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Geometric clustering fuzzy regression based on c-means clustering

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Abstract

In this paper, we present an approach to fit some clustering fuzzy linear regression models based on the fuzzy response variable and fuzzy parameters. In this approach, we first introduce a method for clustering crisp/fuzzy data based on fuzzy c-means clustering, and then, we fit some clustering fuzzy regression models based on the geometric mean. The optimal clustering fuzzy regression models are evaluated under two indices of goodness of fit. The applications of the proposed approach are studied in modeling some real data sets.

Keywords: C-means clustering, fuzzy data, fuzzy regression, geometric mean, goodness of fit.

1 Introduction

An important problem in statistics for formulating and evaluating some models between a dependent variable and some independent variables is the regression analysis. In real world, sometimes, we encounter with some imprecise quantities in available data. In such situations, we wish to fit and evaluate some fuzzy regression models in imprecise (fuzzy) environment. Some recent works on the topic of fuzzy regression are reviewed as follows.

The analysis of fuzzy linear regression model based on least absolutes method is investigated by Kelkinnama and Taheri [24], and Taheri and Kelkinnama [27]. Chen and Nien [13] and Nasibov [26] studied some approaches to model the least squares linear regression based on fuzzy input-output data. Chachi [9], Hesamian and Akbari [17], and Mohammadi and Taheri [25] investigated the least squares fuzzy regression for crisp input-fuzzy output data. Bas and Egrioglu [8], Dotto et al. [14], Yang and Ko [29], and Yang and Lin [30] investigated some approaches to model the fuzzy linear regression based on fuzzy clustering procedures. The regression analysis in intuitionistic/type-2 fuzzy environment studied by Akbari and Hesamian [1], Arefi [2], Arefi and Taheri [6], Hosseinzadeh et al. [18, 19]. The problem of robust estimations for parameters of the linear regression models in imprecise environments are studied by Arefi [3], Arefi and Khammar [4], Asadolahi et al. [7], Chachi and Roozbeh [11], and Khammar et al. [21]. Also, some other methods for analyzing the linear regression models in imprecise environments are investigated by Chachi et al. [10], Chachi and Taheri [12], Hesamian and Akbari [16], Jung et al. [20], Khammar et al. [23], and Zeng et al. [31].

In classical regression analysis, it is supposed that the available data are homogeneous and the parameters of model are as precise quantities. In practice, we should formulate and analyze the regression models in the case of heterogeneity of observations and to exist imprecise quantities. In this paper, the main goal is to apply the new clustering techniques to fuzzy linear regression analysis, when the response variable and the parameters of model are supposed to be as the fuzzy numbers. In this approach, we first introduce a method to cluster the heterogeneous data set based on fuzzy c-means clustering, and then, the parameters of clustering fuzzy regression models are estimated based on an objective function under the geometric mean.

This paper is organized as follows. In Section 2, we first introduce the fuzzy c-means clustering on a crisp heterogeneous observations, and then, we present a new approach to fit some clustering linear regression models based on geometric mean. The extension of approach proposed in Section 2 to fit some geometric clustering fuzzy linear

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regression models based on the fuzzy response variable and fuzzy parameters is investigated in Section 3. The applications of proposed approach are studied on two real data sets (application in soil science and application in effect of the composition Portland cement) in Section 4. Finally, in Section 5, some concluding remarks are provided.

Let us first recall some notations and preliminary concepts of fuzzy sets (see Zimmermann [32]).

Let X be a universal set. A fuzzy set $\widetilde{A}$ on X is defined by a membership function $\widetilde{A} : X \to [0,1]$. The α-cut of $\widetilde{A}$ is given as $\widetilde{A}[\alpha] = \{x \mid \widetilde{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$.

Definition 1.1. *The fuzzy set $\widetilde{A}$ on X is called a fuzzy number if*
(i) $\widetilde{A}(x) = 1$ *for some* $x \in X$,
(ii) $\widetilde{A}[\alpha]$ *is a closed bounded interval for* $0 < \alpha \leq 1$.

Definition 1.2. *A fuzzy number $\tilde{A}$ is called an LR fuzzy number, if the membership function is defined as follows*

$$\tilde{A}(x) = \begin{cases} L\left(\frac{a-x}{l_a}\right) & x \leq a, \\ \\ R\left(\frac{x-a}{r_a}\right) & a < x, \end{cases}$$

where, $L(.)$ and $R(.)$ are strictly decreasing functions from R^+ to $[0,1]$, and $L(0) = R(0) = 1$. $L(.)$ and $R(.)$ are called the reference functions. The LR fuzzy number is denoted by $\tilde{A} = (a; l_a, r_a)_{LR}$.

The triangular fuzzy number is an special case of LR fuzzy numbers with $L(x) = R(x) = \max\{0, 1-x\}$ for all $x \in [0,1]$. It is denoted by $\tilde{A} = (a; l_a, r_a)_T$. Also, if $l_a = r_a$, then it is called a symmetric triangular fuzzy number as $\tilde{A} = (m; l_a)_T$.

Proposition 1.3. *Let $\widetilde{M} = (m; l_m, r_m)_{LR}$ and $\widetilde{N} = (n; l_n, r_n)_{LR}$ be two LR fuzzy numbers and $\lambda \in R - \{0\}$. Then, some arithmetic operations on LR fuzzy numbers based on Extension Principle are as follows*

$$\widetilde{M} \oplus \widetilde{N} = (m+n; l_m + l_n, r_m + r_n)_{LR},$$

$$\lambda \otimes \widetilde{M} = \begin{cases} (\lambda m; \lambda l_m, \lambda r_m)_{LR} & \lambda \geq 0, \\ \\ (\lambda m; -\lambda r_m, -\lambda l_m)_{RL} & \lambda < 0. \end{cases}$$

2 Geometric clustering regression based on crisp data

Suppose that we have a crisp data set. In this section, we first introduce a method to cluster this data set, and then, we present a new approach to fit a clustering regression model.

2.1 Fuzzy c-means clustering on crisp data (FCMCC)

We first recall a distance between two data sets, and then we introduce a clustering method based on fuzzy c-means algorithm (see Gustafson and Kessel [15]).

Definition 2.1. *Let $X_{k\times 1} = (x_1, x_2, ..., x_k)' \in R^k$ and $Y_{k\times 1} = (y_1, y_2, ..., y_k)' \in R^k$ be two vectors of data set. Then, the distance between X and Y is defined as follows*

$$d(X,Y) = \left[(X-Y)'A(X-Y)\right]^{1/2},$$

where $A_{k\times k}$ is the symmetric and positive-definite matrix.

Theorem 2.2. *Let X, Y, and Z be three vectors in R^k. Then, (R,d) is a metric space. It satisfies the following properties*

(i) $d(X,Y) \geq 0$,

(ii) $d(X,Y)=0 \quad \Leftrightarrow \quad X=Y$,

(iii) $d(X,Y)=d(Y,X)$,

(iv) $d(X,Y)\leq d(X,Z)+d(Z,Y)$.

Now, let $G=\{X_1,...,X_n\}=\{(x_{11},...,x_{k1})',...,(x_{1n},...,x_{kn})'\}$ be a set of n vectors. Let $1<c<n$ be a constant integer. A partition of G into c parts can be represented by the membership functions $\mu_{ij}=\mu_i(X_j)$ such that $\mu_{ij}\in[0,1]$ and $\sum_{i=1}^{c}\mu_{ij}=1$ for all $j=1,2,...,n$ in G. Here, μ_{ij} is the membership degree of j^{th} observation into i^{th} cluster.

For estimating the parameters of FCMCC, the objective function is introduced as follows (see Gustafson and Kessel [15]):

$$L_1=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^m d^2(X_j,v_i)=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^m(X_j-v_i)'A_i(X_j-v_i),$$

where, $m\geq 1$ is the degree of fuzziness, $v_i\in R^k,\ i=1,...,c$ are called the cluster centers, and $A_i,\ i=1,...,c$ are the symmetric and positive-definite $k\times k$ matrices with $det(A_i)=\rho_i$.
To minimize the objective function L_1 under μ_{ij} and v_i, we have

$$v_i = \frac{\sum_{j=1}^{n}\mu_{ij}^m X_j}{\sum_{j=1}^{n}\mu_{ij}^m}, \qquad i=1,...,c, \tag{1}$$

$$\mu_{ij} = \Big(\sum_{k=1}^{c}\frac{(d^2(X_j,v_i))^{\frac{1}{m-1}}}{(d^2(X_j,v_k))^{\frac{1}{m-1}}}\Big)^{-1}. \tag{2}$$

Also, in the repeated FCMCC algorithm, we suppose A_i is given as follows:

$$A_i=(\rho_i\det(S_i))^{1/k}S_i^{-1}, \qquad i=1,...,c, \tag{3}$$

where $S_i=\sum_{j=1}^{n}\mu_{ij}^m(X_j-v_i)(X_j-v_i)'$.

Algorithm 1: FCMCC

1- First, we consider some the initial values for the membership degrees μ_{ij}^0 and $A_i^0=I_{k\times k}$.

2- Based on μ_{ij}^0, we calculate the cluster centers $v_i^0,\ i=1,...,c$ under the relation (1).

3- Based on v_i^0, A_i^0 and the relation (2), we update the membership degrees, denoted by μ_{ij}^1.

4- Based on v_i^0 and μ_{ij}^1, we update A_i based on the relation (3), denoted by A_i^1.

5- If $\max|\mu_{ij}^1-\mu_{ij}^0|<\varepsilon$ (ε is a small value), then the algorithm is stopped, otherwise return to step 2 and repeat the steps of algorithm.

Example 2.3. *Consider the data set $X_j=(x_j,y_j)',\ j=1,...,10$ in second column of Table 1. Now, we cluster this data set in $c=2$ clusters. The cluster centers and the membership degrees are obtained based on Algorithm 1 with $\varepsilon=0.0001$ and $c=2$. The results are summarized in third column of Table 1.*

Example 2.4. *Consider the data set $X_j=(x_j,y_j)',\ j=1,...,20$ in second and 5^{th} columns of Table 2. Based on Algorithm 1 with $\varepsilon=0.0001$, the results of clustering with $c=2$ are given in third and 6^{th} of Table 2.*

2.2 Geometric clustering regression based on crisp data

Assume that a crisp data set is observed as $(x_{1j},x_{2j},...,x_{kj},Y_j),\ j=1,2,...,n$. The aim is to fit some clustering regression models to this data set, as follows

$$\hat{Y}_j^i=\hat{a}_{0i}+\hat{a}_{1i}x_{1j}+...+\hat{a}_{ki}x_{kj}, \qquad i=1,...,c,\ \ j=1,...,n,$$

Table 1: Data set and the membership degrees in Example 2.3

No.	$X'_j = (x_j, y_j)$	(μ_{1j}, μ_{2j})
1	(9,3)	$(0.0184, 0.9816)$
2	(11,3)	$(0.0332, 0.9668)$
3	(9,1.5)	$(0.0329, 0.9671)$
5	(11,1.5)	$(0.0222, 0.9778)$
5	(5,3)	$(0.2733, 0.7267)$
6	(1,8)	$(0.9790, 0.0210)$
7	(3,8)	$(0.9690, 0.0310)$
8	(2,7)	$(0.9999, 0.0001)$
9	(1,6)	$(0.9464, 0.0536)$
10	(3,6)	$(0.9626, 0.0374)$
		$v_1 = (2.0542, 6.9504)'$
		$v_2 = (9.3767, 2.3484)'$

Table 2: Data set and the membership degrees in Example 2.4

No.	$X'_j = (x_j, y_j)$	(μ_{1j}, μ_{2j})	No.	$X'_j = (x_j, y_j)$	(μ_{1j}, μ_{2j})
1	$(1, 11)$	$(0.9988, 0.0012)$	11	$(21, 5)$	$(0.0015, 0.9985)$
2	$(2, 13)$	$(0.9983, 0.0017)$	12	$(22, 8)$	$(0.0012, 0.9988)$
3	$(3, 21)$	$(0.9996, 0.0004)$	13	$(23, 18)$	$(0.0008, 0.9992)$
4	$(4, 29)$	$(0.9990, 0.0010)$	14	$(24, 19)$	$(0.0005, 0.9995)$
5	$(5, 31)$	$(1.0000, 0.0000)$	15	$(25, 24)$	$(0.0006, 0.9994)$
6	$(6, 34)$	$(0.9997, 0.0003)$	16	$(26, 35)$	$(0.0005, 0.9995)$
7	$(7, 45)$	$(0.9981, 0.0019)$	17	$(27, 44)$	$(0.0032, 0.9968)$
8	$(8, 47)$	$(0.9994, 0.0006)$	18	$(28, 45)$	$(0.0007, 0.9993)$
9	$(9, 48)$	$(0.9985, 0.0015)$	19	$(29, 48)$	$(0.0010, 0.9990)$
10	$(10, 54)$	$(0.9984, 0.0016)$	20	$(30, 50)$	$(0.0027, 0.9973)$
					$v_1 = (5.4994, 33.2968)'$
					$v_2 = 25.4983, 29.5895)'$

where $(a_{0i}, a_{1i}, ..., a_{ki})$ are the parameters of i^{th} clustering regression model. For estimating the parameters of the clustering regression models, the objective function is defined based on geometric mean as follows:

$$H_1 = \sum_{j=1}^{n} \prod_{i=1}^{c} \left(E_{ij}^2\right)^{\mu_{ij}^m} = \sum_{j=1}^{n} \Big((E_{1j}^2)^{\mu_{1j}^m} \times (E_{2j}^2)^{\mu_{2j}^m} \times ... \times (E_{cj}^2)^{\mu_{cj}^m} \Big),$$

where $m \geq 1$ is the fuzziness constant (in this paper, $m = 2$), μ_{ij} is obtained based on Algorithm 1, and E_{ij} is

$$E_{ij} = Y_j - \hat{Y}_j^i = Y_j - \hat{a}_{0i} - \hat{a}_{1i} x_{1j} - ... - \hat{a}_{ki} x_{kj}. \tag{4}$$

For minimizing the objective function H_1 with respect to c-partitions, we have

$$\frac{\partial H_1}{\partial \hat{a}_{pi}} = \sum_{j=1}^{n} \Big[-2 \frac{\mu_{ij}^m x_{pj}}{E_{ij}} \prod_{i=1}^{c} \left(E_{ij}^2\right)^{\mu_{ij}^m} \Big] = \sum_{j=1}^{n} \Big[-2 \frac{\mu_{ij}^m x_{pj}}{E_{ij}} Prod_j \Big], \tag{5}$$

where $Prod_j = \prod_{i=1}^{c} \left(E_{ij}^2\right)^{\mu_{ij}^m}$. Also, the second derivatives are calculated as follows:

$$\frac{\partial^2 H_1}{\partial \hat{a}_{pi} \partial \hat{a}_{p^* i^*}} = \begin{cases} \sum_{j=1}^{n} \Big[\frac{(2\mu_{ij}^m)(2\mu_{ij}^m - 1) x_{pj} x_{p^* j}}{E_{ij}^2} Prod_j \Big] & \text{if } i = i^*, \\ \sum_{j=1}^{n} \Big[\frac{(2\mu_{ij}^m)(2\mu_{i^* j}^m) x_{pj} x_{p^* j}}{E_{ij} E_{i^* j}} Prod_j \Big] & \text{if } i \neq i^*. \end{cases} \tag{6}$$

Now, we consider some matrix forms as follows:

$$\begin{aligned} A' &= [\hat{a}_{01}, ..., \hat{a}_{k1}, \hat{a}_{02}, ..., \hat{a}_{k2}, ..., \hat{a}_{0c}, ..., \hat{a}_{kc}], \\ F_1' &= \left[\frac{\partial L}{\partial \hat{a}_{01}}, ..., \frac{\partial L}{\partial \hat{a}_{k1}}, \frac{\partial L}{\partial \hat{a}_{02}}, ..., \frac{\partial L}{\partial \hat{a}_{k2}}, ..., \frac{\partial L}{\partial \hat{a}_{0c}}, ..., \frac{\partial L}{\partial \hat{a}_{kc}}\right], \\ F_2 &= \begin{bmatrix} \frac{\partial^2 L}{\partial \hat{a}_{01}\partial \hat{a}_{01}} & \frac{\partial^2 L}{\partial \hat{a}_{01}\partial \hat{a}_{11}} & \cdots & \frac{\partial^2 L}{\partial \hat{a}_{01}\partial \hat{a}_{kc}} \\ \frac{\partial^2 L}{\partial \hat{a}_{11}\partial \hat{a}_{01}} & \frac{\partial^2 L}{\partial \hat{a}_{11}\partial \hat{a}_{11}} & \cdots & \frac{\partial^2 L}{\partial \hat{a}_{11}\partial \hat{a}_{kc}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 L}{\partial \hat{a}_{kc}\partial \hat{a}_{01}} & \frac{\partial^2 L}{\partial \hat{a}_{kc}\partial \hat{a}_{11}} & \cdots & \frac{\partial^2 L}{\partial \hat{a}_{kc}\partial \hat{a}_{kc}} \end{bmatrix}_{(k+1)c\times(k+1)c}. \end{aligned}$$

Now, the parameters of the proposed models is obtained based on the following algorithm.

Algorithm 2: Fuzzy clustering regression models based on crisp data

1) Fix $m = 2$, $c \in \{2, ..., n-1\}$, and $\varepsilon > 0$;
2) For each $i = 1, 2, ..., c$ and $j = 1, 2, ..., n$, calculate the membership degrees μ_{ij} based on Algorithm 1;
3) For each $i = 1, 2, ..., c$, choose the initial values for $(a_{0i}^0, a_{1i}^0, ..., a_{ki}^0)$ and calculate the vector A, denoted by A^0;
4) Based on the relations 5 and 6, we calculate the vector F_1 and the matrix F_2 for A^0. We denote them as F_1^0 and F_2^0.
5) The parameters of clustering regression models are updated as follows
$$A^1 = A^0 - (F_2^0)^{-1} F_1^0.$$
6) Using $\max_{p,i}\{|a_{pi}^1 - a_{pi}^0|\} \leq \varepsilon$, compare A^0 to A^1. If it is correct, then algorithm is stopped, otherwise, replace $A^0 = A^1$, and go to step 4.

Definition 2.5. *Based on the proposed clustering models and the membership degrees μ_{ij}, the estimated value $\widehat{Y}_j$ is defined as follows*

$$\widehat{Y}_j = (\mu_{1j} \times \widehat{Y}_j^1) + (\mu_{2j} \times \widehat{Y}_j^2) + ... + (\mu_{cj} \times \widehat{Y}_j^c).$$

Definition 2.6. *To evaluate the goodness of fit of the proposed models, the mean of distance (MD) and the similarity measure (SM) between the observed values Y_j and the estimated values $\widehat{Y}_j$, $j = 1, 2, ..., n$, are defined as follows*

$$MD = \frac{1}{n}\sum_{j=1}^{n}(Y_j - \widehat{Y}_j)^2,$$

$$SM = 1 - \frac{\sum_{j=1}^{n}|Y_j - \widehat{Y}_j|}{\sum_{j=1}^{n}|Y_j| + \sum_{j=1}^{n}|\widehat{Y}_j|}.$$

Remark 2.7. *Note that the optimal models are the models with the minimum value of MD or with the maximum value of SM.*

Example 2.8. *Consider the data set in Example 2.4. First, based on Algorithm 1 (with $\varepsilon = 0.0001$), the membership degrees for $c = 2$ clusters are given in Table 2. Based on the Algorithm 2 with $\varepsilon = 0.0001$, the geometric clustering regression models on this data set with $c = 2$ are presented as follows (see Figure 1):*

$$\begin{aligned} \hat{Y}_j^1 &= 6.2025 + 4.9264x_j, \qquad j = 1, ..., 20, \\ \hat{Y}_j^2 &= -110.4470 + 5.4918x_j, \qquad j = 1, ..., 20. \end{aligned}$$

Hence, the estimated response variable is obtained as

$$\widehat{Y}_j = (\mu_{1j} \times \widehat{Y}_j^1) + (\mu_{2j} \times \widehat{Y}_j^2), \qquad j = 1, ..., 20.$$

The values of estimated response variable and the goodness of fit indices are summarized in Table 3.

Table 3: Estimated response variable in Example 2.8

j	$\widehat{Y}_j$	j	$\widehat{Y}_j$
1	10.9897	11	5.0366
2	15.8590	12	10.4963
3	20.9358	13	15.9459
4	25.7938	14	21.4062
5	30.8346	15	26.9080
6	35.7270	16	32.3892
7	40.4733	17	38.1544
8	45.5466	18	43.3923
9	50.3729	19	48.9137
10	55.2891	20	54.5744
	$MD = 7.0697$		
	$SM = 0.9657$		

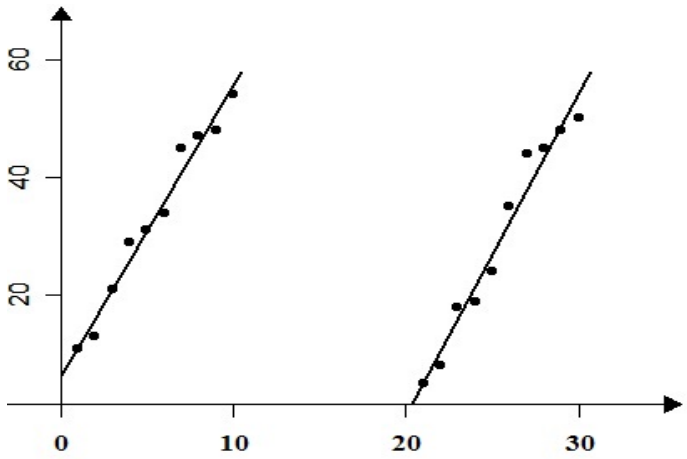


Figure 1: The clustering fuzzy regression models in Example 2.8

3 Geometric clustering regression based on fuzzy data

In this section, we fit some geometric clustering regression models based on fuzzy data. To do this, we first introduce a method to cluster fuzzy data set, and then, a new approach is presented to fit some clustering regression models.

3.1 Fuzzy c-means clustering on fuzzy data (FCMCF)

Assume that $\widetilde{G} = \{(\widetilde{x}_1, \widetilde{y}_1), ..., (\widetilde{x}_n, \widetilde{y}_n)\}$ is a set of n fuzzy data with $\widetilde{x}_j = (x_j, l_{x_j}, r_{x_j})_T$ and $\widetilde{y}_j = (y_j, l_{y_j}, r_{y_j})_T$. Now, we cluster $\widetilde{G}$ into c parts, by using membership functions $\mu_{ij} = \mu_i(\widetilde{x}_j, \widetilde{y}_j)$ such that $\mu_{ij} \in [0, 1]$ and $\sum_{i=1}^{c} \mu_{ij} = 1$ for all $j = 1, 2, ..., n$ in $\widetilde{G}$. Based on the distance given in Definition 2.1, the objective function is introduced as follows

$$\begin{aligned} L_2 &= \sum_{i=1}^{c}\sum_{j=1}^{n} \mu_{ij}^m \Big[d^2(X_j^L, v_i^L) + d^2(X_j, v_i) + d^2(X_j^U, v_i^U) \Big] \\ &= \sum_{i=1}^{c}\sum_{j=1}^{n} \mu_{ij}^m \Big[(X_j^L - v_i^L)' A_i (X_j^L - v_i^L) + (X_j - v_i)' A_i (X_j - v_i) + (X_j^U - v_i^U)' A_i (X_j^U - v_i^U) \Big], \end{aligned}$$

where

$$X_j^L = \begin{bmatrix} x_j - l_{x_j} \\ y_j - l_{y_j} \end{bmatrix}, \quad X_j = \begin{bmatrix} x_j \\ y_j \end{bmatrix}, \quad X_j^U = \begin{bmatrix} x_j + r_{x_j} \\ y_j + r_{y_j} \end{bmatrix},$$
$$v_i^L = \begin{bmatrix} v_i^x - l_{v_i}^x \\ v_i^y - l_{v_i}^y \end{bmatrix}, \quad v_i = \begin{bmatrix} v_i^x \\ v_i^y \end{bmatrix}, \quad v_i^U = \begin{bmatrix} v_i^x + r_{v_i}^x \\ v_i^y + r_{v_i}^y \end{bmatrix}.$$

Here, $(\widetilde{v}_i^x, \widetilde{v}_i^y)$ are called the fuzzy clustering centers, where $\widetilde{v}_i^x = (v_i^x, l_{v_i}^x, r_{v_i}^x)_T$ and $\widetilde{v}_i^y = (v_i^y, l_{v_i}^y, r_{v_i}^y)_T$.

To minimize the objective function L_2 under v_i^L, v_i, v_i^U, and μ_{ij}, we obtain the following equations

$$v_i^L = \frac{\sum_{j=1}^n \mu_{ij}^m X_j^L}{\sum_{j=1}^n \mu_{ij}^m}, \quad v_i = \frac{\sum_{j=1}^n \mu_{ij}^m X_j}{\sum_{j=1}^n \mu_{ij}^m}, \quad v_i^U = \frac{\sum_{j=1}^n \mu_{ij}^m X_j^U}{\sum_{j=1}^n \mu_{ij}^m}, \quad i = 1, ..., c, \tag{7}$$

$$\mu_{ij} = \Big(\sum_{k=1}^{c} \frac{(d_{ij}^2)^{\frac{1}{m-1}}}{(d_{kj}^2)^{\frac{1}{m-1}}} \Big)^{-1}, \tag{8}$$

where, $d_{ij}^2 = d^2(X_j^L, v_i^L) + d^2(X_j, v_i) + d^2(X_j^U, v_i^U)$. Also, in the repeated FCMCF algorithm, we suppose the A_i is given as follows:

$$A_i = (\rho_i det(S_i))^{1/k} S_i^{-1}, \qquad i = 1, ..., c, \tag{9}$$

where

$$S_i = \sum_{j=1}^{n} \mu_{ij}^m \Big[(X_j^L - v_i^L)(X_j^L - v_i^L)' + (X_j - v_i)(X_j - v_i)' + (X_j^U - v_i^U)(X_j^U - v_i^U)' \Big].$$

Algorithm 3: FCMCF

1) First, we consider some initial values for the membership degrees μ_{ij}^0 and $A_i^0 = I_{k\times k}$.
2) Based on μ_{ij}^0, we calculate the fuzzy cluster centers $v_i^{L0},\ v_i^0,\ v_i^{U0},\ i = 1, ..., c$ under the relation (7).
3) Based on the fuzzy centers in item 2 and A_i^0, we update the membership degrees, denoted by μ_{ij}^1.
4) Based on the fuzzy centers in item 2 and μ_{ij}^1, we update A_i based on the relation (9), denoted by A_i^1.
5) If $\max |\mu_{ij}^1 - \mu_{ij}^0| < \varepsilon$ (ε is a small value), then the algorithm is stopped, otherwise return to step 2 and repeat the steps of algorithm.

3.2 Geometric clustering regression based on fuzzy data

Assume that a data set is observed as $(x_{1j}, x_{2j}, ..., x_{kj}, \widetilde{Y}_j)$, $j = 1, 2, ..., n$, where $\widetilde{Y}_j = (y_j, l_{y_j}, r_{y_j})_T$ and $x_{pj} \geq 0$. The aim is to fit some clustering regression models to the data set, as follows

$$\hat{\widetilde{Y}}_j^i = \hat{\widetilde{A}}_{0i} + \hat{\widetilde{A}}_{1i} x_{1j} + ... + \hat{\widetilde{A}}_{ki} x_{kj}, \qquad i = 1, ..., c, \quad j = 1, ..., n,$$

where, $(\hat{\widetilde{A}}_{0i}, \hat{\widetilde{A}}_{1i}, ..., \hat{\widetilde{A}}_{ki})$ are the estimated parameters of i^{th} clustering regression model and $\hat{\widetilde{A}}_{pi} = (a_{pi}, l_{a_{pi}}, r_{a_{pi}})_T$, $i = 1, ..., c,\ p = 0, 1, ..., k$. To estimate the parameters of the geometric clustering models, the objective function is defined as follows:

$$H_2 = \sum_{j=1}^{n} \prod_{i=1}^{c} \Big(D_{ij}^2 \Big)^{\mu_{ij}^m} = \sum_{j=1}^{n} \prod_{i=1}^{c} \Big(\frac{(E_{ij}^L)^2 + (E_{ij})^2 + (E_{ij}^U)^2}{3} \Big)^{\mu_{ij}^m},$$

where

$$\begin{aligned}
E_{ij} &= y_j - \hat{y}_j^i = y_j - \sum_{p=1}^{k} a_{pi} x_{pj}, \\
E_{ij}^L &= y_j^L - \hat{y}_j^{Li} = y_j^L - \sum_{p=1}^{k} a_{pi}^L x_{pj}, \qquad y_j^L = y_j - l_{y_j}, \qquad a_{pi}^L = a_{pi} - l_{a_{pi}}, \\
E_{ij}^U &= y_j^U - \hat{y}_j^{Ui} = y_j^U - \sum_{p=1}^{k} a_{pi}^U x_{pj}, \qquad y_j^U = y_j + r_{y_j}, \qquad a_{pi}^U = a_{pi} + r_{a_{pi}}, \\
D_{ij}^2 &= \frac{(E_{ij}^L)^2 + (E_{ij})^2 + (E_{ij}^U)^2}{3}.
\end{aligned}$$

Note that the membership degrees μ_{ij} are obtained based on Algorithm 3. For minimizing the objective function H_2 with respect to c-partitions, we have

$$\frac{\partial H_2}{\partial \hat{a}_{pi}} = \sum_{j=1}^{n} \Big[-2\frac{\mu_{ij}^m E_{ij} x_{pj}}{D_{ij}^2} Prod_j^* \Big],$$
$$\frac{\partial H_2}{\partial \hat{a}_{pi}^L} = \sum_{j=1}^{n} \Big[-2\frac{\mu_{ij}^m E_{ij}^L x_{pj}}{D_{ij}^2} Prod_j^* \Big],$$
$$\frac{\partial H_2}{\partial \hat{a}_{pi}^U} = \sum_{j=1}^{n} \Big[-2\frac{\mu_{ij}^m E_{ij}^U x_{pj}}{D_{ij}^2} Prod_j^* \Big],$$

where, $Prod_j^* = \prod_{i=1}^{c} \left(D_{ij}^2\right)^{\mu_{ij}^m}$. Also, the second derivatives are calculated as follows:

$$\frac{\partial^2 H_2}{\partial \hat{a}_{pi} \partial \hat{a}_{p^* i^*}} = \begin{cases} \sum_{j=1}^{n} \Big[\big(\frac{4\mu_{ij}^m (\mu_{ij}^m - 1) E_{ij}^2}{D_{ij}^4} + \frac{2\mu_{ij}^m}{D_{ij}^2} \big) x_{pj} x_{p^* j} Prod_j^* \Big] & \text{if } i = i^*, \\ \sum_{j=1}^{n} \Big[\frac{4\mu_{ij}^m \mu_{i^* j}^m E_{ij} E_{i^* j}}{D_{ij}^2 D_{i^* j}^2} x_{pj} x_{p^* j} Prod_j^* \Big] & \text{if } i \neq i^*, \end{cases}$$

$$\frac{\partial^2 H_2}{\partial \hat{a}_{pi}^L \partial \hat{a}_{p^* i^*}^L} = \begin{cases} \sum_{j=1}^{n} \Big[\big(\frac{4\mu_{ij}^m (\mu_{ij}^m - 1) (E_{ij}^L)^2}{D_{ij}^4} + \frac{2\mu_{ij}^m}{D_{ij}^2} \big) x_{pj} x_{p^* j} Prod_j^* \Big] & \text{if } i = i^*, \\ \sum_{j=1}^{n} \Big[\frac{4\mu_{ij}^m \mu_{i^* j}^m E_{ij}^L E_{i^* j}^L}{D_{ij}^2 D_{i^* j}^2} x_{pj} x_{p^* j} Prod_j^* \Big] & \text{if } i \neq i^*, \end{cases}$$

$$\frac{\partial^2 H_2}{\partial \hat{a}_{pi}^U \partial \hat{a}_{p^* i^*}^U} = \begin{cases} \sum_{j=1}^{n} \Big[\big(\frac{4\mu_{ij}^m (\mu_{ij}^m - 1) (E_{ij}^U)^2}{D_{ij}^4} + \frac{2\mu_{ij}^m}{D_{ij}^2} \big) x_{pj} x_{p^* j} Prod_j^* \Big] & \text{if } i = i^*, \\ \sum_{j=1}^{n} \Big[\frac{4\mu_{ij}^m \mu_{i^* j}^m E_{ij}^U E_{i^* j}^U}{D_{ij}^2 D_{i^* j}^2} x_{pj} x_{p^* j} Prod_j^* \Big] & \text{if } i \neq i^*. \end{cases}$$

Now, we consider some matrix forms as follows:

$$\begin{aligned} W &= [\hat{a}_{01}, ..., \hat{a}_{k1}, \hat{a}_{02}, ..., \hat{a}_{k2}, ..., \hat{a}_{0c}, ..., \hat{a}_{kc}]', \\ W^L &= [\hat{a}_{01}^L, ..., \hat{a}_{k1}^L, \hat{a}_{02}^L, ..., \hat{a}_{k2}^L, ..., \hat{a}_{0c}^L, ..., \hat{a}_{kc}^L]', \\ W^U &= [\hat{a}_{01}^U, ..., \hat{a}_{k1}^U, \hat{a}_{02}^U, ..., \hat{a}_{k2}^U, ..., \hat{a}_{0c}^U, ..., \hat{a}_{kc}^U]', \\ F_1 &= \Big[\frac{\partial H}{\partial \hat{a}_{01}}, ..., \frac{\partial H}{\partial \hat{a}_{k1}}, \frac{\partial H}{\partial \hat{a}_{02}}, ..., \frac{\partial H}{\partial \hat{a}_{k2}}, ..., \frac{\partial H}{\partial \hat{a}_{0c}}, ..., \frac{\partial H}{\partial \hat{a}_{kc}} \Big]', \\ F_1^L &= \Big[\frac{\partial H}{\partial \hat{a}_{01}^L}, ..., \frac{\partial H}{\partial \hat{a}_{k1}^L}, \frac{\partial H}{\partial \hat{a}_{02}^L}, ..., \frac{\partial H}{\partial \hat{a}_{k2}^L}, ..., \frac{\partial H}{\partial \hat{a}_{0c}^L}, ..., \frac{\partial H}{\partial \hat{a}_{kc}^L} \Big]', \\ F_1^U &= \Big[\frac{\partial H}{\partial \hat{a}_{01}^U}, ..., \frac{\partial H}{\partial \hat{a}_{k1}^U}, \frac{\partial H}{\partial \hat{a}_{02}^U}, ..., \frac{\partial H}{\partial \hat{a}_{k2}^U}, ..., \frac{\partial H}{\partial \hat{a}_{0c}^U}, ..., \frac{\partial H}{\partial \hat{a}_{kc}^U} \Big]', \end{aligned}$$

$$F_2 = \begin{bmatrix} \frac{\partial^2 H}{\partial \hat{a}_{01} \partial \hat{a}_{01}} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{01} \partial \hat{a}_{kc}} \\ \frac{\partial^2 H}{\partial \hat{a}_{11} \partial \hat{a}_{01}} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{11} \partial \hat{a}_{kc}} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 H}{\partial \hat{a}_{kc} \partial \hat{a}_{01}} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{kc} \partial \hat{a}_{kc}} \end{bmatrix}, \quad F_2^L = \begin{bmatrix} \frac{\partial^2 H}{\partial \hat{a}_{01}^L \partial \hat{a}_{01}^L} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{01}^L \partial \hat{a}_{kc}^L} \\ \frac{\partial^2 H}{\partial \hat{a}_{11}^L \partial \hat{a}_{01}^L} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{11}^L \partial \hat{a}_{kc}^L} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 H}{\partial \hat{a}_{kc}^L \partial \hat{a}_{01}^L} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{kc}^L \partial \hat{a}_{kc}^L} \end{bmatrix},$$

$$F_2^U = \begin{bmatrix} \frac{\partial^2 H}{\partial \hat{a}_{01}^U \partial \hat{a}_{01}^U} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{01}^U \partial \hat{a}_{kc}^U} \\ \frac{\partial^2 H}{\partial \hat{a}_{11}^U \partial \hat{a}_{01}^U} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{11}^U \partial \hat{a}_{kc}^U} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 H}{\partial \hat{a}_{kc}^U \partial \hat{a}_{01}^U} & \cdots & \frac{\partial^2 H}{\partial \hat{a}_{kc}^U \partial \hat{a}_{kc}^U} \end{bmatrix}.$$

Now, the parameters of the proposed models is obtained based on the following algorithm.

Algorithm 4: Fuzzy clustering regression models based on fuzzy data

1) Fix $m = 2$, $c \in \{2, ..., n-1\}$, and $\varepsilon > 0$;

2) Based on the data set $(x_{1j}, x_{2j}, ..., x_{kj}, \widetilde{Y}_j)$, $j = 1, ..., n$ and Algorithm 3, we calculate the membership degrees μ_{ij}, $i = 1, ..., n$ and $j = 1, ..., c$;

3) Choose some initial values for $(\hat{a}_{0i}^L, \hat{a}_{1i}^L, ..., \hat{a}_{ki}^L)$, $(\hat{a}_{0i}, \hat{a}_{1i}, ..., \hat{a}_{ki})$, and $(\hat{a}_{0i}^U, \hat{a}_{1i}^U, ..., \hat{a}_{ki}^U)$, $i = 1, 2, ..., c$ and also construct the vectors W^L, W, and W^U. We denote them by W^{L0}, W^0, and W^{U0};

4) Based on the initial values in item 3, we calculate the vectors F_1^L, F_1, and F_1^U, and also the matrices F_2^L, F_2, and F_2^U. They are denoted as F_1^{L0}, F_1^0, F_1^{U0}, F_2^{L0}, F_2^0, F_2^{U0};

5) The parameters of clustering regression models are updated as follows

$$\begin{aligned} W^1 &= W^0 - (F_2^0)^{-1} F_1^0, \\ W^{L1} &= W^{L0} - (F_2^{L0})^{-1} F_1^{L0}, \\ W^{U1} &= W^{U0} - (F_2^{U0})^{-1} F_1^{U0}. \end{aligned}$$

6) If $\max_{p,i}\{|W^{L1} - W^{L0}|, |W^1 - W^0|, |W^{U1} - W^{U0}|\} \leq \varepsilon$, then the algorithm is stopped, otherwise, replace $W^0 = W^1$, $W^{L0} = W^{L1}$, $W^{U0} = W^{U1}$, and go to step 4.

Definition 3.1. *Based on the proposed clustering models and the membership degrees μ_{ij}, the estimated values $\hat{\widetilde{Y}}_j$ are defined as follows*

$$\hat{\widetilde{Y}}_j = (\mu_{1j} \otimes \hat{\widetilde{Y}}_j^1) \oplus (\mu_{2j} \otimes \hat{\widetilde{Y}}_j^2) \oplus ... \oplus (\mu_{cj} \otimes \hat{\widetilde{Y}}_j^c).$$

Definition 3.2. *Assume that we have the new observation $\underline{x}_{new} = (x_{1,new}, x_{2,new}, ..., x_{k,new})$. The predicted value $\widehat{\widetilde{Y}}_{new}$ is obtained based on the following procedure:*

1) *First, based on the proposed clustering linear regression models, we calculate the response values $\hat{\widetilde{Y}}_{new}^1$, $\hat{\widetilde{Y}}_{new}^2$,...,$\hat{\widetilde{Y}}_{new}^c$ under $\underline{x}_{new}$.*

2) *Calculate $D_1, ..., D_n$, as $D_j = \sqrt{\sum_{i=1}^k (x_{i,new} - x_{ij})^2}$, $j = 1, ..., n$. Also, we order $D_1, D_2, ..., D_n$, denoted by $D_{(1)}, D_{(2)}, ..., D_{(n)}$.*

3) *Estimate the membership degrees $\mu_{i,new}$ as follows*

$$\mu_{i,new} = \frac{\sum_{j \in U} \mu_{ij}}{r}, \qquad i = 1, ..., c,$$

where U is a set containing the numbers of r data that have the lowest values of $D_{(j)}$ (i.e. $D_{(1)} \leq D_{(2)} \leq ... \leq D_{(r)}$). The value of r is chosen by the user.

4) *Finally, the response value $\widehat{\widetilde{Y}}_{new}$ is predicted as*

$$\widehat{\widetilde{Y}}_{new} = (\mu_{1,new} \otimes \hat{\widetilde{Y}}_{new}^1) \oplus ... \oplus (\mu_{c,new} \otimes \hat{\widetilde{Y}}_{new}^c).$$

3.3 Goodness of fit of the model

To evaluate the goodness of fit of the proposed models, two indices are introduced based on the similarity measure and also the distance between the observed values of response variables $\widetilde{Y}_j$ and the estimated values of response variables $\hat{\widetilde{Y}}_j$.

Definition 3.3. (Arefi and Taheri [5]) *The mean of similarity measures (MSM) between* $\widetilde{Y}_j$ *and* $\hat{\widetilde{Y}}_j$, $j = 1, 2, ..., n$ *are defined as*

$$MSM = \frac{1}{n}\sum_{j=1}^{n} SM(\widetilde{Y}_j, \hat{\widetilde{Y}}_j),$$

where, $SM(\widetilde{Y}_j, \hat{\widetilde{Y}}_j) = 1 - \frac{\int_R |\hat{\widetilde{Y}}_j(x) - \widetilde{Y}_j(x)|dx}{\int_R \hat{\widetilde{Y}}_j(x)dx + \int_R \widetilde{Y}_j(x)dx}$, *and* $\hat{\widetilde{Y}}_j(x)$ *and* $\widetilde{Y}_j(x)$ *are the membership functions of* $\hat{\widetilde{Y}}_j$ *and* $\widetilde{Y}_j$, *respectively.*

Definition 3.4. *The index for goodness of fit based on the mean of distance between* $\widetilde{Y}_j$ *and* $\hat{\widetilde{Y}}_j$, $j = 1, 2, ..., n$, *is defined as*

$$MD = \frac{1}{n}\sum_{j=1}^{n} D_{ij} = \frac{1}{n}\sum_{j=1}^{n}\sqrt{\frac{(E_{ij}^L)^2 + (E_{ij})^2 + (E_{ij}^U)^2}{3}}.$$

Remark 3.5. *The optimal model is the model with maximum value of* MSM *and minimum value of* MD.

4 Applied examples

4.1 Application in soil science

One of the problem in soil science is the measurement of physical, chemical and/or biological soil properties. The problem results from the difficulty, time, and cost of direct measurements. Here, some clustering fuzzy models are studied to develop some relationships between different chemical and physical soil properties. Different soil physical and chemical properties were measured using standard procedures. Due to some impreciseness in related experimental environment, the observed response variables were reported as fuzzy numbers. This data set is presented by Mohammadi and Taheri [25] (see Table 4). In this table, the cation exchange capacity (CEC) is the response variable as a triangular fuzzy number, the sand content percentage (SAND) and the organic matter content (OM) are the explanatory variables.

Now, we fit some geometric fuzzy clustering models between the response variable (CEC) and the explanatory variables (SAND and OM) with $c = 3$ clusters. First, based on Algorithm 3 with $\varepsilon = 0.0001$, we calculate the membership degrees μ_{ij}, $i = 1, 2, 3$, $j = 1, ..., 24$. The results of μ_{ij} are summarized in Table 5. Based on Algorithm 4 with $\varepsilon = 0.0001$, the optimal geometric clustering fuzzy regression models with $c = 3$ are obtained as follows:

$$\begin{aligned}
\hat{\widetilde{Y}}_j^1 &= \hat{\widetilde{A}}_{01} \oplus (\hat{\widetilde{A}}_{11} \otimes x_{1j}) \oplus (\hat{\widetilde{A}}_{21} \otimes x_{2j}) \\
&= (17.3634; 0.2673, 0.2276)_T \oplus ((-0.0351; 0.0126, 0.0112)_T \otimes x_{1j}) \oplus ((2.1695; 0.2454, 0.2320)_T \otimes x_{2j}), \\
\hat{\widetilde{Y}}_j^2 &= \hat{\widetilde{A}}_{02} \oplus (\hat{\widetilde{A}}_{12} \otimes x_{1j}) \oplus (\hat{\widetilde{A}}_{22} \otimes x_{2j}) \\
&= (22.6508; 0.2932, 0.2450)_T \oplus ((-0.3459; 0.0086, 0.0134)_T \otimes x_{1j}) \oplus ((4.2367; 0.2274, 0.1849)_T \otimes x_{2j}), \\
\hat{\widetilde{Y}}_j^3 &= \hat{\widetilde{A}}_{03} \oplus (\hat{\widetilde{A}}_{13} \otimes x_{1j}) \oplus (\hat{\widetilde{A}}_{23} \otimes x_{2j}) \\
&= (20.7796; 0.2167, 0.5329)_T \oplus ((-0.3968; 0.0334, 0.0011)_T \otimes x_{1j}) \oplus ((4.5337; 0.1704, 0.1782)_T \otimes x_{2j}.
\end{aligned}$$

Hence, the estimated values of response variables $\hat{\widetilde{Y}}_j$ are obtained as follows (see the third column of Table 5)

$$\hat{\widetilde{Y}}_j = (\mu_{1j} \otimes \hat{\widetilde{Y}}_j^1) \oplus (\mu_{2j} \otimes \hat{\widetilde{Y}}_j^2) \oplus (\mu_{3j} \otimes \hat{\widetilde{Y}}_j^3), \quad j = 1, ..., 24.$$

Now, suppose that we observe the new values $\underline{x}_{new} = (25, 1.5)$, based on Definition 3.2, we obtain the response values for $c = 1, 2, 3$ as

$$\begin{aligned}
\hat{\widetilde{Y}}_{new}^1 &= (19.7401; 0.9504, 0.8556)_T, \\
\hat{\widetilde{Y}}_{new}^2 &= (20.3584; 0.8493, 0.8574)_T, \\
\hat{\widetilde{Y}}_{new}^3 &= (17.6601; 1.3073, 0.8277)_T.
\end{aligned}$$

Also, based on Definition 3.2 with $r = 3$, we obtain $U = \{3, 18, 19\}$, and the membership degrees is presented as $(\mu_{1,new}, \mu_{2,new}, \mu_{3,new}) = (0.0857, 0.8960, 0.0183)$. Hence, the new response value $\widehat{\widetilde{Y}}_{new}$ is predicted as

$$\begin{aligned}\widehat{\widetilde{Y}}_{new} &= (\mu_{1,new} \otimes \hat{\tilde{Y}}^1_{new}) \oplus (\mu_{2,new} \otimes \hat{\tilde{Y}}^2_{new}) \oplus (\mu_{3,new} \otimes \hat{\tilde{Y}}^3_{new}) \\ &= (20.2559; 0.8664, 0.8567)_T.\end{aligned}$$

Table 4: Some of measured soil characteristics (CEC, SAND, and OM)

No.	$SAND$ (%) $\tilde{x}_{1j}$	OM (%) $\tilde{x}_{2j}$	CEC $(Cmol(+)/kg)$ $\tilde{y}_j = (y_j; l_{y_j})_T$	No.	$SAND$ (%) $\tilde{x}_{1j}$	OM (%) $\tilde{x}_{2j}$	CEC $(Cmol(+)/kg)$ $\tilde{y}_j = (y_j; l_{y_j})_T$
1	35	0.88	$(16.5; 0.83)_T$	13	31	3.52	$(24.4; 1.22)_T$
2	37	1.13	$(18.6; 0.93)_T$	14	31	2.33	$(21.8; 1.09)_T$
3	27	1.31	$(19.3; 0.97)_T$	15	17	1.71	$(23.8; 1.19)_T$
4	29	1.98	$(20.3; 1.02)_T$	16	14	1.14	$(20.8; 1.04)_T$
5	38	1.02	$(17.3; 0.87)_T$	17	19	0.99	$(17.5; 0.88)_T$
6	32	1.29	$(20.4; 1.02)_T$	18	28	1.14	$(17.8; 0.89)_T$
7	29	1.52	$(19.3; 0.97)_T$	19	26	1.46	$(20.2; 1.01)_T$
8	18	1.33	$(21.9; 1.09)_T$	20	32	1.81	$(20.0; 1.00)_T$
9	40	1.71	$(15.9; 0.80)_T$	21	10	1.38	$(22.8; 1.14)_T$
10	28	2.00	$(18.3; 0.92)_T$	22	38	0.84	$(19.1; 0.96)_T$
11	13	1.68	$(22.6; 1.13)_T$	23	49	1.48	$(12.1; 0.60)_T$
12	19	2.15	$(23.7; 1.19)_T$	24	42	1.08	$(12.8; 0.64)_T$

Table 5: Membership degrees and the goodness of fit indices on soil characteristics

No.	$(\mu_{1j}, \mu_{2j}, \mu_{3j})$	$\hat{\tilde{Y}}_j = (\hat{y}_j; \hat{l}_{y_j}, \hat{r}_{y_j})_T$	D_{ij}	$SM(\tilde{Y}_j, \hat{\tilde{Y}}_j)$
1	(0.8922, 0.0921, 0.0156)	$(17.5842, 0.9234, 0.8255)_T$	1.0525	0.1498
2	(0.9925, 0.0055, 0.0020)	$(18.4815, 1.0132, 0.9025)_T$	0.1592	0.8492
3	(0.0890, 0.8944, 0.0166)	$(18.8500, 0.8417, 0.8457)_T$	0.4604	0.5664
4	(0.6501, 0.2870, 0.0629)	$(20.5969, 1.1090, 1.0011)_T$	0.2637	0.7598
5	(0.9739, 0.0202, 0.0059)	$(18.1078, 0.9993, 0.8884)_T$	0.7736	0.3323
6	(0.9550, 0.0333, 0.0117)	$(18.9138, 0.9905, 0.8837)_T$	1.5235	0.0468
7	(0.0991, 0.8938, 0.0071)	$(19.0973, 0.9043, 0.9120)_T$	0.2064	0.7978
8	(0.0293, 0.9077, 0.0630)	$(21.8375, 0.7712, 0.7352)_T$	0.2850	0.8118
9	(0.0690, 0.9103, 0.0207)	$(16.2390, 1.0549, 1.0894)_T$	0.4151	0.6617
10	(0.4272, 0.0455, 0.5273)	$(19.7074, 1.3073, 0.9586)_T$	1.3054	0.1470
11	(0.0278, 0.0427, 0.9295)	$(23.2503, 0.9282, 0.8389)_T$	0.6527	0.4783
12	(0.0883, 0.1510, 0.7607)	$(23.1768, 1.1605, 0.9305)_T$	0.6137	0.5382
13	(0.1403, 0.0731, 0.7867)	$(24.5425, 1.7701, 1.2296)_T$	0.2643	0.8860
14	(0.5953, 0.3273, 0.0774)	$(21.3076, 1.2174, 1.0958)_T$	0.5365	0.5840
15	(0.0386, 0.8644, 0.0970)	$(23.6628, 0.8553, 0.7958)_T$	0.3367	0.7960
16	(0.0183, 0.0543, 0.9273)	$(20.4938, 0.8645, 0.7432)_T$	0.3977	0.6635
17	(0.0533, 0.0483, 0.8984)	$(17.9115, 0.9894, 0.7241)_T$	0.3296	0.6546
18	(0.0969, 0.8809, 0.0221)	$(17.8314, 0.8161, 0.8259)_T$	0.0662	0.9510
19	(0.0711, 0.9127, 0.0163)	$(19.7845, 0.8646, 0.8613)_T$	0.4335	0.6047
20	(0.8560, 0.1322, 0.0118)	$(20.0010, 1.1040, 1.0034)_T$	0.0595	0.9739
21	(0.0144, 0.0432, 0.9424)	$(23.1093, 0.7811, 0.7810)_T$	0.4261	0.7003
22	(0.9873, 0.0084, 0.0043)	$(17.7771, 0.9559, 0.8463)_T$	1.3605	0.0695
23	(0.0322, 0.9484, 0.0194)	$(12.1184, 1.0786, 1.1653)_T$	0.4292	0.6966
24	(0.0830, 0.8955, 0.0215)	$(13.0794, 0.9334, 0.9959)_T$	0.4008	0.6628
			$MD = 0.5313$	$MSM = 0.5993$

Table 6: Comparison with some other approaches on soil characteristics

Approach	MD	MSM
Proposed model	0.5313	0.5993
Arefi [3]	1.2633	0.3408
Mohammadi and Taheri [25]	1.4627	0.2895
Khammar et al. [22] (Model 1)	1.1785	0.4107
Khammar et al. [22] (Model 2)	1.1755	0.4105

Note that, Arefi[3], Khammar et al. [22], and Mohammadi and Taheri [25] are presented some fuzzy regression models on this data set. The fuzzy regression models based on these approaches are given as follows:

• Arefi [3]:

$$\hat{\widetilde{Y}}_A = (21.0778; 0.1502, 0.1502)_T \oplus (-0.1949x_1) \oplus ((2.6605; 0.3039, 0.3039)_T \otimes x_2).$$

• Mohammadi and Taheri [25]:

$$\hat{\widetilde{Y}}_{MT} = (22.9810; 0.6383, 0.6383)_T \oplus (-0.2223x_1) \oplus ((2.4741; 0.2059, 0.2059)_T \otimes x_2).$$

• Khammar et al. [22] (in this approach, two models are fitted as follows):

$$Model\ 1: \ \hat{\widetilde{Y}} = (21.1440; 0.8210, 0.8210)_T \oplus (-0.2020x_1) \oplus ((2.7270; 0.1190, 0.1190)_T \otimes x_2).$$

$$Model\ 2: \ \hat{\widetilde{Y}} = (21.2080; 0.7860, 0.7860)_T \oplus (-0.2010x_1) \oplus ((2.6750; 0.1230, 0.1230)_T \otimes x_2).$$

Based on the above models, we compare them based on the goodness of fit indices (see Table 6). Hence, the proposed model in this paper is better for this data set.

4.2 Application in effect of the composition Portland cement

We analyze the effect of the composition Portland cement on heat evolved during hardening ($\widetilde{Y}$: Heat evolved in calories per gram of cement; x_1: Amount of tricalcium aluminate; x_2: Amount of tricalcium silicate; x_3: Amount of tetracalcium aluminoferrite). The data set is given in Table 7 (see Xu and Li [28]).

Now, we wish to fit some geometric clustering fuzzy regression models with $c = 2$ clusters. First, based on Algorithm 3 with $\varepsilon = 0.0001$, we calculate the membership degrees $\mu_{ij},\ i = 1, 2,\ j = 1, ..., 13$ (see Table 8). Then, based on Algorithm 4 with $\varepsilon = 0.0001$, the optimal geometric clustering fuzzy regression models with $c = 2$ are obtained as follows:

$$\begin{aligned}
\hat{\widetilde{Y}}_j^1 &= \hat{\widetilde{A}}_{01} \oplus (\hat{\widetilde{A}}_{11} \otimes x_{1j}) \oplus (\hat{\widetilde{A}}_{21} \otimes x_{2j}) \oplus (\hat{\widetilde{A}}_{31} \otimes x_{3j}) \\
&= (89.9458; 6.9209)_T \oplus ((-2.9937; 0.1786)_T \otimes x_{1j}) \oplus (2.1891x_{2j}) \oplus (-3.8557x_{3j}) \\
\hat{\widetilde{Y}}_j^2 &= \hat{\widetilde{A}}_{02} \oplus (\hat{\widetilde{A}}_{12} \otimes x_{1j}) \oplus (\hat{\widetilde{A}}_{22} \otimes x_{2j}) \oplus (\hat{\widetilde{A}}_{32} \otimes x_{3j}) \\
&= (31.3831; 2.0734)_T \oplus ((3.5564; 0.3647)_T \otimes x_{1j}) \oplus ((0.4012; 0.0408)_T \otimes x_{2j}) \oplus ((1.8524; 0.1858)_T \otimes x_{3j}).
\end{aligned}$$

Hence, the estimated values of response variables $\hat{\widetilde{Y}}_j$ are obtained as follows (see third column of Table 8)

$$\hat{\widetilde{Y}}_j = (\mu_{1j} \otimes \hat{\widetilde{Y}}_j^1) \oplus (\mu_{2j} \otimes \hat{\widetilde{Y}}_j^2), \quad j = 1, ..., 13.$$

Now, suppose that we observe the new values $\underline{x}_{new} = (4, 41, 10)$. Based on Definition 3.2, we obtain the response values for $c = 1, 2$ as

$$\begin{aligned}
\hat{\widetilde{Y}}_{new}^1 &= (129.1671; 7.6353)_T, \\
\hat{\widetilde{Y}}_{new}^2 &= (80.5819; 7.0630)_T.
\end{aligned}$$

Also, based on Definition 3.2 with $r = 3$, we obtain $U = \{2, 4, 5\}$, and the membership degrees is presented as $(\mu_{1,new}, \mu_{2,new}) = (0.3561, 0.6439)$. Hence, the new response value $\hat{\widetilde{Y}}_{new}$ is predicted as

$$\begin{aligned}
\hat{\widetilde{Y}}_{new} &= (\mu_{1,new} \otimes \hat{\widetilde{Y}}_{new}^1) \oplus (\mu_{2,new} \otimes \hat{\widetilde{Y}}_{new}^2) \\
&= (97.8831; 7.2668)_T.
\end{aligned}$$

Table 7: Data set of composition Portland cement

No.	x_{1j}	x_{2j}	x_{3j}	$\tilde{y}_j = (y_j; l_{y_j})_T$
1	7	26	6	$(78.5, 6.9)_T$
2	1	29	15	$(74.3, 6.4)_T$
3	7	26	6	$(104.3, 9.4)_T$
4	11	31	8	$(87.6, 7.8)_T$
5	7	52	6	$(87.6, 7.8)_T$
6	11	55	9	$(109.2, 9.9)_T$
7	3	71	17	$(102.7, 9.3)_T$
8	1	31	22	$(72.5, 6.2)_T$
9	2	54	18	$(93.1, 8.3)_T$
10	21	47	4	$(115.9, 10.6)_T$
11	1	40	23	$(83.8, 7.4)_T$
12	11	66	9	$(113.3, 10.6)_T$
13	10	68	8	$(109.4, 9.9)_T$

Table 8: Membership degrees in composition Portland cement

No.	(μ_{1j}, μ_{2j})	$\hat{\tilde{y}}_j = (\hat{y}_j; \hat{l}_{y_j})_T$	D_{ij}	$SM(\widetilde{Y}_j, \hat{\widetilde{Y}}_j)$
1	$(0.0805, 0.9195)$	$(79.8321; 6.9113)_T$	1.3321	0.8164
2	$(0.1350, 0.8650)$	$(76.8227; 6.5008)_T$	2.5240	0.6471
3	$(0.9863, 0.0137)$	$(102.4300; 8.1527)_T$	2.1293	0.7983
4	$(0.9265, 0.0735)$	$(94.3047; 8.8824)_T$	6.7627	0.3577
5	$(0.0068, 0.9932)$	$(88.7410; 7.8632)_T$	1.1422	0.8596
6	$(0.0178, 0.9822)$	$(109.8380; 9.9797)_T$	0.6413	0.9368
7	$(0.0044, 0.9956)$	$(102.3320; 9.2131)_T$	0.3748	0.9606
8	$(0.9640, 0.0360)$	$(70.6415; 7.1244)_T$	2.0059	0.7405
9	$(0.0093, 0.9907)$	$(93.8693; 8.3389)_T$	0.7700	0.9097
10	$(0.9686, 0.0314)$	$(115.1000; 10.7266)_T$	0.8067	0.9264
11	$(0.9150, 0.0850)$	$(86.4939; 7.2052)_T$	2.6986	0.6651
12	$(0.0065, 0.9935)$	$(114.0010; 10.4379)_T$	0.7134	0.9345
13	$(0.0028, 0.9972)$	$(109.2420; 9.9756)_T$	0.1696	0.9842
			$MD = 1.6977$	$MSM = 0.8105$

Table 9: Comparison with some other approaches in composition Portland cement

Approach	MD	MSM
Proposed model	1.6977	0.8105
Arefi [3]	4.0305	0.7127
Xu and Li [28]	4.0751	0.6992

The other authors presented some fuzzy linear regression models on this data set. By applying the approaches introduced by Arefi [3] and Xu and Li [28], the fuzzy regression models are given as follows:

$$(Arefi): \hat{\tilde{Y}}_A = (47.0142; 4.1465)_T \oplus ((1.7379; 0.1600)_T \otimes x_1) \oplus ((0.6538; 0.0611)_T \otimes x_2) \oplus (0.3867 x_3),$$

$$(Xu\ and\ Li): \hat{\tilde{Y}}_{Xu} = (48.1934; 3.8693)_T \oplus ((1.6959; 0.1688)_T \otimes x_1) \oplus ((0.6569; 0.0651)_T \otimes x_2) \oplus ((0.2500; 0.0233)_T \otimes x_3).$$

Based on the above models, we compare them based on the goodness of fit indices. The results are summarized in Table 9. Hence, the proposed model in this paper is better for this data set.

5 Conclusions

In this paper, a new approach is presented to fit some clustering fuzzy linear regression models. Some certain merits of the proposed approach are provided as follows:

I) It is a new hybrid approach to formulate some clustering fuzzy linear regression models based on the fuzzy c-means clustering and geometric mean. In this approach, we first obtain the membership degrees based on the fuzzy c-means clustering, and then, we fit some fuzzy linear regression models based on geometric mean.

II) It is a new version of clustering fuzzy linear regression models when the data are as crisp or fuzzy (fuzzy response variable and the fuzzy parameters).

III) To evaluate the goodness-of-fit of clustering regression models, two indices are provided based on the similarity measure and the distance between the observed and estimated response variables. Also, the application of the proposed approach on some real data sets is studied.

The proposed approach is suitable for fitting the clustering regression models on the data sets consisting of linear clusters. The extension of proposed approach on non-linear clusters can be studied in the future research. Also, we can extended the proposed approach to formulate the clustering regression models in interval-valued fuzzy environment.

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