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Original paper

Bipolar ordered weighted quasi-averages and induced bipolar ordered weighted averages: g-BIOWA and IBIOWA operators

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Abstract

A generalization of bipolar OWA operators, bipolar ordered weighted quasi-averages, based on the bipolar Choquet g-integrals, so called g-BIOWA operators are introduced and investigated. A generalization of induced OWA operators considering non-negative inputs and order-inducing variables, induced bipolar ordered weighted averages are introduced considering real inputs and order-inducing vectors. Their main properties are considered and some illustrative examples are presented.

Keywords: Symmetric g-operations, bi-capacity, bipolar Choquet g-integral, g-BIOWA, induced BIOWA.

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1 Introduction

Ordered weighted averages (OWA) introduced by Yager [30] have shown their usefulness in muticriteria decision-making, and also in other fields where ranking is important [33]. Later, OWA were generalized to induced OWA (IOWA) by Yager [31]. Recently, several other generalizations of OWA were introduced and studied [4, 14, 16, 18, 21]. Important for our purposes will be also bipolar ordered weighted averages (BIOWA), recently introduced by Stupňanová and Jin [27] and later investigated by Mesiar et al. in [20], that are based on the Choquet integral with respect to bi-capacities. The phenomenon of bipolarity in the context of multi-criteria decision making was studied by Jin et al. in [13]. Of course, there are several ways how to handle decision-making process. Important is, e.g., the approach using decision trees (see [2]). In our paper, we follow the OWA approach.

The non-additive integrals were observed in [5, 6, 7, 10, 17, 19, 25, 29]. The Choquet integral appears as the most used non-additive integral defined with respect to fuzzy measures, with numerous applications in statistics and probability, engineering and economics. The idea of bipolar Choquet integration with respect to bi-capacities, proposed in [9], and the bipolar fuzzy integration considered in [12], was further developed in [11] by introducing the discrete bipolar universal integral based on bi-capacities, whereas some construction methods and orness measure of bi-capacities were considered in [15]. The bipolar pseudo-integrals and bipolar Choquet g-integrals in discrete settings have been introduced and discussed recently in [23, 24, 26]. Besides many in literature known applications of bi-capacities in game theory, applications of the last integrals in actuarial science are related to many different types of premium principles, whereas in decision-making framework they are suitable for mathematical modelling of the preference relation [1, 22].

OWA can well model the preferences under a linearly ordered set, e.g., real line, without differentiating positive numbers and negative numbers. As usually in decision making it is meaningful and necessary to differentiate them and OWA cannot work more on this, however BIOWA can well model and differentiate by considering a symmetrical structure with respect to the real number zero. The main aim of this paper is to introduce and investigate extensions and further generalizations of BIOWA operators that cover a wider spectrum of means from arithmetic to the general

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root-power and quasi-arithmetic means, which are commonly used in both theory and applications, in particular, bipolar generalized ordered weighted averages based on the bipolar Choquet g-integral.

In this paper our considerations will be devoted to a generalization of BIOWA operators, as well as induced aggregation in the bipolar context in terms of induced bipolar ordered weighted averages considering real inputs and order-inducing vectors. In Section 2, a brief overview on the bipolar Choquet integral, BIOWA operators and IOWA operators is given. In Section 3, we present the bipolar Choquet-like integral, based on symmetric g-operations generated by an odd, strictly increasing and continuous function and give some of its properties. Based on the bipolar Choquet g-integrals, in Section 4, the g-BIOWA operators are introduced and investigated and some examples are given. In Section 5, we present induced BIOWA operators (IBIOWA) and show their main properties. Finally, some concluding remarks are given.

2 Preliminaries

The definitions of bi-capacities and the bipolar Choquet integral are recalled, according to [8, 9, 11, 12], whereas the definitions on BIOWA and IOWA operators are presented following [20, 21, 31].

2.1 Bipolar Choquet integral

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a non-empty finite set with $\operatorname{card}(X) = n$ and the power set denoted by $\mathcal{P}(X)$.

Definition 2.1. *A function $m : \mathcal{P}(X) \to [0, \infty]$ is called a capacity (or a fuzzy measure) if $m(\varnothing) = 0$ and $m(A) \le m(B)$ for all $A \subset B \subset X$.*

Denote $\mathcal{Q}(X) = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \cap B = \varnothing\}$.

Definition 2.2. *A function $\mathbf{m} : \mathcal{Q}(X) \to \mathbb{R}$ is called a bi-capacity if besides $\mathbf{m}(\varnothing, \varnothing) = 0$, for all $A \subset B \subset X$, and (A, E), (B, E), (F, A), $(F, B) \in \mathcal{Q}(X)$, the following holds:*

$$\mathbf{m}(A, E) \le \mathbf{m}(B, E) \text{ and } \mathbf{m}(F, A) \ge \mathbf{m}(F, B).$$

A bi-capacity $\mathbf{m}$ is normalized if it satisfies $\mathbf{m}(X, \varnothing) = 1 = -\mathbf{m}(\varnothing, X)$.

Consider $f : X \to \mathbb{R}$ and a bi-capacity $\mathbf{m} : \mathcal{Q}(X) \to [-1, 1]$. We denote $f(x_i) = f_i$, $i = 1, \ldots, n$. The class of functions $f : X \to \mathbb{R}$ is denoted by $\mathcal{S}$, whereas the class of functions $f : X \to [-1, 1]$ is denoted by $\mathcal{S}_1$. The class of all normalized bi-capacities $\mathbf{m} : \mathcal{Q}(X) \to [-1, 1]$ is denoted by $\mathcal{M}$.

According to [11], the definition of the bipolar Choquet integral is given as follows (compare [9, 10, 11, 12]).

Definition 2.3. *The bipolar Choquet integral of $f \in \mathcal{S}$, with respect to $\mathbf{m} \in \mathcal{M}$, is defined regarding any permutation of indexes $\alpha = (\alpha(1), \ldots \alpha(n))$ related to non-decreasing order of values $|f(x_i)| = |f_i|$, i.e., such that $0 \le |f_{\alpha(1)}| \le |f_{\alpha(2)}| \le \cdots \le |f_{\alpha(n)}|$, $|f_{\alpha(0)}| = 0$, by:*

$$BCh(f, \mathbf{m}) = \sum_{i=1}^{n} (|f_{\alpha(i)}| - |f_{\alpha(i-1)}|) \cdot \mathbf{m}(\{f \ge |f_{\alpha(i)}|\}, \{f \le -|f_{\alpha(i)}|\}),$$

with the convention that $(\{f \ge 0\}, \{f \le 0\})$ will be interpreted as $(\{f \ge 0\}, \{f < 0\})$.

Notice that the last convention is necessary in order to $(\{f \ge 0\}, \{f < 0\}) \in \mathcal{Q}(X)$ be satisfied.

Two functions $f, h \in \mathcal{S}$ are comonotone if for any $x_i, x_j \in X$ it holds

$$(f(x_i) - f(x_j))(h(x_i) - h(x_j)) \ge 0.$$

Notice that the previous requirement is equivalent to the following: if $f(x_i) - f(x_j) < 0$ then $h(x_i) - h(x_j) \le 0$ for any $x_i, x_j \in X$.

We say that two functions $f, h \in \mathcal{S}$ are bi-comonotone if the pairs f^+ and h^+, f^- and h^-, $|f|$ and $|h|$ are comonotone, where $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$ and $|f| = f^+ + f^-$.

It has been shown in [20] that the bipolar Choquet integral is bi-comonotone additive, i.e., for any bi-capacity $\mathbf{m}$ and bi-comonotone functions f, h it holds

$$BCh(f + h, \mathbf{m}) = BCh(f, \mathbf{m}) + BCh(h, \mathbf{m}).$$

2.2 Bipolar OWA operators

A BIOWA operator, as a generalization of OWA operator, has been introduced in [20]. The notion of BIOWA operator is based on symmetry (also called commutativity, neutrality or anonymity) of the bipolar Choquet integral. Namely, the bipolar Choquet integral with respect to $\mathbf{m} \in \mathcal{M}$ is symmetric if and only if $\mathbf{m}$ is symmetric, i.e., if it holds

$$\mathbf{m}\left(A, B\right) = \mathbf{m}\left(T^{-1}\left(A\right), T^{-1}\left(B\right)\right),$$

for all $(A, B) \in \mathcal{Q}(X)$ and any permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$, with $T(A) = \{x_{\tau(i)} \,|\, x_i \in A\}$.

Definition 2.4. [20] *Let* $\mathbf{m} \in \mathcal{M}$ *be a symmetric bi-capacity. An operator* BIOWA : $\mathbb{R}^n \to \mathbb{R}$, *defined by*

$$\mathrm{BIOWA}\left(f\right) = BCh(f, \mathbf{m}),$$

where $f \in \mathcal{S}$, *is called a Bipolar Ordered Weighted Average (a BIOWA operator).*

Let us recall Proposition 3.1, presented in [20], where

$$\widetilde{N} = \{(i, j) \mid i, j \in \{0, 1, \ldots, \mathrm{card}(X)\}, i + j \leq \mathrm{card}(X)\}.$$

Proposition 2.5. *A bi-capacity* $\mathbf{m} \in \mathcal{M}$ *is symmetric if and only if there exists a function* $t : \widetilde{N} \to [-1, 1]$ *such that*

$$\mathbf{m}\left(A, B\right) = t\left(\mathrm{card}(A), \mathrm{card}(B)\right),$$

for all $A, B \in \mathcal{P}(X), A \cap B = \emptyset$ *and the following conditions are fulfilled*

(i) $t\left(n, 0\right) = 1$, $t\left(0, 0\right) = 0$, $t\left(0, n\right) = -1$,

(ii) *t is increasing in the first coordinate,*

(iii) *t is decreasing in the second coordinate.*

2.3 Induced OWA operators

While in the case of OWA the order of inputs is determined by the vector itself to be aggregated, in the case of IOWA there is another vector, usually called an order-inducing vector, that is used for the order determination, i.e. the ranking is induced by that vector, which need not to be the input vector itself. Induced OWA operators, or IOWA was introduced in [31]. In IOWA we have an order-inducing vector $u = (u_1, \ldots, u_n)$, an input vector $f = (f_1, \ldots, f_n)$ and a weighting vector $w = (w_1, \ldots, w_n)$. We choose a permutation π ordering the vector u and as it has been remarked in [3, 31], there may be some ties in u. This means that several permutations yield the same order of u. For this reason, the definition of IOWA will be as follows.

Definition 2.6. *Let* $u = (u_1, \ldots, u_n)$ *be an order-inducing vector with* $u_i \geq 0$, $w = (w_1, ..., w_n)$ *a weighting vector such that* $\sum_i w_i = 1$ *and* $w_i \geq 0$, *and* $f_i \in [0, 1]$ *an input function. Let* $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ *be a permutation such that* $u_{\sigma(1)} \leq u_{\sigma(2)} \leq \cdots \leq u_{\sigma(n)}$. *For each* $i \in \{1, \ldots, n\}$, *let*

$$J_i = \{\sigma(j) \in \{1, \ldots, n\} \mid u_{\sigma(i)} = u_{\sigma(j)}\}.$$

We denote $\tilde{w}_i = \frac{1}{\mathrm{card}(J_i)} \sum\limits_{\sigma(j) \in J_i} w_j$. *Then*

$$\mathrm{IOWA}_w(\langle u, f\rangle) = \sum_{i=1}^{n} \tilde{w}_i f_{\sigma(i)}. \tag{1}$$

In what follows we will not distinguish between w and $\tilde{\mathbf{w}}$ keeping in mind that we must use averages of weights if ties in the inducing vector u occur.

3 Bipolar Choquet g-integral

In this section, the definition of the bipolar Choquet g-integrals, introduced in [23], and some of its properties are presented.

Assume a strictly increasing, continuous and odd function $g : [-\infty, \infty] \to [-\infty, \infty]$, with $g(1) = 1$. We will consider the symmetric g-addition on $[-\infty, \infty]^2$ given by

$$x \oplus_g y = g^{-1}(g(x) + g(y)),$$

with $\infty - \infty = \infty$, the symmetric g-multiplication on $[-\infty, \infty]^2$ given by

$$x \odot_g y = g^{-1}(g(x)g(y)),$$

with $0 \cdot \infty = 0$, and the symmetric g-difference on $[-\infty, \infty]^2 \setminus \{(\infty, \infty), (-\infty, -\infty)\}$ given by

$$x \ominus_g y = x \oplus_g (-y) = g^{-1}(g(x) - g(y)).$$

Such a function g will be referred to as a generator for the symmetric g-operations. For $x \in \mathbb{R}$, $\ominus_g x = g^{-1}(-g(x)) = -x$, and it holds $x \oplus_g (-x) = x \oplus_g (\ominus_g x) = 0$.

Definition 3.1. *Let $f \in \mathcal{S}$, $\mathbf{m} \in \mathcal{M}$ and let $g : [-\infty, \infty] \to [-\infty, \infty]$ be a strictly increasing, continuous and odd function with $g(1) = 1$. The bipolar Choquet g-integral of f with respect to $\mathbf{m}$ is defined by:*

$$BCh^g(f, \mathbf{m}) = g^{-1}(BCh(g \circ f, g \circ \mathbf{m})),$$

where $BCh(g \circ f, g \circ \mathbf{m})$ is the bipolar Choquet integral of $g \circ f$ with respect to $g \circ \mathbf{m}$.

Obviously, if $g \equiv id$, then $BCh^{id} = BCh$.

For $f \in \mathcal{S}$, denote

$$X_f^+ = \{x_i \in X \mid f(x_i) > 0\},\ X_f^- = \{x_i \in X \mid f(x_i) < 0\},$$
$$X_f^0 = \{x_i \in X \mid f(x_i) = 0\},\ X_f^+ \cup X_f^- = X \setminus X_f^0.$$

Further, $X_f^{+0} = X_f^+ \cup X_f^0$ and $X_f^{-0} = X_f^- \cup X_f^0$. Let $\mathbf{m} \in \mathcal{M}$ be a bi-capacity. For $f \in \mathcal{S}$, let set functions $\mu_{f^+}, \tilde{\mu}_{f^+} : \mathcal{P}(X) \to [-1, 1]$ be given by

$$\mu_{f^+}(A) = \mathbf{m}(A \cap X_f^{+0},\, A \cap X_f^-), \qquad (2)$$
$$\tilde{\mu}_{f^+}(A) = \mathbf{m}(A \cap X_f^+,\, A \cap X_f^{-0}). \qquad (3)$$

For each $A \subset X_f^+ \cup X_f^-$, it holds $\mu_{f^+}(A) = \tilde{\mu}_{f^+}(A)$. If $f \geq 0$, then for each $A \in \mathcal{P}(X)$ we have $A \cap X_f^{+0} = A \cap X = A$ and $A \cap X_f^- = A \cap \varnothing = \varnothing$, therefore, $\mu_{f^+} : \mathcal{P}(X) \to [0, 1]$ is a normalized capacity. Similarly, if $f \leq 0$, then $-\tilde{\mu}_{f^+} : \mathcal{P}(X) \to [0, 1]$ is a normalized capacity, where $-\tilde{\mu}_{f^+}(A) = -\mathbf{m}(\varnothing, A)$, for all $A \in \mathcal{P}(X)$.

For given symmetric g-operations, $\oplus = \oplus_g$, $\odot = \odot_g$ and g-difference $\ominus = \ominus_g$, each function $f \in \mathcal{S}$, has the following representations:

$$\begin{aligned} f &= \bigoplus_{i=1}^{n} (|f_{\alpha(i)}| \ominus |f_{\alpha(i-1)}|) \odot (\chi_{A_i \cap X_f^{+0}} - \chi_{A_i \cap X_f^-}) \\ &= \bigoplus_{i=1}^{n} (|f_{\alpha(i)}| \ominus |f_{\alpha(i-1)}|) \odot (\chi_{A_i \cap X_f^{+}} - \chi_{A_i \cap X_f^{-0}}), \end{aligned}$$

where α is any permutation of indexes such that $0 \leq |f_{\alpha(1)}| \leq |f_{\alpha(2)}| \leq \cdots \leq |f_{\alpha(n)}|$, $f_{\alpha(0)} = 0$, $A_i = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$ for all i, and for each $A \in \mathcal{P}(X)$, the characteristic function χ_A is defined as usual. The formulas for computing the bipolar Choquet g-integrals are presented in the next proposition. The proof has been given in [23] and it is similar to the proof of Proposition 1 from [24], for $\varphi(x) = x$, $x \in \mathbb{R}$, thus it is omitted.

Proposition 3.2. *Let $\oplus = \oplus_g$ and $\odot = \odot_g$ be given symmetric g-operations. The bipolar Choquet g-integral of $f \in \mathcal{S}$ with respect to $\mathbf{m} \in \mathcal{M}$ can be expressed by:*

$$\begin{aligned} BCh^g(f, \mathbf{m}) &= \bigoplus_{i=1}^{n} \Big(|f_{\alpha(i)}| \ominus |f_{\alpha(i-1)}| \Big) \odot \mu_{f^+}(A_i) && (4) \\ &= \bigoplus_{i=1}^{n} \Big(|f_{\alpha(i)}| \ominus |f_{\alpha(i-1)}| \Big) \odot \tilde{\mu}_{f^+}(A_i) && (5) \\ &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot \Big(\mu_{f^+}(A_i) \ominus \mu_{f^+}(A_{i+1}) \Big), && (6) \end{aligned}$$

where α is any permutation of indexes such that $|f_{\alpha(1)}| \le |f_{\alpha(2)}| \le \dots \le |f_{\alpha(n)}|$, $f_{\alpha(0)} = 0$, $A_i = \{x_{\alpha(i)}, \dots, x_{\alpha(n)}\}$, $i = 1, \dots n$, and $A_{n+1} = \varnothing$.

Notice that ties in $|f|$ do not affect on the bipolar Choquet g-integral of $f \in \mathcal{S}$. If $g(x) = x$, $x \in [-\infty, \infty]$ and $f \ge 0$, we obtain that the Choquet integral of non-negative $f \in \mathcal{S}$ with respect to $\mathbf{m} \in \mathcal{M}$ equals to the Choquet integral of $f : X \to [0, \infty)$ with respect to a capacity $\mu = \mu_{f^+}$, i.e. $BCh(f, \mathbf{m}) = Ch(f, \mu)$ [9]. In general, for $f \ge 0$, we have $BCh^g(f, \mathbf{m}) = Ch^g(f, \mu)$, where $Ch^g(f, \mu)$ denotes the Choquet-like integral (g-Choquet integral) of $f : X \to [0, \infty)$ with respect to $\mu = \mu_{f^+}$, [19].

In the following theorem we present the bi-comonotone $\oplus$-additivity of the bipolar Choquet g-integral.

Theorem 3.3. *If $\oplus = \oplus_g$ and $\odot = \odot_g$ are given symmetric g-operations, $\mathbf{m} \in \mathcal{M}$ and $f, h \in \mathcal{S}$ are bi-comonotone, then it holds*

$$BCh^g(f \oplus h, \mathbf{m}) = BCh^g(f, \mathbf{m}) \oplus BCh^g(h, \mathbf{m}).$$

Proof. Since the function g is odd and strictly increasing, it holds $g(0) = 0$, $g(|x|) = |g(x)|$, for all $x \in \mathbb{R}$. Thus

$$\begin{aligned} (g \circ f)^+ &= (g \circ f) \vee 0 = g \circ (f \vee 0) = g \circ f^+, \\ (g \circ f)^- &= (-(g \circ f)) \vee 0 = g \circ ((-f) \vee 0) = g \circ f^-, \\ |g \circ f| &= g \circ |f|. \end{aligned}$$

According to the previous equalities, we conclude that if f, h are bi-comonotone then $g \circ f$, $g \circ h$ are bi-comonotone. Now, from the definition of the bipolar Choquet g-integral and the bi-comonotone additivity of the bipolar Choquet integral it follows

$$\begin{aligned} BCh^g(f \oplus h, \mathbf{m}) &= g^{-1}(BCh(g \circ (f \oplus h), g \circ \mathbf{m})) \\ &= g^{-1}(BCh(g \circ f + g \circ h, g \circ \mathbf{m})) \\ &= g^{-1}(BCh(g \circ f, g \circ \mathbf{m}) + BCh(g \circ h, g \circ \mathbf{m})) \\ &= BCh^g(f, \mathbf{m}) \oplus BCh^g(h, \mathbf{m}). \end{aligned}$$

□

4 g-Bipolar OWA operators

In this section, we will introduce a generalization of BIOWA related to a generating function g of symmetric g-operations, denoted by $\oplus$ and $\odot$.

Definition 4.1. *Let $\mathbf{m} \in \mathcal{M}$ be a symmetric bi-capacity and let $g : [-\infty, \infty] \to [-\infty, \infty]$ be a strictly increasing, continuous and odd function with $g(1) = 1$. An operator g-BIOWA : $\mathbb{R}^n \to \mathbb{R}$ defined by*

$$g\text{-BIOWA}(f) = BCh^g(f, \mathbf{m}),$$

is called a Bipolar Ordered Weighted Quasi-Average (a g-BIOWA operator).

As a consequence of Proposition 2.5 and the monotonicity of g we have the following result.

Proposition 4.2. *A bi-capacity $\mathbf{m} \in \mathcal{M}$ is symmetric if and only if $g \circ \mathbf{m}$ is a symmetric bi-capacity.*

Let $f \in \mathcal{S}$ and let α be a permutation such that $|f_{\alpha(1)}| \leq |f_{\alpha(2)}| \leq \cdots \leq |f_{\alpha(n)}|$. Let $A_i = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$ and $A_{n+1} = \varnothing$. Assume a function $s : X \to \{(1,0), (0,1)\}$ defined by

$$s(x_i) = \begin{cases} (1,0), \; x_{\alpha(i)} \in X_f^{+0}, \\ (0,1), \; x_{\alpha(i)} \in X_f^{-}. \end{cases} \tag{7}$$

Let

$$w_i^s = t\left(\sum_{j=i}^{n} s(x_j)\right) \ominus t\left(\sum_{j=i+1}^{n} s(x_j)\right), \tag{8}$$

where $\sum_{j=n+1}^{n} s(x_j) = (0,0)$ and t is a generating function of a symmetric bi-capacity $\mathbf{m} \in \mathcal{M}$. The vector $w_s = (w_1^s, .., w_n^s)$ is weighting vector and w_i^s are weights. In general, for a fixed t there can be 2^n possible weighting vectors depending on the function f. Obviously, for $f \geq 0$ or $f < 0$ the weighting vector w_s satisfies $\sum_{i=1}^{n} |w_i^s| = 1$. Although we refer to w_i^s, $i = 1, \ldots, n$ as weights, they need not to be non-negative.

Theorem 4.3. *Let $\mathbf{m} \in \mathcal{M}$ be symmetric and based on a generating function t. Then for g-BIOWA operator $g\text{-BIOWA}_t : \mathbb{R}^n \to \mathbb{R}$ it holds*

$$g\text{-BIOWA}_t(f) = BCh^g(f, \mathbf{m}) = \bigoplus_{i=1}^{n} w_i^s \odot |f_{\alpha(i)}|,$$

where weights w_i^s and s are given by (8) and (7), respectively.

Proof. Using Proposition 3.2 and the above notation we get

$$\begin{aligned} g\text{-BIOWA}_t(f) &= BCh^g(f, \mathbf{m}) \\ &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot \Big(\mu_{f^+}(A_i) \ominus \mu_{f^+}(A_{i+1})\Big) \\ &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot \Big(\mathbf{m}(A_i \cap X_f^{+0}, A_i \cap X_f^{-}) \ominus \mathbf{m}(A_{i+1} \cap X_f^{+0}, A_{i+1} \cap X_f^{-})\Big) \\ &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot \Big(t(\left|A_i \cap X_f^{+0}\right|, \left|A_i \cap X_f^{-}\right|) \ominus t(\left|A_{i+1} \cap X_f^{+0}\right|, \left|A_{i+1} \cap X_f^{-}\right|)\Big) \\ &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot \Big(t\left(\sum_{j=i}^{n} s(x_j)\right) \ominus t\left(\sum_{j=i+1}^{n} s(x_j)\right)\Big) \\ &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot w_i^s, \end{aligned}$$

where we use the notation $|A| = \text{card}(A)$. □

Notice that a choice of a permutation α of indexes of inputs obtained by $|f|$, such that $|f_{\alpha(1)}| \leq |f_{\alpha(2)}| \leq \cdots \leq |f_{\alpha(n)}|$ does not affect on g-BIOWA operator. However, this additional condition for α can be required (see [20]): if for some $i < j$ it holds $|f_{\alpha(i)}| = |f_{\alpha(j)}|$, then either sign $f_{\alpha(i)} = -1$ and sign $f_{\alpha(j)} = 1$, or sign $f_{\alpha(i)}=$ sign $f_{\alpha(j)}$ and $\alpha(i) < \alpha(j)$.

Example 4.4. Let $X = \{1,2,3,4\}$ and function $f \in \mathcal{S}_1$ be given by $f(1) = 0.2$, $f(2) = -0.3$, $f(3) = 0$, $f(4) = -0.4$. Therefore $X_f^{+0} = \{1,3\}$, $X_f^{-} = \{2,4\}$. For $\alpha = (3,1,2,4)$, we have $0 = |f_3| \leq |f_1| \leq |f_2| \leq |f_4|$. Then

$$s(1) = (1,0), \; s(2) = (1,0), \; s(3) = (0,1), \; s(4) = (0,1).$$

Let $t(x,y) = g^{-1}\left(\frac{x-y}{4}\right)$. Then

$$\begin{aligned} w_1^s &= t(2,2) \ominus t(1,2) = g^{-1}(0) \ominus g^{-1}\left(-\frac{1}{4}\right) \\ &= g^{-1}\left(0 - (-\frac{1}{4})\right) = g^{-1}\left(\frac{1}{4}\right), \end{aligned}$$

$$\begin{aligned} w_2^s &= t(1,2) \ominus t(0,2) = g^{-1}\left(-\frac{1}{4}\right) \ominus g^{-1}\left(-\frac{1}{2}\right) \\ &= g^{-1}\left(-\frac{1}{4} - \left(-\frac{1}{2}\right)\right) = g^{-1}\left(\frac{1}{4}\right), \end{aligned}$$

$$\begin{aligned} w_3^s &= t(0,2) \ominus t(0,1) = g^{-1}\left(-\frac{1}{2}\right) \ominus g^{-1}\left(-\frac{1}{4}\right) \\ &= g^{-1}\left(-\frac{1}{2} - \left(-\frac{1}{4}\right)\right) = g^{-1}\left(-\frac{1}{4}\right), \end{aligned}$$

$$\begin{aligned} w_4^s &= t(0,1) \ominus t(0,0) = g^{-1}\left(-\frac{1}{4}\right) \ominus g^{-1}(0) \\ &= g^{-1}\left(-\frac{1}{4}\right), \end{aligned}$$

and by Theorem 4.3, the bipolar Choquet g-integral of a function f with respect to $\mathbf{m}$ is

$$\begin{aligned} g\text{-BIOWA}_t(f) &= w_1^s \odot \left|f_{\alpha(1)}\right| \oplus w_2^s \odot \left|f_{\alpha(2)}\right| \oplus w_3^s \odot \left|f_{\alpha(3)}\right| \oplus w_4^s \odot \left|f_{\alpha(4)}\right| \\ &= g^{-1}\left(\frac{1}{4}\right) \odot 0.2 \oplus g^{-1}\left(-\frac{1}{4}\right) \odot 0.3 \oplus g^{-1}\left(-\frac{1}{4}\right) \odot 0.4 \\ &= g^{-1}\left(\frac{1}{4} g(0.2) - \frac{1}{4} g(0.3) - \frac{1}{4} g(0.4)\right). \end{aligned}$$

If $g(x) = x$, $x \in [-\infty, \infty]$, is a generator for $\oplus = \oplus_g, \odot = \odot_g$ then

$$\begin{aligned} g\text{-BIOWA}_t(f) &= \frac{1}{4} \cdot 0.2 - \frac{1}{4} \cdot 0.3 - \frac{1}{4} \cdot 0.4 \\ &= -0.125, \end{aligned}$$

and it is the arithmetic mean.
If $g(x) = (\operatorname{sign} x)|x|^k$, $x \in [-\infty, \infty]$, for $k > 0$ is a generator for $\oplus = \oplus_g, \odot = \odot_g$, then

$$g\text{-BIOWA}_t(f) = (\operatorname{sign} m)\left|\frac{1}{4}\left(0.2^k - 0.3^k - 0.4^k\right)\right|^{\frac{1}{k}},$$

where $4m = 0.2^k - 0.3^k - 0.4^k$, and for odd $k \in \mathbb{N}$, it is the root-power mean.

A function of n variables $F : [-1,1]^n \to [-1,1]$ that is non-decreasing in each variable and that fulfills the boundary conditions $F(1,1,...,1) = 1$ and $F(-1,-1,...,-1) = -1$ is said to be an aggregation function on $[-1,1]$. Moreover, F is a symmetric aggregation function if

$$F(f_1, f_2, ..., f_n) = F\left(f_{\tau(1)}, f_{\tau(2)}, ..., f_{\tau(n)}\right),$$

for all $(f_1, f_2, ..., f_n) \in [-1,1]^n$ and any permutation of indexes τ.
Some properties of the g-BIOWA operator are considered in the following theorem.

Theorem 4.5. *The g-BIOWA operator is an idempotent, positively $\odot$-homogeneous and symmetric aggregation function on $[-1,1]$.*

Proof. (i) (Monotonicity) Let $f, h \in \mathcal{S}_1$, $f \leq h$. Then $g \circ f \leq g \circ h$. The bipolar Choquet integral is increasing [12]. Therefore, we have

$$\begin{aligned} g\text{-BIOWA}(f) &= g^{-1}(BCh(g \circ f, g \circ \mathbf{m})) \\ &\leq g^{-1}(BCh(g \circ h, g \circ \mathbf{m})) \\ &= g\text{-BIOWA}(h), \end{aligned}$$

i.e., g-BIOWA operator is monotonic.

(ii) (Idempotency) Let $f \in \mathcal{S}_1$ be such that it holds $f_1 = f_2 = ... = f_n$. By Proposition 3.2 we have

$$\begin{aligned} g\text{-BIOWA}\,(f) &= \bigoplus_{i=1}^{n} \left(|f_{\alpha(i)}| \ominus |f_{\alpha(i-1)}|\right) \odot \mu_{f^+}(A_i) \\ &= |f_{\alpha(1)}| \odot \mu_{f^+}(A_1) \\ &= |f_{\alpha(1)}| \odot \mathbf{m}(X \cap X_f^{+0},\, X \cap X_f^-). \end{aligned}$$

If $f_{\alpha(1)} \geq 0$ then $\mathbf{m}(X \cap X_f^{+0},\, X \cap X_f^-) = \mathbf{m}(X, \varnothing)$, if $f_{\alpha(1)} < 0$ then $\mathbf{m}(X \cap X_f^{+0},\, X \cap X_f^-) = \mathbf{m}(\varnothing, X)$ and bi-capacity $\mathbf{m}$ is normalized, therefore g-BIOWA $(f) = f$, i.e., g-BIOWA operator is idempotent.

(iii) (Positively $\odot$-homogeneity) Let $f \in \mathcal{S}_1$ and $c \in (0,1]$. Since the bipolar Choquet integral is positively homogeneous and $g(c) > 0$, we have

$$\begin{aligned} g\text{-BIOWA}\,(c \odot f) &= g^{-1}(BCh(g \circ (c \odot f)\,, g \circ \mathbf{m})) \\ &= g^{-1}(BCh(g(c) \cdot g \circ f, g \circ \mathbf{m})) \\ &= g^{-1}(g(c) \cdot BCh(g \circ f, g \circ \mathbf{m})) \\ &= g^{-1}(g(c) \cdot g\left(g^{-1}\left(BCh(g \circ f, g \circ \mathbf{m})\right)\right)) \\ &= c \odot g\text{-BIOWA}\,(f)\,. \end{aligned}$$

(iv) (Symmetry) Let $\tau : \{1, 2, \ldots, n\} \to \{1, 2 \ldots, n\}$ be arbitrary permutation and α be any permutation such that $|f_{\alpha(1)}| \leq |f_{\alpha(2)}| \leq \cdots \leq |f_{\alpha(n)}|$, $f_{\alpha(0)} = 0$, and $A_i = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$ for all i, where $f \in \mathcal{S}_1$. Let β be a permutation such that $(\tau \circ \beta)(i) = \tau(\beta(i)) = \alpha(i)$ and $B_i = \{x_{\tau^{-1}(\alpha(i))}, \ldots, x_{\tau^{-1}(\alpha(n))}\}$ for all i. Then $T^{-1}(A_i) = B_i$ for all i and according to Proposition 3.2, for $f_\tau = (f_{\tau(1)}, \ldots, f_{\tau(n)})$, $f_\tau(x_i) = f(x_{\tau(i)}) = f_{\tau(i)}$ we have

$$\begin{aligned} g\text{-BIOWA}\,(f_\tau) &= \bigoplus_{i=1}^{n} \left(|\left(f_{(\tau\circ\beta)(i)}\right)| \ominus |\left(f_{(\tau\circ\beta)(i-1)}\right)|\right) \odot \mu_{f_\tau^+}(B_i) \\ &= \bigoplus_{i=1}^{n} \left(|f_{\alpha(i)}| \ominus |f_{\alpha(i-1)}|\right) \odot \mu_{f_\tau^+}(B_i)\,. \end{aligned}$$

Due to symmetry of $\mathbf{m}$ it holds

$$\begin{aligned} \mu_{f_\tau^+}(B_i) &= \mathbf{m}(B_i \cap X_{f_\tau}^{+0},\, B_i \cap X_{f_\tau}^-) \\ &= \mathbf{m}(T^{-1}\left(A_i \cap X_f^{+0}\right), T^{-1}\left(A_i \cap X_f^-\right)) \\ &= \mathbf{m}(A_i \cap X_f^{+0},\, A_i \cap X_f^-) \\ &= \mu_{f^+}(A_i)\,, \end{aligned}$$

which implies g-BIOWA $(f_\tau) = g$-BIOWA (f).

□

Remark 4.6. *Let $\mathbf{m} \in \mathcal{M}$ be symmetric and based on a generating function t and let w_i^s be given by (8). Let $f \in \mathcal{S}$, let α be a permutation such that $|f_{\alpha(1)}| \leq |f_{\alpha(2)|} \leq \cdots \leq |f_{\alpha(n)}|$, $A_i = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$, $i = 1, \ldots n$, $A_{n+1} = \varnothing$ and let γ be a permutation such that $f_{\gamma(1)} \leq f_{\gamma(2)} \leq \cdots \leq f_{\gamma(n)}$. Let $\left|A_i \cap X_f^{+0}\right| = a_i$, $\left|A_i \cap X_f^-\right| = b_i$, for all i.*

(i) *Let $t(x,y) = g^{-1}\left(\frac{x-y}{n}\right)$. If $f_{\alpha(i)} \geq 0$, then*

$$\begin{aligned} w_i^s &= t(\left|A_i \cap X_f^{+0}\right|, \left|A_i \cap X_f^-\right|) \ominus t(\left|A_{i+1} \cap X_f^{+0}\right|, \left|A_{i+1} \cap X_f^-\right|) \\ &= t(a_i, b_i) \ominus t(a_{i+1}, b_{i+1}) \\ &= g^{-1}\left(g\left(t\left(a_i, b_i\right)\right) - g\left(t\left(a_i - 1, b_i\right)\right)\right) \\ &= g^{-1}\left(\frac{a_i}{n} - \frac{a_i - 1}{n}\right) \\ &= g^{-1}\left(\frac{1}{n}\right). \end{aligned}$$

If $f_{\alpha(i)} < 0$, *then*

$$\begin{aligned}
w_i^s &= t(\left|A_i \cap X_f^{+0}\right|, \left|A_i \cap X_f^{-}\right|) \ominus t(\left|A_{i+1} \cap X_f^{+0}\right|, \left|A_{i+1} \cap X_f^{-}\right|) \\
&= t(a_i, b_i) \ominus t(a_{i+1}, b_{i+1}) \\
&= g^{-1}(g(t(a_i, b_i)) - g(t(a_i, b_i - 1))) \\
&= g^{-1}\left(-\frac{b_i}{n} + \frac{b_i - 1}{n}\right) \\
&= g^{-1}\left(-\frac{1}{n}\right) \\
&= -g^{-1}\left(\frac{1}{n}\right).
\end{aligned}$$

Further,

$$\begin{aligned}
g\text{-BIOWA}_t(f) &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot w_i^s = \bigoplus_{i=1}^{n} f_{\alpha(i)} \odot g^{-1}\left(\frac{1}{n}\right) \\
&= g^{-1}\left(\sum_{i=1}^{n} g\left(f_{\alpha(i)}\right) \cdot g\left(g^{-1}\left(\frac{1}{n}\right)\right)\right) \\
&= g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g\left(f_{\gamma(i)}\right)\right) \\
&= g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(f_i)\right),
\end{aligned}$$

i.e., g-BIOWA_t *reduces to a quasi-arithmetic mean* [10], *while Example 4.4 (i) is its special case.*

(ii) *Let* $t(x,y) = g^{-1}(l(x) - 1 + l(n-y))$, *where* $l : \{0, 1, \ldots, n\} \to [0,1]$ *is an increasing function satisfying* $l(0) = 0$ *and* $l(n) = 1$. *Let* $\mu, \overline{\mu} : \mathcal{P}(X) \to [0,1]$ *be symmetric capacities such that for each* $A \in \mathcal{P}(X)$ *it holds* $\mu(A) = l(\mathrm{card}(A))$ *and* $\overline{\mu}(A) = 1 - \mu(X \setminus A)$. *If* $f_{\alpha(i)} \geq 0$, *then*

$$\begin{aligned}
w_i^s &= t(\left|A_i \cap X_f^{+0}\right|, \left|A_i \cap X_f^{-}\right|) \ominus t(\left|A_{i+1} \cap X_f^{+0}\right|, \left|A_{i+1} \cap X_f^{-}\right|) \\
&= t(a_i, b_i) \ominus t(a_{i+1}, b_{i+1}) \\
&= g^{-1}(g(t(a_i, b_i)) - g(t(a_i - 1, b_i))) \\
&= g^{-1}(l(a_i) - 1 + l(n - b_i) - l(a_i - 1) + 1 - l(n - b_i)) \\
&= g^{-1}(l(a_i) - l(a_i - 1)).
\end{aligned}$$

If $f_{\alpha(i)} < 0$, *then*

$$\begin{aligned}
w_i^s &= t(\left|A_i \cap X_f^{+0}\right|, \left|A_i \cap X_f^{-}\right|) \ominus t(\left|A_{i+1} \cap X_f^{+0}\right|, \left|A_{i+1} \cap X_f^{-}\right|) \\
&= t(a_i, b_i) \ominus t(a_{i+1}, b_{i+1}) \\
&= g^{-1}(g(t(a_i, b_i)) - g(t(a_i, b_i - 1))) \\
&= g^{-1}(l(a_i) - 1 + l(n - b_i) - l(a_i) + 1 - l(n - b_i + 1)) \\
&= g^{-1}(-l(n - b_i + 1) + l(n - b_i)).
\end{aligned}$$

Thus,

$$\begin{aligned}
g\text{-BIOWA}_t\,(f) &= \bigoplus_{i=1}^{n} |f_{\alpha(i)}| \odot w_i^s \\
&= \bigoplus_{i=1}^{n} f_{\alpha(i)} \odot |w_i^s| \\
&= g^{-1}\left(\sum_{i=1}^{n} g\left(f_{\alpha(i)}\right) \cdot g\left(|w_i^s|\right)\right) \\
&= g^{-1}\left(\sum_{i=1}^{n} g\left(f_{\alpha(i)}^{+}\right) \cdot g\left(|w_i^s|\right) - \sum_{i=1}^{n} g\left(f_{\alpha(i)}^{-}\right) \cdot g\left(|w_i^s|\right)\right) \\
&= g^{-1}\Big(Ch\left((g\circ f)^{+},\mu\right) - Ch\left((g\circ f)^{-},\overline{\mu}\right)\Big) \\
&= g^{-1}\left(\sum_{i=1}^{n} g\left(f_{\gamma(i)}\right) \cdot \left(l(n-i+1) - l(n-i)\right)\right).
\end{aligned}$$

In this case g-BIOWA$_t$ *is an ordered weighted quasi-arithmetic mean* [10, 18, 32] *and for* $g(x) = x$ *it is a standard OWA operator* [31].

Notice that g-BIOWA$_t$ *from Example 4.4 (ii) is its special case.*

Also, it should be stressed that the main properties of these special g-BIOWA *are herited from ordered weighted arithmetic means.*

(iii) *Using the previous consideration, as an application of statistical methods in insurance, we will present some actuarial premium principles defined for discrete insurance risk* $f \in \mathcal{S}$, *also investigated and studied in* [22], *here in terms of bipolar bipolar Choquet* g*-integrals.*

Net premium principle: *If* $g(x) = x$, *and* $\mathbf{m}(A,B) = \mu(A) - \overline{\mu}(B)$, *for all* $(A,B) \in \mathcal{Q}(X)$, *and* $\mu = \overline{\mu} = P$ *is a probability measure then*

$$BCh^{id}(f,\mathbf{m}) = \Pi_N(f),$$

i.e. it coincides with the mathematical expectation $E(f)$ *of a random variable* f.

Mean premium principle*: If* $\mathbf{m}(A,B) = g^{-1}(\mu(A) - \overline{\mu}(B))$, *for all* $(A,B) \in \mathcal{Q}(X)$, *and* $\mu = \overline{\mu} = P$ *is a probability measure then*

$$BCh^{g}(f,\mathbf{m}) = \Pi_M(f),$$

i.e. it coincides with the mean premium principle of a random variable f, *since* g *is odd and strictly increasing.*

CPT-like premium principle: *If* $\mathbf{m}(A,B) = g^{-1}(\mu(A) - \overline{\mu}(B))$, *for all* $(A,B) \in \mathcal{Q}(X)$, *where* $\mu, \overline{\mu} : \mathcal{P}(X) \to [0,1]$ *are capacities,* $\overline{\mu}(A) = 1 - \mu(X \setminus A)$, *for all* $A \in \mathcal{P}(X)$, *then*

$$BCh^{g}(f,\mathbf{m}) = \Pi_{CPT_g}(f) = g^{-1}\Big(Ch\left((g\circ f)^{+},\mu\right) - Ch\left((g\circ f)^{-},\overline{\mu}\right)\Big).$$

5 Induced bipolar ordered weighted averages: IBIOWA operators

Following the notion of the induced Choquet (I-COA) aggregation [10, 18, 28] we will define the induced BIOWA operator. This operator is based on the order-inducing variables u_i, i.e. the order-inducing function $u : X \to \mathbb{R}$. These variables determine the position of inputs f_i, i.e., the position of the absolute values of components of the input function $f : X \to \mathbb{R}$.

For $u \in \mathcal{S}$, let σ be a permutation such that $|u_{\sigma(1)}| \le |u_{\sigma(2)}| \le \cdots \le |u_{\sigma(n)}|$. Additionally, let σ be such that, besides the previous one, the next condition is satisfied: if $|u_{\sigma(i)}| = |u_{\sigma(j)}|$ for some $i < j$, then either sign $u_{\sigma(i)} = -1$ and sign $u_{\sigma(j)} = 1$, or sign $u_{\sigma(i)}$= sign $u_{\sigma(j)}$ and $\sigma(i) < \sigma(j)$. Let $A_i = \{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\}$ and for $u \in \mathcal{S}$ denote

$$X_u^{+0} = \{x_i \in X \mid u_i \ge 0\}, \ X_u^{-} = \{x_i \in X \mid u_i < 0\}.$$

Assume a function $s^u : X \to \{(1,0),(0,1)\}$ defined by

$$s^u(x_i) = \begin{cases} (1,0)\,, \ x_{\sigma(i)} \in X_u^{+0}, \\ (0,1)\,, \ x_{\sigma(i)} \in X_u^{-}. \end{cases} \tag{9}$$

Let

$$w_i^{s^u} = t\left(\sum_{j=i}^{n} s^u(x_j)\right) - t\left(\sum_{j=i+1}^{n} s^u(x_j)\right), \tag{10}$$

where $\sum_{j=n+1}^{n} s^u(x_j) = (0,0)$ and t is a generating function of a symmetric bi-capacity $\mathbf{m} \in \mathcal{M}$. The vector $w_{s^u} = \left(w_1^{s^u}, \dots, w_n^{s^u}\right)$ is a weighting vector and $w_i^{s^u}$ are weights. Obviously, for $u \geq 0$ or $u < 0$ the weighting vector w_{s^u} fulfils $\sum\limits_{i=1}^{n} |w_i^{s^u}| = 1$. For each $i \in \{1, \dots, n\}$, let

$$K_i = \{x_{\sigma(j)} \in X \mid |u_{\sigma(i)}| = |u_{\sigma(j)}|\}.$$

Denote $K_i^{+0} = K_i \cap X_u^{+0}$, $K_i^{-} = K_i \setminus K_i^{+0}$.

As it has been suggested in [3, 21] for IOWA operators, in order to obtain a bipolar weighting vector suitable for well-defining IBIOWA, for given $u \in \mathcal{S}$, if $|u_{\sigma(i)}| = |u_{\sigma(j)}|$ for some $i < j$, we shell modify the weights on tied positions in accordance with Definition 2.6.

Now, let $f \in \mathcal{S}$. Define a bipolar weighting vector $w = (w_1, w_2, \dots, w_n)$ induced by $u \in \mathcal{S}$:

$$w_i = \begin{cases} \frac{1}{\operatorname{card}(K_i^{+0} \cap X_f^{+0})} \sum\limits_{x_{\sigma(j)} \in K_i^{+0} \cap X_f^{+0}} w_j^{s^u}, & x_{\sigma(i)} \in X_f^{+0} \cap K_i^{+0}, \\ \frac{1}{\operatorname{card}(K_i^{-} \cap X_f^{+0})} \sum\limits_{x_{\sigma(j)} \in K_i^{-} \cap X_f^{+0}} (-w_j^{s^u}), & x_{\sigma(i)} \in X_f^{+0} \cap K_i^{-}, \\ \frac{1}{\operatorname{card}(K_i^{+0} \cap X_f^{-})} \sum\limits_{x_{\sigma(j)} \in K_i^{+0} \cap X_f^{-}} (-w_j^{s^u}), & x_{\sigma(i)} \in X_f^{-} \cap K_i^{+0}, \\ \frac{1}{\operatorname{card}(K_i^{-} \cap X_f^{-})} \sum\limits_{x_{\sigma(j)} \in K_i^{-} \cap X_f^{-}} w_j^{s^u}, & x_{\sigma(i)} \in X_f^{-} \cap K_i^{-}. \end{cases} \tag{11}$$

It is said that vectors $f = (f_1, f_2, \dots, f_n)$ and $h = (h_1, h_2, \dots, h_n)$ are *cosigned* (*well tied* or *sign tied*) if $f_i \cdot h_i \geq 0$ for all $i = 1, \dots, n$. The vectors f and h are *strictly cosigned* if they are cosigned and for all i it holds $f_i \geq 0$ if and only if $h_i \geq 0$.

Definition 5.1. *Let* $\mathbf{m} \in \mathcal{M}$ *be a symmetric bi-capacity generated by a generator* t. *Let* $f \in \mathcal{S}$ *be an input function and* $u \in \mathcal{S}$ *be an order inducing function. An operator* $\mathrm{IBIOWA}_t : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$ *is defined by*

$$\mathrm{IBIOWA}_t(\langle u, f\rangle) = \sum_{i=1}^{n} |f_{\sigma(i)}| \cdot w_i,$$

where $w = (w_1, w_2, \dots, w_n)$ *is a weighting vector,* $\sum\limits_{i=1}^{n} |{w_i}^{s^u}| = 1$, *obtained by* (9), (10) *and* (11). *The* IBIOWA_t *operator is called an Induced Bipolar Ordered Weighted Average (an* IBIOWA *operator).*

The properties of an IBIOWA operator are observed in the following theorem.

Theorem 5.2. *For a fixed order-inducing function* $u \in \mathcal{S}_1$ *such that* $\sum\limits_{i=1}^{n} |w_i^{s^u}| = 1$, *the* IBIOWA *operator is an idempotent and positively homogeneous operator on* $[-1, 1]$. *Moreover, if we denote* $\mathcal{U} = \{f \in \mathcal{S}_1; f$ *is strictly cosigned with* $u \in \mathcal{S}_1\}$, *then for all* $f, h \in \mathcal{U}$ *it holds*

$$f \leq h \ \Rightarrow\ \mathrm{IBIOWA}(\langle u, f\rangle) \leq \mathrm{IBIOWA}(\langle u, h\rangle).$$

Proof. Let t be a generating function of a symmetric bi-capacity. For a fixed function $u \in \mathcal{S}_1$, let σ be a permutation such that $\left|u_{\sigma(1)}\right| \leq \left|u_{\sigma(2)}\right| \leq \dots \leq \left|u_{\sigma(n)}\right|$ and if $|u_{\sigma(i)}| = |u_{\sigma(j)}|$ for some $i < j$ then either sign $u_{\sigma(i)} = -1$ and sign $u_{\sigma(j)} = 1$, or sign $u_{\sigma(i)}$= sign $u_{\sigma(j)}$ and $\sigma(i) < \sigma(j)$ and $\sum\limits_{i=1}^{n} |w_i^{s^u}| = 1$, obtained by (9), (10) and (11).

(i) (Idempotency) Let $f \in \mathcal{S}_1$ be such that it holds $f_1 = f_2 = ... = f_n$. We have

$$\begin{aligned} \mathrm{IBIOWA}(\langle u, f\rangle) &= \sum_{i=1}^{n} |f_{\sigma(i)}| \cdot w_i \\ &= f_1 \cdot \sum_{i=1}^{n} |{w_i}^{s^u}|, \end{aligned}$$

where $\sum\limits_{i=1}^{n} |w_i^{s^u}| = 1$, therefore IBIOWA $(\langle u, f\rangle) = f_1$, i.e., IBIOWA operator is idempotent.

(ii) (Positively homogeneity) Let $f \in \mathcal{S}_1$ and $c \in (0, 1]$. Since

$$X_f^{+} = \{x_i \in X \mid f(x_i) > 0\} = \{x_i \in X \mid (c \cdot f)(x_i) > 0\} = X_{c \cdot f}^{+},$$

$$X_f^{-} = \{x_i \in X \mid f(x_i) < 0\} = \{x_i \in X \mid (c \cdot f)(x_i) < 0\} = X_{c \cdot f}^{-},$$

$$X_f^{0} = \{x_i \in X \mid f(x_i) = 0\} = \{x_i \in X \mid (c \cdot f)(x_i) = 0\} = X_{c \cdot f}^{0},$$

we have

$$\begin{aligned} \text{IBIOWA}\left(\langle u, c \cdot f\rangle\right) &= \sum_{i=1}^{n} |(c \cdot f)_{\sigma(i)}| \cdot w_i \\ &= c \cdot \text{IBIOWA}\left(\langle u, f\rangle\right). \end{aligned}$$

(iii) (Monotonicity on $\mathcal{U}$) Assume $f, h \in \mathcal{U}$, $f \le h$. Let $i \in \{1, \ldots, n\}$. It holds $f_{\sigma(i)} \ge 0$ or $f_{\sigma(i)} < 0$. Let us remark that, since both, f and h are strictly cosigned with u, also f and h are strictly cosigned.

$(*)$ First, assume that $h_{\sigma(i)} \ge f_{\sigma(i)} \ge 0$, then $x_{\sigma(i)} \in K_i^{+0}$. For all $x_{\sigma(j)} \in K_i^{+0}$ it holds $w_j^{s^u} \ge 0$, where $w_j^{s^u}$ are given by (10). Therefore, since f and h are strictly cosigned, we obtain

$$\begin{aligned} w_i &= \frac{1}{\operatorname{card}(K_i^{+0} \cap X_f^{+0})} \sum_{x_{\sigma(j)} \in K_i^{+0} \cap X_f^{+0}} w_j^{s^u} \\ &= \frac{1}{\operatorname{card}(K_i^{+0} \cap X_h^{+0})} \sum_{x_{\sigma(j)} \in K_i^{+0} \cap X_h^{+0}} w_j^{s^u} \ge 0. \end{aligned}$$

$(**)$ If $f_{\sigma(i)} \le h_{\sigma(i)} < 0$, then $x_{\sigma(i)} \in K_i^{-}$. For all $x_{\sigma(j)} \in K_i^{-}$ it holds $w_j^{s^u} \le 0$. Therefore, in this case, since f and h are strictly cosigned, we have

$$\begin{aligned} w_i &= \frac{1}{\operatorname{card}(K_i^{-} \cap X_f^{-})} \sum_{x_{\sigma(j)} \in K_i^{-} \cap X_f^{-}} w_j^{s^u} \\ &= \frac{1}{\operatorname{card}(K_i^{-} \cap X_h^{-})} \sum_{x_{\sigma(j)} \in K_i^{-} \cap X_h^{-}} w_j^{s^u} \le 0, \end{aligned}$$

By $(*)$ and $(**)$ for strictly cosigned $f, h \in \mathcal{U}$, such that $f \le h$, clearly, for each i, it holds $|f_{\sigma(i)}| \cdot w_i \le |h_{\sigma(i)}| \cdot w_i$. Finally, taking the sum over i, we obtain IBIOWA$(\langle u, f\rangle) \le$ IBIOWA$(\langle u, h\rangle)$.

□

Proposition 5.3. *For an arbitrary permutation* $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *and for all* $f, u \in \mathcal{S}$ *it holds*

$$\text{IBIOWA}\left(\langle u_\tau, f_\tau\rangle\right) = \text{IBIOWA}\left(\langle u, f\rangle\right).$$

Proof. Let $u = (u_1, ..., u_n)$ and $u_\tau = \left(u_{\tau(1)}, ..., u_{\tau(n)}\right)$ be the vectors of order-inducing values for an arbitrary permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$. For the permutation σ such that $|u_{\sigma(1)}| \le |u_{\sigma(2)}| \le \cdots \le |u_{\sigma(n)}|$ and if $|u_{\sigma(i)}| = |u_{\sigma(j)}|$ for some $i < j$ then either sign $u_{\sigma(i)} = -1$ and sign $u_{\sigma(j)} = 1$, or sign $u_{\sigma(i)} =$ sign $u_{\sigma(j)}$ and $\sigma(i) < \sigma(j)$, with $\sum\limits_{i=1}^{n} |w_i^{s^u}| = 1$ and $A_i = \{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\}$, $A_{n+1} = \varnothing$, there exists a permutation β such that $|u_{\tau(\beta(1))}| \le |u_{\tau(\beta(2))}| \le \cdots \le |u_{\tau(\beta(n))}|$ and $B_i = \{x_{\beta(i)}, \ldots, x_{\beta(n)}\}$, $B_{n+1} = \varnothing$, such that it holds $\sigma(i) = \tau\left(\beta(i)\right)$, and $T^{-1}(A_i) = B_i$, for all i. Therefore, due to the symmetry of $\mathbf{m}$, for each $f \in \mathcal{S}$, if K_i is a singleton set for each i, we have

$$\begin{aligned}
\text{IBIOWA}\left(\langle u_\tau, f_\tau \rangle\right) &= \sum_{i=1}^{n} f_{\tau(\beta(i))} \cdot |w_i^{s^{u_\tau}}| \\
&= \sum_{i=1}^{n} f_{\tau(\beta(i))} \cdot |\mu_{u_\tau^+}\left(B_i\right) - \mu_{u_\tau^+}\left(B_{i+1}\right)| \\
&= \sum_{i=1}^{n} f_{\sigma(i)} \cdot |\mu_{u^+}\left(A_i\right) - \mu_{u^+}\left(A_{i+1}\right)| \\
&= \sum_{i=1}^{n} |f_{\sigma(i)}| \cdot w_i \\
&= \text{IBIOWA}\left(\langle u, f \rangle\right).
\end{aligned}$$

In the other cases, when there are ties in u, the proof is similar, by using (11). $\square$

We conclude with the following facts.

- Let $t\left(x,y\right) = l(x) - 1 + l\left(n-y\right)$, where $l : \{0,1,\ldots,n\} \to [0,1]$ is an increasing function satisfying $l(0) = 0$ and $l\left(n\right) = 1$.

  If $u \geq 0$, with no ties, then for all i

  $$\begin{aligned}
  w_i^{s^u} &= t\left(n-i+1,0\right) - t\left(n-i,0\right) \\
  &= l(n-i+1) - l(n-i),
  \end{aligned}$$

  therefore, for $f \geq 0$

  $$\begin{aligned}
  \text{IBIOWA}_t\left(\langle u, f \rangle\right) &= \sum_{i=1}^{n} f_{\sigma(i)} \cdot w_i \\
  &= \sum_{i=1}^{n} f_{\sigma(i)} \cdot \left(l(n-i+1) - l(n-i)\right).
  \end{aligned}$$

  In this case, IBIOWA_t is $\text{IOWA}_{\mathbf{v}}$ operator acting on $[0,1]$ with the weights $v_i = w_i^{s^u} \geq 0$ such that $\sum_{i=1}^{n} v_i = 1$, and for $u = f$ it is a standard OWA operator [21].

Example 5.4. Let $X = \{1,2,3,4,5\}$, $t\left(x,y\right) = l(x) - 1 + l\left(5-y\right)$, where $l\left(x\right) = \left(\frac{x}{5}\right)^3$, for $x \in \{0,1,2,3,4,5\}$, and the vector of order-inducing values $u = \left(0.3,-0.2,0.2,0.2,-0.1\right)$. Therefore $X_u^{+0} = \{1,3,4\}$, $X_u^- = \{2,5\}$. For $\sigma = (5,2,3,4,1)$, it holds $|u_5| \leq |u_2| \leq |u_3| \leq |u_4| \leq |u_1|$. We have

$$a_3 = |\{1,3,4\}| = 3, \quad a_4 = |\{1,4\}| = 2, \quad a_5 = |\{1\}| = 1,$$

$$b_1 = |\{2,5\}| = 2, \quad b_2 = |\{2\}| = 1,$$

and

$$\begin{aligned}
w_1^{s^u} &= \left(\frac{5-b_1}{5}\right)^3 - \left(\frac{5-b_1+1}{5}\right)^3 = -\frac{37}{125}, \\
w_2^{s^u} &= \left(\frac{5-b_2}{5}\right)^3 - \left(\frac{5-b_2+1}{5}\right)^3 = -\frac{61}{125}, \\
w_3^{s^u} &= \left(\frac{a_3}{5}\right)^3 - \left(\frac{a_3-1}{5}\right)^3 = \frac{19}{125}, \\
w_4^{s^u} &= \left(\frac{a_4}{5}\right)^3 - \left(\frac{a_4-1}{5}\right)^3 = \frac{7}{125}, \\
w_5^{s^u} &= \left(\frac{a_5}{5}\right)^3 - \left(\frac{a_5-1}{5}\right)^3 = \frac{1}{125}.
\end{aligned}$$

Notice that it holds $\sum_{i=1}^{5}\left|w_i^{s^u}\right| = \frac{37}{125} + \frac{61}{125} + \frac{19}{125} + \frac{7}{125} + \frac{1}{125} = 1.$

(i) Let f be given by

$$f(1) = 0.2,\ f(2) = -0.6,\ f(3) = 0.8,\ f(4) = 0.1,\ f(5) = -0.4,$$

f and u are strictly cosigned. We observe five pairs:

$$\langle 0.3, 0.2\rangle, \langle -0.2, -0.6\rangle, \langle 0.2, 0.8\rangle, \langle 0.2, 0.1\rangle, \langle -0.1, -0.4\rangle,$$

where the first component is the order-inducing variable. Let we observe the following reordering

$$\langle -0.1, -0.4\rangle, \langle -0.2, -0.6\rangle, \langle 0.2, 0.8\rangle, \langle 0.2, 0.1\rangle, \langle 0.3, 0.2\rangle,$$

induced by the vector of order-inducing values u. There exists the ties between $\langle -0.2, -0.6\rangle$, $\langle 0.2, 0.8\rangle$ and $\langle 0.2, 0.1\rangle$. By (11) we have

$$w_1 = -\frac{37}{125},\quad w_2 = -\frac{61}{125},\quad w_3 = \frac{13}{125},\quad w_4 = \frac{13}{125},\quad w_5 = \frac{1}{125}.$$

Using the definition of IBIOWA operator we obtain the following result:

$$\begin{aligned}\mathrm{IBIOWA}_t(\langle u, f\rangle) &= -0.4\cdot\frac{37}{125} - 0.6\cdot\frac{61}{125} + 0.8\cdot\frac{13}{125} + 0.1\cdot\frac{13}{125} + 0.2\cdot\frac{1}{125}\\ &= -0.316.\end{aligned}$$

(ii) Let h be given by

$$h(1) = 0.2,\ h(2) = -0.8,\ h(3) = 0.4,\ h(4) = 0.1,\ h(5) = -0.7,$$

h and u are strictly cosigned. We observe the following pairs:

$$\langle 0.3, 0.2\rangle, \langle -0.2, -0.8\rangle, \langle 0.2, 0.4\rangle, \langle 0.2, 0.1\rangle, \langle -0.1, -0.7\rangle,$$

where the first component is the order-inducing variable. Let us observe the following reordering

$$\langle -0.1, -0.7\rangle, \langle -0.2, -0.8\rangle, \langle 0.2, 0.4\rangle, \langle 0.2, 0.1\rangle, \langle 0.3, 0.2\rangle,$$

induced by vector u. Due to the ties between $\langle -0.2, -0.8\rangle$, $\langle 0.2, 0.4\rangle$ and $\langle 0.2, 0.1\rangle$, we have

$$w_1 = -\frac{37}{125},\quad w_2 = -\frac{61}{125},\ w_3 = \frac{13}{125},\ w_4 = \frac{13}{125},\ w_5 = \frac{1}{125}.$$

We obtain the following result:

$$\begin{aligned}\mathrm{IBIOWA}_t(\langle u, h\rangle) &= -0.7\cdot\frac{37}{125} - 08\cdot\frac{61}{125} + 0.4\cdot\frac{13}{125} + 0.1\cdot\frac{13}{125} + 0.2\cdot\frac{1}{125}\\ &= -0.544.\end{aligned}$$

(iii) Let p be given by

$$p(1) = 1,\ p(2) = -0.1,\ p(3) = 1,\ p(4) = 0.3,\ p(5) = -0.4,$$

we observe the following pairs:

$$\langle 0.3, 1\rangle, \langle -0.2, -0.1\rangle, \langle 0.2, 1\rangle, \langle 0.2, 0.3\rangle, \langle -0.1, -0.4\rangle,$$

where the first component is the order-inducing variable. Let we observe the following reordering

$$\langle -0.1, -0.4\rangle, \langle -0.2, -0.1\rangle, \langle 0.2, 1\rangle, \langle 0.2, 0.3\rangle, \langle 0.3, 1\rangle,$$

induced by u. Due to the ties between $\langle -0.2, -0.1\rangle$, $\langle 0.2, 1\rangle$ and $\langle 0.2, 0.3\rangle$, we obtain by (11)

$$w_1 = -\frac{37}{125},\ w_2 = -\frac{61}{125},\ w_3 = \frac{13}{125},\ w_4 = \frac{13}{125},\ w_5 = \frac{1}{125}.$$

We obtain the following result:

$$\begin{aligned}\mathrm{IBIOWA}_t(\langle u, p\rangle) &= -0.4\cdot\frac{37}{125} - 0.1\cdot\frac{61}{125} + 1\cdot\frac{13}{125} + 0.3\cdot\frac{13}{125} + 1\cdot\frac{1}{125}\\ &= -0.024.\end{aligned}$$

Hence, we get that for the functions f, h, p satisfying $h \le f \le p$ it holds

$$\mathrm{IBIOWA}_t(\langle u, h\rangle) \le \mathrm{IBIOWA}_t(\langle u, f\rangle) \le \mathrm{IBIOWA}_t(\langle u, p\rangle).$$

6 Conclusions

As a generalisation of recently introduced BIOWA operators in [20], we have introduced g-BIOWA operators defined with respect to an odd, strictly increasing and continuous function g. These operators are also an extension of GOWA operators defined on $[0, 1]$ with non-negative weights. Secondly, we have introduced IBIOWA operators, as extensions of IOWA operators defined on $[0, 1]$ with non-negative weights. The both operators extend the OWA operators with non-negative weights, considering non-negative inputs. The theoretical results regarding the main properties of g-BIOWA and IBIOWA operators have been obtained. In the future work, further investigations on numerical characteristics of BIOWA and on generalisations of IBIOWA operators will be provided. Also, we intend to introduce bipolarity on bounded lattices replacing bipolar Choquet integral by bipolar Sugeno integral.

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