Iranian Journal of Fuzzy Systems

Volume 22, Number 3, (2025), pp. 125-138

Original paper

University of Sistan and Baluchestan

Some results on Stonean filters in equality algebras

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Abstract

The present paper aims to study the notion of Stonean filters on equality algebras and discusses the connection between these filters and other types of filters on equality algebras. We show that any implicative filter, positive implicative filter and Boolean filter is a Stonean filter on equality algebras and by some counterexample, we show that the vice versa do not generally hold. In addition, it is checked that the relations of Stonean filter with implicative filter, fantastic filter, Boolean filter, prime filter, maximal filter, ultra filter and positive implicative filter on equality algebras, while the implicative, positive implicative, Boolean and Stonean filters are equivalent together under the condition involutive on an equality algebra. The concept of primary filter is introduced, and its relationship with the prime filter is investigated. Finally, Stonean radical filter ($Rad_s(F)$) is introduced according to the concept of Stonean filter on equality algebras, and we prove that under a suitable condition if F is a prime filter, then $Rad_s(F) = F$. Also, in this article, we provide several examples to make the definitions clearer.

Keywords: Equality algebra, (Stonean, implicative, fantastic, Boolean, prime, maximal, primary, Stonean radical) filter.

1 Introduction

The concept of equality algebras was introduced for the first time in 2012 by Jenei [6]. This algebra is actually an extension of EQ-algebra which was introduced, by Novák and De Baets [8]. Indeed, the equality algebra is similar to EQ-algebras, but without a product [7]. Every good EQ-algebra is an equality algebra but the converse is not true, in general [10]. The relation between equality algebra and some other algebraic structures was obtained in [10], and also, under special conditions, they are equivalent. Jenei and Kóródi [7] proposed the notions of filters and congruence relations on equality algebras and obtained some of their properties. Filter theory plays an important role in studying ordered algebras and accordingly in equality algebras. Moreover, some types of filters, such as (positive) implicative, fantastic, Boolean, and prime filters, on equality algebras were introduced by Borzooei et al. [3]. From a logical point of view, various filters have a natural interpretation as various sets of provable formulas, which have a very close relationship with decision-making; see [11]. There is a one-to-one correspondence between the set of all congruence relations and the set of all filters in equality algebra [4]. It is proved that positive implicative filters, implicative filters, and Boolean filters are equivalent on bounded commutative equality algebras [3]. The concept of Stonean radical on commutative algebras was presented by Zariski and Atiyah [2, 9]. They have worked on the Stonean radical of ideals in commutative ring. Inspired by them, we present the concept of Stonean radical in equality algebras and derive some results.

Our motivation for writing this paper is to introduce Stonean filter as a new filter on equality algebras. Our goal is to investigate the relationship between Stonean filters and some other filters by assuming conditions and restrictions on equality algebras.

Corresponding Author: M. M. Zahedi

Received: March 2025; Accepted: June 2025.

https://doi.org/10.22111/ijfs.2025.9268

In this paper, it has been tried to present numerous examples while making the presented concepts more concrete, and they have also been used to state that the converse of a theorem is not true. In section 2, we review some basic definitions and results about equality algebras, which have been mentioned in other articles. In section 3, we examine the relationship between Stonean filters and other defined filters on equality algebras. Furthermore, we show that, in general, Stonean filters are not prime, and also vice versa. Then we prove that the implicative filters and positive implicative filters are Stonean. Also, we derive that filters of Stonean, implicative, positive implicative, and Boolean are equivalent on involutive equality algebras. In section 4, we introduce the concept of primary filters and examine its relationship with Stonean and prime filters. Moreover, we check under what condition Stonean filters and primary filters are equivalent. In section 5, we introduce the concept of Stonean radical filters according to the concept of Stonean filters on equality algebras, and we prove that under certain conditions, Stonean radical of a prime filter is a prime filter. At the end, we have tried to summarize the obtained results in a formal of diagrams.

2 Preliminaries

In this section, we state some concepts and theorems related to equality algebra, that we will use or refer them in the forthcoming sections.

Definition 2.1. [6] *An equality algebra is an algebraic structure* $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$ *of type* $(2, 2, 0)$ *such that the following axioms are fulfilled, for all* $x, y, z \in X$*:*

(E1) $\langle X, \wedge, 1\rangle$ *is a commutative idempotent integral monoid (i.e.* $\wedge$*-semi-lattice with top element 1),*
(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ *implies* $x \sim z \leq y \sim z$ *and* $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called *meet* (infimum) and $\sim$ is an *equality* operation. We write $x \leq y$ if and only if $x \wedge y = x$, as usual. Define the following two derived operations, the *implication* and the *equivalence* operation of the equality algebra $\mathcal{X}$, by

$$x \to y = x \sim (x \wedge y) \quad \text{and} \quad x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

An equality algebra $\mathcal{X}$ is a *lattice equality algebra* if X is a lattice.
A *bounded equality algebra* is an equality algebra that has the least element 0 such that $0 \leq x$, for all $x \in X$. If X is bounded, then we define the negation "$'$" on X, by $x' = x \to 0 = x \sim 0$, for all $x \in X$. If $(x')' = x$, for all $x \in X$, then the bounded equality algebra $\mathcal{X}$ is called *involutive*. An equality algebra $\mathcal{X}$ is called *prelinear*, if 1 is the unique upper bound of the set $\{x \to y, y \to x\}$, for all $x, y \in X$.

Proposition 2.2. [6] *Let* $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$ *be an equality algebra. Then the following statements hold, for all* $x, y, z \in X$*:*

(*i*) $x \sim y \leq x \leftrightarrow y \leq x \to y$,
(*ii*) $x \leq (x \sim y) \sim y$,
(*iii*) $x \sim y = 1$ *if and only if* $x = y$,
(*iv*) $x \to y = 1$ *if and only if* $x \leq y$,
(*v*) $x \to y = 1$ *and* $y \to x = 1$ *imply* $x = y$,
(*vi*) $1 \to x = x$, $x \to 1 = 1$, $x \to x = 1$,
(*vii*) $x \leq y \to x$,
(*viii*) $x \leq (x \to y) \to y$,
(*ix*) $x \to y \leq (y \to z) \to (x \to z)$,
(*x*) $x \leq y \to z$ *if and only if* $y \leq x \to z$,
(*xi*) $x \to (y \to z) = y \to (x \to z)$,
(*xii*) $y \leq x$ *implies* $x \leftrightarrow y = x \to y = x \sim y$.

Proposition 2.3. [10] *Let* $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$ *be an equality algebra. Then the following statements hold, for all* $x, y, z \in X$*:*

(*i*) $x \leq y$ *implies* $y \to z \leq x \to z$, $z \to x \leq z \to y$,
(*ii*) $x \to y = x \to (x \wedge y)$,
(*iii*) $x \sim y \leq (z \to x) \sim (z \to y)$,
(*iv*) $x \sim y \leq (z \to x) \to (z \to y)$,

(v) $x \to y \leq (z \to x) \to (z \to y)$,
(vi) $x \to y \leq (x \wedge z) \to (y \wedge z)$,
(vii) $x \to y = ((x \to y) \to y) \to y$.

Theorem 2.4. [10] *Let $\mathcal{X} = \langle X, \sim, \wedge, 0, 1\rangle$ be a bounded lattice equality algebra. Then for all $x, y \in X$, the following properties hold:*
(i) $(x \vee y)' = x' \wedge y'$,
(ii) $x \leq (x')'$,
(iii) $x \to y \leq y' \to x'$, *and if it is involutive, then* $x \to y = y' \to x'$.

Definition 2.5. [7] *Let $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra and $F \subseteq X$. Then F is called a* deductive system *or* filter *of $\mathcal{X}$ if for all $x, y \in X$, we have:*
(i) $1 \in F$,
(ii) *If $x \in F$ and $x \leq y$, then $y \in F$,*
(iii) *If $x \in F$ and $x \sim y \in F$, then $y \in F$.*

A filter F of $\mathcal{X}$ is called a *proper filter* if $F \neq X$, and the set of all filters of $\mathcal{X}$ is denote by $\mathcal{F}(\mathcal{X})$.

Proposition 2.6. [7] *Let $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$ be an equality algebra. Then $F \in \mathcal{F}(\mathcal{X})$ if and only if, for all $x, y \in X$:*
(i) $1 \in F$,
(ii) *If $x \in F$ and $x \to y \in F$, then $y \in F$.*

Note. The definitions of filters implicative, positive implicative, fantastic, and maximal on equality algebras can be seen in [3].

Proposition 2.7. ([3], Proposition 3.16)
Let F be a filter of bounded equality algebra $\mathcal{X}$. Then F is an implicative filter on $\mathcal{X}$ if and only if $x' \to x \in F$ implies $x \in F$, for all $x \in X$.

Proposition 2.8. [3] *Let F be a filter of $\mathcal{X}$. Then F is a positive implicative filter if and only if, for all $x, y \in X$, $(x \wedge (x \to y)) \to y \in F$.*

Corollary 2.9. [3] *In every involutive equality algebra, implicative filters and positive implicative filters coincide.*

Definition 2.10. [3] *Let $\mathcal{X}$ be a bounded lattice equality algebra. Then a filter F of $\mathcal{X}$ is called a Boolean filter if $x \vee x' \in F$, for all $x \in X$.*

Theorem 2.11. [3] *Suppose that $\mathcal{X}$ is a bounded lattice equality algebra and F is a filter of $\mathcal{X}$. Then F is a Boolean filter if and only if F is an implicative filter.*

Definition 2.12. [3] *A proper filter F of equality algebra $\mathcal{X}$ is called a prime filter if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in X$.*

Theorem 2.13. [3] *Let F be a filter of equality algebra $\mathcal{X}$. Then the following conditions are equivalent:*
(i) *F is a fantastic filter on $\mathcal{X}$,*
(ii) *$y \to x \in F$ implies $((x \to y) \to y) \to x \in F$, for all $x, y \in X$,*
(iii) *If $\mathcal{X}$ is a lattice, then $((x \to y) \to y) \to ((y \to x) \to x) \in F$, for all $x, y \in X$.*

3 Stonean filters

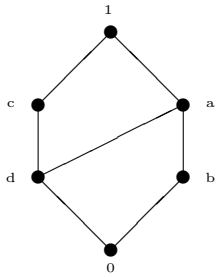
In this section, we will introduce the concept of Stonean filters on equality algebras and then examine its relationship with some other filters. We show that every implicative filter, positive implicative filter, and Boolean filter are Stonean filters, but in general, the converse is not true.

Note. From now on, the symbol $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$ is a lattice equality algebra unless otherwise stated.

Definition 3.1. *Let $\mathcal{X}$ be bounded. Then a filter F on $\mathcal{X}$ is called a* Stonean filter *if $x^{'} \vee x^{''} \in F$, for all $x \in X$.*

Denote by $\mathcal{ST}(\mathcal{X})$ the set of all Stonean filters on $\mathcal{X}$.

Example 3.2. *Let $\mathcal{X} = \langle X, \sim, \wedge, 0, 1\rangle$ be a bounded with the following Hasse diagram. Define the operations $\sim$ and $\rightarrow$ on X as follows:*

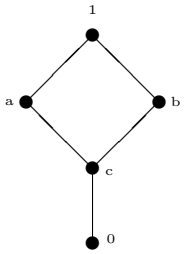


$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

We show that $F = \{a, b, 1\}$ is a Stonean filter, because $0^{'} \vee 0^{''} = 1 \vee 0 = 1 \in F$, $a^{'} \vee a^{''} = d \vee a = a \in F$, $b^{'} \vee b^{''} = c \vee b = 1 \in F$, $c^{'} \vee c^{''} = b \vee c = 1 \in F$, $d^{'} \vee d^{''} = a \vee d = a \in F$, $1^{'} \vee 1^{''} = 0 \vee 1 = 1 \in F$.

Example 3.3. *Let $(X = \{0, c, a, b, 1\}, \leq)$ be a bounded lattice with the following Hasse diagram. Define the operations $\sim$ and $\rightarrow$ on X as follows:*



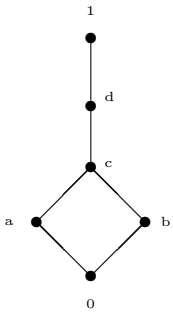
$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

By routine calculations, we can see $\langle X, \sim, \wedge, 0, 1\rangle$ is an equality algebra and $F = \{b, 1\}$ is a Stonean filter.

In the next example, we give a filter which is not Stonean.

Example 3.4. *Let $(X = \{0, c, a, b, d, 1\}, \leq)$ be a bounded lattice with the following Hasse diagram. Define the operations $\sim$ and $\rightarrow$ on X as follows:*



$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

Then, by routine calculations, we can see that $\langle X, \sim, \wedge, 0, 1\rangle$ is a bounded equality algebra. It is not difficult to check that $F = \{1\}$ is a filter on X, while F is not a Stonean filter because $d' \vee d'' = c \vee d = d \notin F$.

Theorem 3.5. *(Extension theorem for Stonean filters)*
Let $\mathcal{X}$ be bounded, F be a Stonean filter and let G be a filter of $\mathcal{X}$ such that $F \subseteq G$. Then so is G.

Remark 3.6. *If $\mathcal{X}$ is bounded, then the intersection of Stonean filters on $\mathcal{X}$ is a Stonean filter.*

Note. It is not necessary that the union of two or more Stonean filters is a Stonean filter.

Example 3.7. *Let $\mathcal{X}$ be the equality algebra of Example 3.3 and $F_1 = \{a, 1\}$, $F_2 = \{b, 1\}$. It is clear that $F_1, F_2 \in \mathcal{ST}(\mathcal{X})$, while*

$$F_1 \cup F_2 = \{a, b, 1\},$$

is not Stonean, because $a, a \rightarrow c \in \{a, b, 1\}$ but $c \notin \{a, b, 1\}$.

Theorem 3.8. *Let $\mathcal{X}$ be bounded and let F be an implicative filter. Then F is a Stonean filter.*

Proof. Since $x' \wedge x'' \leq x' \leq x' \vee x$, so $x' \wedge x'' \leq x' \vee x$ and hence $(x' \wedge x'') \to (x' \vee x) = 1$, by Proposition 2.2$(iv)$. By Theorem 2.4$(i)$, we have $(x \vee x')' \to (x' \vee x) = 1$. Because F is a filter and $1 \in F$, thus $(x \vee x')' \to (x' \vee x) \in F$. Since F is an implicative filter, so $x' \vee x \in F$, by Proposition 2.7. Because $x' \vee x \leq x' \vee x''$ and $x' \vee x \in F$, so $x' \vee x'' \in F$, for all $x \in X$, by Theorem 2.4(ii). Therefore F is a Stonean filter. □

In the next example, we show that the converse of Theorem 3.8 does not hold, in general.

Example 3.9. *Let $\mathcal{X}$ be the equality algebra of Example 3.3. We know that $F = \{b, 1\}$ is a Stonean filter. For $c \in X$, we have $c' \to c = 0 \to c = 1 \in F$, where $c \notin F$. Thus by Proposition 2.7, F is not implicative.*

Next, we will examine the connection between Stonean filters and Boolean filters.

Proposition 3.10. *Let $\mathcal{X}$ be bounded and let F be a Boolean filter. Then F is a Stonean filter.*

Proof. For any $x \in X$ also $x' \in X$. Since F is a Boolean filter, $x' \vee x'' \in F$. Thus F is a Stonean filter. □

In the following example, we show that the converse of Proposition 3.10 is not satisfied, in general.

Example 3.11. *Let $\mathcal{X}$ be the equality algebra of Example 3.3 and let $F = \{b, 1\}$. Then F is a Stonean filter, but it is not Boolean, because for $c \in X$, we have $c \vee c' = c \vee 0 = c \notin F$.*

Theorem 3.12. *Let F be a positive implicative filter and let $\mathcal{X}$ be bounded. Then F is a Stonean filter.*

Proof. We show that $x' \vee x'' \in F$, for all $x \in X$. Put $y = 0$, in Proposition 2.8, so $(x \wedge (x \to 0)) \to 0 \in F$. Thus:

$$\begin{aligned}(x \wedge (x \to 0)) \to 0 &= (x \wedge x') \to 0 \\ &= (x \wedge x')' \\ &= x' \vee x'', \qquad \text{by Theorem 2.4}(i).\end{aligned}$$

Therefore $x' \vee x'' \in F$, and F is a Stonean filter on $\mathcal{X}$. □

In the following example, we show that the converse of Theorem 3.12 is not true in general. Furthermore, we show that in the general case, a Stonean filter is not a prime filter and vice versa. In other words, there is no relationship between these two classes of filters.

Example 3.13. (i) *Let $(X = \{0, a, b, c, 1\}, \leq)$ be a chain. Define the operations $\sim$ and $\to$ on $\mathcal{X}$, by*

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	c	b
c	0	b	c	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	c	1	1
1	0	a	b	c	1

By routine calculations, we can see that $\mathcal{X} = \langle X, \sim, \wedge, 0, 1\rangle$ is a bounded equality algebra. We know that $F = \{1\}$ is a Stonean filter, but F is not positive implicative filter because $(c \wedge (c \to b)) \to b = (c \wedge c) \to b = c \to b = c \notin F$, for $b, c \in X$, by Proposition 2.8.
(ii) *In Example 3.3, let $F = \{1\}$. Thus F is a Stonean filter, while F is not a prime filter because $a \to b = b \notin F$ and $b \to a = a \notin F$, for $a, b \in X$, by Definition 2.12.*
(iii) *In Example 3.2, $F = \{c, 1\}$ is a prime filter, but it is not a Stonean filter because $d' \vee d'' = a \vee d = a \notin F$, for $d \in X$.*

Corollary 3.14. Let $F \in \mathcal{F}(\mathcal{X})$ and $\mathcal{X}$ be bounded such that $x' = 0$, for all $0 \neq x \in X$. Then F is a Stonean filter.

Proof. Indeed, let $x \in X$, and $x = 0$. Then $x' \vee x'' = 0' \vee 0'' = 1 \vee 0 = 1 \in F$. If $x \neq 0$, then

$$x' \vee x'' = 0 \vee 0' = 0 \vee 1 = 1 \in F.$$

Thus F is a Stonean filter. □

Note. In Example 3.3, all filters are Stonean.

In the following example, we show that, in general, the Stonean filter and the fantastic filter are not related to each other.

Example 3.15. (*i*) *Let* $(X = \{0, a, b, 1\}, \leq)$ *be a chain. Define the operations* $\sim$ *and* $\to$ *on* $\mathcal{X}$*, by*

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then, by some computations, we can see that $\mathcal{X} = \langle X, \sim, \wedge, 0, 1\rangle$ *is a bounded equality algebra, and* $F = \{b, 1\}$ *is a fantastic filter on* $\mathcal{X}$*. For* $a \in X$*, we have* $a' \vee a'' = a \vee a = a \notin F$*, thus* F *is not a Stonean filter.*
(*ii*) *In Example 3.3,* $G = \{b, 1\}$ *is a Stonean filter. Since* $((c \to 0) \to 0) \to ((0 \to c) \to c) = (0 \to 0) \to (1 \to c) = 1 \to c = c \notin G$*, so* G *is not a fantastic filter, for* $0, c \in X$*, by Theorem 2.13*(*iii*) $\Rightarrow$ (*i*).

Proposition 3.16. *Let* $\mathcal{X}$ *be bounded and let* F *be a proper filter. Then if for any* $x \in X$*, we have* $x \in F$ *or* $x' \in F$*, then* F *is a prime.*

Proof. Let $x \in X$. By hypothesis $x \in F$ or $x' \in F$. Now, let $y \in X$ be an arbitrary element. If $x \in F$, then by Proposition 2.2(*vii*) we have $x \leq y \to x$. Therefore by Definition 2.5, we get that $y \to x \in F$. On the other hand, if $x' \in F$, then by Proposition 2.3(*i*) we conclude that $x' \leq x \to y$. Thus $x \to y \in F$. Therefore $x \to y \in F$ or $y \to x \in F$, for all $x, y \in X$ and hence F is a prime filter, by Definition 2.12. □

In the next example, we show that in general case maximal filters and Stonean filters are not related to each other.

Example 3.17. (*i*) *In Example 3.2,* $F = \{c, 1\}$ *is a maximal filter. Since* $a' \vee a'' = d \vee a = a \notin F$ *for* $a \in X$*, so* F *is not Stonean.*
(*ii*) *Let* $\mathcal{X}$ *be the equality algebra of Example 3.3. Then* $G_1 = \{b, 1\}$ *is a Stonean filter on* $\mathcal{X}$*. It is clear that* $G_2 = \{a, b, c, 1\}$ *is a filter on* $\mathcal{X}$*. Since* $G_1 \subseteq G_2$*, therefore* G_1 *is not maximal.*

Definition 3.18. [5] *A proper filter of a bounded equality algebra* $\mathcal{X}$ *is called an ultra filter, whenever* $x \in F$ *if and only if* $x' \notin F$*, for any* $x \in X$*.*

Example 3.19. *Consider the filter* $F = \{c, 1\}$ *on the equality algebra of Example 3.2. Then* F *is an ultra filter.*

In the following, we will examine the relationship between Stonean filters and the ultra filters. We show every Stonean filter is an ultra filter, but the converse is not true, in general.

Theorem 3.20. *Let* $\mathcal{X}$ *be bounded and* F *be a proper Stonean filter. Then* F *is an ultra filter.*

Proof. On the contrary, F is not an ultra filter, then there exists $x \in F$, such that $x' \in F$. Since $x' = x \to 0$ and $x \in F$, so $0 \in F$, by Proposition 2.6. Therefore $F = X$, which is a contradiction. □

Note. In Example 3.19, F is an ultra filter while it is not Stonean.

Remark 3.21. *Although in Definition* 2.4 *of* [1]*, the ultra filter is considered equivalent to the maximal filter, but in the following example we show that it is not true in general.*

Example 3.22. *In Example 3.3, let* $F_1 = \{b, 1\}$ *and* $F_2 = \{a, b, c, 1\}$*. We know that* $F_1, F_2 \in \mathcal{F}(\mathcal{X})$*. Since* $b' = 1' = 0 \notin F_1$*, we conclude that* F_1 *is an ultra filter. But clearly it is not a maximal filter, because* $F_1 \subseteq F_2 \neq X$*.*

In the following, by putting a condition on equality algebras or filters of an equality algebra, we find some relationship between Stonean filters and some other filters.

Theorem 3.23. *Let* $\mathcal{X}$ *be involutive. Then Stonean filters, Boolean filters, implicative filters and positive implicative filters coincide.*

Proof. According to Corollary 2.9, Theorem 2.11 and Proposition 3.10, it is enough to prove that every Stonean filter is a Boolean filter, on involutive equality algebra.
Let F be a Stonean filter. Then we show that F is a Boolean filter. It suffices to show that $x' \vee x \in F$, for all $x \in X$, by Definition 2.10. Since F is a Stonean filter, so $x' \vee x'' \in F$, for all $x \in X$. Because $\mathcal{X}$ is involutive, so $x' \vee x'' = x' \vee x$, for all $x \in X$. Thus $x' \vee x \in F$, for all $x \in X$. □

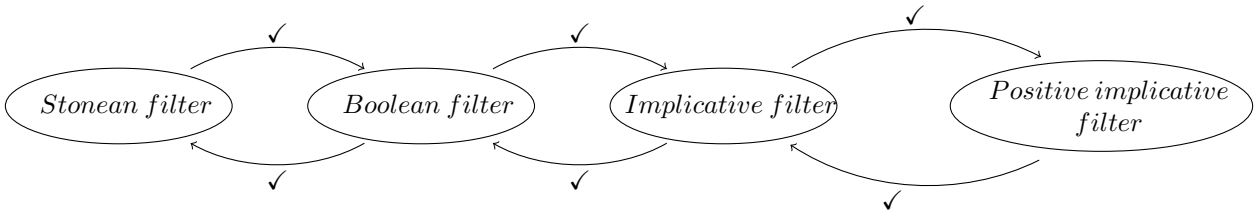


Figure 1: The relationship between Stonean filter and other types of filters on involutive equality algebra.

Theorem 3.24. *Let $\mathcal{X}$ be bounded such that $x \to x' \leq x$, for all $0 \neq x \in X$ and let F be a prime filter on $\mathcal{X}$. Then F is a Stonean filter.*

Proof. It is enough to prove that $x' \vee x'' \in F$, for all $x \in X$. If $x = 0$, then clearly $x' \vee x'' = 1 \in F$. Let $0 \neq x \in X$. Since $0 \neq x \in X$ and $x \leq x''$, so $x'' \neq 0$, by Proposition 2.4(ii). Hence we get:

$$\begin{aligned} x'' \to (x'')' \leq x'' &\Rightarrow x'' \to x' \leq x'', && \text{by Proposition 2.3}(vii) \\ &\Rightarrow x'' \to x' \leq x' \vee x''. && (3.1) \end{aligned}$$

On the other hand, since F is a prime filter, so $x'' \to x' \in F$ or $x' \to x'' \in F$. Now, if $x'' \to x' \in F$, then from (3.1), we conclude that $x' \vee x'' \in F$ and in this case the theorem is solved.
If $x' \to x'' \in F$ and $x' = 0$, then clearly $x' \vee x'' = 1 \in F$. So let $x' \to x'' \in F$ and $x' \neq 0$. Then $x' \to x'' \leq x'$, by hypothesis. Since $x' \to x'' \leq x' \vee x''$ and $x' \to x'' \in F$, so $x' \vee x'' \in F$. Therefore we conclude that F is Stonean. □

In the following example, we show that there is an equality algebra that satisfies the condition of Theorem 3.24, and this condition is also necessary for this theorem.

Example 3.25. (i) *In Example 3.15(i), $F = \{b, 1\}$ is a prime filter, while it is not a Stonean filter because for $a \in X$, $a' \vee a'' = a \vee a = a \notin F$. Note that the condition $x \to x' \leq x$, for all $0 \neq x \in X$, is not satisfied in Theorem 3.24. Because for $0 \neq a \in X$, we have:*

$$a \to a' = a \to a = 1 \not\leq a.$$

Now, we give an example to show that the condition Theorem 3.24 is satisfied.
(ii) *Let $\mathcal{X}$ be the equality algebra of Example 3.3. It is not difficult to show that $x \to x' \leq x$, for all $0 \neq x \in X$. It is clear that $F = \{a, 1\}$ is a prime filter on this equality algebra. Since $x' \vee x'' = 0 \vee 1 = 1 \in F$, for all $x \in X$, therefore F is a Stonean filter.*

Theorem 3.26. *Let F be a Stonean filter on $\mathcal{X}$ such that $x \to y \in F$ implies $y \to x \in F$, for all $x, y \in X$. Then F is a fantastic filter.*

Proof. Let $y \to x \in F$. By Theorem 2.13$(ii) \Rightarrow (i)$, it is enough to show that $((x \to y) \to y) \to x \in F$, for all $x, y \in X$. We have:

$$\begin{aligned} y \to x &\leq ((x \to y) \to y) \to ((x \to y) \to x), && \text{by Proposition 2.3}(v) \\ &= (x \to y) \to (((x \to y) \to y) \to x). && \text{by Proposition 2.2}(xi) \end{aligned}$$

Since $y \to x \in F$, so $(x \to y), (x \to y) \to (((x \to y) \to y) \to x) \in F$, by the assumption of the theorem and Definition 2.5(ii). Therefore $((x \to y) \to y) \to x \in F$, for all $x, y \in X$, by Proposition 2.6(ii). Thus F is a fantastic filter. □

In the next example, we show that the condition "if $x \to y \in F$, then $y \to x \in F$, for all $x, y \in X$" in Theorem 3.26 is necessary.

Example 3.27. *In Example 3.15(ii), $G = \{b, 1\}$ is a Stonean filter, while it is not a fantastic filter. For $a, b \in X$, we have:*

$$a \to b = b \in F, \text{ but } b \to a = a \notin F.$$

As we see that the above mention condition that not hold, thus this condition is necessary in Theorem 3.26.

Now, we give an example to show that the condition of Theorem 3.26 is satisfied.

Example 3.28. *Let $\mathcal{X}$ be the equality algebra of Example 3.13(i) and $F = \{a, b, c, 1\}$. It is not difficult to show that the condition of Theorem 3.26 is satisfied and F is a Stonean filter. We know that F is fantastic.*

4 Primary filters

In this section, we define the concept of a primary filter on equality algebras and examine the relationship of this filter with some other filters. Also, we show that if $\mathcal{X}$ is prelinear equality algebra, then every prime filter is a primary filter, while the converse is not true, in general.

Note. From now on, we assume that $\mathcal{X} = \langle X, \sim, \wedge, \vee, 0, 1\rangle$ is a bounded lattice equality algebra.

Definition 4.1. *Let F be a proper filter on $\mathcal{X}$. Then F is a primary filter if $x \vee y \in F$ implies $x \in F$ or $y^n \in F$, for all $x, y \in X$. Symbol y^n is defined inductively for $n \in N$, $y^1 = y$ and $y^n = y^{n-1} \to y'$, for $n \geq 2$.*

Example 4.2. (*i*) *Let $\mathcal{X}$ be the equality algebra of Example 3.2. Then for every $n, k \in \mathbb{N}$, we get:*

$$x^n = \begin{cases} 0, & \text{if} \quad x = 1 \quad \text{and} \quad n = 2k, \\ b, & \text{if} \quad x = c \quad \text{and} \quad n = 2k, \\ 1, & \text{if} \quad x = d \quad \text{and} \quad n = 2k, \\ a, & \text{if} \quad x = a \text{ or } d \quad \text{and} \quad n = 2k+1, \\ c, & \text{if} \quad x = a \text{ or } b \quad \text{and} \quad n = 2k, \\ 1, & \text{if} \quad x = 0, b, c \text{ or } 1 \quad \text{and} \quad n = 2k+1. \end{cases}$$

(*ii*) *Let $\mathcal{X}$ be the equality algebra of Example 3.13(i). Then, for any $n, k \in \mathbb{N}$, we have:*

$$x^n = \begin{cases} 1, & \text{if} \quad x = 0 \quad \text{for all} \quad n \geq 2, \\ 0, & \text{if} \quad x \neq 0 \quad \text{and} \quad n = 2k, \\ 1, & \text{if} \quad x \neq 0 \quad \text{and} \quad n = 2k+1. \end{cases}$$

(*iii*) *Let $\mathcal{X}$ be the equality algebra of Example 3.4. Then, for any $n, k \in \mathbb{N}$, we get:*

$$x^n = \begin{cases} 1, & \text{if} \quad x = 0 \quad \text{and for all} \quad n \geq 2, \\ d, & \text{if} \quad x = d \quad \text{and for all} \quad n \in \mathbb{N}, \\ 1, & \text{if} \quad x = a, b \text{ or } c \quad \text{and} \quad n = 2k, \\ d, & \text{if} \quad x = a, b \text{ or } c \quad \text{and} \quad n = 2k+1, \\ 0, & \text{if} \quad x = 1 \quad \text{and} \quad n = 2k, \\ 1, & \text{if} \quad x = 1 \quad \text{and} \quad n = 2k-1. \end{cases}$$

(*iv*) *In Example 3.3, $F = \{a, 1\}$ is a primary filter. It is easy to check that for all $x, y \in X$, if $x \vee y \in F$ implies $x \in F$ or $y^n \in F$, for some $n \in \mathbb{N}$.*

Proposition 4.3. Let $\mathcal{X}$ be prelinear. Then every prime filter is a primary filter.

Proof. Let $x \vee y \in F$. Since F is prime, so by Theorem 5.11 of [3], we get that $x \in F$ or $y \in F$. Thus F is primary by Definition 4.1. □

Open Problem. Is the condition " $\mathcal{X}$ to be prelinear " necessary in the above Proposition?

In next example, we show that in general the converse of Proposition 4.3 is not true.

Example 4.4. *Let $\mathcal{X}$ be the equality algebra of Example 3.3 and let $F = \{1\}$. It is not difficult to show that $\mathcal{X}$ is prelinear and F is a primary filter, while it is not a prime filter, because $a \vee b = 1$, but $a, b \notin F$.*

Next, we will examine the relationship between primary filters and Stonean filters on $\mathcal{X}$. In the next example, we show that a primary filter is not generally a Stonean filter.

Example 4.5. *Let $\mathcal{X}$ be the equality algebra of Example 3.2 and let $F = \{c, 1\}$. It is clear that F is a primary filter. Since $d \in X$ and $d' \vee d'' = a \vee d = a \notin F$, so F is not Stonean.*

Theorem 4.6. *Let F be a proper filter of $\mathcal{X}$ and let $x' = 0$, for all $0 \neq x \in X$. Then F is a primary filter.*

Proof. Let $x \vee y \in F$ and $x \notin F$. Then $y \neq 0$, otherwise we obtain a contradiction

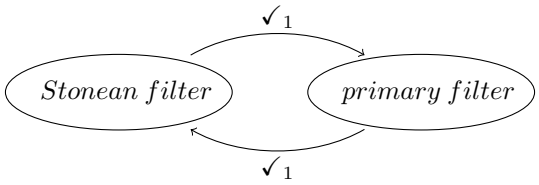
$$x \vee y = x \vee 0 = x \in F.$$

Computing powers of y, we get that $y^1 = y$, $y^2 = y \to y' = y' = 0$ and $y^3 = y^2 \to y' = 0 \to 0 = 1 \in F$. So $y^3 \in F$ and F is a primary filter. □

In the following example, we show that in general the converse of Theorem 4.6 is not true.

Example 4.7. (*i*) *In Example 3.2, let* $F = \{1\}$. *It is clear that* F *is a primary filter while the condition* $x' = 0$, *for all* $0 \neq x \in X$ *is not satisfied for* $\mathcal{X}$.
(*ii*) *Let* $\mathcal{X}$ *be the equality algebra of Example 3.3 and let* $F = \{a, 1\}$. *We know that* $x' = 0$, *for all* $0 \neq x \in X$. *It is clear that* F *is a primary filter.*

Note. Although in the general case it is not necessary that the filters Stonean, Boolean, positive implicative, implicative and fantastic are proper, nevertheless, in the rest of this section, we assume these filters to be proper.



✓$_1$ Considering the condition $\mathcal{X}$ be bounded and $x' = 0$, for all $0 \neq x \in X$.

Figure 2: The relationship between primary filters and Stonean filters on bounded lattice equality algebra.

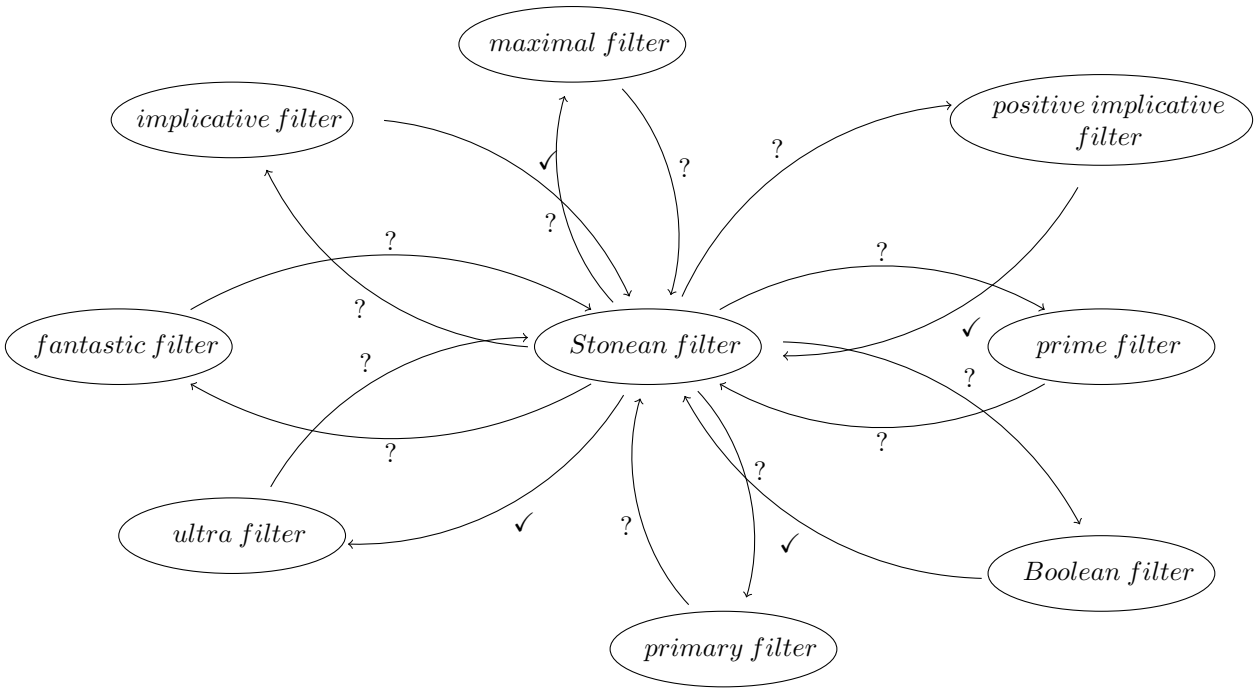
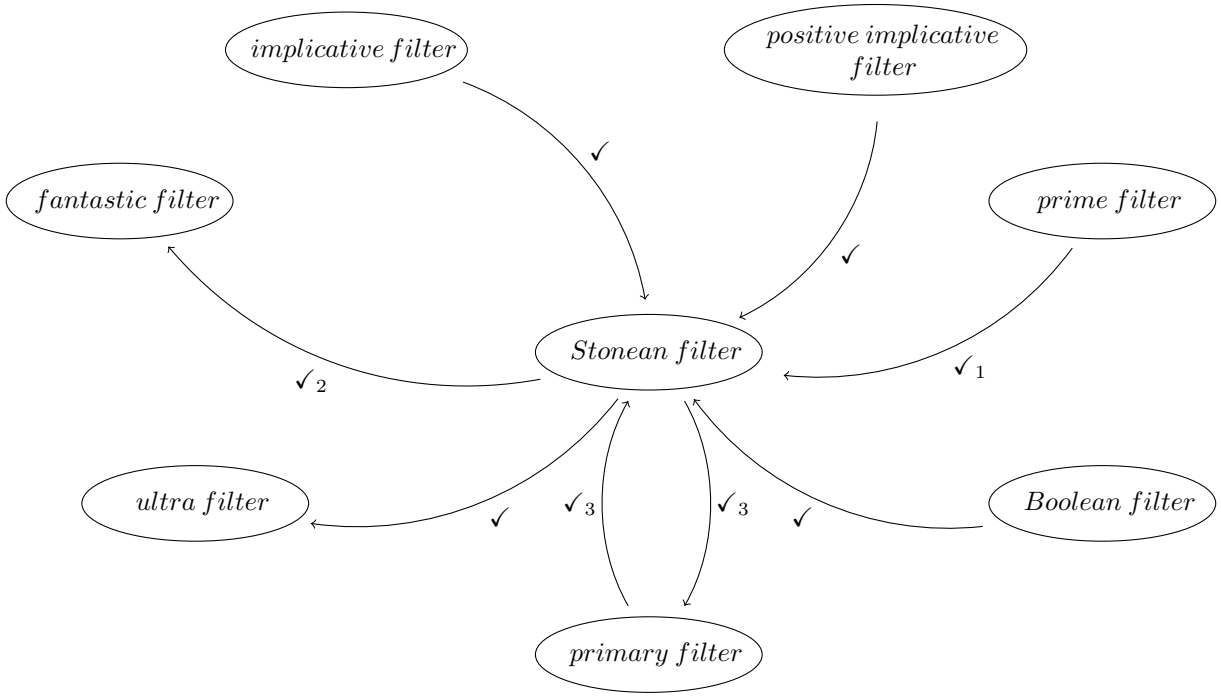


Figure 3: Connection of other filters with Stonean filters on bounded lattice equality algebra.

In the following, we will try to answer some of the questions raised with question marks.



✓$_1$ Considering the condition that $x \to x' \leq x$, for all $0 \neq x \in X$.
✓$_2$ Considering the condition that if $x \to y \in F$, then $y \to x \in F$ for all $x, y \in X$ and $y \neq 1$.
✓$_3$ Considering the condition that $x' = 0$, for all $0 \neq x \in X$.

Figure 4: The connection between Stonean filters and other filters considering conditions on F or $\mathcal{X}$.

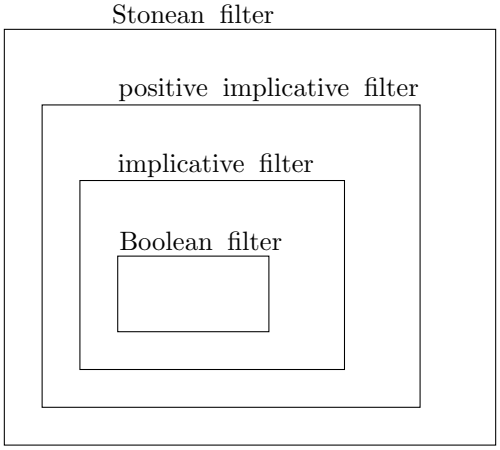


Figure 5: The connection between Stonean, positive implicative, implicative and Boolean filters.

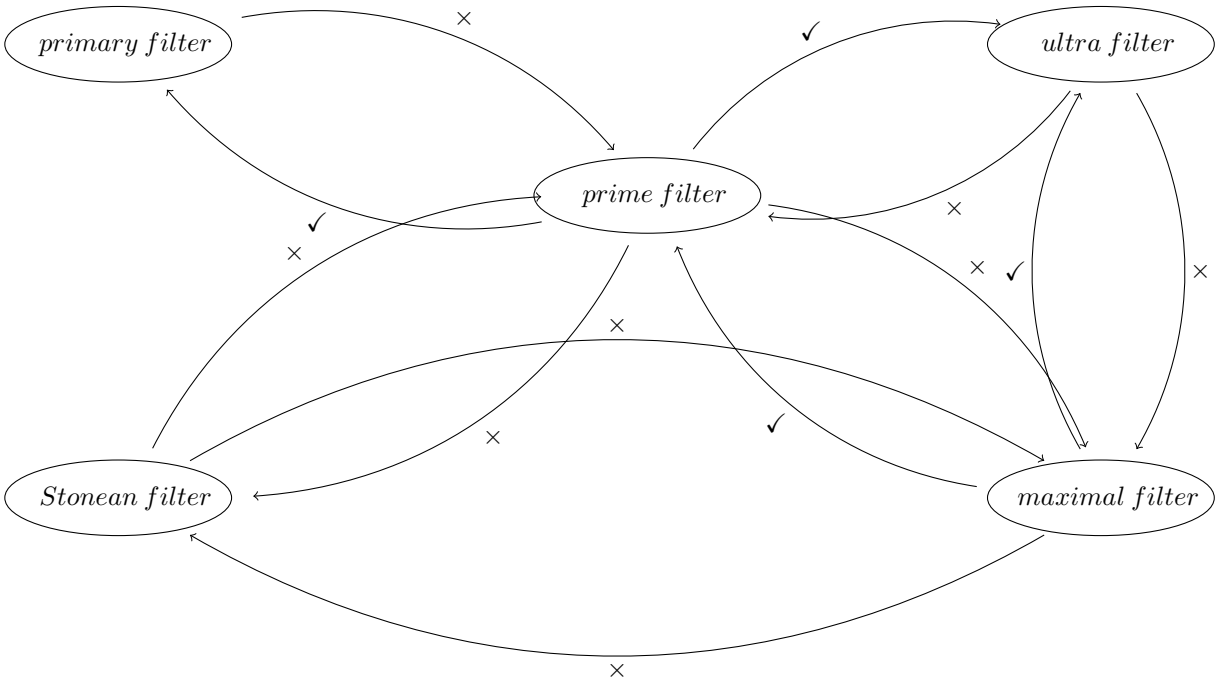


Figure 6: The connection between Stonean, prime, maximal, ultra, and primary filters.

5 Stonean radical filter

In this section, we will introduce Stonean radical filters on equality algebras, then we check its properties. We also study the Stonean radicals of different filters. In the end, we show that under certain conditions, the Stonean radical of a prime filter is a prime filter.

Definition 5.1. *Let F be an arbitrary non-empty set of X. Then we define the Stonean radical of F on X as follows:*

$$Rad_s(F) = \bigcap_{S \in \mathcal{ST}(\mathcal{X}),\ F \subseteq S} S.$$

In other words, $Rad_s(F)$ is equal to the smallest Stonean filter containing F (by inclusion relation).

Example 5.2. *Let $\mathcal{X}$ be the equality algebra of Example 3.2 and let $F = \{1\}$. We have:*

$$Rad_s(F) = Rad_s(\{1\}) = \{a, b, 1\} \cap \{0, a, b, c, d, 1\} = \{a, b, 1\},$$

and for $a, d \in X$,

$$Rad_s(\{a\}) = \{a, b, 1\} \cap \{0, a, b, c, d, 1\} = \{a, b, 1\},$$
$$Rad_s(\{d\}) = X.$$

Remark 5.3. *Indeed, if F is a filter on $\mathcal{X}$, then $Rad_s(F)$ is an extension of F, which generalizes it to a Stonean filter.*

Theorem 5.4. *Let F be a proper filter on $\mathcal{X}$. Then the following properties hold:*
(i) If F is a Stonean (implicative, Boolean, positive implicative) filter, then $Rad_s(F) = F$,
(ii) If F is a maximal filter, then $Rad_s(F) = F$ or X,
(iii) If F is a fantastic filter, then $Rad_s(F)$ is also a fantastic filter.

Proof. (i) By Definition 5.1 (Theorem 3.8, Proposition 3.10 and Theorem 3.12), it is clear.
(ii) If F is also Stonean filter, then $Rad_s(F) = F$, by (i). Otherwise, since F is maximal, so there is not a proper filter containing F. It means that $Rad_s(F) = X$.
(iii) Since $F \subseteq Rad_s(F)$, the proof is clear, by Proposition 4.7 of [3]. □

Corollary 5.5. Let F be a prime filter on $\mathcal{X}$ such that $x \to x' \leq x$, for all $0 \neq x \in X$. Then $Rad_s(F) = F$.

Proof. By considering Theorems 3.24 and 5.4(i), the proof is straightforward. □

In the following example, we show that the condition $x \to x' \leq x$, for all $0 \neq x \in X$, is necessary in Corollary 5.5.

Example 5.6. *Let $\mathcal{X}$ be the equality algebra as in Example 3.15(i) and let $F = \{b, 1\}$. It is clear that F is a prime filter and the condition $x \to x' \leq x$, for all $0 \neq x \in X$ is not satisfied, because for $a \in X$, $a \to a' = a \to a = 1 \not\leq a$. Since $a \in X$, we get $a' \vee a'' = a \vee a = a \notin F$, then F is not a Stonean filter. Therefore $Rad_s(F) = X \neq F$.*

Theorem 5.7. *Let $F \in \mathcal{F}(\mathcal{X})$ and let $P_F = \{x \mid x' \to x \in F\}$ be a filter. Then P_F is a Stonean filter and $Rad_s(F) \subseteq P_F$.*

Proof. We show that P_F is a Stonean filter including the filter F. Let $x \in F$. Since $x' \leq 1$, so $1 \to x \leq x' \to x$ and hence $x \leq x' \to x$, by Propositions 2.3(i) and 2.2(vi). Because F is a filter and $x \in F$, so $x' \to x \in F$. By the definition of P_F, we get $x \in P_F$. Therefore $F \subseteq P_F$.
Now, we prove that P_F is a Stonean filter. Let x be an arbitrary element of X. Then we prove that $x' \vee x'' \in P_F$. Since $x' \wedge x'' \leq x' \vee x''$, so $(x' \wedge x'') \to (x' \vee x'') = 1 \in F$, by Proposition 2.2$(iv)$. Thus:

$$(x' \vee x'')' \to (x' \vee x'') = (x''' \vee x'')' \to (x' \vee x'') = (x'''' \wedge x''') \to (x' \vee x'') = (x'' \wedge x') \to (x' \vee x'') = 1 \in F \subseteq P_F.$$

Because $(x' \vee x'')' = x' \wedge x''$, from definition of P_F and the fact that $(x' \vee x'')' \to (x' \vee x'') \in F$, we conclude $x' \vee x'' \in P_F$, for all $x \in X$. Therefore P_F is a Stonean filter and hence $Rad_s(F) \subseteq P_F$, by Theorem 5.4(i). □

The following example shows that it is not necessary $Rad_s(F) = P_F$, in general.

Example 5.8. *Let $\mathcal{X}$ be the equality algebra of Example 3.3 and let $F = \{b, 1\}$. It is not difficult to show that the F is a Stonean filter, so $Rad_s(F) = F$. It is easy to check that $P_F = \{a, b, c, 1\}$ and P_F is Stonean filter. Therefore $Rad_s(F) \neq P_F$.*

Proposition 5.9. *Let F be an implicative filter on $\mathcal{X}$ and let $P_F = \{x \in X \mid x' \to x \in F\}$ be a filter. Then P_F is a Stonean filter and $F = P_F = Rad_s(F) = Rad_s(P_F)$.*

Proof. The proof follows from Propositions 2.7, 3.8 and Theorem 5.4(i). □

In the following example, we show that the condition of implicativity, in Proposition 5.9, is necessary.

Example 5.10. *Let $\mathcal{X}$ be the equality algebra of Example 3.3, $F = \{1\}$, and $M = \{b, 1\}$. We know that $F, M \in \mathcal{ST}(\mathcal{X})$, by Corollary 3.14. Since $a' \to a = 1 \in F$, but $a \notin F$, for $a \in X$, so F is not implicative, by Proposition 2.7. Now, it is easy to see that $P_F = \{a, b, c, 1\}$. Thus $F \subsetneq M \subsetneq P_F$, that is $Rad_s(F) \neq P_F$.*

Proposition 5.11. *Let $\mathcal{X} = \langle X, \sim, \wedge, 1\rangle$, and let $F, G \subseteq X$. Then the following statements hold:*
(i) If $F \subseteq G$, then $Rad_s(F) \subseteq Rad_s(G)$,
(ii) $Rad_s(F) = Rad_s(Rad_s(F)) = Rad_s(Rad_s(Rad_s(F))) = \cdots$,
(iii) $Rad_s(F \cap G) \subseteq Rad_s(F) \cap Rad_s(G) \subseteq Rad_s(F \cup G)$,
(iv) $Rad_s(F \cap G) \subseteq Rad_s(Rad_s(F) \cup Rad_s(G))$,
(v) If $F \in \mathcal{F}(\mathcal{X})$ and $x' = 0$, for all $0 \neq x \in X$, then $Rad_s(F) = F$.

Proof. (i) Since $F \subseteq G$, so $\{S \in ST(X); G \subseteq S\} \subseteq \{S \in ST(X); F \subseteq S\}$, thus $Rad_s(F) = \cap\{S \in ST(X); F \subseteq S\} \subseteq \cap\{S \in ST(X); G \subseteq S\} = Rad_s(G)$.
(ii) Since $Rad_s(F)$ is a Stonean filter, by Theorem 5.4(i) it follows $Rad_s(F) = Rad_s(Rad_s(F)) = \ldots$.
(iii) Since $F \cap G \subseteq F, G \subseteq F \cup G$, thus by point (i), $Rad_s(F \cap G) \subseteq Rad_s(F)$, $Rad_s(G) \subseteq Rad_s(F \cup G)$ and so $Rad_s(F \cap G) \subseteq Rad_s(F) \cap Rad_s(G) \subseteq Rad_s(F \cup G)$.
(iv) $Rad_s(F \cap G) \subseteq Rad_s(F) \cap Rad_s(G) \subseteq Rad_s(F) \cup Rad_s(G) \subseteq Rad_s(Rad_s(F) \cup Rad_s(G))$.
(v) By Corollary 3.14, it is clear that F is a Stonean filter. Thus $Rad_s(F) = F$, by Theorem 5.4(i). □

Proposition 5.12. *Let $F \in \mathcal{ST}(\mathcal{X})$, $P_F = \{x \in X \mid x' \to x \in F\}$, $H_F = \{x \in X \mid x'' \in F\}$ and $P_F, H_F \in \mathcal{F}(\mathcal{X})$. Then $H_F, P_F \in \mathcal{ST}(\mathcal{X})$ and $Rad_s(F) \subseteq H_F \subseteq P_F$.*

Proof. Since $F \in \mathcal{ST}(\mathcal{X})$, so $Rad_s(F) = F$ by Theorem 5.4(i). It suffice to show that $F \subseteq H_F \subseteq P_F$. By Theorem 2.4$(ii)$, it is clear $F \subseteq H_F$. Let $x \in H_F$. Then $x'' \in F$. Since $0 \leq x$, so $x' \to 0 \leq x' \to x$ and hence $x'' \leq x' \to x$, by Proposition 2.3(i). Because F is a filter and $x'' \in F$, thus $x' \to x \in F$ and hence $x \in P_F$. Therefore $H_F \subseteq P_F$ and $F \subseteq H_F \subseteq P_F$. Since $F \in \mathcal{ST}(\mathcal{X})$ and $F \subseteq H_F \subseteq P_F$, so $H_F, P_F \in \mathcal{ST}(\mathcal{X})$, by Theorem 3.5. □

In the next examples, we show that the converses of Theorem 5.11(iii), (iv) and Proposition 5.12 do not hold, in general.

Example 5.13. (i) *In Example 3.2, let* $F = \{c, 1\}$ *and* $G = \{d, 1\}$. *Since* $Rad_s(F \cap G) = Rad_s(\{1\}) = \{a, b, 1\}$ *and* $Rad_s(F) \cap Rad_s(G) = X \cap X = X$, *so* $Rad_s(F) \cap Rad_s(G) \nsubseteq Rad_s(F \cap G)$.
(ii) *In Example 3.3, let* $F = \{a, 1\}$ *and* $G = \{b, 1\}$. *Then we get* $Rad_s(F \cup G) = Rad_s(\{a, b, 1\}) = \{a, b, c, 1\}$, $Rad_s(F) = Rad_s(\{a, 1\}) = \{a, 1\}$, *and* $Rad_s(G) = Rad_s(\{b, 1\}) = \{b, 1\}$. *Thus* $Rad_s(F) \cap Rad_s(G) = \{1\}$. *Therefore* $Rad_s(F \cup G) \nsubseteq Rad_s(F) \cap Rad_s(G)$.
(iii) *In Example 3.3, let* $F = \{1\}$ *and* $G = \{a, 1\}$. *It is clear that* $Rad_s(F) = Rad_s(\{1\}) = \{1\}$, $Rad_s(G) = Rad_s(\{a, 1\}) = \{a, 1\}$, *and* $Rad_s(F \cap G) = Rad_s(\{1\}) = \{1\}$. *Thus* $Rad_s(F) \cup Rad_s(G) = \{a, 1\}$ *and* $Rad_s(Rad_s(F) \cup Rad_s(G)) = Rad_s(\{a, 1\}) = \{a, 1\}$. *Thus:*

$$Rad_s(Rad_s(F) \cup Rad_s(G)) \nsubseteq Rad_s(F \cap G).$$

(iv) *In Example 3.3, let* $F = \{a, 1\}$ *and* $H_F = \{x \in X \mid x'' \in F\}$. *Then* $H_F = \{c, a, b, 1\}$ *and* $Rad_s(F) = \{a, 1\}$. *Thus* $H_F \nsubseteq Rad_s(F)$.
(v) *In Example 3.2, let* $F = \{c, 1\}$. *It is not difficult to check that* $H_F = \{c, 1\}$ *and* $P_F = \{a, c, d, 1\}$. *Thus* $P_F \nsubseteq H_F$.

Theorem 5.14. *Let* F *be a prime filter on* $\mathcal{X}$. *Then* $T = \{G \mid F \subseteq G,$ *where* G *is a proper filter on* $\mathcal{X}\}$ *is a totally ordered set (by inclusion).*

Proof. Let $G_1, G_2 \in T$. Then we show that $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$. Assume $G_1 \nsubseteq G_2$ and $G_2 \nsubseteq G_1$. Thus there exist $x \in G_1 \backslash G_2$ and $y \in G_2 \backslash G_1$. Since F is a prime filter, so we have $x \to y \in F$ or $y \to x \in F$, by Definition 2.12. If $x \to y \in F$, then $x \to y \in G_1$. Since G_1 is a filter and $x \in G_1$, so $y \in G_1$, by Proposition 2.6(ii), which is a contradiction. By a similarly reasoning, if $y \to x \in F$, then $y \to x \in G_2$. Since G_2 is a filter and $y \in G_2$, so $x \in G_2$, which is a contradiction. Therefore T is a totally order set (by inclusion). □

Corollary 5.15. Let F be a prime filter on $\mathcal{X}$. Then

$$T = \{G \mid Rad_s(F) \subseteq G, \text{ where } G \text{ is a proper filter on } \mathcal{X}\},$$

is a totally ordered set (by inclusion).

Theorem 5.16. *Let* F *be a proper Stonean filter on* $\mathcal{X}$. *Then for any* $a \in X \backslash F$, *there exists a Stonean filter* S *such that* $F \subseteq S$ *and* $a \notin S$.

Proof. Let $\Gamma = \{G \in \mathcal{F}(\mathcal{X}) \mid F \subseteq G,\ a \notin G\}$. It is clear that $F \in \Gamma$, thus Γ not ϕ. Let $\{G_i\}_{i \in I}$ be a chain in Γ. It is easy to check that $G = \bigcup_{i \in I} G_i$ is a Stonean filter and $a \notin G$ such that G is an upper bound of $\{G_i\}_{i \in I}$. Therefore Γ satisfies in Zorn's lemma conditions, thus Γ has a maximal element denoted S. Because $S \in \Gamma$, it is clear that S is a filter and that $a \notin S$. Since F is a Stonean filter and $F \subseteq S$, so S is a Stonean filter, by Theorem 3.5. □

6 Conclusion

In this paper, the concept of Stonean filters on equality algebras has been introduced. Then we have checked its relationship with some other filters. After that, we introduced the concept of primary and Stonean radical filters on equality algebras and showed that under suitable conditions, the Stonean radical of a prime filter is a prime filter. In our future work, we are going to study and investigate more results and features on Stonean radical and primary filters and try to introduce new filters, such as equality filters on equality algebras.

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