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Original paper

Optimizing linear functions over novel fuzzy relation equations: Structure, feasibility, and global solutions

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Abstract

We investigate the linear objective function optimization problem constrained by a new system of fuzzy relation equations, utilizing the minimum t-norm for fuzzy compositions. Our findings reveal that the feasible region is characterized as a finite union of closed convex cells. We provide necessary and sufficient conditions to determine the problem's feasibility. To streamline optimization, seven novel rules are proposed, on which an algorithm is based to achieve a global optimum. Notably, a specific instance of our problem is shown to be equivalent to the well-known minimal vertex cover problem. The efficacy of our algorithm is demonstrated through a concrete example.

Keywords: Linear optimization, fuzzy relational equations, minimal vertex covering.

1 Introduction

The theory of Fuzzy Relational Equations (FRE), first proposed by Sanchez [41] as a generalization of Boolean relation equations, was initially applied to problems in medical diagnosis. Pedrycz later categorized and expanded the generalizations of FRE in two primary directions: the nature of the sets under consideration and the variety of operations employed [38]. Since its inception, FRE has found numerous applications in diverse fields, including fuzzy control, fuzzy system prediction, fuzzy decision-making, fuzzy pattern recognition, image compression and reconstruction, and fuzzy clustering. In many scenarios where inference rules are applied and their consequences are known, the problem of determining the antecedents can be mathematically formulated as solving an FRE [35]. It is now well-established that many problems related to knowledge representation can be effectively addressed as FRE problems [36]. The significant practical applications of FRE have motivated extensive theoretical research, focusing on solution methodologies and optimization problems subject to FRE constraints.

A primary challenge in the study of FRE is the identification of solvability and the characterization of the solution set. Di Nola et al. [6] demonstrated that the solution set of an FRE defined by a continuous max-t-norm composition, if non-empty, is typically a non-convex set. This set is fully determined by a single maximal solution and a finite number of minimal solutions. The non-convexity of the solution space is a major bottleneck that

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contributes to the complexity of FRE-related problems, especially in optimization. Another significant challenge lies in identifying the minimal solutions. Chen and Wang [2] presented an algorithm to derive the logical representation of all minimal solutions and concluded that a polynomial-time algorithm for finding all minimal solutions of FRE with max-min composition is unlikely to exist. Furthermore, Markovskii [33] established a connection between solving max-product FRE and the NP-hard covering problem. This complexity result extends to more general t-norms beyond the minimal and product operators [3, 28, 31]. Over the years, the solvability of FREs defined with various max-t compositions has been extensively investigated [14, 17, 19, 23, 43, 45, 46, 49, 51].

In parallel, researchers have introduced and refined the theoretical and practical aspects of Fuzzy Relational Inequalities (FRI) [13, 16, 19, 20, 26, 53]. For instance, Li and Yang [26] investigated FRI with addition-min composition, developing an algorithm to find minimal solutions and applying it to data transmission in BitTorrent-like peer-to-peer networks. In [16], a mixed fuzzy system comprising two fuzzy relational inequalities, $A\phi x \leq b^1$ and $D\phi x \geq b^2$, was studied, where ϕ is an operator yielding convex solution sets.

Optimization problems constrained by FRE and FRI remain a prominent and active area of research [1, 10, 14, 15, 16, 21, 25, 30, 39, 42, 47, 53]. Many solution approaches involve transforming the original problem into an integer linear programming problem, which can then be solved using established techniques. Alternatively, other algorithms leverage the structure of the feasible region, employing necessary and sufficient optimality conditions and simplification processes, often building upon the analytical foundations laid by Sanchez [40] and Pedrycz [37]. For example, Fang and Li [11] converted a linear optimization problem with max-min FRE constraints into an integer programming problem, which they solved using a branch-and-bound method with a jump-tracking technique. Wu et al. [48] improved upon this method by reducing the search domain and introducing three simplification rules derived from a necessary condition. Chang and Shieh [1] presented new theoretical results for linear optimization problems constrained by max-min FRE, including an improved upper bound on the optimal objective value and rules for problem simplification and solution tree reduction. An application of linear optimization with max-min composition was demonstrated in [24] for a streaming media provider minimizing costs while meeting service requirements. Linear optimization with the max-product operation has also been extensively studied [21, 34]. Loetamonphong and Fang [34] addressed such problems by separating the objective function's negative and non-negative coefficients into two sub-problems and combining their optimal solutions. Generalizations have been explored by replacing max-min and max-product compositions with others, such as max-average [47] and general max-t-norm compositions [14, 17, 19, 22, 25, 42]. Li and Fang [25], for instance, solved a linear optimization problem with sup-t equation constraints by reducing it to a 0-1 integer optimization problem. In [22], a method for solving linear optimization problems with max-Archimedean t-norm FRE constraints was presented, while [42] tackled the same problem for continuous Archimedean t-norms, utilizing the covering problem to find optimal variables instead of branch-and-bound methods.

Recently, novel generalizations of linear programming over systems of fuzzy relations have been introduced, driven by developments in composite operations, the nature of fuzzy relations in constraints, and modifications to the objective function [5, 7, 12, 19, 27, 30, 50, 55]. For example, Wu et al. [50] proposed an efficient method for optimizing a linear fractional programming problem under max-Archimedean t-norm FRE constraints. Dempe and Ruziyeva [5] generalized the fuzzy linear optimization problem by incorporating fuzzy coefficients. Dubey et al. [7] studied linear programming problems with interval uncertainty modeled by intuitionistic fuzzy sets. Yang [55] investigated the minimization of a linear objective function subject to an FRI defined by constraints of the form $\sum_{j=1}^{n} \min\{a_{ij}, x_j\} \geq b_i$ for $i = 1, 2, ..., m$. In [53], the latticized linear programming problem subject to max-product FRI was introduced and applied to an optimization management model for wireless communication base stations. This problem was formulated as minimizing $z(x) = \min_{j=1}^{n}\{x_j\}$ over the feasible region $X(A, b) = \{x \in [0, 1]^n : A \circ x \geq b\}$, where '$\circ$' denotes the fuzzy max-product composition, and was solved using an algorithm based on the resolution of the feasible region.

The concept of FRE has also been extended to bipolar fuzzy relational equations, reflecting the human capacity to process and represent positive and negative information separately [8]. Dubois and Prade [9] provided an overview of the asymmetric bipolar representation of information within possibility theory, demonstrating its suitability for distinguishing between negative and positive aspects in preference modeling [8, 9]. Linear optimization over bipolar FREs has been explored with various compositions, including max-min (with applications to product awareness for suppliers) [12, 29], max-product [4], and max-Łukasiewicz [27, 30, 52, 56]. The concept of bipolar FRE was first introduced with the max-min composition, featuring constraints of the form $\max_{j=1}^{n}\{\max\{\min\{a_{ij}^{+}, x_j\}, \min\{a_{ij}^{-}, 1-x_j\}\}\}$ for $i = 1, 2, ..., m$, where $a_{ij}^{+}, a_{ij}^{-}, x_j \in [0, 1]$ [12]. Similarly, [27] introduced a linear optimization problem constrained by a system of bipolar FREs defined as $X(A^{+}, A^{-}, b) = \{x \in [0, 1]^{m} : x \circ A^{+} \vee \tilde{x} \circ A^{-} = b\}$, where $\tilde{x_i} = 1 - x_i$ for each component of $\tilde{x} = (\tilde{x_i})_{1\times m}$, and '$\circ$' and '$\vee$' denote the max-Lukasiewicz composition and the max operation, respectively. This problem was transformed into a 0-1 integer linear program. An alternative analytical method, based on the resolution and structural properties of the feasible region, was later proposed [30]. However, a resolution method for obtaining the complete solution set of bipolar max-Łukasiewicz FREs was not provided in these works. Yang [52] addressed this gap by showing that the complete solution set is fully determined by a finite number of conservative bipolar paths.

The classical problem of solving max-min fuzzy relational equations is defined by the following system:

$$\begin{aligned} &\mathbf{A} \circ \mathbf{x} = \mathbf{b} \\ &x \in [0, 1]^{n} \end{aligned} \tag{1}$$

where $\mathbf{A} = (a_{ij})_{m\times n}$ is the coefficient matrix and $\mathbf{b} = (b_i)_{m\times 1}$ is the vector on the right side, with components in $[0, 1]$. The operator "$\circ$" denotes the max-min composition, such that the i-th constraint is $\max_{j=1}^{n}\{\min\{a_{ij}, x_j\}\} = b_i$, for $i = 1, 2, ..., m$. Given $\mathbf{A}$ and $\mathbf{b}$, the resolution problem is to find all vectors $\mathbf{x}$ that satisfy the constraints in (1). Figure 1 provides a schematic representation of the feasible region for max-min FRE problems. In this paper,

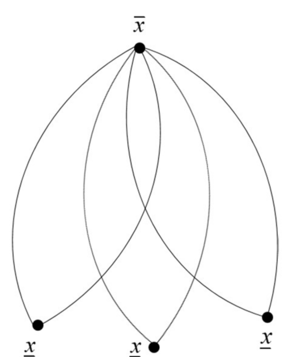

Figure 1: The feasible domain of the classical max-min FRE.

we investigate the following optimization problem:

$$\begin{aligned} &\min Z = \mathbf{c}^{T}\mathbf{x} \\ &\mathbf{A} \otimes \mathbf{x} = \mathbf{b} \\ &\mathbf{x} \in [0, 1]^{n} \end{aligned} \tag{2}$$

where $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$. The coefficient matrix is $\mathbf{A} = (a_{ij})_{m\times n}$ with $a_{ij} \in [0, 1]$, the right-hand-side vector is $\mathbf{b} = (b_i)_{m\times 1}$ with $b_i \in [0, 1]$, and $\mathbf{c}$ is a vector in $\mathbb{R}^{n}$. If $\mathbf{a}_i$ denotes the i-th row of matrix $\mathbf{A}$, the

i-th constraint of Problem (2) is defined as:

$$\mathbf{a}_i \otimes \mathbf{x} = \max_{j=1}^{n}\{\min\{a_{ij}, x_i, x_j\}\} = b_i, \quad i \in I \tag{3}$$

Without loss of generality, we can assume that $\mathbf{A}$ is a square matrix (i.e., $m = n$). If $m > n$, we can augment $\mathbf{A}$ with $m - n$ columns $j_{n+1}, ..., j_m$ having coefficients $a_{ij_k} = 0$ for $i \in I$ and $k \in \{n+1, ..., m\}$. If $m < n$, we can add $n - m$ equations of the form $\mathbf{a}_i \otimes \mathbf{x} = \max_{j=1}^{n}\{\min\{0, x_i, x_j\}\} = 0$ for $i \in \{m+1, ..., n\}$. Therefore, we assume $m = n$ throughout this paper.

The remainder of this paper is organized as follows. Section 2 presents preliminary definitions and concepts, and characterizes the feasible solutions set of the problem. It also derives necessary and sufficient conditions for feasibility. Section 3 introduces seven rules to expedite the resolution process by eliminating special cases and reducing the problem size. In Section 4, the optimization of the linear objective function is addressed. The existence of a global optimal solution is proven for the general case. It is also shown that for the special case where $\mathbf{b}$ is the zero vector and all coefficients and variables are binary, the optimal solutions are also binary. Section 5 summarizes the results in an algorithm and demonstrates that the well-known minimal vertex cover problem is a special case of our problem. Finally, Section 6 provides a numerical example to illustrate the proposed algorithm.

2 Feasible solutions set of the problem

This section describes the characterization of the feasible region of Problem (2). For this purpose, we firstly determine the feasible solutions set of the i'th constraint of Problem (2), i.e., Relation (3) for each $i \in I$.

Definition 2.1. *For each $i \in I$, let $S(\boldsymbol{a}_i, b_i)$ denotes the feasible solutions set of i'th equation, i.e., $S(\boldsymbol{a}_i, b_i) = \{\boldsymbol{x} \in [0,1]^n : \boldsymbol{a}_i \otimes \boldsymbol{x} = b_i\} = \{\boldsymbol{x} \in [0,1]^n : \max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i\}$. Also let $S(\boldsymbol{A}, \boldsymbol{b}) = \{\boldsymbol{x} \in [0,1]^n : \boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b}\}$. So, $S(\boldsymbol{A}, \boldsymbol{b})$ denotes the feasible solutions set of Problem (2). A solution $\bar{\boldsymbol{x}} \in S(\boldsymbol{A}, \boldsymbol{b})$ ($\underline{\boldsymbol{x}} \in S(\boldsymbol{A}, \boldsymbol{b})$) is said to be a maximal (minimal) solution when $\bar{\boldsymbol{x}} \le \boldsymbol{x}$ ($\boldsymbol{x} \le \underline{\boldsymbol{x}}$) implies $\bar{\boldsymbol{x}} = \boldsymbol{x}$ ($\boldsymbol{x} = \underline{\boldsymbol{x}}$) for any $\boldsymbol{x} \in S(\boldsymbol{A}, \boldsymbol{b})$.*

From Definition 2.1, it is clear that $S(\mathbf{A}, \mathbf{b}) = \bigcap_{i\in I} S(\mathbf{a}_i, b_i)$. Moreover, this definition together with Relation (3) result in Lemma 2.2 below that provides a necessary and sufficient condition for the feasibility of $S(\mathbf{a}_i, b_i)$, $\forall i \in I$.

Lemma 2.2. *For a fixed $i \in I$, $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$ iff $\boldsymbol{x} \in [0,1]^n$ and the following two conditions are satisfied:*

$$\begin{aligned} &(i) \quad \min\{a_{ij}, x_i, x_j\} \le b_i, \quad \forall j \in J. \\ &(ii) \quad \min\{a_{ij_0}, x_i, x_{j_0}\} = b_i \quad \textit{for some } j_0 \in J. \end{aligned} \tag{4}$$

Definition 2.3. *For each $i \in I$, we define $J_i^1 = \{j \in J : a_{ij} > b_i\}$ and $J_i^2 = \{j \in J : a_{ij} = b_i\}$. Also let $J_i = J_i^1 \bigcup J_i^2$, i.e., $J_i = \{j \in J : a_{ij} \ge b_i\}$.*

Based on Lemma 2.2 and Definition 2.3, we have also a necessary feasibility condition for Problem (2) as follows:

Corollary 2.4. *If $S(\boldsymbol{A}, \boldsymbol{b}) \neq \emptyset$, then $J_i \neq \emptyset$, $\forall i \in I$.*

Proof. Let $S(\mathbf{A}, \mathbf{b}) \neq \emptyset$. By contradiction, suppose that there exists some $i_0 \in I$ such that $J_{i_0} = \emptyset$. So, according to Definition 2.3, for each $j \in J$ we have $a_{i_0 j} < b_{i_0}$, and therefore $\min\{a_{i_0 j}, x_{i_0}, x_j\} < b_{i_0}$ for each $j \in J$ and each $\mathbf{x} \in [0,1]^n$. Consequently, from Lemma 2.2 we conclude that $S(\mathbf{a}_i, b_i) = \emptyset$ which contradicts $S(\mathbf{A}, \mathbf{b}) \neq \emptyset$. □

In contrast to the feasible solutions set of Problem (1) (depicted in Figure 1), it will be proved that the feasible region of (3) (also, that of Problem (2)) interestingly comes in three different shapes depending on the value of the diagonal element a_{ii}. Figure 2 below schematically shows these differences in three cases $a_{ii} > b_i$, $a_{ii} = b_i$, and

$a_{ii} < b_i$.

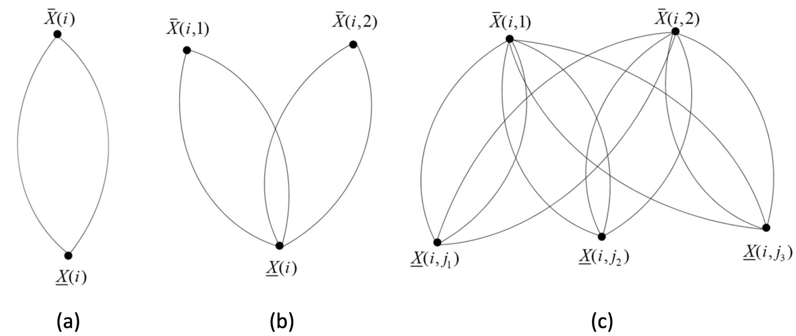

Figure 2: The shape of the feasible solutions set for equation $\max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i$, (a) if $a_{ii} > b_i$, (b) if $a_{ii} = b_i$, and (c) if $a_{ii} < b_i$.

As shown in Figure 2, when $a_{ii} > b_i$, set $S(\mathbf{a}_i, b_i)$ has exactly one maximal solution, $\bar{\mathbf{X}}(i)$, and one minimal solution, $\underline{\mathbf{X}}(i)$. In the case that $a_{ii} = b_i$, $S(\mathbf{a}_i, b_i)$ has yet a unique minimal, but exactly two maximal solutions $\bar{\mathbf{X}}(i,1)$ and $\bar{\mathbf{X}}(i,2)$. Finally, if $a_{ii} < b_i$, set $S(\mathbf{a}_i, b_i)$ is determined by two maximal solutions and a finite number of minimal ones. It will be shown that the maximal solutions are always constructed in two specific ways, whether $a_{ii} = b_i$ or $a_{ii} < b_i$. For this reason, we interpret these two ways (possibilities) as two types for maximal solutions and this is why they are denoted by $\bar{\mathbf{X}}(i,1)$ and $\bar{\mathbf{X}}(i,2)$. So, solution $\bar{\mathbf{X}}(i,1)$ (solution $\bar{\mathbf{X}}(i,2)$) means the maximal solution type 1 (type 2) obtained by the first (second) way. As mentioned above, in the case that $a_{ii} \le b_i$, $S(\mathbf{a}_i, b_i)$ includes exactly two maximal solutions $\bar{\mathbf{X}}(i,1)$ and $\bar{\mathbf{X}}(i,2)$. Moreover, it will be slightly later shown that if $a_{ii} > b_i$, the unique maximal solution $\bar{\mathbf{X}}(i)$ is also obtained by the same way used for constructing the maximal solutions type 1. In order to prove the correctness of the foregoing results, we firstly give some necessary definitions.

Definition 2.5. *Let* $I_1 = \{i \in I : a_{ii} > b_i\}$, $I_2 = \{i \in I : a_{ii} = b_i\}$, *and* $I_3 = \{i \in I : a_{ii} < b_i\}$.

Lemma 2.6. *Let* $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$.
(a) $x_i \ge b_i$
(b) *If* $i \in I_1 \bigcup I_2$ *and* $x_i = b_i$, *then* $x_j \in [0,1], \forall j \in J - \{i\}$.
(c) *If* $i \in I_3$ *and* $x_i = b_i$, *then there exists at least one* $j_0 \in J_i$ *such that* $x_{j_0} \ge b_i$.
(d) *If* $i \in I_1$, *then* $x_i = b_i$.
(e) *If If* $i \in I_2 \bigcup I_3$ *and* $x_i > b_i$, *then* $x_j \le b_i$ *for each* $j \in J_i^1$.
(f) *If* $i \in I_3$, *and* $x_i > b_i$, *then there exist some* $j_0 \in J_i$ *such that either* $j_0 \in J_i^1$ *and* $x_{j_0} = b_i$ *or* $j_0 \in J_i^2$ *and* $x_{j_0} \ge b_i$.

Proof. By contradiction, suppose that $x_i < b_i$. So, $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \in J$. Hence, $\mathbf{x}$ violates Part (II) of the necessary and sufficient feasibility conditions (4) and therefore $\mathbf{x} \notin S(\mathbf{a}_i, b_i)$ that is a contradiction. **(b)** By the assumptions, we have $\min\{a_{ii}, x_i, x_i\} = \min\{a_{ii}, b_i, b_i\} = b_i$ and $\min\{a_{ij}, x_i, x_j\} = \min\{a_{ij}, b_i, x_j\} \le b_i$, for each $j \in J - \{i\}$ and each $x_j \in [0,1]$. Hence, conditions (4) hold true for each value $x_j \in [0,1]$ where $J - \{i\}$. **(c)** The result is obtained by contradiction, i.e., by assuming that $x_j < b_i, \forall j \in J_i$. Hence, $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \in J_i$. On the other hand, we have $\min\{a_{ii}, x_i, x_i\} < b_i$ (because, $a_{ii} < b_i$ if $i \in I_3$) and and $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \notin J_i$ (because, $a_{ij} < b_i$ if $j \notin J_i$). Consequently, $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \in J$, that contradicts Part (II) of conditions (4). (d) According to Part (a), $x_i \ge b_i$. If $x_i > b_i$, then $\min\{a_{ii}, x_i, x_i\} > b_i$ which contradicts Part (I) of conditions (4). Therefore, we must have $x_i = b_i$. (e) By contradiction, suppose that $x_{j_0} > b_i$ for some $j_0 \in$

J_i^1. Thus, $\min\{a_{i_0}, x_i, x_{j_0}\} > b_i$ which violates Part (I) of conditions (4). (f) Since $i \in I_3$, then $a_{ii} < b_i$ that implies $\min\{a_{ii}, x_i, x_i\} < b_i$. By contradiction, suppose that $x_j \neq b_i, \forall j \in J_i^1$, and $x_j < b_i, \forall j \in J_i^2$. Therefore, $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \in J_i^2$. Now, if $x_j < b_i, \forall j \in J_i^1$, then we also have $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \in J_i^1$. So, $\min\{a_{ij}, x_i, x_j\} < b_i, \forall j \in J$, that contradicts Part (II) of conditions (4). Otherwise, if $x_{j_0} > b_i$ for some $j_0 \in J_i^1$, then $\min\{a_{ij_0}, x_i, x_{j_0}\} > b_i$ which violates Part (I) of conditions (4). □

Definition 2.7. *For each $i \in I_1$, define two $n \times 1$ vectors*
$\bar{X}(i)^T = \left(\bar{X}(i)_1, \bar{X}(i)_2, \ldots, \bar{X}(i)_n\right)$ *and* $\underline{\boldsymbol{X}}(i)^T = (\underline{X}(i)_1, \underline{X}(i)_2, \ldots, \underline{X}(i)_n)$ *such that*

$$\bar{X}(i)_j = \begin{cases} b_i & j = i \\ 1 & j \neq i \end{cases}, \quad \underline{X}(i)_j = \begin{cases} b_i & j = i \\ 0 & j \neq i \end{cases}$$

for each $j \in J$.

Definition 2.8. *For each $i \in I_2$, define three $n \times 1$ vectors*

$$\begin{aligned} &\bar{X}(i,1)^T = \left(\bar{X}(i,1)_1, \bar{X}(i,1)_2, \ldots, \bar{X}(i,1)_n\right), \\ &\overline{\boldsymbol{X}}(i,2)^T = \left(\bar{X}(i,2)_1, \bar{X}(i,2)_2, \ldots, \bar{X}(i,2)_n\right), \\ &\underline{\boldsymbol{X}}(i)^T = (\underline{X}(i)_1, \underline{X}(i)_2, \ldots, \underline{X}(i)_n) \end{aligned}$$

such that

$$\bar{X}(i,1)_j = \begin{cases} b_i & j = i \\ 1 & j \neq i \end{cases}, \quad \bar{X}(i,2)_j = \begin{cases} b_i & a_{ij} > b_i \\ 1 & a_{ij} \leq b_i \end{cases}, \quad \underline{X}(i)_j = \begin{cases} b_i & j = i \\ 0 & j \neq i \end{cases}$$

for each $j \in J$.

Definition 2.9. *For each $i \in I_3$, two $n \times 1$ vectors $\bar{X}(i,1)$ and $\bar{X}(i,2)$ are defined as in Definition 2.8. Moreover, for each $i \in I_3$ and each $j \in J_i$, we define an $n \times 1$ vector $\underline{X}(i,j)^T = (\underline{X}(i,j)_1, \underline{X}(i,j)_2, \ldots, \underline{X}(i,j)_n)$ such that*

$$\underline{X}(i,j)_k = \begin{cases} b_i & k = i \text{ or } k = j \\ 0 & \text{otherwise} \end{cases}$$

for each $k \in J$.

Lemma 2.10 describes the shape of the feasible solutions set of $S(\boldsymbol{a}_i, b_i)$ for each value of the diagonal element a_{ii}, or equivalently for each $i \in I\,(I = I_1 \cup I_2 \cup I_3)$. In all the cases, notation $[\boldsymbol{Y}, \boldsymbol{Z}]$ (where $\boldsymbol{Y} = (Y_j)_{n\times1}$ and $\boldsymbol{Z} = (Z_j)_{n\times1}$ are two $n \times 1$ vectors) means the closed convex cell including all the $n \times 1$ vectors $\boldsymbol{x} = (x_j)_{n\times1}$ such that $\boldsymbol{Y} \leq \boldsymbol{x} \leq \boldsymbol{Z}$, i.e., $Y_j \leq x_j \leq Z_j$ for each $j \in J$.

Lemma 2.10. (a) *If $i \in I_1$, then $S(\boldsymbol{a}_i, b_i) = [\underline{X}(i), \bar{X}(i)]$.*
(b) *If $i \in I_2$, then $S(\boldsymbol{a}_i, b_i) = [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,1)] \cup [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,2)]$.*
(c) *If $i \in I_3$, then $S(\boldsymbol{a}_i, b_i) = \left\{\bigcup_{j\in J_i}[\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,1)]\right\} \bigcup \left\{\bigcup_{j\in J_i}[\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,2)]\right\}$.*

Proof. (a) Since $i \in I_1$, then $a_{ii} > b_i$. Let $\boldsymbol{x} \in [\underline{X}(i), \overline{\boldsymbol{X}}(i)]$. From Definition 2.7, we have $\underline{X}(i)_i = x_i = \bar{X}(i)_i = b_i$. So, $\min\{a_{ii}, x_i, x_i\} = \min\{a_{ii}, b_i, b_i\} = b_i$ and $\min\{a_{ij}, x_i, x_j\} = \min\{a_{ij}, b_i, x_j\} \leq b_i$ for each $j \in J$ such that $j \neq i$. Hence, $\max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i$ that implies $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$. Conversely, let $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$. So, from Lemma 2.6(d), we have $x_i = b_i$, and then the result follows from Lemma 2.6(b) and Definition 2.7. (b) By the assumption, $i \in I_2$ and therefore $a_{ii} = b_i$. Let $\boldsymbol{x} \in [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,1)] \cup [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,2)]$. If $\boldsymbol{x} \in [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,1)]$, then the argument stated in Part (a) results in $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$. Otherwise, suppose $\boldsymbol{x} \in [\underline{X}(i), \overline{\boldsymbol{X}}(i,2)]$. Then, by Definition **??** we have $b_i \leq x_i \leq 1, 0 \leq x_j \leq b_i$ when $j \in J_i^1$ and $0 \leq x_j \leq 1$ when $j \in J_i^2 - \{i\}$ or $j \notin J_i$. Hence, $\min\{a_{ii}, x_i, x_i\} =$

$b_i, \min\{a_{ij}, x_i, x_j\} \le b_i$ if $j \in J_i - \{i\}$, and $\min\{a_{ij}, x_i, x_j\} < b_i$ if $j \notin J_i$ (because, $a_{ii} < b_i$ if $j \notin J_i$). Consequently, $\max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i$, i.e., $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$. The converse statement is resulted from Lemma 2.6 (Parts (a), (b) and (e)) and Definition 2.8. (c) Let $i \in I_3$ and $\boldsymbol{x} \in \left\{\bigcup_{j\in J_i}[\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,1)]\right\} \cup \left\{\bigcup_{j\in J_i}[\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,2)]\right\}$. Therefore, $a_{ii} < b_i$. If $\boldsymbol{x} \in \bigcup_{j\in J_i}[\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,1)]$, then there exists at least one $j' \in J_i$ such that $\boldsymbol{x} \in \left[\underline{\boldsymbol{X}}(i,j'), \overline{\boldsymbol{X}}(i,1)\right]$. Thus, by Definition 2.9, $x_i = b_i$ and $b_i \le x_{j'} \le 1$. Therefore, we have $\min\{a_{ii}, x_i, x_i\} < b_i$, $\min\{a_{i'}, x_i, x_{j'}\} = b_i$, $\min\{a_{ij}, x_i, x_j\} \le b_i$, $\forall j \in J_i - \{j'\}$, and $\min\{a_{ij}, x_i, x_j\} < b_i$ when $j \notin J_i$. Hence, $\max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i$ that means $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$. On the other hand, if $\boldsymbol{x} \in \bigcup_{j\in J_i}[\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,2)]$, then $\boldsymbol{x} \in \left[\underline{\boldsymbol{X}}(i,j'), \overline{\boldsymbol{X}}(i,2)\right]$ for at least one $j' \in J_i$. In this case, Definition 2.9 implies $b_i \le x_i \le 1$; also, we have $x_{j'} = b_i$ if $j' \in J_i^1$ and $b_i \le x_{j'} \le 1$ if $j' \in J_i^2$; additionally, $0 \le x_j \le b_i$ if $j \in J_i^1 - \{j'\}$ and $0 \le x_j \le 1$ if $j \in J_i^2 - \{j'\}$. Therefore, $\min\{a_{ii}, x_i, x_i\} < b_i, \min\{a_{ij'}, x_i, x_{j'}\} = b_i, \min\{a_{ij}, x_i, x_j\} \le b_i, \forall j \in J_i - \{j'\}$, and $\min\{a_{ij}, x_i, x_j\} < b_i$ if $j \notin J_i$. So, we obtain the same result, i.e., $\max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i$. Conversely, let $\boldsymbol{x} \in S(\boldsymbol{a}_i, b_i)$. From Lemma 2.6(a), $x_i \ge b_i$. If $x_i = b_i$, then by Lemma 2.6(c) and Definition 2.9, $\boldsymbol{x} \in \left[\underline{X}(i,j_0), \overline{\boldsymbol{X}}(i,1)\right]$ for some $j_0 \in J_i$ such that $x_{j_0} \ge b_i$. Otherwise, if $x_i > b_i$, then Lemma 2.6(e) and Definition 2.9 imply $\boldsymbol{x} \le \overline{\boldsymbol{X}}(i,2)$. Moreover, from Lemma 2.6(f) and Definition 2.9 , we have $\underline{X}(i,j_0) \le \boldsymbol{x}$ for some $j_0 \in J_i$ such that either $j_0 \in J_i^1$ and $x_{j_0} = b_i$ or $j_0 \in J_i^2$ and $x_{j_0} \ge b_i$. As a result, we have $\boldsymbol{x} \in \left[\underline{\boldsymbol{X}}(i,j_0), \overline{\boldsymbol{X}}(i,2)\right]$ where $j_0 \in J_i$.

Corollary 2.11. *According to Lemma 2.10, solutions $\bar{X}(i)$ and $\underline{X}(i)$ are the maximal and minimal solutions of $S(\boldsymbol{a}_i, b_i), \forall i \in I_1$, respectively. Also, there are two maximal solutions $\overline{\boldsymbol{X}}(i,1)$ and $\overline{\boldsymbol{X}}(i,2)$, and a unique minimal solution $\underline{\boldsymbol{X}}(i)$ for $S(\boldsymbol{a}_i, b_i), \forall i \in I_2$. Moreover, for each $i \in I_3$, $S(\boldsymbol{a}_i, b_i)$ has two maximal solutions $\bar{X}(i,1)$ and $\bar{X}(i,2)$, and a finite number of minimal solutions $\underline{X}(i,j), \forall j \in J_i$. Additionally, from Definitions 2.7 and 2.9, it is clear that the maximal solution $\bar{X}(i)$ is defined in the same way as the maximal solutions $\overline{\boldsymbol{X}}(i,1)$ are defined. Lemma 2.10 determines the feasible region of the single equation $\boldsymbol{a}_i \otimes \boldsymbol{x} = b_i$ in all the cases, where the value of a_{ii} may be greater than b_i, less than b_i or equal to b_i. Based on the differences between the feasible regions of these single equations depending on the values of a_{ii}, it seems reasonable to categorize the equations of Problem (2) into three groups of constraints such that k'th group includes each equation $\boldsymbol{a}_i \otimes \boldsymbol{x} = b_i$ (i.e., $\max_{j\in J}\{\min\{a_{ij}, x_i, x_j\}\} = b_i$) with $i \in I_k$, for $k = 1, 2, 3$. The following definition formally expresses the feasible regions of these three groups of the constraints. In what follows, we separately investigate these feasible regions and determine their shapes as well as their extreme solutions (maximal, minimal, maximal and minimal solutions). Obviously, the intersection of these feasible regions will finally give the feasible solutions set of Problem (2).*

Definition 2.12. *Let $S_k(\boldsymbol{A}, \boldsymbol{b}) = \bigcap_{i\in I_k} S(\boldsymbol{a}_i, b_i), k = 1, 2, 3$.*

Based on the notation used in Definition 2.12, $S(\boldsymbol{A}, \boldsymbol{b}) = S_1(\boldsymbol{A}, \boldsymbol{b}) \cap S_2(\boldsymbol{A}, \boldsymbol{b}) \cap S_3(\boldsymbol{A}, \boldsymbol{b})$. The following definition gives a useful tool enabling us to determine the minimal and maximal solutions of sets $S_2(\boldsymbol{A}, \boldsymbol{b})$ and $S_3(\boldsymbol{A}, \boldsymbol{b})$.

Definition 2.13. *Let $e' : I_2 \to \{1, 2\}$ be a function from I_2 into $\{1, 2\}$ and E' denote the set of all the functions e' on I_2. Similarly, the notation E'' is used to denote the set of all functions $e'' : I_3 \to \{1, 2\}$. Moreover, let $\underline{e} : I_3 \to \bigcup_{i\in I_3} J_i$ be a function from I_3 into $\bigcup_{i\in I_3} J_i$ such that $\underline{e}(i) \in J_i, \forall i \in I_3$, and let $\underline{E}$ denote the set of all the functions $\underline{e}$. For the sake of simplicity, each $\underline{e} \in \underline{E}$ can be also presented as the vector $\underline{e} = [j_1, j_2, \ldots, j_w]$ in which $j_k = \underline{e}(k), k = 1, 2, \ldots, w$.*

Remark 2.14. *According to Definition 2.13, for each $e' \in E'$ and each $i \in I_2, e'(i)$ can be interpreted as a variable that takes values in $\operatorname{dom}(e'(i)) = \{1, 2\}$. By the same interpretation, $\operatorname{dom}(e''(i)) = \{1, 2\}$ for each $e'' \in E''$ and each $i \in I_3$. Similarly, for each $\underline{e} \in \underline{E}$ and each $i \in I_3$, variable $\underline{e}(i)$ takes values in $\operatorname{dom}(\underline{e}(i)) = J_i$. Furthermore, $|E'| = 2^{|I_2|}$, $|E''| = 2^{|I_3|}$ and $|\underline{E}| = \prod_{i\in I_3} |J_i|$, where $|\cdot|$ denotes the cardinality of the sets.*

Definition 2.15. *Define $\underline{\boldsymbol{X}}_1 = \max_{i\in I_1}\{\underline{\boldsymbol{X}}(i)\}$ and $\overline{\boldsymbol{X}}_1 = \min_{i\in I_1}\{\overline{\boldsymbol{X}}(i)\}$. Also, let $\underline{\boldsymbol{X}}_2 = \max_{i\in I_2}\{\underline{\boldsymbol{X}}(i)\}$ and $\overline{\boldsymbol{X}}_2(e') = \min_{i\in I_2}\left\{\overline{\boldsymbol{X}}(i, e'(i))\right\}, \forall e' \in E'$. Additionally, for each $\underline{e} \in \underline{E}$ and each $e'' \in E''$, we define $\underline{\boldsymbol{X}}_3(\underline{e}) = \max_{i\in I_3}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))\}$ and $\overline{\boldsymbol{X}}_3(e'') = \min_{i\in I_3}\left\{\overline{\boldsymbol{X}}(i, e''(i))\right\}$, respectively.*

The following lemma shows that the vectors defined in Definition 2.15 are indeed the extreme solutions determining the feasible regions of sets $S_1(\boldsymbol{A},\boldsymbol{b}), S_2(\boldsymbol{A},\boldsymbol{b})$ and $S_3(\boldsymbol{A},\boldsymbol{b})$.

Lemma 2.16. (a) $S_1(\boldsymbol{A},\boldsymbol{b}) = \left[\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1\right]$.

(b) $S_2(\boldsymbol{A},\boldsymbol{b}) = \bigcup_{e'\in E'} \left[\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2\,(e')\right]$.

(c) $S_3(\boldsymbol{A},\boldsymbol{b}) = \bigcup_{\underline{e}\in \underline{E}} \bigcup_{e''\in E''} \left[\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_3\,(e'')\right]$.

Proof. **(a)** From Lemma 2.10(a) and Definitions 2.12 and 2.15, we have

$$S_1(\boldsymbol{A},\boldsymbol{b}) = \bigcap_{i\in I_1} [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i)] = \left[\max_{i\in I_1}\{\underline{\boldsymbol{X}}(i)\}, \min_{i\in I_1}\{\overline{\boldsymbol{X}}(i)\}\right] = \left[\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1\right].$$

(b) The result is obtained from the following equalities, Lemma 2.10(b) and Definitions 2.12-2.15:

$$S_2(\boldsymbol{A},\boldsymbol{b}) = \bigcap_{i\in I_2} \{[\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,1)] \cup [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}(i,2)]\} = \bigcup_{e'\in E'} \left\{\bigcap_{i\in I_2} [\underline{\boldsymbol{X}}(i), \overline{\boldsymbol{X}}\,(i, e'(i))]\right\}$$
$$= \bigcup_{e'\in E'} \left[\max_{i\in I_2}\{\underline{\boldsymbol{X}}(i)\}, \min_{i\in I_2}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))\right\}\right] = \bigcup_{e'\in E'} \left[\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2\,(e')\right].$$

(c) Similarly, by Lemma 2.10(c) and Definitions 2.12-2.15, we have

$$S_3(\boldsymbol{A},\boldsymbol{b}) = \bigcap_{i\in I_3} \left\{\left\{\bigcup_{j\in J_i} [\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,1)]\right\} \bigcup \left\{\bigcup_{j\in J_i} [\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}(i,2)]\right\}\right\}$$
$$= \bigcup_{e''\in E''} \left\{\bigcap_{i\in I_3} \left\{\bigcup_{j\in J_i} [\underline{\boldsymbol{X}}(i,j), \overline{\boldsymbol{X}}\,(i, e''(i))]\right\}\right\}$$
$$= \bigcup_{e''\in E''} \left\{\bigcup_{\underline{e}\in \underline{E}} \left\{\bigcap_{i\in I_3} [\underline{\boldsymbol{X}}(i, \underline{e}(i)), \overline{\boldsymbol{X}}\,(i, e''(i))]\right\}\right\}$$
$$= \bigcup_{e''\in E''} \bigcup_{\underline{e}\in \underline{E}} \{\bigcap_{i\in I_3} [\underline{\boldsymbol{X}}(i, \underline{e}(i)), \overline{\boldsymbol{X}}\,(i, e''(i))]\}$$
$$= \bigcup_{e''\in E''} \bigcup_{\underline{e}\in \underline{E}} [\max_{i\in I_3}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))\}, \min_{i\in I_3}\{\overline{\boldsymbol{X}}(i, e''(i))\}] = \bigcup_{e''\in E''} \bigcup_{\underline{e}\in \underline{E}} [\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_3(e'')].$$

Thus Part (c) holds, and the proof is complete. □

Based on Lemma 2.16(b), $S_2(\boldsymbol{A},\boldsymbol{b})$ is formed as the union of a finite number of nonempty cells $\left[\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2\,(e')\right]$ $(e' \in E')$. Clearly, $\left[\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2\,(e')\right] \neq \varnothing$ iff $\underline{\boldsymbol{X}}_2 \leq \overline{\boldsymbol{X}}_2\,(e')$. So, $\underline{\boldsymbol{X}}_2$ is the unique minimal solution of $S_2(\boldsymbol{A},\boldsymbol{b})$, and also each $\overline{\boldsymbol{X}}_2\,(e')$ $(e' \in E')$ such that $\underline{\boldsymbol{X}}_2 \leq \overline{\boldsymbol{X}}_2\,(e')$ is a maximal solution of $S_2(\boldsymbol{A},\boldsymbol{b})$. Similarly, from Lemma 2.16(c), if $\left[\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_3\,(e'')\right] \neq \varnothing$ for any $\underline{e} \in \underline{E}$ and $e'' \in E''$, then $\underline{\boldsymbol{X}}_3(\underline{e})$ and $\overline{\boldsymbol{X}}_3\,(e'')$ are minimal and maximal solutions of $S_3(\boldsymbol{A},\boldsymbol{b})$, respectively. From Parts (a) - (c) of Lemma 2.16, the following necessary and sufficient conditions can be directly resulted for the feasibility of sets $S_k(\boldsymbol{A},\boldsymbol{b}), k = 1, 2, 3$:

Corollary 2.17. (a) $S_1(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$ *iff* $\underline{\boldsymbol{X}}_1 \leq \overline{\boldsymbol{X}}_1$.

(b) $S_2(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$ *iff there exists some* $e' \in E'$ *such that* $\underline{\boldsymbol{X}}_2 \leq \overline{\boldsymbol{X}}_2\,(e')$.

(c) $S_3(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$ *iff there exist some* $\underline{e} \in \underline{E}$ *and* $e'' \in E''$ *such that* $\underline{\boldsymbol{X}}_3(\underline{e}) \leq \overline{\boldsymbol{X}}_3\,(e'')$.

The following theorem provides a necessary condition for the feasibility of Problem (2) that is used in the next section, where some rules are introduced for reducing the problem.

Theorem 2.18. *Suppose that* $S(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$*. Then,* $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2\} \leq \overline{\boldsymbol{X}}_1$.

Proof. Let $\boldsymbol{x} \in S(\boldsymbol{A},\boldsymbol{b})$. Since $S(\boldsymbol{A},\boldsymbol{b}) = S_1(\boldsymbol{A},\boldsymbol{b}) \cap S_2(\boldsymbol{A},\boldsymbol{b}) \cap S_3(\boldsymbol{A},\boldsymbol{b})$, then $\boldsymbol{x} \in S_1(\boldsymbol{A},\boldsymbol{b}) \cap S_2(\boldsymbol{A},\boldsymbol{b})$. So, Lemma 2.16 implies $\boldsymbol{x} \in [\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \cap \{\bigcup_{e' \in E'} [\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2(e')]\}$, i.e., $\boldsymbol{x} \in \bigcup_{e' \in E'} \{[\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \cap [\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2(e')]\}$. Therefore, $\boldsymbol{x} \in [\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \cap [\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2(e')]$ for at least one $e' \in E'$. Subsequently, we can conclude that $[\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \neq \varnothing$ and $[\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \cap [\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2(e')] \neq \varnothing$. Thus, we must have $\underline{\boldsymbol{X}}_1 \leq \overline{\boldsymbol{X}}_1$ and $\underline{\boldsymbol{X}}_2 \leq \overline{\boldsymbol{X}}_1$, and then the result follows. □

Theorem 2.19 determines the extreme solutions of Problem (2) that are also the bounds of the feasible region.

Theorem 2.19. *Suppose that* $S(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$*. Then,*

$$S(\boldsymbol{A},\boldsymbol{b}) = \bigcup_{\underline{e} \in \underline{E}} \bigcup_{e'' \in E''} \bigcup_{e' \in E'} [\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(e)\}, \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}].$$

Proof. Since $S(\boldsymbol{A},\boldsymbol{b}) = S_1(\boldsymbol{A},\boldsymbol{b}) \cap S_2(\boldsymbol{A},\boldsymbol{b}) \cap S_3(\boldsymbol{A},\boldsymbol{b})$, Lemma 2.16 implies that

$$S(\boldsymbol{A},\boldsymbol{b}) = [\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \cap \{\bigcup_{e' \in E'} [\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2(e')]\} \cap \left\{\bigcup_{\underline{e} \in \underline{E}} \bigcup_{e'' \in E''} [\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_3(e'')]\right\}.$$

Therefore, we have

$$\begin{aligned} S(A,b) &= \bigcup_{\underline{e} \in \underline{E}} \bigcup_{e'' \in E''} \bigcup_{e' \in E'} \{[\underline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_1] \cap [\underline{\boldsymbol{X}}_2, \overline{\boldsymbol{X}}_2(e')] \cap [\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_3(e'')]\} \\ &= \bigcup_{\underline{e} \in \underline{E}} \bigcup_{e'' \in E''} \bigcup_{e' \in E'} \left[\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\}, \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}\right], \end{aligned}$$

that completes the proof. □

Based on Theorem 2.19, if $\left[\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\}, \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}\right] \neq \varnothing$ for any $\underline{e} \in \underline{E}$, $e' \in E'$ and $e'' \in E''$, then

$$\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\},$$

and

$$\min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\},$$

are minimal and maximal solutions of $S(\boldsymbol{A},\boldsymbol{b})$, respectively. Theorem 2.19 also gives the following necessary and sufficient condition for the feasibility of Problem (2):

Theorem 2.20. $S(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$ *if and only if there exist* $\underline{e} \in \underline{E}, e' \in E'$ *and* $e'' \in E''$ *such that* $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(e)\} \leq \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}$.

Proof. The proof is directly resulted from Theorem 2.19. □

3 Simplification rules

Based on Theorem 2.19, the feasible solutions set of Problem (2) is completely determined by a finite number of non-empty closed convex cells. Furthermore, by Theorem 2.20, $S(\boldsymbol{A},\boldsymbol{b}) \neq \varnothing$ if and only if

$$\left[\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(e)\}, \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}\right] \neq \varnothing,$$

for some $\underline{e} \in \underline{E}, e' \in E'$ and $e'' \in E''$. For this reason, the resolution of $S(\boldsymbol{A}, \boldsymbol{b})$ can be accelerated by focusing only on those functions $\underline{e} \in \underline{E}, e' \in E'$ and $e'' \in E''$ such that

$$\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(e)\} \leq \min\left\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\right\}. \tag{5}$$

For the sake of expository reference, such functions are formally defined in the following definition:

Definition 3.1. *Suppose that $S(\boldsymbol{A}, \boldsymbol{b}) \neq \varnothing$ (and therefore, $J_i \neq \varnothing, \forall i \in I$, from Corollary 2.4), $I_2 \neq \varnothing$ and $I_3 \neq \varnothing$. Triple $(\underline{e}, e', e'')$ of functions $\underline{e} \in \underline{E}, e' \in E'$ and $e'' \in E''$ is called admissible, if $\underline{e}, e'$ and e'' satisfy Relation (5). Also, the set of all the admissible triples is denoted by T, i.e.,*

$$T = \left\{(\underline{e}, e', e'') : \underline{e} \in \underline{E}, e' \in E', e'' \in E'', \max\left\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\right\} \leq \min\left\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\right\}\right\}.$$

In contrast, an inadmissible triple is a triple $(\underline{e}, e', e'')$ that violates Relation (5) and therefore leads to an empty cell $\left[\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\}, \min\left\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\right\}\right]$. Consequently, the inadmissible triples can be considered as redundant or irrelevant selections of functions $\underline{e}, e'$ and e'' in the sense that if they are disregarded, the feasible region $S(\boldsymbol{A}, \boldsymbol{b})$ is not affected. Due to this fact, some simplification rules are presented that restrict the selection possibilities of triples (e, e', e'') by removing the inadmissible triples as far as possible. These rules are initially applied to Problem (2) to reduce the problem before starting to solve it, i.e., before selecting any triple $(\underline{e}, e', e'')$ to construct solutions $\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_2(e')$ and $\overline{\boldsymbol{X}}_3(e'')$.

Definition 3.2. *Suppose that $I_2 = \{i'_1, \dots, i'_p\}$ and $I_3 = \{i''_1, \dots, i''_q\}$. Let $\overline{\boldsymbol{M}}^1 = (\bar{m}^1_{ij})_{p\times n}$ and $\overline{\boldsymbol{M}}^2 = (\bar{m}^2_{ij})_{p\times n}$ be two $p \times n$ matrices, and $\overline{\boldsymbol{N}}^1 = (\bar{n}^1_{ij})_{q\times n}$ and $\overline{\boldsymbol{N}}^2 = (\bar{n}^2_{ij})_{q\times n}$ be two $q \times n$ matrices whose k 'th rows (denoted by $\overline{\boldsymbol{M}}^1_k, \overline{\boldsymbol{M}}^2_k, \bar{N}^1_k$ and $\bar{N}^2_k$, respectively) are defined as follows:*

$$\overline{\boldsymbol{M}}^1_{\boldsymbol{k}} = \overline{\boldsymbol{X}}(i'_k, 1), \overline{\boldsymbol{M}}^2_k = \overline{\boldsymbol{X}}(i'_k, 2), k = 1, \dots, p,$$
$$\overline{\boldsymbol{N}}^1_{\boldsymbol{k}} = \overline{\boldsymbol{X}}(i''_k, 1), \overline{\boldsymbol{N}}^2_{\boldsymbol{k}} = \overline{\boldsymbol{X}}(i''_k, 2), k = 1, \dots, q,$$

where $\overline{\boldsymbol{X}}(i'_k, 1), \overline{\boldsymbol{X}}(i'_k, 2), \overline{\boldsymbol{X}}(i''_k, 1)$ and $\overline{\boldsymbol{X}}(i''_k, 2)$ are given by Definitions 2.8 and 2.9 . Moreover, let $\underline{N} = (\underline{n}_{ij})_{q\times n}$ be a $q \times n$ matrix whose entries are defined as follows:

$$\underline{n}_{kj} = \begin{cases} b_{i''_k} & j \in J_{i''_k} \cup \{i''_k\} \\ -\infty & otherwise \end{cases} \quad , k = 1, \dots, q \quad , \quad j = 1, \dots, n. wehaveeight.$$

Remark 3.3. *For each $i'_k \in I_2 (k = 1, \dots, p)$, the k' th row of matrix $\bar{M}^1$ $(\bar{M}^2)$ is exactly the maximal solution $\overline{\boldsymbol{X}}(i'_k, 1)$ $(\overline{\boldsymbol{X}}(i'_k, 2))$ of $S\left(\boldsymbol{a}_{i'_k}, b_{i_k}\right)$. Similarly, for each $i''_k \in I_3 (k = 1, \dots, q)$, the k' th row of matrix $\bar{N}^1$ $(\bar{N}^2)$ is exactly the maximal solution $\bar{X}(i''_k, 1)$ $(\bar{X}(i''_k, 2))$ of $S\left(a_{i'_k}, b_{i^*_k}\right)$. Moreover, Matrix $\underline{N}$ basically plays the same role as do matrix M^{1^*} defined in* [16]*, the minimal solution matrix $\breve{\Gamma}$ introduced in* [44] *and the simplified matrix presented in* [32]*. We refer the reader to* [16] *in which some details were provided about the relationships between the three mentioned matrices.*

Lemma 3.4. *(Rule 1). Let $I_2 = \{i'_1, \dots, i'_p\}$ and consider a fixed $i'_k \in I_2$. Also, suppose that there exists some $j \in J$ such that $\max\left\{(\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j\right\} > \bar{m}^1_{kj}$ $\left(\max\left\{(\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j\right\} > \bar{m}^2_{kj}\right)$, where $(\underline{X}_1)_j$ and $(\underline{X}_2)_j$ denote the j 'th components of $\underline{X}_1$ and $\underline{\boldsymbol{X}}_2$, respectively. Then, each triple $(\underline{e}, e', e'')$ such that $\vec{e}'(i'_k) = 1$ $(\vec{e}'(i'_k) = 2)$ is inadmissible.*

Proof. Let $\max\left\{(\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j\right\} > \bar{m}^1_{kj}$ for some $j \in J$. Also, consider a triple $(\underline{e}, e', e'')$ in which $\underline{e}$ and e'' are two arbitrary functions in $\underline{E}$ and E'', respectively, and $e' \in E'$ such that $e'(i'_k) = 1$. From Definition 2.15 , we have

$$\overline{\boldsymbol{X}}_2(e') = \min_{i \in I_2} \left\{ \overline{\boldsymbol{X}}(i, e'(i)) \right\} = \min \left\{ \overline{\boldsymbol{X}}(i'_k, 1), \min_{i \in I_2 - \{i'_k\}} \left\{ \overline{\boldsymbol{X}}(i, e'(i)) \right\} \right\}.$$

Therefore, j' th component of $\overline{\boldsymbol{X}}_2(e')$ is obtained as

$$\overline{\boldsymbol{X}}_2(e')_j = \min \left\{ \overline{\boldsymbol{X}}(i'_k, 1)_j, \min_{i \in I_2 - \{i'_k\}} \left\{ \overline{\boldsymbol{X}}(i, e'(i))_j \right\} \right\},$$

that implies $\overline{\boldsymbol{X}}_2(e')_j \le \overline{\boldsymbol{X}}(i'_k, 1)_j$. Since $\overline{\boldsymbol{X}}(i'_k, 1)_j = \bar{m}^1_{kj}$ (from Definition 3.2) and $\bar{m}^1_{kj} < \max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\}$, we have $\overline{\boldsymbol{X}}_2(e')_j < \max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\}$. Thus, $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2\} \nleq \overline{\boldsymbol{X}}_2(e')$ that implies $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\} \nleq \overline{\boldsymbol{X}}_2(e')$. Consequently, $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\} \nleq \min\left\{ \overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'') \right\}$ that violates Relation (5). Hence, by Definition 3.1, it follows that $(\underline{e}, e', e'') \notin T$. The proof is similar if $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{m}^2_{kj}$ for some $j \in J$ and $e'(i'_k) = 2$. □

Remark 3.5. *By Definition 2.13 and Remark 2.14,* $e'(i'_k) \in \operatorname{dom}(e'(i'_k)) = \{1, 2\}$ *for each* $e' \in E'$ *and each* $i'_k \in I_2$*. In the case that* $e'(i'_k) = 1$*, we have* $\overline{\boldsymbol{X}}(i'_k, e'(i'_k)) = \overline{\boldsymbol{X}}(i'_k, 1)$ *and therefore according to Definition 2.15, the maximal solution* $\overline{\boldsymbol{X}}(i'_k, 1)$ *participates in the construction of* $\overline{\boldsymbol{X}}_2(e') = \min_{i \in I_2}\left\{ \overline{\boldsymbol{X}}(i, e'(i)) \right\}$*. However, if* $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{m}^1_{kj}$*, Lemma 3.4 implies that each triple* $(\underline{e}, e', e'')$ *such that* $e'(i'_k) = 1$ *is inadmissible and it leads to an empty cell*

$$\left[\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(e)\}, \min\left\{ \overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'') \right\} \right].$$

Therefore, based on Rule 1 , if $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{m}^1_{kj}$ *for some* $i'_k \in I_2$ *and* $j \in J$*, we set* $\overline{\boldsymbol{M}}^1_{\boldsymbol{k}} = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *and* $\operatorname{dom}(e'(i'_k)) = \{2\}$*, that is, the domain of* $e'(i'_k)$ *is decreased from* $\{1, 2\}$ *to* $\{2\}$*. The replacement* $\overline{\boldsymbol{M}}^1_k = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *means that the maximal solution* $\overline{\boldsymbol{X}}(i'_k, 1)$ *(i.e.,* $\overline{\boldsymbol{X}}(i'_k, e'(i'_k))$ *with* $e'(i'_k) = 1$*) is not allowed to be selected in the construction of* $\overline{\boldsymbol{X}}_2(e')$*. Similar notation is used if* $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{m}^2_{kj}$ *for some* $i'_k \in I_2$ *and* $j \in J$*. In this case, we set* $\overline{\boldsymbol{M}}^2_{\boldsymbol{k}} = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *and* $\operatorname{dom}(e'(i'_k)) = \{1\}$*.*

Corollary 3.6. *If* $\overline{\boldsymbol{M}}^1_k = \overline{\boldsymbol{M}}^2_k = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *for some* $k \in \{1, \ldots, p\}$*, then Problem (2) is infeasible.*

Proof. From Remark 3.5, the assignments $\overline{\boldsymbol{M}}^1_{\boldsymbol{k}} = \overline{\boldsymbol{M}}^2_{\boldsymbol{k}} = (\infty, \infty, \ldots, \infty)_{1 \times n}$ mean that each value of $e'(i'_k)$, whether $e'(i'_k) = 1$ or $e'(i'_k) = 2$, leads to an inadmissible triple. Therefore, each triple $(\underline{e}, e', e'')$ violates Relation (5). Now, Theorem 2.20 implies $S(\boldsymbol{A}, \boldsymbol{b}) \neq \varnothing$. □

Lemma 3.7. *(Rule 2). Let* $I_3 = \left\{ i''_1, \ldots, i''_q \right\}$ *and consider a fixed* $i''_k \in I_3$*. Also, suppose that there exists some* $j \in J$ *such that* $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{n}^1_{kj}$ $\left(\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{n}^2_{kj} \right)$*. Then, each triple* $(\underline{e}, e', e'')$ *such that* $e''(i''_k) = 1\, (e''(i''_k) = 2)$ *is inadmissible.*

Proof. The proof is similar to the proof of Lemma 3.4 by replacing I_2, E', e' and $\overline{\boldsymbol{X}}_2(e')$ with I_3, E'', e'' and $\overline{\boldsymbol{X}}_3(e'')$, respectively. □

Remark 3.8. *According to Rule 2, if* $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{n}^1_{kj}$ *for some* $i''_k \in I_3$ *and* $j \in J$*, we set* $\overline{\boldsymbol{N}}^1_{\boldsymbol{k}} = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *and* $\operatorname{dom}(e''(i''_k)) = \{2\}$*. Similarly, if* $\max\left\{ (\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j \right\} > \bar{n}^2_{kj}$ *for some* $i''_k \in I_3$ *and* $j \in J$*, we set* $\bar{N}^2_k = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *and* $\operatorname{dom}(e''(i''_k)) = \{1\}$*.*

Corollary 3.9. *If* $\overline{\boldsymbol{N}}^1_k = \overline{\boldsymbol{N}}^2_k = (\infty, \infty, \ldots, \infty)_{1 \times n}$ *for some* $k \in \{1, \ldots, q\}$*, then Problem (2) is infeasible.*

Proof. The proof is similar to the proof of Corollary 3.6 by replacing I_2, E' and e' with I_3, E'' and e'', respectively. □

Lemma 3.10. *(Rule 3). Let $I_3 = \{i''_1, \ldots, i''_q\}$ and consider a fixed $i''_k \in I_3$. Also, suppose that there exists some $j \in J_{i''_k}$ such that $\underline{n}_{kj} > \left(\overline{\boldsymbol{X}}_1\right)_j$, where $\left(\overline{\boldsymbol{X}}_1\right)_j$ denotes the j 'th component of $\bar{X}_1$. Then, each triple $(\underline{e}, e', e'')$ such that $\underline{e}\,(i''_k) = j$ is inadmissible.*

Proof. Suppose that $(\underline{e}, e', e'')$ in which e' and e'' are two arbitrary functions in E' and E'', respectively, and $\underline{e} \in \underline{E}$ such that $\underline{e}\,(i''_k) = j$. From Definition 2.15, we have

$$\underline{\boldsymbol{X}}_3(\underline{e}) = \max_{i \in I_3}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))\} = \max\left\{\underline{\boldsymbol{X}}\,(i''_k, j)\,,\ \max_{i \in I_3 - \{i'_k\}}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))\}\right\}.$$

Therefore, j 'th component of $\underline{\boldsymbol{X}}_3(\underline{e})$ is obtained as

$$\underline{\boldsymbol{X}}_3(\underline{e})_j = \max\left\{\underline{\boldsymbol{X}}\,(i''_k, j)_j\,,\ \max_{i \in I_3 - \{k''_k\}}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))_j\}\right\},$$

that implies $\underline{\boldsymbol{X}}_3(\underline{e})_j \geq \underline{\boldsymbol{X}}\,(i''_k, j)_j$. On the other hand, since $\underline{\boldsymbol{X}}\,(i''_k, j)_j = \underline{n}_{kj}$ (from Definitions 2.9 and 3.2) and $\underline{n}_{kj} > \left(\overline{\boldsymbol{X}}_1\right)_j$, it follows that $\underline{\boldsymbol{X}}_3(\underline{e})_j > \left(\overline{\boldsymbol{X}}_1\right)_j$. Thus, $\underline{\boldsymbol{X}}_3(\underline{e}) \nleq \overline{\boldsymbol{X}}_1$, and therefore $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\} \nleq \overline{\boldsymbol{X}}_1$. Hence, we have $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\} \nleq \min\left\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2\,(e')\,, \overline{\boldsymbol{X}}_3\,(e'')\right\}$ that violates Relation (5). Now, from Definition 3.1, we have $(\underline{e}, e', e'') \notin T$ that completes the proof. □

Remark 3.11. *Based on Rule 3, if $\underline{n}_{kj} > \left(\overline{\boldsymbol{X}}_1\right)_j$ for some $i''_k \in I_3$ and $j \in J_{i''_k}$, then we set $\underline{n}_{kj} = -\infty$ and* $\operatorname{dom}(\underline{e}\,(i''_k)) = J_{i''_k} - \{j\}$*. The assignment $\underline{n}_{kj} = -\infty$ means that the minimal solution $\underline{\boldsymbol{X}}\,(i''_k, j)$ (i.e., $\underline{\boldsymbol{X}}\,(i''_k, \underline{e}\,(i''_k))$ with $\underline{e}\,(i''_k) = j$) is not allowed to be selected in the construction of $\underline{\boldsymbol{X}}_3(\underline{e}) = \max_{i \in I_3}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))\}$ as defined in Definition 2.15.*

Corollary 3.12. *Suppose that $i''_k \in I_3$ and $\underline{n}_{kj} = -\infty, \forall j \in J_{i''_k}$. Then, Problem (2) is infeasible.*

Proof. By Definition 2.13 and Remark 2.14, $\operatorname{dom}(\underline{e}\,(i''_k)) = J_{i''_k}$. Moreover, since $\underline{n}_{kj} = -\infty$, $\forall j \in J_{i''_k}$, Lemma 3.10 implies that any possible value of $\underline{e}\,(i''_k)$ leads to an inadmissible triple $(\underline{e}, e', e'')$ that violates Relation (5). Thus, from Theorem 2.20, we have $S(\boldsymbol{A}, \boldsymbol{b}) = \varnothing$. □

Lemma 3.13. *(Rule 4). Let $I_2 = \{i'_1, \ldots, i'_p\}$ and $I_3 = \{i''_1, \ldots, i''_q\}$. Also, suppose that $i'_r \in I_2$ and $i''_s \in I_3$ such that $a_{i'_r r''_s} > b_{i'_r}$ and $b_{i''_r} < b_{i''_s}$. Then, each triple $(\underline{e}, e', e'')$ such that $e'\,(i'_r) = 2$ is inadmissible.*

Proof. Consider a triple $(\underline{e}, e', e'')$ in which $\underline{e}$ and e'' are two arbitrary functions in $\underline{E}$ and E'', respectively, and $e' \in E'$ such that $e'\,(i'_r) = 2$. Similar to the proof of Lemmas 3.4 and 3.10, i''_s th components of $\overline{\boldsymbol{X}}_2\,(e')$ and $\underline{\boldsymbol{X}}_3(\underline{e})$ are obtained as follows:

$$\overline{\boldsymbol{X}}_2\,(e')_{i'_s} = \min_{i \in I_2}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))_{i''_s}\right\} = \min\left\{\overline{\boldsymbol{X}}\,(i'_r, 2)_{i_{i'}}\,,\ \min_{i \in I_2 - \{i'_r\}}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))_{i''_s}\right\}\right\},$$

$$\underline{\boldsymbol{X}}_3(\underline{e})_{i''_s} = \max_{i \in I_3}\left\{\underline{\boldsymbol{X}}(i, \underline{e}(i))_{i''_s}\right\} = \max\left\{\underline{\boldsymbol{X}}\,(i''_s, j)_{i''_s}\,,\ \max_{i \in I_3 - \{i''_s\}}\left\{\underline{\boldsymbol{X}}(i, \underline{e}(i))_{i''_s}\right\}\right\}.$$

Therefore, we have

$$\overline{\boldsymbol{X}}_2\,(e')_{i''_s} \leq \overline{\boldsymbol{X}}\,(i'_r, 2)_{i''_s} \tag{6}$$

$$\underline{\boldsymbol{X}}_3(\underline{e})_{i''_s} \geq \underline{\boldsymbol{X}}\,(i''_s, j)_{i''_s}\,. \tag{7}$$

Also, we have $\overline{\boldsymbol{X}}\,(i'_r, 2)_{i''_s} = \bar{m}^2_{rr''_s} = b_{i'_r}$ (Definitions 2.8 and 3.2) and $\underline{\boldsymbol{X}}\,(i''_s, j)_{i_{i''}} = \underline{n}_{s''_i} = b_{i''_s}$ (Definitions 2.9 and 3.2). The last equalities together with (6) and (7) imply $\overline{\boldsymbol{X}}_2\,(e')_{i''_s} \leq b_{i'_t}$ and $\underline{\boldsymbol{X}}_3(\underline{e})_{i'_s} \geq b_{i''_s}$. Therefore, $\overline{\boldsymbol{X}}_2\,(e')_{i''_s} \leq$

$b_{i'_i} < b_{i''_s} \leq \underline{\boldsymbol{X}}_3(\underline{e})_{i''_s}$, and then $\underline{\boldsymbol{X}}_3(\underline{e}) \nleq \overline{\boldsymbol{X}}_2(e')$. So, $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\} \nleq \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}$ that violates Relation (5). Now, from Definition 3.1, we conclude $(\underline{e}, e', e'') \notin T$, and the result is obtained. □

Lemma 3.14. *(Rule 5). Let* $I_3 = \{i''_1, \ldots, i''_q\}$*. Also, suppose that* $i''_r, i''_s \in I_3$ *such that* $i''_r \neq i''_s$, $a_{i''_r i''_s} > b_{i''_r}$ *and* $b_{i''_r} < b_{i''_s}$*. Then, each triple* $(\underline{e}, e', e'')$ *such that* $e''(i''_r) = 2$ *is inadmissible.*

Proof. The proof is similar to the proof of Lemma 3.13. □

Remark 3.15. *According to Rule 4, if there exist* $i'_r \in I_2$ *and* $i''_s \in I_3$ *such that* $a_{i'r''_s} > b_{i'_r}$ *and* $b_{i'_r} < b_{i''_s}$*, then we set* $\overline{\boldsymbol{M}}_r^2 = (\infty, \infty, \ldots, \infty)_{1\times n}$ *and* $\operatorname{dom}(e'(i'_r)) = \{1\}$*. Similarly, based on Rule 5, if there exist* $i''_r, i''_s \in I_3$ *such that* $i''_r \neq i''_s, \quad a_{i''_r s''_s} > b_{i''_r}$ *and* $b_{i''_r} < b_{i''_s}$*, then we set* $\bar{N}_r^2 = (\infty, \infty, \ldots, \infty)_{1\times n}$ *and* $\operatorname{dom}(e''(i''_r)) = \{1\}$*.*

Lemma 3.16. *(Rule 6). Suppose that Rules* $1-5$ *have been already applied. Also, suppose that* $\operatorname{dom}(e'(i'_r)) = \{1\}$ *for some* $i'_r \in I_2$ *(or equivalently,* $\overline{\boldsymbol{M}}_r^1 \neq (\infty, \infty, \ldots, \infty)_{1\times n}$ *and* $\overline{\boldsymbol{M}}_r^2 = (\infty, \infty, \ldots, \infty)_{1\times n}$*). If there exists* $i''_s \in I_3$ *such that* $i'_r \in J_{i''_s}$ *and* $b_{i'_r} < b_{i''_s}$*, Then, each triple* $(\underline{e}, e', e'')$ *such that* $\underline{e}(i''_s) = i'_r$ *is inadmissible.*

Proof. Suppose that $(\underline{e}, e', e'')$ in which e' and e'' are two arbitrary functions in E' and E'', respectively, and $\underline{e} \in \underline{E}$ such that $\underline{e}(i''_s) = i'_r$. Since $\operatorname{dom}(e'(i'_r)) = \{1\}$, then there exists only one option for the value of $e'(i'_r)$, i.e., $e'(i'_r) = 1$. The i'_r th components of $\overline{\boldsymbol{X}}_2(e')$ and $\underline{\boldsymbol{X}}_3(\underline{e})$ are obtained as follows:

$$\overline{\boldsymbol{X}}_2(e')_{i_{r'}} = \min_{i\in I_2}\left\{\overline{\boldsymbol{X}}(i, e'(i))_{i_{r'}}\right\} = \min\left\{\overline{\boldsymbol{X}}(i'_r, 1)_{i'_r}, \min_{i\in I_2-\{i_{r'}\}}\left\{\bar{X}(i, e'(i))_{i_{i'_r}}\right\}\right\},$$

$$\underline{\boldsymbol{X}}_3(\underline{e})_{i'_r} = \max_{i\in I_3}\left\{\underline{\boldsymbol{X}}(i, \underline{e}(i))_{i_{i'_r}}\right\} = \max\left\{\underline{\boldsymbol{X}}(i''_s, i'_r)_{i'_r}, \max_{i\in I_3-\{i''_s\}}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))_{i'_r}\}\right\}.$$

Therefore, we have

$$\overline{\boldsymbol{X}}_2(e')_{i'_r} \leq \overline{\boldsymbol{X}}(i'_r, 1)_{i'_r}, \tag{8}$$

$$\underline{\boldsymbol{X}}_3(\underline{e})_{i'_r} \geq \max \underline{\boldsymbol{X}}(i''_s, i'_r)_{i'_r}. \tag{9}$$

Also, we have $\overline{\boldsymbol{X}}(i'_r, 1)_{i'_r} = \bar{m}^1_{ri'_r} = b_{i'_r}$ (Definitions 2.8 and 3.2) and $\underline{\boldsymbol{X}}(i''_s, i'_r)_{i'_r} = \underline{n}_{si'_r} = b_{i''_s}$ (Definitions 2.9 and 3.2). The latter equalities together with (8) and (9) imply $\overline{\boldsymbol{X}}_2(e')_{i'_r} \leq b_{i'_r}$ and $\underline{\boldsymbol{X}}_3(\underline{e})_{i'_r} \geq b_{i''_s}$. Therefore, $\overline{\boldsymbol{X}}_2(e')_{i'_r} \leq b_{i'_r} < b_{i''_s} \leq \underline{\boldsymbol{X}}_3(\underline{e})_{i'_r}$, and then $\underline{\boldsymbol{X}}_3(\underline{e}) \nleq \overline{\boldsymbol{X}}_2(e')$. So, $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\} \nleq \min\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\}$ that violates Relation (5). Hence, we have $(\underline{e}, e', e'') \notin T$ from Definition 3.1 that completes the proof. □

Lemma 3.17. *(Rule 7). Suppose that Rules* $1-5$ *have been already applied. Also, suppose that* $\operatorname{dom}(e''(i''_r)) = \{1\}$ *for some* $i''_r \in I_3$ *(or equivalently,* $\bar{N}_r^1 \neq (\infty, \infty, \ldots, \infty)_{1\times n}$ *and* $\bar{N}_r^2 = (\infty, \infty, \ldots, \infty)_{1\times n}$*). If there exists* $i''_s \in I_3$ *such that* $i''_r \neq i''_s, i''_r \in J_{i''_s}$ *and* $b_{i''_r} < b_{i''_s}$*, then each triple* $(\underline{e}, e', e'')$ *such that* $\underline{e}(i''_s) = i''_r$ *is inadmissible.*

Proof. The proof is similar to the proof of Lemma 3.16. □

Remark 3.18. *According to Rule 6 , if there exist* $i'_r \in I_2$ *and* $i''_s \in I_3$ *such that* $\operatorname{dom}(e'(i'_r)) = \{1\}, \quad i'_r \in J_{i''_s}$ *and* $b_{i'_r} < b_{i''_s}$*, we set* $\underline{n}_{si'_r} = -\infty$ *and* $\operatorname{dom}(\underline{e}(i''_s)) = J_{i''_s} - \{i'_r\}$*. Similarly, based on Rule 7 , if there exist* $i''_r, i''_s \in I_3$ *such that* $i''_r \neq i''_s, \operatorname{dom}(e''(i''_r)) = \{1\}$, $i''_r \in J_{i''_s}$ *and* $b_{i''_r} < b_{i''_s}$*, we set* $\underline{n}_{si''_r} = -\infty$ *and* $\operatorname{dom}(\underline{e}(i''_s)) = J_{i''_s} - \{i''_r\}$*.*

4 Optimal solution of the problem

This section describes the optimal solutions of Problem (2). Also, it is proved that under some assumptions on the components of matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, Problem (2) automatically yields binary optimal solutions.

Definition 4.1. *Let* $J^+ = \{j \in J : c_j \geq 0\}$ *and* $J^- = \{j \in J : c_j < 0\}$*. Associated with each triple* $e = (\underline{e}, e', e'')$ *in* T*, we define a vector* $\boldsymbol{x}_e = ((x_e)_1, (x_e)_2, \ldots, (x_e)_n)$ *as follows:*

$$(x_e)_j = \begin{cases} \max\left\{(\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j, \underline{\boldsymbol{X}}_3(\underline{e}_j)\right\} & , j \in J^+ \\ \min\left\{\left(\overline{\boldsymbol{X}}_1\right)_j, \overline{\boldsymbol{X}}_2(e')_j, \overline{\boldsymbol{X}}_3(e'')_j\right\} & , j \in J^- \end{cases} \tag{10}$$

According to the following theorem, the optimal solution of Problem (2) is always attained as $\boldsymbol{x}_e$ *for some* $e = (\underline{e}, e', e'') \in T$*.*

Theorem 4.2. *Let* $\boldsymbol{c}^T \boldsymbol{x}_{e^*} = \min\left\{\boldsymbol{c}^T \boldsymbol{x}_e : e \in T\right\}$*. Then,* $\boldsymbol{x}_{e^*}$ *is the optimal solution of Problem (2).*

Proof. Let $\boldsymbol{x} \in S(\boldsymbol{A}, \boldsymbol{b})$. Therefore, Theorem 2.19 implies that

$$\boldsymbol{x} \in \left[\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \underline{\boldsymbol{X}}_3(\underline{e})\}, \min\left\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')\right\}\right],$$

for some $e = (\underline{e}, e', e'') \in T$. Therefore, from Definition 4.1, $(x_e)_j \leq x_j, \quad \forall j \in J^+$, and $x_j \leq (x_e)_j, \forall j \in J^-$. Hence, $\sum_{j \in J^+} c_j (x_e)_j \leq \sum_{j \in J^+} c_{j'} x_j$ and $\sum_{j \in J^-} c_j (x_e)_j \leq \sum_{j \in J^-} c_j x_j$, and therefore $\boldsymbol{c}^T \boldsymbol{x}_e = \sum_{j \in J^+} c_j (x_e)_j + \sum_{j \in J^-} c_j (x_e)_j \leq \sum_{j \in J^+} c_j x_j + \sum_{j \in J^-} c_j x_j = \boldsymbol{c}^T \boldsymbol{x}$. Now, by the assumption of the theorem, it follows that $\boldsymbol{c}^T \boldsymbol{x}_{e^*} \leq \boldsymbol{c}^T \boldsymbol{x}_e \leq \boldsymbol{c}^T \boldsymbol{x}$. □

Corollary 4.3. *If Problem (2) is expressed as a maximization problem, the global optimal solution* $\boldsymbol{x}^*$ *is obtained by* $\boldsymbol{c}^T \boldsymbol{x}^* = \max\left\{\boldsymbol{c}^T \boldsymbol{x}_e : e \in T\right\}$ *where solutions* $\boldsymbol{x}_e = ((x_e)_1, (x_e)_2, \ldots, (x_e)_n)$*, where* $e = (\underline{e}, e', e'') \in T$*, are defined as follows*

$$(x_e)_j = \begin{cases} \min\left\{\left(\overline{\boldsymbol{X}}_1\right)_j, \overline{\boldsymbol{X}}_2(e')_j, \overline{\boldsymbol{X}}_3(e'')_j\right\} & , j \in J^+ \\ \max\left\{(\underline{\boldsymbol{X}}_1)_j, (\underline{\boldsymbol{X}}_2)_j, \underline{\boldsymbol{X}}_3(\underline{e})_j\right\} & , j \in J^- \end{cases} \tag{11}$$

The following theorem shows that in the special cases of Problem (2), optimal solutions are binary valued.

Theorem 4.4. *Consider Problem (2) where* $\boldsymbol{A} = (a_{ij})_{m \times n}$ *is a matrix such that* $a_{ij} \in \{0, 1\}$ $(\forall i \in I$ *and* $\forall j \in J)$ *and* $\boldsymbol{b} = \boldsymbol{0}_{m \times 1}$ *is a zero vector. If* $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ *is the optimal solution of the problem, then* $x_j^* \in \{0, 1\}, \forall j \in J$*.*

Proof. By contradiction, suppose that $0 < x_{j_0}^* < 1$ for some $j_0 \in J$. Since $\boldsymbol{b} = \boldsymbol{0}$, then from Relation (3), the constraints of the problem are expressed as $\max_{j=1}^n \{\min\{a_{ij}, x_i, x_j\}\} = 0, \forall i \in I$. Therefore, $\min\{a_{ij}, x_i, x_j\} = 0, \forall i \in I$ and $\forall j \in J$. Particularly, we have

$$\begin{array}{l} \min\left\{a_{ij_0}, x_i^*, x_{j_0}^*\right\} = 0, \forall i \in I, \\ \min\left\{a_{j_0 j}, x_{j_0}^*, x_j^*\right\} = 0, \forall j \in J \text{ such that } j_0 \leq j. \end{array} \tag{12}$$

The assumption $0 < x_{j_0}^* < 1$ and (12) result in the following two conditions:

$$\begin{array}{l} (I) \ \text{ for each } \ i \in I, \ \text{ either } \ a_{ij_0} = 0 \ \text{ or } \ x_i^* = 0. \\ (II) \ \text{ for each } \ j \in J \ \text{ such that } \ j_0 \leq j, \ \text{ either } \ a_{j_j j} = 0 \ \text{ or } \ x_j^* = 0. \end{array} \tag{13}$$

At the first case, consider $j_0 \in J^-$and define solution x' such that $x'_{j_0} = 1$ and $x'_j = x_j^*$, $\forall j \in J - \{j_0\}$. Based on (13), it is clear that $\min\left\{a_{ij_0}, x'_i, x'_{j_0}\right\} = \min\{a_{ij_0}, x_i^*, 1\} = 0, \forall i \in I$, and $\min\left\{a_{j_0 j}, x'_{j_0}, x'_j\right\} = \min\left\{a_{j_0 j}, 1, x_j^*\right\} = 0, \forall j \in J$ such that $j_0 \leq j$; that is, x' satisfies all the equations stated in (12). Moreover, since $x'_j = x_j^*, \forall j \in J - \{j_0\}$, x' also satisfies the other equations in which $x_{j_0}^*$ does not appear. So, $\boldsymbol{x}'$ is a feasible solution to the problem. On the other hand, we have $c_{j_0} x'_{j_0} < c_{j_0} x_{j_0}^*$ and $\sum_{j \in J - \{j_0\}} c_j x'_j = \sum_{j \in J - \{j_0\}} c_j x_j^*$. Therefore,

Algorithm 1 Solution of Problem (2)

Given: Problem (2) and suppose that $I_2 = \{i'_1, \ldots, i'_p\}$ and $I_3 = \{i''_1, \ldots, i''_q\}$.

if $J_i = \varnothing$ for some $i \in I$ **then**
  return; Problem (2) is infeasible (Corollary 2.4)
end if
Compute solutions $\underline{X}_1, \bar{X}_1$ and $\underline{X}_2$ (Definition 2.15).
if $\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2\} \not\leq \overline{\boldsymbol{X}}_1$ **then**
  return; Problem (2) is infeasible (Theorem 2.18)
end if
Apply Rule 1 in order to reduce matrices $\overline{\boldsymbol{M}}^1$ and $\overline{\boldsymbol{M}}^2$.
if $\overline{\boldsymbol{M}}^1_{\boldsymbol{k}} = \overline{\boldsymbol{M}}^2_{\boldsymbol{k}} = (\infty, \infty, \ldots, \infty)$ for some $k \in \{1, \ldots, p\}$ **then**
  return; Problem (2) is infeasible (Corollary 3.6)
end if
Apply Rule 2 to reduce matrices $\bar{N}^1$ and $\bar{N}^2$.
if $\bar{N}^1_k = \bar{N}^2_k = (\infty, \infty, \ldots, \infty)$ for some $k \in \{1, \ldots, q\}$ **then**
  return; Problem (2) is infeasible (Corollary 3.9)
end if
Apply Rule 3 to reduce matrix $\underline{N}$.
if there exists some $k \in \{1, \ldots, q\}$ such that $\underline{n}_{kj} = -\infty, \forall j \in J_{i''_k}$ **then**
  return; Problem (2) is infeasible (Corollary 3.12)
end if
Apply Rules 4 and 5 to reduce matrices $\bar{M}^2$ and $\bar{N}^2$.
Apply Rules 6 and 7 to reduce matrix $\underline{N}$.
For each triple $(\underline{e}, e', e'')$ based on the reduced matrices $\bar{M}^1, \bar{M}^2, \bar{N}^1, \overline{\boldsymbol{N}}^2$, and $\underline{\boldsymbol{N}}$, compute solutions $\overline{\boldsymbol{X}}_2(e'), \overline{\boldsymbol{X}}_3(e'')$ and $\underline{\boldsymbol{X}}_3(\underline{e})$ (Remarks 3.3-3.18).
If $\underline{\boldsymbol{X}}_3(\underline{e}), \overline{\boldsymbol{X}}_2(e')$ and $\overline{\boldsymbol{X}}_3(e'')$ satisfy Relation (5), generate solution $\boldsymbol{x}_e$.
Find the optimal solution $\boldsymbol{x}_{e^*}$ by $\boldsymbol{c}^T\boldsymbol{x}_{e^*} = \min\{\boldsymbol{c}^T\boldsymbol{x}_e : e \in T\}$ (Theorem 4.2).

$\boldsymbol{c}^T\boldsymbol{x}' = c_{j_0}x'_{j_0} + \sum_{j \in J-\{j_0\}} c_j x'_j < c_{j_0}x^*_{j_0} + \sum_{j \in J-\{j_0\}} c_j x^*_j = \boldsymbol{c}^T\boldsymbol{x}^*$ that violates the optimality of $\boldsymbol{x}^*$. If $j_0 \in J^+$, the proof is simpler by defining solution $\boldsymbol{x}'$ such that $x'_{j_0} = 0$ and $x'_j = x^*_j$,$\forall j \in J - \{j_0\}$. □

Corollary 4.5. *Suppose that Problem (2) is expressed as a maximization problem where $\boldsymbol{A} = (a_{ij})_{m \times n}$ is a matrix such that $a_{ij} \in \{0,1\}$ $(\forall i \in I$ and $\forall j \in J)$ and $\boldsymbol{b} = \boldsymbol{0}_{m \times 1}$ is a zero vector. If $\boldsymbol{x}^* = (x^*_1, x^*_2, \ldots, x^*_n)$ is the optimal solution of the problem, then $x^*_j \in \{0,1\}, \forall j \in J$.*

The following algorithm summarizes the preceding discussion.

5 An important especial case of Problem (2); minimal Vertex Cover Problem

Consider an undirected simple connected graph $G = (V, E)$ consisting of a set V of n vertices (nodes) and a set E of edges whose elements are unordered pairs of the distinct vertices. Let (i, j) denote an undirected edge between two vertices i and j. Formally, a vertex cover V' of an undirected graph $G = (V, E)$ is a subset of V such that $(i, j) \in E$

implies $i \in V'$ and $j \in V'$; that is, every edge has at least one endpoint in V'. Such a set is said to cover the edges of G. Figure 3 shows two examples of vertex covers, where vertices of V' have been marked in black. A minimal vertex cover is a vertex cover with the smallest possible size. Figure 3(b) shows an example of minimal vertex covers.

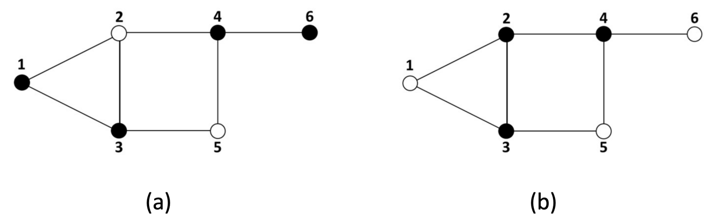

Figure 3: (a) Example of a vertex cover, (b) Example of a minimal vertex cover.

5.1 Formulation of the minimal vertex cover problem

Let the node-node adjacency matrix $\boldsymbol{A} = (a_{ij})_{n\times n}$ have a row and a column corresponding to every vertex, and its (i,j) 'th entry equals 1 (i.e., $a_{ij} = 1$) if $(i,j) \in E$ and equals 0 (i.e., $a_{ij} = 0$) if $(i,j) \notin E$. Particularly, we have $a_{ii} = 0$ for each $i \in I$. Moreover, associate with each vertex $j \in V$ a binary variable $v'_j \in \{0,1\}$ such that $v'_j = 1$ if $j \in V'$ (black vertices in Fig. 3) and $v'_j = 0$ otherwise (white vertices). Based on the above statements, it is clear that V' is a vertex cover if and only if there are no vertices i and j such that $a_{ij} = 1$ and $v'_i = v'_j = 0$. So, the minimal vertex cover problem can be formulated as follows:

$$\begin{aligned}&\min Z = \textstyle\sum_{j\in J} v'_j \\ &\max_{j=1}^{n}\left\{\min\left\{a_{ij}, 1-v'_i, 1-v'_j\right\}\right\} = 0 \quad , \quad i\in I, j\in J \\ &v'_j \in \{0,1\} \quad , \quad j \in J\end{aligned} \tag{14}$$

where $a_{ij} \in \{0,1\}(\forall i \in I$ and $\forall j \in J)$. It is worth noting that if $a_{ij} = 1$ for any vertices i and j, the above equations prevent the variables v'_i and v'_j from having zero value at the same time. Therefore, each feasible solution of (14) corresponds to a vertex cover V', and vice versa (vertex j belongs to V' iff $v'_j = 1$). So, the goal of the above problem is to find a solution with the maximal number of variables v'_j with zero values (or equivalently, the minimal number of variables v'_j with one values). By setting $x_j = 1 - v'_j$, Problem (14) is converted into an equivalent problem as follows:

$$\begin{aligned}&\min Z = \textstyle\sum_{j\in J} (1-x_j) \\ &\max_{j=1}^{n}\left\{\min\left\{a_{ij}, x_i, x_j\right\}\right\} = 0 \quad , \quad i\in I, j\in J \\ &x_j \in \{0,1\} \quad , \quad j \in J\end{aligned} \tag{15}$$

Now, consider the following problem derived from (15) by manipulating the objective function and replacing the constraints $x_j \in \{0,1\}$ by $x_j \in [0,1] (j \in J)$:

$$\begin{aligned}&\max Z = \textstyle\sum_{j\in J} x_j \\ &\max_{j=1}^{n}\left\{\min\left\{a_{ij}, x_i, x_j\right\}\right\} = 0 \quad , \quad i\in I, j\in J \\ &\boldsymbol{x} \in [0,1]^n\end{aligned} \tag{16}$$

Clearly, Problem (16) is a special case of Problem (2) (see Corollaries 4.3 and 4.5) in which $b_i = 0 (i \in I)$ and $c_j = 1 (j \in J)$. Hence, according to Corollary 4.5, all optimal solutions of Problem (16) are binary, and therefore problems (15) and (16) have the same optimal solutions. As a consequence, the minimal vertex cover problem is indeed a special case of Problem (2).

5.2 Properties of Problem (16)

As mentioned before, $G=(V,E)$ is a simple graph (i.e., a graph without loops and multiple edges). So, in Problem (16) we have $a_{ii}=0$ for each $i\in I$, and therefore it is concluded from Definition 2.5 that $I_1=I_3=\varnothing$ and $I=I_2$. Consequently, $E''=\underline{E}=\varnothing$ (see Definition 2.13) and $S(a_i,b_i)=\varnothing, \forall i\in I_1\cup I_2$, which in turn implies that $S_1(\boldsymbol{A},\boldsymbol{b})=S_3(\boldsymbol{A},\boldsymbol{b})=\varnothing$ (see Definition 2.12). Moreover, since $I=I_2$ and $S(\boldsymbol{A},\boldsymbol{b})=\bigcap_{i\in I} S(\boldsymbol{a}_i,b_i)$, from Definition 2.12 we have

$$S_2(\boldsymbol{A},\boldsymbol{b})=\bigcap_{i\in I_2} S(\boldsymbol{a}_i,b_i)=\bigcap_{i\in I} S(\boldsymbol{a}_i,b_i)=S(\boldsymbol{A},\boldsymbol{b}), \tag{17}$$

which together with Lemma 2.16(b) imply

$$S(\boldsymbol{A},\boldsymbol{b})=S_2(\boldsymbol{A},\boldsymbol{b})=\bigcup_{e'\in E'}\left[\underline{\boldsymbol{X}}_2,\overline{\boldsymbol{X}}_2(e')\right]. \tag{18}$$

According to (18), $S(\boldsymbol{A},\boldsymbol{b})\neq\varnothing$ iff $\left[\underline{\boldsymbol{X}}_2,\overline{\boldsymbol{X}}_2(e')\right]$ for some $e'\in E'$. Hence, function $e'\in E'$ is admissible, if $\left[\underline{\boldsymbol{X}}_2,\overline{\boldsymbol{X}}_2(e')\right]\neq\varnothing$ or equivalently $\underline{\boldsymbol{X}}_2\leq\overline{\boldsymbol{X}}_2(e')$. By noting this fact and the equalities $E''=\underline{E}=\varnothing$, (5) is reduced to the following relation:

$$\underline{\boldsymbol{X}}_2\leq\overline{\boldsymbol{X}}_2(e'), \tag{19}$$

and set T (defined in Definition 3.1) is also modified as follows:

$$T=\left\{e'\in E' : \underline{\boldsymbol{X}}_2\leq\overline{\boldsymbol{X}}_2(e')\right\}. \tag{20}$$

Additionally, from Definition 2.8, the following results are directly obtained for each $i\in I_2$ and $j\in J$:

$$\bar{X}(i,1)_j=\begin{cases}0 & j=i\\ 1 & j\neq i\end{cases}\quad,\quad \bar{X}(i,2)_j=\begin{cases}0 & a_{ij}=1\\ 1 & a_{ij}=0\end{cases}\quad,\quad \underline{X}(i)_j=0 \tag{21}$$

Furthermore, the equalities $I_1=I_3=\varnothing$ and Definition 3.2 imply$\overline{\boldsymbol{N}}^1=\overline{\boldsymbol{N}}^2=\underline{\boldsymbol{N}}=\varnothing$. Also, from (21) and Definition 3.2, we have $\overline{\boldsymbol{M}}^{\boldsymbol{1}}=\mathbf{1}-\mathbf{I}$ where $\mathbf{1}_{n\times n}$ is a matrix of ones and $\mathbf{I}_{n\times n}$ is the identity matrix, and

$$\overline{\boldsymbol{M}}^2=\begin{pmatrix}1-a_{11} & \cdots & 1-a_{1n}\\ \vdots & \ddots & \vdots\\ 1-a_{n1} & \cdots & 1-a_{nn}\end{pmatrix}. \tag{22}$$

For the sake of expository reference, some especial results have been summarized in the following corollary:

Corollary 5.1. *Consider Problem (16) and let $i\in I_2=I$.*
(a) $\overline{\boldsymbol{X}}(i,1)$ *contains a zero in the i' th position and ones everywhere else.* (b) $\overline{\boldsymbol{X}}(i,2)^T=(1-a_{i1},1-a_{i2},\ldots,1-a_{in})$. (c) $\underline{\boldsymbol{X}}(i)=\mathbf{0}$ *where* $\mathbf{0}$ *denotes the zero vector.* (d) $\overline{\boldsymbol{X}}(i,e'(i))_j\in\{0,1\},\forall e'\in E'$ *and* $\forall j\in J$. (e) $\underline{\boldsymbol{X}}_2=\mathbf{0}$. (f) $T=E'$. (g) $S(\boldsymbol{A},\boldsymbol{b})=\bigcup_{e'\in E'}\left[\mathbf{0},\overline{\boldsymbol{X}}_2(e')\right]$.

Proof. **(a)-(c)** The results follow from (21). **(d)** Since $e'(i)\in\{1,2\}$ (see Definition 2.13), the proof is directly resulted from Parts (a) and (b). **(e)** The result is easily attained from Definition 2.15 and (21). **(f)** From Part (e) and (20), we have $T=\left\{e'\in E' : \mathbf{0}\leq\overline{\boldsymbol{X}}_2(e')\right\}$. However, since the condition $\mathbf{0}\leq\overline{\boldsymbol{X}}_2(e')$ is satisfied for each $e'\in E'$, then $T=E'$. **(g)** The result follows from Part (e) and (18).
Since $T=E'$ (from Corollary 5.1(f)), then for each $e\in T$ we can consider $e=e'$ where $e'\in E'$. On the other hand, for Problem (16), it is easily found that $J^-=\varnothing$ and $J^+=J$ (because, in the objective function we have

$c_j = 1, \forall j \in J$). Consequently, by setting $\boldsymbol{c} = \mathbf{1}$ where $\mathbf{1}_{n\times 1}$ denotes the sum vector (i.e., a vector having each component equal to one), Corollary 4.3 is reduced to Corollary 5.2 below. □

Corollary 5.2. *Consider Problem (16) and let $\mathbf{1}_{n\times 1}$ denote the sum vector. Then, the global optimal solution $\boldsymbol{x}^*$ is obtained by $\mathbf{1}^T\boldsymbol{x}^* = \max\left\{\mathbf{1}^T\boldsymbol{x}_{e'} : e' \in E'\right\}$ where solutions $\boldsymbol{x}_{e'} = \left[(x_{e'})_1, (x_{e'})_2, \ldots, (x_{e'})_n\right](e' \in E')$ are defined as follows:*

$$(x_{e'})_j = \overline{\boldsymbol{X}}_2\,(e')_j \quad , j \in J \tag{23}$$

Lemma 5.3. *Let x^* denote the global optimal solution of Problem (16). Then,* (a). $\mathbf{1}^T\boldsymbol{x}^* = \max\left\{\mathbf{1}^T\overline{\boldsymbol{X}}_2\,(e') : e' \in E'\right\}$. (b) $\boldsymbol{x}^* = \overline{\boldsymbol{X}}_2\,(e^*)$ *for some* $e^* \in E'$. (c). $x_j^* \in \{0,1\}, \forall j \in J$.

Proof. **(a)** From Corollary 5.2, we have $\mathbf{1}^T\boldsymbol{x}^* = \max\left\{\mathbf{1}^T\boldsymbol{x}_{e'} : e' \in E'\right\}$. Now, the result follows from (23). **(b)** It is a direct consequence of Part (a). **(c)** The proof is directly resulted from Part (b), Definition 2.15 and Corollary 5.1(d). □

Lemma 5.4. *Suppose that $\boldsymbol{x}^* = \overline{\boldsymbol{X}}_2\,(e^*)\,(e^* \in E')$ is the global optimal solution of Problem (16) and $e' \in E'$.*

(a). *If $e'(i) = 1, \forall i \in I_2$, then $\overline{\boldsymbol{X}}_2\,(e') = \mathbf{0}$.*

(b). *Let $e'(i_0) = 2$ for some $i_0 \in I_2$ and $a_{i_0 j} = 1$ (or equivalently, $\bar{m}^2_{i_0 j} = 0$). Then, $\overline{\boldsymbol{X}}_2\,(e')_j = 0$.*

(c). *Let $e^*(i_0) = 2$ for some $i_0 \in I_2$. Then, $e^*(i) = 1$ for each $i \in I_2 - \{i_0\}$ such that $a_{i_0 i} = 1$ (or equivalently, $\bar{m}^2_{i_0 i} = 0$).*

Proof. **(a)** It is sufficient to show that $\overline{\boldsymbol{X}}_2\,(e')_j = 0, \forall j \in J$. From Definition 2.15, $\overline{\boldsymbol{X}}_2\,(e') = \min_{i\in I_2}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))\right\}$. Since $I_2 = I$ and $e'(i) = 1, \forall i \in I_2$, then $\overline{\boldsymbol{X}}_2\,(e') = \min_{i\in I}\{\overline{\boldsymbol{X}}(i,1)\}$. Thus, for each $j \in J, \overline{\boldsymbol{X}}_2\,(e')_j = \min_{i\in I}\left\{\overline{\boldsymbol{X}}(i,1)_j\right\}$. Now, Corollary 5.1, (a) implies that $\overline{\boldsymbol{X}}(j,1)_j = 0$, and therefore $\overline{\boldsymbol{X}}_2\,(e')_j = \min_{i\in I}\left\{\overline{\boldsymbol{X}}(i,1)_j\right\} = \min\left\{\overline{\boldsymbol{X}}(j,1)_j, \min_{i\in I-\{j\}}\left\{\overline{\boldsymbol{X}}(i,1)_j\right\}\right\} = \min\left\{0, \min_{i\in I-\{j\}}\left\{\overline{\boldsymbol{X}}(i,1)_j\right\}\right\} = 0$. **(b)** Similar to Part (a), from Definition 2.15 and the equality $I_2 = I$, we have

$$\begin{aligned}\overline{\boldsymbol{X}}_2\,(e')_j &= \min_{i\in I_2}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))_j\right\} = \min_{i\in I}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))_j\right\}\\ &= \min\left\{\overline{\boldsymbol{X}}\,(i_0, 2)_j\,, \min_{i\in I-\{i_0\}}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))_j\right\}\right\}.\end{aligned}$$

But, Corollary 5.1(b) requires that $\overline{\boldsymbol{X}}\,(i_0, 2)_j = 1 - a_{i_0 j} = 0$. Therefore, $\overline{\boldsymbol{X}}_2\,(e')_j = \min\left\{0, \min_{i\in I-\{i_0\}}\left\{\overline{\boldsymbol{X}}\,(i, e'(i))_j\right\}\right\} = 0$. **(c)** Let $e^*(i_0) = 2$ and $e^*(i) = 1$ for each $i \in I_2 - \{i_0\}$ such that $a_{i_0} = 1$. Also, suppose that $i' \in I_2 - \{i_0\}$ and $a_{i_0 i'} = 1$. Now, consider $e' \in E'$ such that $e'(i') = 2$ and $e'(i) = e^*(i)$, $\forall i \in I_2 - \{i'\}$. We show that $\mathbf{1}^T\overline{\boldsymbol{X}}_2\,(e^*) \geq \mathbf{1}^T\overline{\boldsymbol{X}}_2\,(e')$; that is, $\overline{\boldsymbol{X}}_2\,(e')$ cannot be a better solution than $\overline{\boldsymbol{X}}_2\,(e^*)$. Firstly, we note that since $e'(i_0) = e^*(i_0) = 2$ and $a_{i_0 i'} = 1$, we have from Part (b) that $\overline{\boldsymbol{X}}_2\,(e')_{i'} = \overline{\boldsymbol{X}}_2\,(e^*)_{i'} = 0$. Otherwise, assume that $j \neq i'$. So, since $\overline{\boldsymbol{X}}\,(i', 2)_j = 1 - a_{i'}$ (Corollary 5.1(b)), then $\overline{\boldsymbol{X}}\,(i', 2)_j = 1$ if $a_{i'j} = 0$, and $\overline{\boldsymbol{X}}\,(i', 2)_j = 0$ if $a_{i'j} = 1$.

Therefore, we have

$$\begin{aligned}\overline{\boldsymbol{X}}_2(e^*)_j &= \min_{i\in I}\left\{\overline{\boldsymbol{X}}(i,e^*(i))_j\right\} = \min\left\{\overline{\boldsymbol{X}}(i',e^*(i'))_j, \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e^*(i))_j\right\}\right\}\\ &= \min\left\{\overline{\boldsymbol{X}}(i',1)_j, \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e^*(i))_j\right\}\right\} = \min\left\{1, \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e^*(i))_j\right\}\right\}\\ &= \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e^*(i))_j\right\} = \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e'(i))_j\right\}\\ &\geq \min\left\{\overline{\boldsymbol{X}}(i',e'(i'))_j, \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e'(i))_j\right\}\right\}\\ &\geq \min\left\{\overline{\boldsymbol{X}}(i',2)_j, \min_{i\in I-\{i'\}}\left\{\overline{\boldsymbol{X}}(i,e'(i))_j\right\}\right\}\\ &= \overline{\boldsymbol{X}}_2(e')_j.\end{aligned}$$

Therefore, $\overline{\boldsymbol{X}}_2(e^*)_j \geq \overline{\boldsymbol{X}}_2(e')_j, \forall j \in J$, that implies $\mathbf{1}^T\overline{\boldsymbol{X}}_2(e^*) \geq \mathbf{1}^T\overline{\boldsymbol{X}}_2(e')$. □

Theorem 5.5. *Suppose that $\boldsymbol{x}^* = \overline{\boldsymbol{X}}_2(e^*)\left(e^* \in \bar{E}_2\right)$ is the global optimal solution of Problem (16).*

(a). *There exists at least one $i_0 \in I_2$ such that $e^*(i_0) = 2$.*

(b). *If $e^*(i_1) = e^*(i_2) = 2$, then $a_{i_1 i_2} = a_{i_2 i_1} = 0$ (or equivalently, $\bar{m}^2_{i_i i_2} = \bar{m}^2_{i_2 i_1} = 1$).*

Proof. **(a)** By contradiction, suppose that $e^*(i) = 1, \forall i \in I_2$. Hence, Lemma 5.4(a) implies that $\boldsymbol{x}^* = \overline{\boldsymbol{X}}_2(e^*) = \mathbf{0}$ which is clearly not optimal; because, for example, by introducing $x' \in [0,1]^n$ such that $x'_1 = 1$ and $x'_j = x^*_j = 0, \forall j \in \{2,\ldots,n\}$, it is concluded that $\boldsymbol{x}'$ is feasible to (16) and $\mathbf{1}^T\boldsymbol{x}' = 1 > 0 = \mathbf{1}^T\boldsymbol{x}^*$. **(a)** The proof is obtained from Lemma 5.4(c). □

6 Numerical example

Example 6.1. *Consider the problem $A \otimes x = b$, where*

$$A = \begin{pmatrix} 0.81 & 0.15 & 0.65 & 0.70 & 0.43 & 0.27 & 0.75 & 0.84 & 0.35 & 0.07\\ 0.90 & 0.57 & 0.03 & 0.98 & 0.38 & 0.67 & 0.25 & 0.25 & 0.83 & 0.05\\ 0.12 & 0.95 & 0.84 & 0.27 & 0.76 & 0.14 & 0.50 & 0.81 & 0.58 & 0.53\\ 0.19 & 0.40 & 0.93 & 0.40 & 0.29 & 0.16 & 0.99 & 0.94 & 0.54 & 0.87\\ 0.63 & 0.80 & 0.67 & 0.09 & 0.45 & 0.11 & 0.89 & 0.92 & 0.91 & 0.93\\ 0.09 & 0.14 & 0.75 & 0.82 & 0.48 & 0.79 & 0.95 & 0.85 & 0.28 & 0.12\\ 0.87 & 0.42 & 0.74 & 0.69 & 0.44 & 0.95 & 0.54 & 0.19 & 0.75 & 0.56\\ 0.74 & 0.91 & 0.39 & 0.71 & 0.64 & 0.34 & 0.73 & 0.25 & 0.75 & 0.46\\ 0.95 & 0.79 & 0.65 & 0.95 & 0.70 & 0.58 & 0.14 & 0.61 & 0.38 & 0.01\\ 0.96 & 0.95 & 0.17 & 0.03 & 0.75 & 0.22 & 0.25 & 0.47 & 0.56 & 0.03 \end{pmatrix},$$

$$b^T = \begin{pmatrix} 0.66, 0.57, 0.14, 0.40, 0.45, 0.79, 0.55, 0.62, 0.04, 0.53 \end{pmatrix},$$

$$c^T = \begin{pmatrix} -8.36, 0.92, 4.11, 2.36, -9.66, -8.87, 5.75, -4.78, 8.10, 5.84 \end{pmatrix}.$$

By Definition 2.3, we have

$$\begin{aligned}&J_1 = \{1,4,7,8\}, J_2 = \{1,2,4,6,9\},\\&J_3 = \{2,3,4,5,6,7,8,9,10\}, J_4 = \{2,3,4,7,8,9,10\},\\&J_5 = \{1,2,3,5,7,8,9,10\}, J_6 = \{4,6,7,8\},\\&J_7 = \{1,3,4,6,9,10\}, J_8 = \{1,2,4,5,7,9\},\\&J_9 = \{1,2,3,4,5,6,7,8,9\} \quad \textit{and} \quad J_{10} = \{1,2,5,9\}.\end{aligned}$$

Also, from Definition 2.5, we obtain $I_1 = \{1,3,9\}, I_2 = \{i'_1, i'_2, i'_3, i'_4\} = \{2,4,5,6\}, I_3 = \{i''_1, i''_2, i''_3\} = \{7,8,10\}$.

For each $i \in I_1$*, the minimal solution* $\underline{\boldsymbol{X}}(i)$ *and maximal solution* $\overline{\boldsymbol{X}}(i)$ *of* $S(\boldsymbol{a}_i, b_i)$ *are obtained by Definition 2.7 as follows:*

$$\bar{X}(1) = (0.66,1,1,1,1,1,1,1,1,1),$$

$$\underline{X}(1) = (0.66,0,0,0,0,0,0,0,0,0),$$

$$\bar{X}(3) = (1,1,0.14,1,1,1,1,1,1,1),$$

$$\underline{X}(3) = (0,0,0.14,0,0,0,0,0,0,0),$$

$$\bar{X}(9) = (1,1,1,1,1,1,1,1,0.04,1),$$

$$\underline{X}(9) = (0,0,0,0,0,0,0,0,0.04,0).$$

According to Definition 2.8, for each $i'_k \in I_2 (k = 1,\ldots,4)$*, the minimal solution* $\underline{\boldsymbol{X}}(i)$ *and the maximal solutions* $\overline{\boldsymbol{X}}(i,1)$ *and* $\overline{\boldsymbol{X}}(i,2)$ *of* $S\left(\boldsymbol{a}_{i'_k}, b_{i_k}\right)$ *are attained as follows:*

$$\begin{aligned}&\bar{X}(2,1) = (1,0.57,1,1,1,1,1,1,1,1),\\&\overline{\boldsymbol{X}}(2,2) = (0.57,1,1,0.57,1,0.57,1,1,0.57,1),\\&\underline{\boldsymbol{X}}(2) = (0,0.57,0,0,0,0,0,0,0,0),\\&\overline{\boldsymbol{X}}(4,1) = (1,1,1,0.40,1,1,1,1,1,1),\\&\overline{\boldsymbol{X}}(4,2) = (1,1,0.40,1,1,1,0.40,0.40,0.40,0.40),\\&\underline{X}(4) = (0,0,0,0.40,0,0,0,0,0,0),\\&\overline{\boldsymbol{X}}(5,1) = (1,1,1,1,0.45,1,1,1,1,1),\\&\overline{\boldsymbol{X}}(5,2) = (0.45,0.45,0.45,1,1,1,0.45,0.45,0.45,0.45),\\&\underline{\boldsymbol{X}}(5) = (0,0,0,0,0.45,0,0,0,0,0),\\&\overline{\boldsymbol{X}}(6,1) = (1,1,1,1,1,0.79,1,1,1,1),\\&\overline{\boldsymbol{X}}(6,2) = (1,1,1,0.79,1,1,0.79,0.79,1,1),\\&\underline{X}(6) = (0,0,0,0,0,0.79,0,0,0,0).\end{aligned}$$

Also, for each $i''_k \in I_3 (k = 1,2,3)$*, the minimal solutions* $\underline{X}(i,j)$ *and the maximal solutions* $\overline{\boldsymbol{X}}(i,1)$ *and* $\overline{\boldsymbol{X}}(i,2)$ *of*

$S\left(\boldsymbol{a}_{i_k}, b_{i_k}\right)$ *are attained by Definition 2.9 as follows:*

$$\begin{aligned}
&\bar{X}(7,1) = (1,1,1,1,1,1,0.55,1,1,1),\\
&\bar{X}(7,2) = (0.55,1,0.55,0.55,1,0.55,1,1,0.55,0.55),\\
&\underline{X}(7,1) = (0.55,0,0,0,0,0,0.55,0,0,0),\\
&\underline{X}(7,3) = (0,0,0.55,0,0,0,0.55,0,0,0),\\
&\underline{X}(7,4) = (0,0,0,0.55,0,0,0.55,0,0,0),\\
&\underline{X}(7,6) = (0,0,0,0,0,0.55,0.55,0,0,0),\\
&\underline{X}(7,9) = (0,0,0,0,0,0,0.55,0,0.55,0),\\
&\underline{X}(7,10) = (0,0,0,0,0,0,0.55,0,0,0.55).
\end{aligned}$$

$$\begin{aligned}
&\bar{X}(8,1) = (1,1,1,1,1,1,1,0.62,1,1),\\
&\bar{X}(8,2) = (0.62,0.62,1,0.62,0.62,1,0.62,1,0.62,1),\\
&\underline{X}(8,1) = (0.62,0,0,0,0,0,0,0.62,0,0),\\
&\underline{X}(8,2) = (0,0.62,0,0,0,0,0,0.62,0,0),\\
&\underline{X}(8,4) = (0,0,0,0.62,0,0,0,0.62,0,0),\\
&\underline{X}(8,5) = (0,0,0,0,0.62,0,0,0.62,0,0),\\
&\underline{X}(8,7) = (0,0,0,0,0,0,0.62,0.62,0,0),\\
&\underline{X}(8,9) = (0,0,0,0,0,0,0,0.62,0.62,0),\\
&\bar{X}(10,1) = (1,1,1,1,1,1,1,1,1,0.53),\\
&\bar{X}(10,2) = (0.53,0.53,1,1,0.53,1,1,1,0.53,1),\\
&\underline{X}(10,1) = (0.53,0,0,0,0,0,0,0,0,0.53),\\
&\underline{X}(10,2) = (0,0.53,0,0,0,0,0,0,0,0.53),\\
&\underline{X}(10,5) = (0,0,0,0,0.53,0,0,0,0,0.53),\\
&\underline{X}(10,9) = (0,0,0,0,0,0,0,0,0.53,0.53).
\end{aligned}$$

Therefore, from Definition 2.15, we have

$$\begin{aligned}
&\overline{\boldsymbol{X}}_1 = \min_{i\in I_1}\{\overline{\boldsymbol{X}}(i)\} = (0.66,1,0.14,1,1,1,1,1,0.04,1),\\
&\underline{\boldsymbol{X}}_1 = \max_{i\in I_1}\{\underline{\boldsymbol{X}}(i)\} = (0.66,0,0.14,0,0,0,0,0,0.04,0),\\
&\underline{\boldsymbol{X}}_2 = \max_{i\in I_2}\{\underline{\boldsymbol{X}}(i)\} = (0,0.57,0,0.40,0.45,0.79,0,0,0,0),\\
&\max\{\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2\} = (0.66,0.57,0.14,0.40,0.45,0.79,0,0,0.04,0).
\end{aligned}$$

Based on Remark 2.14, we have $\left|\bar{E}_2\right| = 2^{|I_2|} = 2^4$. *For example, by considering* $\bar{e} \in \bar{E}_2$ *such that* $\vec{e}(2) = 2, \vec{e}(4) = 1, \vec{e}(5) = 2$ *and* $\vec{e}(6) = 1$, *the corresponding maximal solution is obtained as follows:*

$$\begin{aligned}
\overline{\boldsymbol{X}}_2\left(\vec{e}'\right) &= \min_{i\in I_2}\left\{\overline{\boldsymbol{X}}\left(i, \vec{e}'(i)\right)\right\}\\
&= \min\left\{\overline{\boldsymbol{X}}(2,\vec{e}(2)), \overline{\boldsymbol{X}}(4,\vec{e}(4)), \overline{\boldsymbol{X}}(5,\vec{e}(5)), \overline{\boldsymbol{X}}\left(6,\vec{e}'(6)\right)\right\}\\
&= \min\{\overline{\boldsymbol{X}}(2,2), \overline{\boldsymbol{X}}(4,1), \overline{\boldsymbol{X}}(5,2), \overline{\boldsymbol{X}}(6,1)\}\\
&= (0.45,0.45,0.45,0.40,1,0.57,0.45,0.45,0.45,0.45).
\end{aligned}$$

Similarly, $\left|\bar{E}_3\right| = 2^{|I_3|} = 2^3$. *For instance, if* $\vec{e}'' \in \bar{E}_3$ *such that* $\vec{e}''(7) = 1, \vec{e}''(8) = 1$ *and* $\vec{e}''(10) = 2$ *, then the corresponding maximal solution is resulted as follows:*

$$\begin{aligned}
\overline{\boldsymbol{X}}_3\left(\vec{e}''\right) &= \min_{i \in I_3}\left\{\overline{\boldsymbol{X}}\left(i, \vec{e}''(i)\right)\right\} \\
&= \min\left\{\overline{\boldsymbol{X}}\left(7, \vec{e}''(7)\right), \overline{\boldsymbol{X}}\left(8, \vec{e}''(8)\right), \overline{\boldsymbol{X}}\left(10, \vec{e}''(10)\right)\right\} \\
&= \min\{\overline{\boldsymbol{X}}(7,1), \overline{\boldsymbol{X}}(8,1), \overline{\boldsymbol{X}}(10,2)\} \\
&= (0.53, 0.53, 1, 1, 0.53, 1, 0.55, 0.62, 0.53, 1).
\end{aligned}$$

The cardinality of $\underline{E}$ *is equal to* $|\underline{E}| = \prod_{i \in I_3}|J_i| = |J_7| \times |J_8| \times |J_{10}| = 6 \times 6 \times 4 = 144$. *As an example, by selecting* $\underline{e} \in \underline{E}$ *such that* $\underline{e}(7) = 9, \underline{e}(8) = 3$ *and* $\underline{e}(10) = 5$, *we have*

$$\begin{aligned}
\underline{\boldsymbol{X}}_3(\underline{e}) &= \max_{i \in I_3}\{\underline{\boldsymbol{X}}(i, \underline{e}(i))\} \\
&= \max\{\underline{\boldsymbol{X}}(7, \underline{e}(7)), \underline{\boldsymbol{X}}(8, \underline{e}(8)), \underline{\boldsymbol{X}}(10, \underline{e}(10))\} \\
&= \max\{\underline{\boldsymbol{X}}(7,9), \underline{\boldsymbol{X}}(8,2), \underline{\boldsymbol{X}}(10,5)\} \\
&= (0, 0.62, 0, 0, 0.53, 0, 0.55, 0.62, 0.55, 0.53).
\end{aligned}$$

Moreover, the number of all the triples $(\underline{e}, \vec{e}, \vec{e}) \in \underline{E} \times \bar{E}_2 \times \bar{E}_3$ *is equal to* $|\underline{E}| \times \left|\bar{E}_2\right| \times \left|\bar{E}_3\right| = 18432$. *According to Definition 3.2, matrices* $\overline{\boldsymbol{M}}^1, \overline{\boldsymbol{M}}^2, \overline{\boldsymbol{N}}^1, \overline{\boldsymbol{N}}^2$ *and* $\underline{N}$ *are computed as follows:*

$$\overline{\boldsymbol{M}}^1 = \begin{pmatrix} 1 & 0.57 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0.40 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0.45 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0.79 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\overline{\boldsymbol{M}}^2 = \begin{pmatrix} 0.57 & 1 & 1 & 0.57 & 1 & 0.57 & 1 & 1 & 0.57 & 1 \\ 1 & 1 & 0.40 & 1 & 1 & 1 & 0.40 & 0.40 & 0.40 & 0.40 \\ 0.45 & 0.45 & 0.45 & 1 & 1 & 1 & 0.45 & 0.45 & 0.45 & 0.45 \\ 1 & 1 & 1 & 0.79 & 1 & 1 & 0.79 & 0.79 & 1 & 1 \end{pmatrix},$$

$$\overline{\boldsymbol{N}}^1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0.55 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.62 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.53 \end{pmatrix},$$

$$\overline{\boldsymbol{N}}^2 = \begin{pmatrix} 0.55 & 1 & 0.55 & 0.55 & 1 & 0.55 & 1 & 1 & 0.55 & 0.55 \\ 0.62 & 0.62 & 1 & 0.62 & 0.62 & 1 & 0.62 & 1 & 0.62 & 1 \\ 0.53 & 0.53 & 1 & 1 & 0.53 & 1 & 1 & 1 & 0.53 & 1 \end{pmatrix},$$

$$\underline{\boldsymbol{N}} = \begin{pmatrix} 0.55 & -\infty & 0.55 & 0.55 & -\infty & 0.55 & 0.55 & -\infty & 0.55 & 0.55 \\ 0.62 & 0.62 & -\infty & 0.62 & 0.62 & -\infty & 0.62 & 0.62 & 0.62 & -\infty \\ 0.53 & 0.53 & -\infty & -\infty & 0.53 & -\infty & -\infty & -\infty & 0.53 & 0.53 \end{pmatrix}.$$

Since $\max\left\{(\underline{\boldsymbol{X}}_1)_1, (\underline{\boldsymbol{X}}_2)_1\right\} = 0.66 > 0.57 = \bar{m}_{11}^2$ *and* $\max\left\{(\underline{\boldsymbol{X}}_1)_1, (\underline{\boldsymbol{X}}_2)_1\right\} = 0.66 > 0.45 = \bar{m}_{31}^2$, *by Rule 1 (Lemma 3.4) and Remark 3.5, we set* $\overline{\boldsymbol{M}}_1^2 = \overline{\boldsymbol{M}}_3^2 = (\infty, \infty, \ldots, \infty)_{1 \times n}, \mathrm{dom}\left(\vec{e}\left(i_1'\right)\right) = \mathrm{dom}\left(\vec{e}'(2)\right) = \{1\}$ *and* $\mathrm{dom}\left(\vec{e}'\left(i_3'\right)\right) = \mathrm{dom}\left(\vec{e}'(5)\right) = \{1\}$. *After applying Rule 1,* $\left|\bar{E}_2\right|$ *is decreased from* $2^4 = 16$ *to* $2^2 = 4$ *and the matrix*

$\overline{\boldsymbol{M}}^2$ *is reduced as follows:*

$$\overline{\boldsymbol{M}}^2 = \begin{pmatrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 1 & 1 & 0.40 & 1 & 1 & 1 & 0.40 & 0.40 & 0.40 & 0.40 \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 1 & 1 & 1 & 0.79 & 1 & 1 & 0.79 & 0.79 & 1 & 1 \end{pmatrix}.$$

Similarly, by applying Rule 2 (Lemma 3.7), we set

$$\overline{\boldsymbol{N}}_1^2 = \overline{\boldsymbol{N}}_2^2 = \overline{\mathbf{N}}_3^2 = (\infty, \infty, \dots, \infty)_{1\times n},$$

$\operatorname{dom}(\vec{e}''(i_1'')) = \operatorname{dom}(\vec{e}''(7)) = \{1\}, \operatorname{dom}(\vec{e}''(i_2'')) = \operatorname{dom}(\vec{e}''(8)) = \{1\}$ *and*

$$\operatorname{dom}(\vec{e}''(i_3'')) = \operatorname{dom}(\vec{e}''(10)) = \{1\};$$

because, $\bar{n}_{11}^2 = 0.55, \bar{n}_{21}^2 = 0.62$ *and* $\bar{n}_{31}^2 = 0.53$*, and therefore*

$$\bar{n}_{11}^2, \bar{n}_{21}^2, \bar{n}_{31}^2 < \max\left\{(\underline{\boldsymbol{X}}_1)_1, (\underline{\boldsymbol{X}}_2)_1\right\}.$$

So, $\left|\bar{E}_3\right|$ *is decreased from* $2^3 = 8$ *to 1 and the reduced matrix* $\bar{N}^2$ *is obtained as follows:*

$$\bar{N}^2 = \begin{pmatrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix}.$$

By considering the matrix $\underline{N}$*, we note that* $\underline{n}_{13} = 0.55$ *that is greater than* $\left(\overline{\boldsymbol{X}}_1\right)_3 = 0.14$ *. Also,* $\underline{n}_{19} = 0.55, \underline{n}_{29} = 0.62$ *and* $\underline{n}_{39} = 0.53$ *are greater than* $\left(\overline{\boldsymbol{X}}_1\right)_9 = 0.04$*. So, by applying Rule 3 (Lemma 3.10), we set* $\underline{n}_{13} = \underline{n}_{19} = \underline{n}_{29} = \underline{n}_{39} = -\infty$ *and*

$$\begin{aligned} \operatorname{dom}(\underline{e}(i_1'')) &= \operatorname{dom}(\underline{e}(7)) = J_7 - \{3,9\} = \{1,4,6,10\}, \\ \operatorname{dom}(\underline{e}(i_2'')) &= \operatorname{dom}(\underline{e}(8)) = J_8 - \{9\} = \{1,2,4,5,7\}, \\ \operatorname{dom}(\underline{e}(i_3'')) &= \operatorname{dom}(\underline{e}(10)) = J_{10} - \{9\} = \{1,2,5\}. \end{aligned}$$

By this simplification rule, $|\underline{E}|$ *is decreased from 144 to* $|J_7| \times |J_8| \times |J_{10}| = 4 \times 5 \times 3 = 60$ *and the new matrix* $\bar{N}^2$ *is obtained as follows:*

$$\underline{\boldsymbol{N}} = \begin{pmatrix} 0.55 & -\infty & -\infty & 0.55 & -\infty & 0.55 & 0.55 & -\infty & -\infty & 0.55 \\ 0.62 & 0.62 & -\infty & 0.62 & 0.62 & -\infty & 0.62 & 0.62 & -\infty & -\infty \\ 0.53 & 0.53 & -\infty & -\infty & 0.53 & -\infty & -\infty & -\infty & -\infty & 0.53 \end{pmatrix}.$$

Now, consider $i_2' \in I_2 \quad (i_2' = 4)$ *and* $i_1'' \in I_3\,(i_1'' = 7)$*. It is clear that* $a_{i'i_1''} = a_{47} = 0.99 > 0.4 = b_{i_2'} = b_4$ *and* $b_{i_2'} = b_4 = 0.4 < 0.55 = b_{i_1''} = b_7$*. Hence, by applying Rule 4, we set* $\overline{\boldsymbol{M}}_2^2 = (\infty, \infty, \dots, \infty)_{1\times n}$ *and* $\operatorname{dom}(\vec{e}'(i_2')) = \operatorname{dom}(\vec{e}'(4)) = \{1\}$*. After applying Rule 4,* $\left|\vec{E}_2\right|$ *is decreased from 4 to 2 and the new matrix* $\overline{\boldsymbol{M}}^2$ *is obtained as follows:*

$$\overline{\boldsymbol{M}}^2 = \begin{pmatrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 1 & 1 & 1 & 0.79 & 1 & 1 & 0.79 & 0.79 & 1 & 1 \end{pmatrix}.$$

In this example, Rule 5 cannot reduce the matrix $\overline{\boldsymbol{N}}^2$*. However, by considering* $i_1' \in I_2 \quad (i_1' = 2)$ *and* $i_2'' \in$

I_3 $(i''_2 = 8)$, we note that $\mathrm{dom}\,(\vec{e}'\,(i'_1)) = \mathrm{dom}(\vec{e}(2)) = \{1\}$, $i'_1 = 2 \in J_{i''_2} = J_8 = \{1, 2, 4, 5, 7\}$ *and* $b_{i'_1} = b_2 = 0.57 < 0.62 = b_{i''_2} = b_8$. *Therefore, according to Rule 6, we set* $\underline{n}_{2i'_1} = \underline{n}_{22} = -\infty$ *and* $\mathrm{dom}\,(\underline{e}\,(i''_2)) = \mathrm{dom}(\underline{e}(8)) = J_{i''_2} - \{i'_1\} = J_8 - \{2\} = \{1, 4, 5, 7\}$. *By the same argument, we set* $\underline{n}_{22} = \underline{n}_{14} = \underline{n}_{24} = \underline{n}_{25} = \underline{n}_{35} = -\infty$ *and*

$$\begin{aligned}
&\mathrm{dom}\,(\underline{e}\,(i''_2)) = \mathrm{dom}(\underline{e}(8)) = J_{i''_2} - \{i'_2, i'_3\} = J_8 - \{4, 5\} = \{1, 7\},\\
&\mathrm{dom}\,(\underline{e}\,(i''_1)) = \mathrm{dom}(\underline{e}(7)) = J_{i''_1} - \{i'_2\} = J_7 - \{4\} = \{1, 6, 10\},\\
&\mathrm{dom}\,(\underline{e}\,(i''_3)) = \mathrm{dom}(\underline{e}(10)) = J_{i''_3} - \{i'_3\} = J_{10} - \{5\} = \{1, 2\}.
\end{aligned}$$

Hence, through use of Rule $6, |\underline{E}|$ *is decreased from 60 to* $|J_7| \times |J_8| \times |J_{10}| = 3 \times 2 \times 2 = 12$ *and the new matrix* $\bar{N}^2$ *is reduced further as follows:*

$$\underline{\boldsymbol{N}} = \begin{pmatrix} 0.55 & -\infty & -\infty & -\infty & -\infty & 0.55 & 0.55 & -\infty & -\infty & 0.55\\ 0.62 & -\infty & -\infty & -\infty & -\infty & -\infty & 0.62 & 0.62 & -\infty & -\infty\\ 0.53 & 0.53 & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty & 0.53 \end{pmatrix}.$$

Finally, for $i''_1, i''_2 \in I_3$ $(i''_1 = 7$ *and* $i''_2 = 8)$ *we have* $\mathrm{dom}\,(\vec{e}''\,(i''_1)) = \mathrm{dom}\,(\vec{e}''(7)) = \{1\}$, $i''_1 = 7 \in J_{i''_2} = J_8 = \{1, 7\}$ *and* $b_{i''_1} = b_7 = 0.55 < 0.62 = b_{i''_2} = b_8$. *Thus, based on Rule 7, we set* $\underline{n}_{2i''_1} = \underline{n}_{27} = -\infty$ *and* $\mathrm{dom}\,(\underline{e}\,(i''_2)) = \mathrm{dom}(\underline{e}(8)) = J_{i''_2} - \{i''_1\} = J_8 - \{7\} = \{1\}$. *By the same argument, we set* $\underline{n}_{1,10} = -\infty$ *and* $\mathrm{dom}\,(\underline{e}\,(i''_1)) = \mathrm{dom}(\underline{e}(7)) = J_{i''_1}\,\{i''_3\} = J_7 - \{10\} = \{1, 6\}$. *So, Rule 7 decreases* $|\underline{E}|$ *from 12 to* $|J_7| \times |J_8| \times |J_{10}| = 2 \times 1 \times 2 = 4$ *and the new matrix* $\bar{N}^2$ *is obtained as follows:*

$$\underline{\boldsymbol{N}} = \begin{pmatrix} 0.55 & -\infty & -\infty & -\infty & -\infty & 0.55 & 0.55 & -\infty & -\infty & -\infty\\ 0.62 & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty & 0.62 & -\infty & -\infty\\ 0.53 & 0.53 & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty & 0.53 \end{pmatrix}.$$

Consequently, after applying the simplification rules, it follows that $|\bar{E}_2| = 2, |\bar{E}_3| = 1$ *and* $|\underline{E}| = 4$. *Additionally, we note that the simplification rules decreased the number of all the triples* $(\underline{e}, \vec{e}', \vec{e}'') \in \underline{E} \times \bar{E}_2 \times \bar{E}_3$ *from 18432 to* $|\underline{E}| \times |\bar{E}_2| \times |\bar{E}_3| = 4 \times 2 \times 1 = 8$.

Therefore, according to Theorem 2.19, the feasible region $S(\boldsymbol{A}, \boldsymbol{b})$ *can be found using two maximal solutions* $\overline{\boldsymbol{X}}_2\,(\vec{e}_1)\,, \overline{\boldsymbol{X}}_2\,(\vec{e}'_2)\,(\vec{e}'_1, \vec{e}'_2 \in \bar{E}_2)$, *one maximal solution* $\overline{\boldsymbol{X}}_3\,(\vec{e}'')\,(\ \vec{e}'' \in \bar{E}_3)$, *and four minimal solutions*

$$\underline{\boldsymbol{X}}_3\,(\underline{e}_1)\,, \ldots, \underline{\boldsymbol{X}}_3\,(\underline{e}_4)\,(\underline{e}_1, \ldots, \underline{e}_4 \in \underline{E})\,.$$

These solutions are summarized as follows:

$$\begin{aligned}
&\vec{e}'_1(2) = \vec{e}_1(4) = \vec{e}'_1(5) = \vec{e}_1(6) = 1 \quad \Rightarrow \overline{\boldsymbol{X}}_2\,(\vec{e}'_1) = (1, 0.57, 1, 0.40, 0.45, 0.79, 1, 1, 1, 1),\\
&\vec{e}_2(2) = \vec{e}_2(4) = \vec{e}_2(5) = 1, \vec{e}_2(6) = 2 \Rightarrow \overline{\boldsymbol{X}}_2\,(\vec{e}'_2) = (1, 0.57, 1, 0.40, 0.45, 1, 0.79, 0.79, 1, 1),\\
&\vec{e}''(7) = \vec{e}''(8) = \vec{e}''(10) = 1 \qquad\qquad \Rightarrow \overline{\boldsymbol{X}}_3\,(\vec{e}'') = (1, 1, 1, 1, 1, 1, 0.55, 0.62, 1, 0.53),\\
&\underline{e}_1 = (1, 1, 1) \Rightarrow \underline{\boldsymbol{X}}_3\,(\underline{e}_1) = (0.62, 0, 0, 0, 0, 0, 0.55, 0.62, 0, 0.53),\\
&\underline{e}_2 = (1, 1, 2) \Rightarrow \underline{\boldsymbol{X}}_3\,(\underline{e}_2) = (0.62, 0.53, 0, 0, 0, 0, 0.55, 0.62, 0, 0.53),\\
&\underline{e}_3 = (6, 1, 1) \Rightarrow \underline{\boldsymbol{X}}_3\,(\underline{e}_3) = (0.62, 0, 0, 0, 0, 0.55, 0.55, 0.62, 0, 0.53),\\
&\underline{e}_4 = (6, 1, 2) \Rightarrow \underline{\boldsymbol{X}}_3\,(\underline{e}_4) = (0.62, 0.53, 0, 0, 0, 0.55, 0.55, 0.62, 0, 0.53).
\end{aligned}$$

So, we have

$$\min\left\{\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2\,(\vec{e}'_1)\,, \overline{\boldsymbol{X}}_3\,(\vec{e}'')\right\} = (0.66, 0.57, 0.14, 0.40, 0.45, 0.79, 0.55, 0.62, 0.04, 0.53),$$

$$\min\left\{\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2\left(\vec{e}_2'\right),\overline{\boldsymbol{X}}_3\left(\vec{e}''\right)\right\}=(0.66,0.57,0.14,0.40,0.45,1,0.55,0.62,0.04,0.53),$$

Since $\min\left\{\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2\left(\vec{e}_1'\right),\overline{\boldsymbol{X}}_3\left(\vec{e}''\right)\right\}\leq\min\left\{\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2\left(\vec{e}_2'\right),\overline{\boldsymbol{X}}_3\left(\vec{e}''\right)\right\}$, *then*

$$\min\left\{\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2\left(\vec{e}_2'\right),\overline{\boldsymbol{X}}_3\left(\vec{e}''\right)\right\},$$

is the unique maximal solution of $S(\boldsymbol{A},\boldsymbol{b})$. *Moreover, since* $\underline{\boldsymbol{X}}_3\left(\underline{e}_1\right)\leq\underline{\boldsymbol{X}}_3\left(\underline{e}_k\right),k=2,3,4$ *, then*

$$\max\left\{\underline{\boldsymbol{X}}_1,\underline{\boldsymbol{X}}_2,\underline{\boldsymbol{X}}_3\left(\underline{e}_1\right)\right\}=(0.66,0.57,0.14,0.40,0.45,0.79,0.55,0.62,0.04,0.53),$$

is the unique minimal (minimal) solution of $S(\boldsymbol{A},\boldsymbol{b})$. *Hence, according to Theorem 2.19, we have* $\boldsymbol{S}(\boldsymbol{A},\boldsymbol{b})=\left[\max\left\{\underline{\boldsymbol{X}}_1,\underline{\boldsymbol{X}}_2,\underline{\boldsymbol{X}}_3\left(\underline{e}_1\right)\right\},\min\left\{\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2\left(\vec{e}_2'\right),\overline{\boldsymbol{X}}_3\left(\vec{e}''\right)\right\}\right]$. *Additionally,* $J^+=\{2,3,4,7,9,10\}$ *and* $J^-=\{1,5,6,8\}$. *Thus, from Theorem 4.2, the optimal solution of the problem is obtained as*

$$\boldsymbol{x}_{e^*}=(0.66,0.57,0.14,0.40,0.45,1,0.55,0.62,0.04,0.53),$$

where $e^*=(\underline{e}_1,\vec{e}_2,\vec{e}'')$, *and then* $\boldsymbol{c}^T\boldsymbol{x}_{e^*}=-13.07$.

7 Conclusion

This paper presented an algorithm for finding a global optimal solution to linear objective problems constrained by a novel system of fuzzy relation equations using the minimum t-norm. We rigorously proved that the feasible solution set is a finite union of closed convex cells. Additionally, we established necessary and sufficient conditions for problem feasibility. To enhance computational efficiency, seven simplification rules were introduced. We demonstrated that if all coefficients and variable values are binary, the problem exhibits a binary optimum. Furthermore, we showed that the well-known minimal vertex cover problem is a special case of the problem addressed in this study.

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