, we have*

$$
{}^{*}_{ii.g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau) = \left((2\tau-6)\zeta^2\left(\tau+\frac{1}{\Gamma(\frac{1}{4})}\right)^{\frac{3}{3}}, 8(2\tau-6)\zeta^2\left(\tau+\frac{1}{\Gamma(\frac{1}{4})}\right)^{\frac{3}{3}}, 16(2\tau-6)\zeta^2\left(\tau+\frac{1}{\Gamma(\frac{1}{4})}\right)^{\frac{3}{3}}\right).
$$

*The fuzzy-valued function $\check{\psi}(\zeta,\tau)$ is $[beta]_{i.g}-$time differentiable on the domain $(\zeta,\tau) \in [0,2]\times[0,3)$ and is $[beta]_{ii.g}-$time differentiable for all $(\zeta,\tau) \in [0,2]\times[3,6]$. Consequently, the locations $(\zeta,3)$, for all $\zeta \in [0,2]$, are designated as switching points of **Type I** for the $[beta]_g$-time differentiability of $\check{\psi}(\zeta,\tau)$.*

*The fuzzy functions $\check{\psi}(\zeta,\tau)$ and ${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)$ are depicted in Figures 4(a) and (b), respectively. Figure 4(b) illustrates a modification in the positions of the lower cut (orange) and upper cut (green) of the fuzzy function ${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)$ for $\tau > 3$. The positions $(\zeta,3)$, for any $\zeta \in [0,2]$, are classified as switching points of **Type I**.*




(a) (b)

Figure 4: Graphs of $\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$ and $\left[{}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$ for Example 3.15 when $\beta = \frac{1}{4}$, and $r = 0.8$. (a).$\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$. (b).$\left[{}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$.

**Theorem 3.16.** *Let $\check{\psi}(\zeta,\tau)$ and $\check{\varphi}(\zeta,\tau)$ be two fuzzy $[beta]_g$-time differentiable functions, and their type of $[beta]_g$-time differentiability is unchanged in $\Omega$. Additionally, if $\lambda_1$ and $\lambda_2$ are both positive real values, then*

i. ${}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\left(\lambda_1\check{\psi}(\zeta,\tau)\oplus\lambda_2\check{\varphi}(\zeta,\tau)\right) = \lambda_1\ {}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)\oplus\lambda_2{}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\varphi}(\zeta,\tau)$.

ii. ${}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\left(\lambda_1\check{\psi}(\zeta,\tau)\ominus_{gH}\lambda_2\check{\varphi}(\zeta,\tau)\right) = \lambda_1\ {}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)\ominus_{gH}\lambda_2\ {}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\varphi}(\zeta,\tau)$.

*Proof.* Using Theorem 3.1 in [7],

$$\begin{aligned}
{}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\left(\lambda_1\check{\psi}(\zeta,\tau)\oplus\lambda_2\check{\varphi}(\zeta,\tau)\right) &= \lim_{\epsilon\to 0}\frac{\left(\lambda_1\check{\psi}\oplus\lambda_2\check{\varphi}\right)\left(\zeta,\tau+\epsilon\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\left(\lambda_1\check{\psi}\oplus\lambda_2\check{\varphi}\right)(\zeta,\tau)}{\epsilon}\\
&= \lambda_1\lim_{\epsilon\to 0}\frac{\check{\psi}\left(\zeta,\tau+\epsilon\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\psi}(\zeta,\tau)}{\epsilon}\\
&\oplus\ \lambda_2\lim_{\epsilon\to 0}\frac{\check{\varphi}(\zeta,\left(\tau+\epsilon\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)\ominus_{gH}\check{\varphi}(\zeta,\tau)}{\epsilon}\\
&= \lambda_1\ {}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\psi}(\zeta,\tau)\oplus\lambda_2{}^{*}_{g}\mathbb{D}^{\beta}_{\mathfrak{a},\tau}\check{\varphi}(\zeta,\tau).
\end{aligned}$$

The other instance may similarly be substantiated. □

# 4 Fuzzy beta-Laplace transform

This part examines the fuzzy beta-Laplace transform, performing a comprehensive study to reveal novel aspects of this transform. Our aim is to enhance our comprehension of its mathematical properties and possible practical uses.

A piecewise continuous fuzzy function on $[0,\infty)$ of exponential order $\sigma$ is denoted as $\check{\psi}(\zeta,\tau)$. The fuzzy Laplace transform of $\check{\psi}(\zeta,\tau)$ about $\tau$ is defined as follows:

$$\mathbb{L}_{\tau}\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \int_0^{\infty}e^{-\mathfrak{s}\tau}\check{\psi}(\zeta,\tau)d\tau, \tag{6}$$

assuming the limit exists[20].

Assuming that $\check{\psi}(\zeta,\tau)$ and $\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\tau}$ are fuzzy piecewise continuous functions across every finite closed interval of $[0,\infty)$ and exhibit exponential order, devoid of switching points in their domain, we derive the following results [20]:

i. If $\check{\psi}(\zeta,\tau)$ is $[i-p]$-differentiable with respect to $\tau$, then:

$$\mathbb{L}_\tau\left\{\frac{\partial_{i.gH}\check{\psi}(\zeta,\tau)}{\partial\tau}\right\}(\mathfrak{s}) = \mathfrak{s}\mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) \ominus \check{\psi}(\zeta,0). \tag{7}$$

ii. If $\check{\psi}(\zeta,\tau)$ is $[ii-p]$-differentiable concerning $\tau$, then:

$$\mathbb{L}_\tau\left\{\frac{\partial_{ii.gH}\check{\psi}(\zeta,\tau)}{\partial\tau}\right\}(\mathfrak{s}) = (-1)\check{\psi}(\zeta,0) \ominus (-1)\mathfrak{s}\mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}). \tag{8}$$

These characteristics enhance our comprehension of the fuzzy beta-Laplace transform and its ramifications in diverse circumstances. We now provide a novel operator referred to as the fuzzy Atangana transform, or fuzzy beta-Laplace transform.

**Definition 4.1.** *Let $\check{\psi}$ denote a fuzzy function defined on the interval $(0,\infty)$. For any $\beta \in (0,1)$, if the Laplace transform of $\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\check{\psi}(\zeta,\tau)$ exists, the fuzzy beta-Laplace transform of $\check{\psi}$ is defined as follows:*

$$\mathcal{L}_{\beta,\tau}\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \int_0^\infty \left(\tau+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} e^{-\mathfrak{s}\tau}\check{\psi}(\zeta,\tau)d\tau = \check{\mathcal{V}}(\zeta,\mathfrak{s}).$$

**Lemma 4.2.** *Examine two fuzzy functions defined on the interval $[0,\infty)$, referred to as $\check{\psi}(\zeta,\tau)$ and $\check{\varphi}(\zeta,\tau)$, both possessing a clearly defined fuzzy beta-Laplace transform. Let $\lambda_1$ and $\lambda_2$ be non-negative or non-positive real constants. Under these circumstances, the subsequent attributes are valid:*

i. $\mathcal{L}_{\beta,\tau}\left\{\lambda_1\check{\psi}(\zeta,\tau)\oplus\lambda_2\check{\varphi}(\zeta,\tau)\right\}(\mathfrak{s}) = \lambda_1\mathcal{L}_{\beta,\tau}\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s})\oplus\lambda_2\mathcal{L}_{\beta,\tau}\left\{\check{\varphi}(\zeta,\tau)\right\}(\mathfrak{s})$.

ii. $\mathcal{L}_{\beta,\tau}\left\{\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s})$.

*Proof.* The demonstration of the initial property is unambiguous. To establish the second property, we apply Definition 4.1 to compute the beta-Laplace transform of $\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\check{\psi}(\zeta,\tau)$:

$$\mathcal{L}_{\beta,\tau}\left\{\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \int_0^\infty e^{-\mathfrak{s}\tau}\check{\psi}(\zeta,\tau)d\tau.$$

By juxtaposing this outcome with Eq.(6), we finalize the proof. □

**Theorem 4.3.** *Let $\check{\psi}(\zeta,\tau)$ and ${}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)$. Fuzzy functions are designed such that switching points do not exist within their domain, and each possesses a well-defined fuzzy beta-Laplace transform. Under these circumstances, the subsequent relationship is established:*

$$\mathcal{L}_{\beta,\tau}\left\{{}^{*}_{g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \mathbb{L}_\tau\left\{\frac{\partial_g\check{\psi}(\zeta,\tau)}{\partial\tau}\right\}(\mathfrak{s}). \tag{9}$$

*Moreover, depending on the nature of $[beta]_g$-time differentiability, the subsequent assertions can be articulated:*

i. *If $\check{\psi}(\zeta,\tau)$ is $[beta]_{i.g}$-time differentiable, then the following holds*

$$\mathcal{L}_{\beta,\tau}\left\{{}^{*}_{i.g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \mathfrak{s}\mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s})\ominus\check{\psi}(\zeta,0).$$

ii. *If $\check{\psi}(\zeta,\tau)$ is $[beta]_{ii.g}$-time differentiable, then*

$$\mathcal{L}_{\beta,\tau}\left\{{}^{*}_{ii.g}\mathbb{D}^{\beta}_{0,\tau}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = (-1)\check{\psi}(\zeta,0)\ominus(-1)\mathfrak{s}\mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}).$$

*Proof.* To demonstrate the theorem, we utilize Definition 4.1 to compute the beta-Laplace transform of ${}^{*}_{g}\mathbb{D}^{ß}_{0,\tau}\check{\psi}(\zeta,\tau)$:

$$\mathcal{L}_{ß,\tau}\left\{{}^{*}_{g}\mathbb{D}^{ß}_{0,\tau}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) = \int_0^\infty \left(\tau+\frac{1}{\Gamma(ß)}\right)^{ß-1} e^{-\mathfrak{s}\tau}{}^{*}_{g}\mathbb{D}^{ß}_{0,\tau}\check{\psi}(\zeta,\tau)d\tau.$$

Substituting $\mathtt{h}=\epsilon\left(\tau+\frac{1}{\Gamma(ß)}\right)^{1-ß}$ into Definition 3.4 yields:

$$\begin{aligned}{}^{*}_{g}\mathbb{D}^{ß}_{0,\tau}\check{\psi}(\zeta,\tau) &= \lim_{\epsilon\to 0}\frac{\check{\psi}\left(\tau+\epsilon\left(\tau+\frac{1}{\Gamma(ß)}\right)^{1-ß}\right)\ominus_{gH}\check{\psi}(\zeta,\tau)}{\epsilon}\\ &= \left(\tau+\frac{1}{\Gamma(ß)}\right)^{1-ß}\lim_{\mathtt{h}\to 0}\frac{\check{\psi}(\zeta,\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}}.\end{aligned}$$

As a result, we ascertain:

$$\begin{aligned}\mathcal{L}_{ß,\tau}\left\{{}^{*}_{g}\mathbb{D}^{ß}_{0,\tau}\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s}) &= \int_0^\infty\left(\tau+\frac{1}{\Gamma(ß)}\right)^{ß-1}e^{-\mathfrak{s}\tau}\left(\left(\tau+\frac{1}{\Gamma(ß)}\right)^{1-ß}\lim_{\mathtt{h}\to 0}\frac{\check{\psi}(\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}}\right)d\tau\\ &= \int_0^\infty e^{-\mathfrak{s}\tau}\left(\lim_{\mathtt{h}\to 0}\frac{\check{\psi}(\zeta,\tau+\mathtt{h})\ominus_{gH}\check{\psi}(\zeta,\tau)}{\mathtt{h}}\right)d\tau\\ &= \int_0^\infty e^{-\mathfrak{s}\tau}\frac{\partial_g\check{\psi}(\zeta,\tau)}{\partial\tau}d\tau\\ &= \mathbb{L}_\tau\left\{\frac{\partial_g\check{\psi}(\zeta,\tau)}{\partial\tau}\right\}(\mathfrak{s}),\end{aligned}$$

so establishing Eq. (9). The validity of the results, dependent on the type of $[beta]_g$-time differentiability, is clearly illustrated using Eqs.(7) and (8). □

# 5 An overview of fuzzy fuzzy beta-Laplace iterative method

This section analyzes a fuzzy linear time-fractional equation characterized by the fuzzy $[beta]_g$-derivative, constrained by a starting condition inside a designated interval. We will also present an overview of the fuzzy beta-Laplace transform iterative method, a numerical technique for approximating fuzzy solutions to this issue.

The examined fuzzy linear time-fractional equation is defined in terms of the fuzzy $[beta]_g$-derivative:

$${}^{*}_{g}\mathbb{D}^{ß}_{\tau_0,\tau}\check{\psi}(\zeta,\tau)=\mathfrak{S}\left\{\check{\psi}(\zeta,\tau)\right\}\oplus\check{\Bbbk}(\zeta,\tau). \tag{10}$$

This equation is accompanied by the initial condition:

$$\check{\psi}(\zeta,\tau_0)=\check{\phi}(\zeta). \tag{11}$$

$\check{\psi}(\zeta,\tau)$ denotes the concentration of a contaminant at a certain location and time. The notation $\mathfrak{S}\{\check{\psi}(\zeta,\tau)\}$ represents a linear operator, whereas the source term $\check{\Bbbk}(\zeta,\tau)$ indicates the addition or elimination of the contaminant at a specific location and time.

This article will examine the iterative method of the fuzzy beta-Laplace transform as a numerical technique for estimating solutions to the fuzzy time-fractional diffusion Eq. (10). We will examine two strategies predicated on the nature of $[beta]_g$-time differentiability.

- In **Scheme 1**, we presume that $\check{\psi}(\zeta,\tau)$ is $[beta]_{i.g}$-time differentiable. Employing Theorem 4.3 in conjunction with the starting condition (11), we implement the fuzzy beta-Laplace transform on the Eq. (10) about $\tau$. This yields the subsequent equation:

$$\mathfrak{s}\mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s})\ominus\check{\psi}(\zeta,\tau_0)=\mathcal{L}_{ß,\tau}\left\{\mathfrak{S}\left\{\check{\psi}(\zeta,\tau)\right\}\right\}\oplus\mathcal{L}_{ß,\tau}\left\{\check{\Bbbk}(\zeta,\tau)\right\}(\mathfrak{s}). \tag{12}$$

From this, we derive:

$$\mathbb{L}_\tau\left\{\check{\psi}(\zeta,\tau)\right\}(\mathfrak{s})=\mathcal{L}_{ß,\tau}\left\{\mathfrak{S}\left\{\check{\psi}(\zeta,\tau)\right\}\right\}\oplus\frac{1}{\mathfrak{s}}\mathcal{L}_{ß,\tau}\left\{\check{\Bbbk}(\zeta,\tau)\right\}(\mathfrak{s})\oplus\frac{1}{\mathfrak{s}}\check{\phi}(\zeta). \tag{13}$$

By employing the Laplace inverse transform on both sides of Eq. (13), we obtain the approximate solution:

$$\check{\psi}(\zeta,\tau) = \check{\mathcal{Q}}(\zeta,\mathfrak{s}) \oplus \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\mathcal{L}_{\text{ß},\tau}\left\{\mathfrak{S}\left\{\check{\psi}(\zeta,\tau)\right\}\right\}\right\}, \tag{14}$$

where $\check{\mathcal{Q}}(\zeta,\tau) = \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\mathcal{L}_{\text{ß},\tau}\left\{\frac{1}{\mathfrak{s}}\check{\phi}(\zeta) \oplus \check{\Bbbk}(\zeta,\tau)\right\}(\mathfrak{s})\right\}$.

Assume that the solution to Eq. (10) can be expressed as a fuzzy series, $\check{\psi}(\zeta,\tau) = \sum\limits_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau)$. By substituting this into Eq.(14), we derive:

$$\sum_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau) = \check{\mathcal{Q}}(\zeta,\mathfrak{s}) \oplus \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\mathcal{L}_{\text{ß},\tau}\left\{\mathfrak{S}\left\{\sum_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau)\right\}\right\}\right\}. \tag{15}$$

Utilizing Eq.(15), we establish the subsequent iterations:

$$\check{\psi}_0(\zeta,\tau) = \check{\mathcal{Q}}(\zeta,\tau),$$
$$\check{\psi}_{n+1}(\zeta,\tau) = \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\mathcal{L}_{\text{ß},\tau}\left\{\mathfrak{S}\left\{\check{\psi}_n(\zeta,\tau)\right\}\right\}\right\},$$

for $n = 0,1,2,\cdots$. for $n = 0,1,2,\cdots$. Consequently, we may estimate the $[beta]_{i.g}$-time differentiable solution $\check{\psi}(\zeta,\tau)$ as follows:

$$\check{\psi}(\zeta,\tau) = \sum_{n=0}^{k}\check{\psi}_n(\zeta,\tau), \qquad k \to \infty. \tag{16}$$

- In **Scheme 2**, we posit that the $[beta]_{ii.g}$-time differentiable solution $\check{\psi}(\zeta,\tau)$ can be expressed as:

$$\check{\psi}(\zeta,\tau) = \mathcal{Q}(\zeta,\mathfrak{s}) \ominus (-1)\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{\aleph}{\mathfrak{s}}\mathcal{L}_{\text{ß},\tau}\left\{\mathfrak{S}\left\{\check{\psi}(\zeta,\tau)\right\}\right\}\right\}, \tag{17}$$

where $\mathcal{Q}(\zeta,\tau) = \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\check{\psi}(\zeta,\tau_0) \ominus (-1)\frac{1}{\mathfrak{s}}\mathcal{L}_{\text{ß},\tau}\left\{\check{\Bbbk}(\zeta,\tau)\right\}(\mathfrak{s})\right\}$.

We subsequently posit that $\check{\psi}(\zeta,\tau)$ can be represented as an infinite series of fuzzy functions:

$$\check{\psi}(\zeta,\tau) = \sum_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau).$$

By substituting this series into Eq.(17), we derive:

$$\sum_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau) = \mathcal{Q}(\zeta,\mathfrak{s}) \ominus (-1)\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\mathcal{L}_{\text{ß},\tau}\left\{\mathfrak{S}\left\{\sum_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau)\right\}\right\}\right\}.$$

By defining $\check{\psi}_0(\zeta,\tau) = \mathcal{Q}(\zeta,\mathfrak{s})$, we can ascertain the other unknown fuzzy functions $\check{\psi}_n(\zeta,\tau)$ using the subsequent relations:

$$\check{\psi}_{n+1}(\zeta,\tau) = \ominus(-1)\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\mathcal{L}_{\text{ß},\tau}\left\{\mathfrak{S}\left\{\check{\psi}_n(\zeta,\tau)\right\}\right\}\right\}, \qquad n = 0,1,\cdots.$$

Consequently, we may obtain the $[beta]_{ii.g}$-time differentiable solution of $\check{\psi}(\zeta,\tau)$ for the problem (10).

**Theorem 5.1.** *The infinite series $\check{\psi}(\zeta,\tau) = \sum\limits_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau)$ converges to the solution of Eq.(10) if a constant $\lambda \in (0,1]$ exists such that $D\left(\check{\psi}_{n+1}(\zeta,\tau),0\right) \leq \lambda D\left(\check{\psi}_n(\zeta,\tau),0\right)$ for all $n$.*

*Proof.* Let $S_n = \sum\limits_{i=0}^{n}\check{\psi}_i(\zeta,\tau)$. To establish convergence, it is necessary to demonstrate that the sequence $S_n$ constitutes a fuzzy Cauchy sequence. Let $\mathbb{H}$ be the Hausdorff distance. Utilizing the features from [7, 26], we ascertain:

$$\begin{aligned}
\mathbb{H}\left(S_{n+1} \ominus_{gH} S_n, 0\right) &= \mathbb{H}\left(\sum_{i=0}^{n+1}\check{\psi}_i(\zeta,\tau) \ominus_{gH} \sum_{i=0}^{n}\check{\psi}_i(\zeta,\tau), 0\right) \\
&= \mathbb{H}\left(\check{\psi}_{n+1}(\zeta,\tau),0\right) \leq \lambda\mathbb{H}\left(\check{\psi}_n(\zeta,\tau),0\right) \leq \lambda^2\mathbb{H}\left(\check{\psi}_{n-1}(\zeta,\tau),0\right) \leq \cdots \\
&\leq \lambda^{n+1}\mathbb{H}\left(\check{\psi}_0(\zeta,\tau),0\right).
\end{aligned}$$

For each $m, n \in \mathbb{N}$ where $n > m$, the following holds:

$$\begin{aligned}
\mathbb{H}(S_n \ominus_{gH} S_m, 0) &= \mathbb{H}((S_n \ominus_{gH} S_{n-1}) \oplus (S_{n-1} \ominus_{gH} S_{n-2}) \oplus \cdots \oplus (S_{m+1} \ominus_{gH} S_m), 0) \\
&\leq \mathbb{H}(S_n \ominus_{gH} S_{n-1}, 0) + \mathbb{H}(S_{n-1} \ominus_{gH} S_{n-2}, 0) + \cdots + \mathbb{H}(S_{m+1} \ominus_{gH} S_m, 0) \\
&\leq \lambda^n \mathbb{H}\left(\check{\psi}_0(\zeta,\tau), 0\right) + \lambda^{n-1} \mathbb{H}\left(\check{\psi}_0(\zeta,\tau), 0\right) + \cdots + \lambda^{m+1} \mathbb{H}\left(\check{\psi}_0(\zeta,\tau), 0\right) \\
&\leq \frac{1-\lambda^{n-m}}{1-\lambda} \lambda^{m+1} \mathbb{H}\left(\check{\psi}_0(\zeta,\tau), 0\right) \leq \frac{\lambda^{m+1}}{1-\lambda} \mathbb{H}\left(\check{\psi}_0(\zeta,\tau), 0\right).
\end{aligned}$$

Given that $0 < \lambda \leq 1$, it follows that $\mathbb{H}(S_n \ominus_{gH} S_m, 0) = \mathbb{H}(S_n, S_m) \to 0$ as $n, m \to \infty$. Consequently, the sequence $\{S_n\}$ is a Cauchy sequence and so converges. □

# 6 Application of fuzzy beta-Laplace iterative method in fluid dynamics

This section demonstrates the efficacy of the fuzzy beta Laplace transform iterative method in deriving approximate solutions for the fuzzy linear time-fractional issue in fluid dynamics. By presenting complete examples, such as the fuzzy time-fractional Advection-Dispersion and Diffusion equations, we illustrate the practical application of the approach to resolve real-world issues. These examples elucidate the proposed technique by clarifying the sequential solution process. The method's ability to enhance the accuracy of the collected data underscores its practical significance.

## 6.1 Fuzzy mathematical modeling of diffusion equation

In fluid dynamics, the diffusion of substances such as pollutants or heat is characterized by a fuzzy time- fractional diffusion, which illustrates the temporal and spatial variations in the concentration of the substance. A fuzzy time-fractional diffusion equation can represent such processes as:

$$ {}^{*}_{g}\mathbb{D}^{\beta}_{\tau_0,\tau} \check{\psi}(\zeta,\tau) = \aleph \frac{\partial^2 \check{\psi}(\zeta,\tau)}{\partial \zeta^2} \oplus \check{\Bbbk}(\zeta,\tau), \qquad (\zeta,\tau) \in \mathbb{R} \times [0,\infty). \tag{18}$$

In this equation, $\aleph$ denotes the diffusion coefficient, which dictates the rate of diffusion of a substance (such as a pollutant or heat) within the fluid. The function $\check{\psi}(\zeta,\tau)$ denotes the concentration of the substance at position $\zeta$ and time $\tau$, commonly quantified in units such as $kg/m^3$ or $J/m^3$, contingent upon the specific diffusion being modeled. The notation ${}^{*}_{g}\mathbb{D}^{\beta}_{\tau_0,\tau}$ represents a $[beta]_g$-time fractional derivative that incorporates memory effects inside the system, rendering the equation appropriate for anomalous diffusion. Ultimately, $\check{\Bbbk}(\zeta,\tau)$ may denote external sources or sinks within the system.

This equation represents diffusion in intricate fluid systems when classical diffusion fails to accurately characterize the substance's behavior.

**Example 6.1.** *Examine a uniform tissue where oxygen is diffusing at a constant pace. The initial concentration of oxygen is spatially distributed throughout the tissue, and its dispersion is regulated by a diffusion process. The diffusion coefficient, which measures the rate of oxygen diffusion through tissue, is 1 square meter per second. At time $\tau = 0$, the oxygen content at various locations within the tissue is defined by the fuzzy function:*

$$\check{\psi}(\zeta,0) = \left(\frac{1-\zeta^2}{2}, 4(1-\zeta^2), 8(1-\zeta^2)\right) g/m^3.$$

*We utilize the following fuzzy time-fractional diffusion equation to represent oxygen transport in the tissue:*

$$ {}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{\tau_0,\tau} \check{\psi}(\zeta,\tau) = \frac{\partial^2 \check{\psi}(\zeta,\tau)}{\partial \zeta^2} \oplus \check{\Bbbk}(\zeta,\tau).$$

*The source function $\check{\Bbbk}(\zeta,\tau)$, denoting the external oxygen supply, progresses as follows:*

$$\check{\Bbbk}(\zeta,\tau) = (1+\tau, 8(1+\tau), 16(1+\tau)) \oplus \left(\frac{(1-\zeta^2)}{2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau}, 4(1-\zeta^2)\sqrt{\frac{1}{\sqrt{\pi}}+\tau}, 8(1-\zeta^2)\sqrt{\frac{1}{\sqrt{\pi}}+\tau}\right).$$

*Utilizing the methodology outlined in Section 5, we derive the $[beta]_{i.g}-$time differentiable solution for this fuzzy time-fractional diffusion equation. According to this technique, the solution is represented as an infinite series of functions:*

$\check{\psi}(\zeta,\tau) = \sum_{n=0}^{\infty} \check{\psi}_n(\zeta,\tau)$, *with* $\check{\psi}_0(\zeta,\tau)$ *computed as follows:*

$$\begin{aligned}
&\check{\psi}_0(\zeta,\tau) = \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\check{\psi}(\zeta,0) \oplus \frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{\check{\Bbbk}(\zeta,\tau)\right\}(\mathfrak{s})\right\}\\
&= \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\left(\frac{(1-\zeta^2)}{2}, 4(1-\zeta^2), 8(1-\zeta^2)\right)\right\} \oplus \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{(1+\tau, 8(1+\tau), 16(1+\tau))\right\}(\mathfrak{s})\right\}\\
&\quad \oplus \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{\left(\frac{(1-\zeta^2)}{2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau}, 4(1-\zeta^2)\sqrt{\frac{1}{\sqrt{\pi}}+\tau}, 8(1-\zeta^2)\sqrt{\frac{1}{\sqrt{\pi}}+\tau}\right)\right\}(\mathfrak{s})\right\}\\
&= \left(\frac{1}{6\pi^{\frac{3}{4}}}, \frac{4}{3\pi^{\frac{3}{4}}}, \frac{8}{3\pi^{\frac{3}{4}}}\right)\left[8 - 12\sqrt{\pi} - 8\sqrt{1+\sqrt{\pi}\tau} + 12\sqrt{\pi+\pi^{\frac{3}{2}}\tau} + 4\tau\sqrt{\pi+\pi^{\frac{3}{2}}\tau} - 3\pi^{\frac{3}{4}}(1+\tau)(\zeta^2-1)\right].
\end{aligned}$$

*A comparable procedure is employed to calculate* $\check{\psi}_1(\zeta,\tau)$*, as seen below:*

$$\check{\psi}_1(\zeta,\tau) = \mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{\frac{\partial^2\check{\vartheta}_0(\zeta,\tau)}{\partial\zeta^2}\right\}(\mathfrak{s})\right\}.$$

*This reduces to:*

$$\check{\psi}_1(\zeta,\tau) = \left(\frac{2}{3\pi^{\frac{3}{4}}}, \frac{16}{3\pi^{\frac{3}{4}}}, \frac{32}{3\pi^{\frac{3}{4}}}\right)\left[3\sqrt{\pi} - 2 + 2\sqrt{1+\sqrt{\pi}\tau} - 3\sqrt{\pi+\pi^{\frac{3}{2}}\tau} - \tau\sqrt{\pi+\pi^{\frac{3}{2}}\tau}\right].$$

*Furthermore, it can be established that* $\check{\psi}_n(\zeta,\tau) = (0,0,0)$ *for* $n = 2,3,\ldots$.

*The comprehensive fuzzy solution is articulated as:*

$$\check{\psi}(\zeta,\tau) = \check{\psi}_0(\zeta,\tau) \oplus \check{\psi}_1(\zeta,\tau) \oplus \check{\psi}_2(\zeta,\tau) \oplus \cdots.$$

*Upon simplification, we obtain:*

$$\check{\psi}(\zeta,\tau) = \left(\frac{(1-\zeta^2)}{2}(\tau+1), 4(1-\zeta^2)(\tau+1), 8(1-\zeta^2)(\tau+1)\right),$$

*yielding the precise fuzzy solution.*

*Figure 5 illustrates the* $r$*-cut of the solution, accompanied by* ${}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{0,\tau}\check{\psi}(\zeta,\tau)$*. The picture distinctly demonstrates that* $\check{\psi}(\zeta,\tau)$ *is a* $[beta]_{i.g}-$ *time differentiable function, as indicated by the modified locations of the lower cut (Green) and higher cut (Orange).*

**Example 6.2.** *Analyze a homogeneous solid in which thermal diffusion occurs at a steady pace. The initial temperature distribution is spatially established within the solid, and the heat transfer is governed by a diffusion mechanism. The thermal diffusivity, quantifying the rate of heat diffusion within the material, is* $1\,m^2/s$*. At time* $\tau = 0$*, the temperature at different locations within the solid is represented by the fuzzy function:*

$$\check{\psi}(\zeta,0) = (3\zeta, 5\zeta, 9\zeta).$$

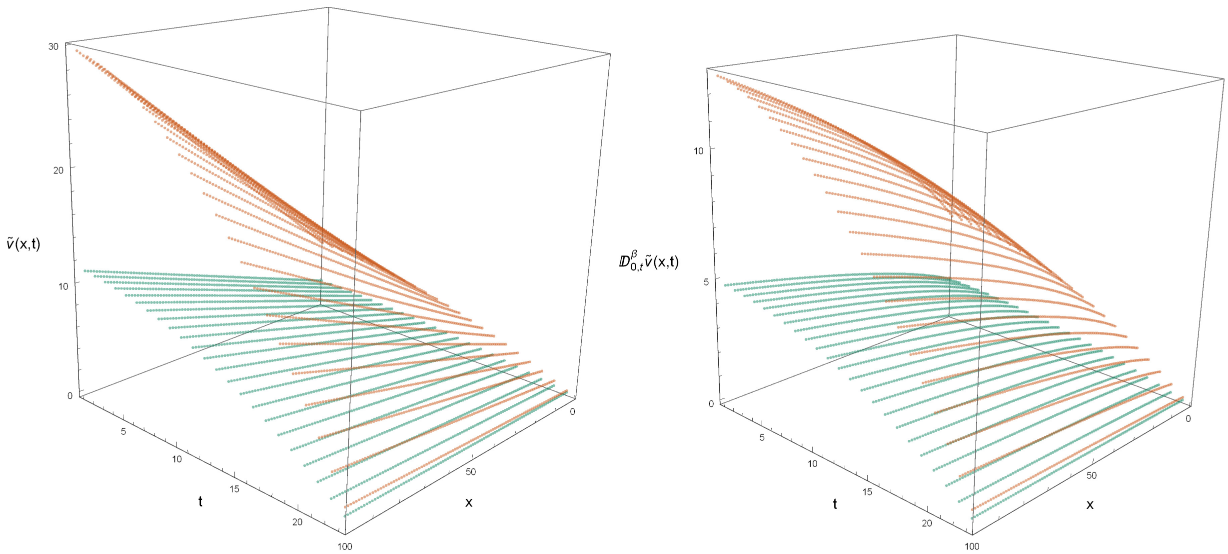
*We utilize the subsequent fuzzy time-fractional heat diffusion equation to characterize heat transmission within the material:*

$${}^{*}_{g}\mathbb{D}^{\beta}_{\tau_0,\tau}\check{\psi}(\zeta,\tau) = \frac{\partial^2\check{\psi}(\zeta,\tau)}{\partial\zeta^2} \oplus (-1)\check{\Bbbk}(\zeta,\tau), \qquad (\zeta,\tau) \in \mathbb{R}\times[0,\infty). \tag{19}$$

*The source function* $\check{\Bbbk}(\zeta,\tau)$*, denoting an external heat source, progresses as follows:*

$$\check{\Bbbk}(\zeta,\tau) = \left(6\tau e^{-t^2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau\zeta}, 10\tau e^{-t^2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau\zeta}, 18\tau e^{-t^2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau\zeta}\right).$$

*This equation represents the temporal evolution of temperature within the solid, influenced by heat diffusion and an external heat source.*




(a) (b)

Figure 5: Plots of $\check{\psi}(\zeta,\tau)$ and ${}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{0,\tau}\check{\psi}(\zeta,\tau)$ for Eq.(6.1) in $\mathfrak{r}=0.5$. (a).$\left[\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$. (b).$\left[{}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{0,\tau}\check{\psi}(\zeta,\tau)\right]^{\mathfrak{r}}$.

*Utilizing the methodology outlined in Section 5, we derive the* $[beta]_{ii.g}-$*times differentiable solution for the fuzzy time-fractional diffusion equation. This approach articulates the answer as an infinite series of functions:* $\check{\psi}(\zeta,\tau)=\sum\limits_{n=0}^{\infty}\check{\psi}_n(\zeta,\tau)$*, with the initial term* $\check{\psi}_0(\zeta,\tau)$ *specified as follows:*

$$
\begin{aligned}
&\check{\psi}_0(\zeta,\tau)=\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\check{\psi}(\zeta,0)\ominus(-1)\frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{\Bbbk(\zeta,\tau)\right\}(\mathfrak{s})\right\}\\
&=\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\left(3\zeta+12,5\zeta+20,9\zeta+36\right)\right\}\\
&\quad\ominus\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{\left(6\tau e^{-t^2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau\zeta},10\tau e^{-t^2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau\zeta},18\tau e^{-t^2}\sqrt{\frac{1}{\sqrt{\pi}}+\tau\zeta}\right)\right\}(\mathfrak{s})\right\},\\
&=\left(3\zeta+12,5\zeta+20,9\zeta+36\right)\ominus\left(3\zeta(1-e^{-t^2}),5\zeta(1-e^{-t^2}),9\zeta(1-e^{-t^2})\right),
\end{aligned}
$$

*and for the ensuing terms:*

$$
\check{\psi}_n(\zeta,\tau)=\ominus(-1)\mathbb{L}_{\mathfrak{s}}^{-1}\left\{\frac{1}{\mathfrak{s}}\mathcal{L}_{\beta,\tau}\left\{\frac{\partial^2\check{\psi}_{n-1}(\zeta,\tau)}{\partial\zeta^2}\right\}(\mathfrak{s})\right\}=(0,0,0)\,,\qquad n=1,2,\cdots.
$$

*As* $n$ *tends to infinity, the precise* $[beat]_{ii.g}-$*solution of Eq.(19) is:*

$$
\check{\psi}(\zeta,\tau)=\left(3\zeta e^{-t^2},5\zeta e^{-t^2},9\zeta e^{-t^2}\right).
$$

## 6.2 Fuzzy time-fractional advection-dispersion equation

The time-fractional Advection-Dispersion equation is a prevalent partial differential equation utilized for modeling solute transport in groundwater flow, tackling issues such as pollutant transport and groundwater recharge. Predicting solute concentration in practical situations is frequently challenging due to the intrinsic uncertainties present in environmental systems.

A fuzzy model has been designed to address these issues. This model use fuzzy logic to depict the temporal evolution of particle concentration, integrating fuzzy sets and rules to enhance predictive accuracy. The fuzzy mathematical model is articulated as:

$$
{}^{*}_{g}\mathbb{D}^{\beta}_{\tau_0,\tau}\check{\psi}(\zeta,\tau)=\mathcal{D}\frac{\partial^2\check{\psi}(\zeta,\tau)}{\partial\zeta^2}\oplus\nu\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta}\oplus\Bbbk(\zeta,\tau).
$$

where $\check{\psi}(\zeta,\tau)$ represents the contaminant concentration, $\mathcal{D}$ signifies the dispersion coefficient, $\nu$ denotes the fluid velocity, and $\Bbbk(\zeta,\tau)$ pertains to contaminant injection or withdrawal.

Subsequently, we will expand the numerical method from Section 5 to derive a $[beta]_{i.g}-$ differentiable solution for this model.

**Example 6.3.** *Examine a uniform aquifer where a pollutant plume measures* 80 *meters in length and progresses at an average velocity of* 0.05 *meters per second. The dispersion coefficient, indicating the rate of pollutant dissemination, is* 0.002 *square meters per second. At time* $\tau = 0$*, the pollutant concentration at different locations within the plume is defined by the fuzzy function:*

$$\check{\psi}(\zeta,0) = (3\zeta+10, 4\zeta+15, 8\zeta+25).$$

*We utilize the following fuzzy time-fractional Advection-Dispersion equation to elucidate the transport of the contaminant:*

$$ {}^{*}_{g}\mathbb{D}^{\frac{1}{2}}_{\tau_0,\tau}\check{\psi}(\zeta,\tau) = 0.002\frac{\partial^2\check{\psi}(\zeta,\tau)}{\partial\zeta^2} \oplus 0.05\frac{\partial\check{\psi}(\zeta,\tau)}{\partial\zeta} \oplus \Bbbk(\zeta,\tau). \tag{20}$$

*The source function* $\Bbbk(\zeta,\tau)$ *progresses as outlined below:*

$$\begin{aligned}\Bbbk(\zeta,\tau) &= \left(2\tau\sqrt{\frac{1}{\sqrt{\pi}+\tau}}(10+3\zeta), 2\tau\sqrt{\frac{1}{\sqrt{\pi}+\tau}}(15+4\zeta), 2\tau\sqrt{\frac{1}{\sqrt{\pi}+\tau}}(25+8\zeta)\right)\\ &\ominus \left(0.15(1+\tau^2), 0.2(1+\tau^2), 0.4(1+\tau^2)\right).\end{aligned}$$

*Employing the methods delineated in Section 5, we obtain the* $[beta]_{i.g}-$*times differentiable solution for the fuzzy time-fractional Advection-Dispersion Eq.(20) as follows*

$$\check{\psi}(\zeta,\tau) = \left(10+3\zeta+\tau^2(10+3\zeta), 15+4\zeta+\tau^2(15+4\zeta), 25+8\zeta+\tau^2(25+8\zeta)\right).$$

# 7 Conclusion

This study presented the fuzzy beta generalized Hukuhara time-fractional derivative as an appropriate fractional derivative for the qualitative examination of fractional differential equations in fuzzy space. This methodology proficiently modelled and analysed intricate systems defined by uncertain or ambiguous data, especially in the domain of fluid dynamics. A numerical methodology, the fuzzy beta Laplace transform iterative method, was devised to derive approximate solutions for fuzzy linear time-fractional equations in fluid dynamics. The efficacy of the suggested method was evidenced by case studies, encompassing the fuzzy time-fractional diffusion equation and the advection-dispersion equation, thereby showcasing its relevance to practical issues.

This study advanced mathematical modelling tools for uncertain complex systems by merging fuzzy calculus with fractional differential equations. The proposed methodology maintained the intrinsic ambiguity of data, augmented numerical stability, and increased computational efficiency, providing a resilient framework for the analysis of fractional dynamical systems amidst uncertainty.

Future study may investigate the application of this methodology to additional domains, including financial modelling and bioengineering, where uncertainty and fractional dynamics are pivotal. Moreover, enhancing the numerical method for higher-dimensional systems could augment computing efficiency and scalability, facilitating its application to progressively intricate issues.

# Declarations


- Author contributions

  M.R. Mastani Shabestari conceptualized the research, designed the methodology, and performed the theoretical analysis. Author M. R. Mastani Shabestari also wrote the initial draft of the manuscript. N. Mikaeilvand contributed to the refinement of the mathematical models, assisted with data analysis, and reviewed and edited the manuscript. Both authors read and approved the final version of the manuscript.

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- Data availability

  No data was used for the research described in the article.

- Competing interests

  The authors have no relevant financial or non-financial interests to disclose.

- Ethical approval

  This article does not contain any studies involving human participants or animals conducted by the author.

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