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Development of the hybrid BWM-fuzzy preference programming approach using a two-phase model: A case study in the automotive after-sales services

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Abstract

Multi-attribute decision-making (MADM) methods are essential tools for solving complex decision problems by offering systematic approaches for evaluating and ranking multiple criteria and alternatives. Despite considerable progress in MADM techniques, challenges persist in ensuring the reliability and consistency of outcomes, particularly under fuzzy environments. A recent advancement the hybrid best-worst method with fuzzy preference programming approach (the BWM-FPP model) has demonstrated promise, notably in hospital performance evaluations. However, the weight vector derived from the BWM-FPP model is a locally weakly efficient, and but not necessarily a locally efficient weight vector, which limits its robustness in practical applications. To address this shortcoming, we propose an enhanced two-phase model that guarantees the derivation of a locally efficient weight vector. In Phase 1, the model maximizes the minimum membership degree, ensuring a strong baseline of reliability, while Phase 2 improves overall robustness by maximizing the mean membership degree. The proposed model's effectiveness is demonstrated through theoretical validation, four numerical examples, and a real-world case study in the automotive after-sales service sector. Results show significant improvements in membership degrees and reliability, validating the model's practical applicability. In the case study, key service quality criteria such as Reliability, Tangibles, and Assurance emerged as most influential, underscoring their critical role in after-sales service evaluation.

Keywords: BWM-FPP model, two-phase model, locally efficient weight vector, membership degree, after-sales service criteria.

1 Introduction

Multi-attribute decision-making (MADM) stands as a cornerstone methodology in decision sciences, providing systematic frameworks to address complex problems characterized by multiple, often conflicting criteria. Its ability to incorporate uncertainty and ambiguity into decision-making processes has made it indispensable in real-world applications, particularly when integrated with advanced techniques such as fuzzy logic or artificial intelligence-driven models. For instance, Debnath and Roy [7] proposed T-spherical fuzzy aggregation operators to enhance unbiased and robust MADM, demonstrating their effectiveness in a hydrogen station site selection case. Debnath and Roy [8] highlight the theoretical foundations and practical applications of MADM, emphasizing its role in facilitating robust decision-making under uncertain conditions. Similarly, Wang et al. [39] propose a hybrid framework that integrates Pythagorean fuzzy sets with MADM techniques to enhance decision accuracy in sustainable supplier selection. Widely adopted methods like the Best-Worst Method (BWM), TOPSIS, and VIKOR have proven invaluable across diverse fields, including supply chain management, resource optimization, and performance evaluation. As noted by Mondal and Roy [28], these

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methodologies excel in balancing qualitative and quantitative factors, enabling comprehensive evaluations. Furthermore, Mondal et al. [29] stress the importance of integrating MADM with complementary approaches to yield more robust solutions for intricate challenges. Dowlatshahi and Hashemi [11] proposed a novel unsupervised feature selection method using a fuzzy multi-criteria decision-making approach, integrating an ensemble of filter rankers with fuzzy TOPSIS to enhance classification performance on high-dimensional datasets. Their method outperforms competing strategies in accuracy, F-score, and computational efficiency across various real-world datasets.

The automotive industry exemplifies the practical application of MADM, particularly in evaluating after-sales service performance. After-sales services play a pivotal role in enhancing customer satisfaction, fostering brand loyalty, and maintaining long-term relationships. Factors such as service quality, cost-effectiveness, operational efficiency, and customer feedback significantly influence customer retention and satisfaction throughout the product lifecycle. Jap [20] emphasizes the importance of managing these factors effectively, while Sivadas and Baker-Prewitt [36] highlight the critical role of after-sales services in building trust and loyalty. Kurata and Nam [22] further argue that feedback mechanisms are essential for adapting to market demands and sustaining competitive advantage. In today's data-driven organizations, accurately measuring product and service quality is paramount for continuous improvement. According to Li et al. [24], precision in evaluating performance allows managers to identify areas for enhancement and implement targeted strategies. Davies [6] underscores the significance of understanding these relationships to achieve sustainable competitive advantage.

The BWM (Rezaei [33]) is typically used in deterministic settings to capture expert opinions. However, given today's uncertain and dynamic conditions, deterministic BWM may not accurately reflect human reasoning and explanations, as it relies on precise numerical values for pairwise comparisons. In many cases, assigning exact values is challenging. Decision-makers (DMs) often find that judgments expressed with fuzzy values are more reliable than those using fixed, precise numbers because they struggle to articulate their judgments explicitly due to the fuzzy nature of comparisons. Therefore, fuzzy set theory should be incorporated to address this issue. Given these considerations, the applicability of the proposed method to BWM would depend on whether it can accommodate the inherent uncertainty and vagueness in decision-making. This study is motivated by a key concern:

Although the BWM-FPP model proposed by Amiri et al. [1] improves decision-making under uncertainty, it suffers from computational inefficiencies and produces weight vectors that are not necessarily locally efficient. As noted by Li et al. [25], the max–min approach employed in the BWM-FPP model does not ensure an efficient solution, which may compromise the consistency and robustness of the results. To address this limitation, we propose a novel two-phase model that enhances the BWM-FPP model by optimizing weight derivation and membership degree calculation. Phase 1 maximizes the minimum membership degree to establish a strong baseline of reliability, while Phase 2 maximizes the mean membership degrees to improve overall robustness and achieve locally efficient solutions. To address these challenges, this study introduces a novel two-phase model that enhances the BWM-FPP model by optimizing weight determination and membership degree calculations. In Phase 1, the model maximizes the minimum membership degree, ensuring a strong baseline of reliability, while Phase 2 improves overall robustness by maximizing the mean membership degree.

The main contributions of this study are as follows:

Introduction of key definitions and metrics for weight vector assessment: This research introduces concepts such as dominance, reliability, locally weakly efficient weight vector, locally efficient weight vector and mean membership degree. These definitions establish a robust framework for evaluating weight vectors. By providing both theoretical and practical foundations, these metrics enhance MADM by offering deeper insights into the quality, reliability, and robustness of decision-making processes.

Development of new theoretical insights and theorems: This study presents and analyzes novel theorems that form the backbone of the proposed framework, strengthening its theoretical foundation. These theorems connect the new framework with existing methodologies, advancing consistency refinement and optimization techniques. By expanding the theoretical landscape of BWM-FPP model, this research contributes to the broader MADM field, uncovering novel relationships and insights that pave the way for future studies and practical applications.

Enhancements in the BWM-FPP model: We propose a two-phase model to systematically address the BWM-FPP model. In Phase 1, the BWM-FPP model is solved to obtain an initial weight vector by maximizing the minimum membership degree. In Phase 2, this weight vector is refined by maximizing the mean membership degree. This ensures that the final weight vector is a locally efficient while simultaneously increasing the mean membership degree. By integrating this secondary refinement process, the proposed method enhances the accuracy, reliability, and consistency of results, making it a robust solution for complex MADM problems.

This paper is structured as follows: Section 2 reviews prior studies that have utilized fuzzy methods, particularly in the context of after-sales service evaluation. Section 3 describes the research methodology, including the details of the BWM-FPP model in a fuzzy environment. Section 4 presents the proposed two-phase fuzzy BWM approach and

validates by some theorems and four numerical examples. Section 5 applies the proposed model to evaluate criteria for automotive after-sales service performance. Sections 6 and 7 provide a theoretical discussion of the findings and their managerial implications, as well as the study's conclusions and directions for future research.

2 Literature review

2.1 After-sales services

After-sales services refer to the support provided by manufacturers or suppliers after the delivery of goods, aimed at enhancing consumers' use of the product. Gaiardelli et al. [14] highlight that these services are designed to assist customers in maximizing the value of their purchases. Given the competitive advantage such services can offer, they have become a significant factor influencing customer satisfaction and, subsequently, the organization's profitability. Miranda et al. [27] emphasize the critical role these services play in building long-term relationships with customers and driving business success. Various methods have been proposed for evaluating the quality of after-sales services in different parts of the world, businesses, and industries. Prior research on marketing has examined the effect of after-sales services on customer satisfaction and the relationship between service level and customer satisfaction in various service industries. Therefore, we will first review the articles centering on after-sales services and then summarize the main features of these works in Table 1. The following is a summary of the literature on after-sales services.

Pakdil et al. [31] used models based on SERVQUAL and Quality Function Deployment (QFD) to analyze both the quantitative and qualitative after-sales services of a construction equipment manufacturer. At first, content analysis and SERVQUAL were employed to identify key customer needs and expectations. Then, QFD and the Quality House model were used to rank customer expectations in terms of importance. Levesque and Boeck [23] demonstrated how companies can use Proximity Marketing to enhance service experiences through customized and personalized advertising. The authors collected data using a multi-case approach and the analysis identified the factors that contribute to improved customer service experience. Wang et al. [38] investigated the impact of customer experience levels on satisfaction in the Chinese auto industry. Numerous online surveys were studied to examine the effects of customer experience on satisfaction with products. After identifying the influential factors between satisfaction and customer experience levels, the authors proposed solutions and management concepts for consumers to better understand satisfaction, as well as for companies to enhance customer satisfaction.

Shokouhyar et al. [35] presented a framework for evaluating the impact of after-sales service tools on customer satisfaction in the auto industry. The researchers performed customer analysis and classified factors that contribute to customer satisfaction with after-sales services by combining the Kano model, SERVQUAL, and clustering techniques. After identifying 21 quality elements, the researchers used a method based on RFM (Recency, Frequency, Monetary) models to cluster customer priorities and tendencies. Rebelo et al. [32] focused on the development and improvement of after-sales service processes in a Latvian company specializing in spare parts management. The study was conducted in two phases. In phase one, after-sales process data were collected through surveys and subsequently analyzed to identify areas in need of improvement. In phase two, the identified improvements such as introducing new services, developing new process charts, and enhancing spare parts management strategies, were implemented and provided to customers.

Chen et al. [5] developed an approach for the evaluation system of automotive after-sales services based on a comprehensive fuzzy evaluation. Moreover, the authors proposed a more efficient evaluation system for after-sales services of modern automobiles, aiming to continuously monitor, optimize, and improve the level of after-sales services. Balinado et al. [3] tried to determine factors affecting customer satisfaction with automotive after-sales services using the SERVQUAL approach. For this purpose, five key factors, namely tangibles, reliability, responsiveness, assurance, and empathy, were simultaneously analyzed. Structural Equation Modeling (SEM) indicated that of the five service quality factors, reliability and empathy had a remarkable correlation with customer satisfaction. Contrastingly, no meaningful correlation was found between customer satisfaction and tangibles, responsiveness, and assurance. Gupta and Raman [17] investigated customer satisfaction with after-sales service experiences in the automotive sector. The study focused on understanding customer experiences in the automotive sector and their possible impact on satisfaction. Ebrahimi et al. [13] employed specific classification methods and predictive analytics to examine factors that influence the vendor-manufacturer relationship and to predict the continuation or termination of manufacturers' relationship with their sales representatives in the after-sales service network. Furthermore, they used predictive analysis tools and classification methods to predict sales representatives' cooperation as a dependent variable in two classes based on real-world data collected over a two-year period. Lastly, the researchers proposed a managerial solution to help manufacturers make informed decisions regarding whether to continue or cease cooperation with representatives at the appropriate time. Baskir [4] proposed a novel management tool using AHP (Analytic Hierarchy Process), QFD, and interval type-2 fuzzy sets to convert customer feedback into technical and operational features. The author also suggested a flexible decision-

Table 1: Summary of studies on after-sales services.

Author(s)	Methods/Models	Field	Uncertainty
Pakdil et al. [31]	SERVQUAL, QFD	Construction machinery industry	
Levesque and Boeck [23]	Proximity Marketing	Fashion industry, Event Industry, Cosmetic Industry, Sports Industry, Restauration Industry	
Wang et al. [38]	Conducted the research based on big data collected from online reviews	Auto industry	
Shokouhyar et al. [35]	KANO, SERVQUAL, clustering	Auto industry	*
Rebelo et al. [32]	CRM, SWOT	Spare parts management	
Chen et al. [5]	Fuzzy Comprehensive evaluation method	Auto industry	*
Balinado et al. [3]	SERVQUAL	Auto industry	
Gupta and Raman [17]	Questionnaire analysis on customer experiences	Auto industry	
Ebrahimi et al. [13]	Predictive analytics	Auto industry	
Baskir [4]	QFD-AHP (BQFD-AHP) model in interval type-2 fuzzy (IT2F) environment	Auto industry	*

making framework for automotive after-sales services. The ranking produced by the proposed model was then compared with those produced by the classical and type-1 fuzzy versions of itself.

2.2 The BWM

The BWM, much like after-sales services, has been utilized in various fields for different purposes. In the context of quality evaluation, numerous innovative approaches have been developed, including applications in fuzzy environments. As our technique and model are being constructed in a fuzzy environment, it is necessary to review studies that have utilized the BWM or introduced new fuzzy-based extensions to this method. The key attributes of the reviewed studies is presented in Table 2.

Mou et al. [30] introduced a novel method called IFMBWM (intuitionistic fuzzy multiplicative BWM), which is essentially a combination of BWM and Intuitionistic Fuzzy Multiplicative Preference Relations (IFMPRs) to solve management problems in uncertain conditions. Several maximin programming models have been proposed by the authors for group multi-criteria decision-making, and new relationships are introduced to calculate the consistency rate, observe the consistency of comparisons, and assess the reliability of the results. Guo and Zhao [16] proposed BWM in a fuzzy environment. Instead of using crisp numbers, this method uses linguistic terms expressed by triangular fuzzy numbers for the comparisons made by DMs. They modeled FBWM under uncertainty using Graded Mean Integration Representation (GMIR) and nonlinear programming. Three case studies were also carried out to demonstrate the effectiveness and feasibility of the proposed FBWM. The results showed that the proposed method could not only obtain logical priority rankings for the solutions but also exhibited higher consistency compared to the regular BWM. Torbati and Sayadi [37] presented an expertly fuzzy model to weight the performance evaluation criteria of insurance representative offices in Iran. They developed a combination of BWM for determining the importance of each criterion and Fuzzy Inference System (FIS) to evaluate and rank the investigated insurance offices. The results pointed to the superiority of the proposed model over the other methods in the literature in terms of simplicity and accuracy. Haeri and Rezaei [18] introduced a new weight allocation model for green supplier selection based on BWM and Grey Numbers Theory. Another weight allocation model was developed by combining BWM and Fuzzy Grey Cognitive Maps to capture the interdependencies among criteria. The authors contributed to Decision Theory by addressing the limitations of existing green supplier selection models. A real-world case study was also provided to demonstrate the applicability and effectiveness of the proposed model. The results showed that the proposed model could effectively solve the green supplier selection problem by considering the interdependencies among criteria and the uncertainties associated with expert judgments. Wu et al. [41] developed a model based on BWM and Interval Type-2 Fuzzy Sets (IT2FSs) for Multi-Criteria Group Decision Making (MCGDM) in green supplier selection. They proposed a hybrid approach consisting of BWM and VIKOR in an Interval Type-2 Fuzzy environment, which produced more reliable results compared to BWM and FBWM. Maghsoodi et al. [26] proposed a novel approach called HBWFAD (Hybrid Best-Worst Fuzzy Axiomatic Design) for selecting an initial design for loudspeakers. The approach, which also featured a hierarchical group decision-making structure, has been applied to a case study to help analyze the advantages and disadvantages in real-world conditions. Karimi et al. [21] developed a fully fuzzy linear mathematical model for obtaining the weights of criteria using an extension of BWM. The model also utilized to calculate the score recorded by each option. The proposed model offered several advantages, such as requiring minimal data for decision-making, high capability for achieving reliable and definitive solutions, and compatibility for integration with other methods. Amiri et al. [1] combined the Group Best-Worst Method (GBWM) and FPP to form a hybrid approach called GBWM-FPP for Hospital Performance Evaluation (HPE). The approach determined the significance of each criterion and selected

the best option among other alternatives. The proposed approach's advantages include its capacity to integrate Group Decision-Making (GDM) into a unified model and eliminate the need for separate consistency ratio (CR) calculations. Finally, the efficiency and effectiveness of the proposed approach have been demonstrated through a case study. Amiri et al. [2] next created a fuzzy linear programming model using trapezoidal fuzzy numbers. The model is based on the three criteria of possibility, necessity, and credibility, and derives its weights from the probabilistic chance-constrained programming (PCCP) framework. Developing the BWM on the basis of probability distributions, allows DMs to consider the uncertainty in weight calculations, as well as optimistic, pessimistic, and composite perspectives in determining the weights of decision criteria. Dong et al. [10] introduced a novel FBWM based on triangular fuzzy numbers. In this method, four linear programming models (i.e. optimistic approach, pessimistic approach, hybrid approach I, and hybrid approach II) have been defined to extract the optimal fuzzy weights of criteria. Furthermore, the researchers proposed the two concepts of Fuzzy Compatibility Index and Fuzzy Compatibility Ratio. Roy and Shaw [34] proposed a sustainable multi-criteria credit scoring model for promoting green finance strategies, considering economic, environmental, social, and managerial features. The authors attempted to simplify the complex current ranking methods by developing a Multi-Criteria Decision Analysis (MCDA) system in three simple stages: (1) identifying the most important financial, managerial, social, and environmental aspects; (2) determining the parameter weights using the proposed FBWM; and (3) calculating a sustainable credit score using fuzzy TOPSIS (the Technique for Order of Preference by Similarity to Ideal Solution). Dehshiri et al. [9] proposed a new Group Decision-Making method based on the BWM to evaluate the criteria for block chain technology implementation in supply chains. This study also led to the development of two nonlinear programming-based optimization methods: Nonlinear Goal Programming based on BWM II (NGPBWM II) and Linear Goal Programming based on BWM II (LGPBWM II), which can be applied in individual and group decision-making problems. The proposed model presents advantages including reduced complexity and computational efforts. Huang et al. [19] presented new methods for extracting weights from the FBWM matrix. These methods are evaluated using Monte Carlo simulations and practical examples. Wu et al. [40] mentioned that the BWM can yield multiple optimal solutions, and its linear model produces a unique solution. The research aims to solve the BWM's nonlinear model mathematically to derive the analytical forms of optimal solutions. The model introduces an equivalent BWM formulation and a secondary objective function, which simplifies the solution process without compromising computational accuracy.

Table 2: Summary of BWM-related studies conducted in fuzzy environments.

Author(s)	Subject	Method
Mou et al. [30]	Numerical examples	Intuitionistic fuzzy multiplicative best-worst method (IFMBWM)
Guo & [16]	Optimal transport mode, car purchase, supplier performance	Fuzzy multi-criteria BWM
Torbati & Sayadi [37]	Insurance branches in Iran	BWM and Fuzzy Inference System
Haeri & Rezaei [18]	Supplier selection	BWM with Grey Numbers Theory (novel weight allocation model)
Wu et al. [41]	Green supplier selection	Interval type-2 FBWM + extended VIKOR
Maghsoodi et al. [26]	Prototype design selection	HBWFAD: hybrid of FAD and FBWM; hierarchical group decision-making
Karimi et al. [21]	Maintenance assessment in hospitals	Fully fuzzy BWM with triangular fuzzy numbers
Amiri et al. [1]	Hospital performance evaluation	Hybrid GBWM-FPP method
Amiri et al. [2]	Numerical examples	Fuzzy LP model based on BWM using trapezoidal fuzzy numbers
Dong et al. [10]	Numerical examples	FBWM with triangular fuzzy numbers
Roy & Shaw [34]	Financial institutions	FBWM and fuzzy TOPSIS
Dehshiri et al. [9]	Blockchain criteria in auto supply chains	Novel group BWM approach
Huang et al. [19]	Numerical examples	Alternative methods to derive priorities from FBWM matrix
Wu et al. [40]	Numerical examples	Analytical solution of the nonlinear FBWM model

2.3 Research gap: Methodological perspectives

Despite the extensive development of fuzzy MADM methods, several methodological limitations remain, particularly in the derivation of reliable and efficient weight vectors under uncertainty. Among these methods, the hybrid best-worst method with the fuzzy preference programming approach has emerged as a significant advancement, offering a more flexible approach to preference elicitation and decision modeling in fuzzy environments. However, a critical methodological shortcoming of the BWM-FPP model lies in the nature of its derived weight vector—it ensures only local weak efficiency, not local efficiency. This distinction is not merely theoretical; it has tangible implications for the robustness and reliability of decision outcomes, especially in high-stakes or real-world applications.

The lack of local efficiency implies that there may exist alternative weight vectors that could perform better in satisfying the fuzzy constraints without violating the decision maker's preferences. Consequently, the solution obtained from the BWM-FPP model may be suboptimal in terms of membership degree maximization, reducing confidence in the results' stability and consistency. Furthermore, existing extensions of BWM or fuzzy MADM methods have largely focused on improving preference elicitation or computational simplicity, while overlooking the structural inefficiency in

the solution space. Few studies have tackled the efficiency of the weight vector from a fuzzy mathematical programming perspective—specifically, no prior work has proposed a structured, two-phase optimization model that explicitly enhances local efficiency within the BWM-FPP model.

Therefore, there is a clear methodological gap in developing a fuzzy MADM model that systematically transforms a locally weakly efficient solution into a locally efficient one. Addressing this gap would not only enhance the theoretical rigor of fuzzy MADM models but also significantly improve their applicability and reliability in practical decision-making contexts.

3 The proposed two-phase model

In this section, the model in a fuzzy environment is introduced. First, the hybrid decision-making method proposed in Amiri et al. [1] is presented. By combining the strengths of these two approaches, the method enhances the overall decision-making process. Next, a novel fuzzy BWM approach that determines the weights of criteria using a two-phase nonlinear fuzzy model is developed.

3.1 Steps of the BWM

Here, the nonlinear formulation of the BWM including six steps, is presented.

Step 1. A set of n decision-making/evaluation criteria is selected.
Step 2. The DM/evaluator chooses the best (i.e., the most important) and worst (i.e., the least important) criterion out of the criteria set mentioned in Step 1.
Step 3. The DM/evaluator makes pairwise comparisons between the best criterion and the other criteria. The comparisons assign a score from 1 to 9, with 1 meaning "equally important" and 9 meaning "much more important." The resulting vector is expressed as $A_B = (a_{B1}, a_{B2}, \ldots, a_{Bn})$.
Step 4. The DM/evaluator compares the worst criterion with the other criteria. The resulting vector is expressed as $A_W = (a_{1W}, a_{2W}, \ldots, a_{nW})^T$.
Step 5. In this step, the objective is to minimize the maximum absolute errors that arise from inconsistencies. In other words, the method aims to minimize the maximum absolute error: $(|w_B/w_j - a_{Bj}|, |w_j/w_W - a_{jW}|)$. The multi-objective model is described as Model (1) with $2n-3$ objective functions:

$$\begin{aligned} &\min_j \left(|\frac{w_B}{w_j} - a_{Bj}|,\ |\frac{w_j}{w_W} - a_{jW}| \right), \quad j = 1, 2, \ldots, n \\ &\text{s.t.} \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \end{aligned} \tag{1}$$

By using the min-max operator, Model (1) is converted to Model (2) as follows:

$$\begin{aligned} &\min \max_j \left(|\frac{w_B}{w_j} - a_{Bj}|,\ |\frac{w_j}{w_W} - a_{jW}| \right), \quad j = 1, 2, \ldots, n \\ &\text{s.t.} \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \end{aligned} \tag{2}$$

By introducing $\xi = \max_j \left(|\frac{w_B}{w_j} - a_{Bj}|,\ |\frac{w_j}{w_W} - a_{jW}| \right)$, Model (2) is converted to Model (3) as follows:

$$\begin{aligned} &\min\ \xi \\ &\text{s.t.} \quad \left| \frac{w_B}{w_j} - a_{Bj} \right| \leq \xi, \quad j = 1, 2, \ldots, n,\ (j \neq B) \\ &\quad \left| \frac{w_j}{w_W} - a_{jW} \right| \leq \xi, \quad j = 1, 2, \ldots, n,\ (j \neq B, W) \\ &\quad \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \end{aligned} \tag{3}$$

By solving Model (3), the optimal weight vector $W^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ and ξ^* are calculated.

Step 6. After obtaining the criteria weights, the Consistencies of the results should be examined. For this purpose, we use ξ^* obtained in Step 5 and the Consistency Index (CI) presented in Table 3, which is calculated based on the priority of the best criterion with respect to the worst criterion, to extract the Consistency Ratio (CR). Rezaei [33] proposed a formula to obtain the CR for the BWM, which is presented in Expression (4):

$$\mathrm{CR} = \frac{\xi^*}{\mathrm{CI}}. \tag{4}$$

Table 3 presents the values of the Consistency Index (CI) corresponding to different values of a_{BW}.

Table 3: Consistency indices based on a_{BW} Rezaei [33].

a_{BW}	1	2	3	4	5	6	7	8	9
CI	0.00	0.44	1.00	1.63	2.30	3.00	3.73	4.47	5.23

3.2 The hybrid of BWM-FPP model

Similar to the steps discussed in Subsection 3.1, the DM selects a set of n decision-making criteria. Then, the best (the most important) criterion and the worst (the least important) criterion are chosen from the set.

In the third step, pairwise comparisons are made between the best criterion and the other criteria, and then between the other criteria and the worst criterion. These pairwise comparisons are made through fuzzy numbers, as presented in Table 4.

Table 4: Linguistic terms for criteria comparisons, Amiri et al. [1].

Linguistic Term	Fuzzy Scale (TFN)
Equal Importance (EI)	(1, 1, 1)
Weak Importance (WI)	(1, 2, 3)
Moderate Importance (MI)	(2, 3, 4)
Moderate Plus Importance (MP)	(3, 4, 5)
Strong Importance (SI)	(4, 5, 6)
Strong Plus Importance (SP)	(5, 6, 7)
Very Strong Importance (VS)	(6, 7, 8)
Extreme Importance (EX)	(7, 8, 9)

The resulting relation is expressed as follows:

$$\tilde{A} = \begin{cases} \tilde{a}_{Bj} = (l_{Bj}, m_{Bj}, u_{Bj}), & j = 1, 2, \ldots, n \quad (j \neq B) \\ \tilde{a}_{jW} = (l_{jW}, m_{jW}, u_{jW}), & j = 1, 2, \ldots, n \quad (j \neq B, W) \end{cases}$$

The membership function of $\mu_{Bj}(w_B/w_j)$ determines the priority of the best criterion with respect to the jth criterion, and the membership function of $\mu_{jW}(w_j/w_W)$ determines the priority of any criterion with respect to the worst criterion for $\tilde{A}$ (Expressions 5 and 6).

$$\mu_{Bj}(w_B/w_j) = \begin{cases} \frac{(w_B/w_j) - l_{Bj}}{m_{Bj} - l_{Bj}}, & \text{if } w_B/w_j \leq m_{Bj}, \quad j = 1, 2, \ldots, n \ (j \neq B) \\ \frac{u_{Bj} - (w_B/w_j)}{u_{Bj} - m_{Bj}}, & \text{if } w_B/w_j > m_{Bj}, \quad j = 1, 2, \ldots, n \ (j \neq B) \end{cases} \tag{5}$$

$$\mu_{jW}(w_j/w_W) = \begin{cases} \frac{(w_j/w_W) - l_{jW}}{m_{jW} - l_{jW}}, & \text{if } w_j/w_W \leq m_{jW}, \quad j = 1, 2, \ldots, n \ (j \neq B, W) \\ \frac{u_{jW} - (w_j/w_W)}{u_{jW} - m_{jW}}, & \text{if } w_j/w_W > m_{jW}, \quad j = 1, 2, \ldots, n \ (j \neq B, W) \end{cases} \tag{6}$$

The multi-objective programming corresponding to the BWM-FPP model (termed the MOP-BWM-FPP model) is described as Model (7):

$$\max_j \left(\mu_{Bj}(w_B/w_j), \mu_{jW}(w_j/w_W)\right), \quad j = 1, 2, \ldots, n \quad \text{s.t.} \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \tag{7}$$

By using the max-min operator, Model (7) is converted to Model (8) as follows:

$$\max \min_j \left(\mu_{Bj}(w_B/w_j), \mu_{jW}(w_j/w_W)\right), \quad j = 1, 2, \ldots, n \quad \text{s.t.} \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \tag{8}$$

By introducing $\lambda = \min_j \left(\mu_{Bj}(w_B/w_j), \mu_{jW}(w_j/w_W)\right)$, Model (8) is converted to Model (9). The optimal value of λ^* and the optimal weight vector $W^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ can be calculated by Model (9) as follows:

$$\begin{aligned}
&\max \ \lambda \\
&\text{s.t.} \ \mu_{Bj}(w_B/w_j) \geq \lambda, \quad j = 1, 2, \ldots, n \quad (j \neq B) \\
&\quad \mu_{jW}(w_j/w_W) \geq \lambda, \quad j = 1, 2, \ldots, n \quad (j \neq B, W) \\
&\quad \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \\
&\quad \lambda \text{ free variable}
\end{aligned} \tag{9}$$

By substituting the membership functions presented in Model (9), the model can be rewritten as Model (10):

$$\begin{aligned}
&\max \ \lambda \\
&\text{s.t.} \quad \frac{w_B/w_j - l_{Bj}}{m_{Bj} - l_{Bj}} \geq \lambda, \quad \frac{u_{Bj} - w_B/w_j}{u_{Bj} - m_{Bj}} \geq \lambda, \quad j = 1, 2, \ldots, n \ (j \neq B) \\
&\quad \frac{w_j/w_W - l_{jW}}{m_{jW} - l_{jW}} \geq \lambda, \quad \frac{u_{jW} - w_j/w_W}{u_{jW} - m_{jW}} \geq \lambda, \quad j = 1, 2, \ldots, n \ (j \neq B, W) \\
&\quad \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \\
&\quad \lambda \text{ free variable}
\end{aligned} \tag{10}$$

and equivalently, we have:

$$\begin{aligned}
&\max \ \lambda \\
&\text{s.t.} \ -w_B + l_{Bj} w_j + \lambda(m_{Bj} - l_{Bj}) w_j \leq 0, \quad j = 1, 2, \ldots, n \ (j \neq B) \\
&\quad w_B - u_{Bj} w_j + \lambda(u_{Bj} - m_{Bj}) w_j \leq 0, \quad j = 1, 2, \ldots, n \ (j \neq B) \\
&\quad -w_j + l_{jW} w_W + \lambda(m_{jW} - l_{jW}) w_W \leq 0, \quad j = 1, 2, \ldots, n \ (j \neq B, W) \\
&\quad w_j - u_{jW} w_W + \lambda(u_{jW} - m_{jW}) w_W \leq 0, \quad j = 1, 2, \ldots, n \ (j \neq B, W) \\
&\quad \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1, 2, \ldots, n \\
&\quad \lambda \text{ free variable}
\end{aligned} \tag{11}$$

Note. If the optimal value of the objective function λ^* is equal to or larger than 0, there is consistency in the pairwise comparisons. However, if $\lambda^* < 0$, no pairwise comparisons can be made between the variables, and the questionnaire should be redesigned with more precision and completed again.

3.3 A two-phase model for the development of the BWM-FPP model

This section introduces a novel two-phase optimization model for determining the criteria weights within the framework of the best-worst method with the fuzzy preference programming approach. The proposed two-phase model achieves this by initially maximizing the minimum membership degree (Phase 1), followed by maximizing the mean membership

degree (Phase 2).

Phase 1: Maximization of the minimum membership degree
In Phase 1, the BWM-FPP model proposed by Amiri et al. [1]-formulated as Model (9) (or equivalently Model (11))-is used to determine a locally optimal weight vector as $W^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$. This model aims to maximize the minimum membership degree among all pairwise comparisons.

Phase 2: Maximization of the mean membership degree
Based on the locally optimal vector W^* obtained in Phase 1, the second phase introduces an optimization model-referred to as Model (16)-which seeks to maximize the mean membership degree. This enhancement is carried out without compromising any individual membership degree achieved in the previous phase. That is, the membership degrees $\mu_{Bj}(w_B^*/w_j^*)$ and $\mu_{jW}(w_j^*/w_W^*)$ must be preserved or improved. The resulting refined weight vector is denoted by $W^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})^T$.

To achieve this, the constraints in Model (16) are derived as follows:

Constraint Type 1: $\mu_{Bj}(w_B/w_j) \geq \lambda_{Bj} \geq \mu_{Bj}(w_B^*/w_j^*)$ is formulated based on Case A or Case B, depending on the relative value of w_B^*/w_j^*.

Constraint Type 2: $\mu_{jW}(w_j/w_W) \geq \lambda_{jW} \geq \mu_{jW}(w_j^*/w_W^*)$ is derived according to Case C or Case D, based on the value of w_j^*/w_W^*.

Case-based reformulation of membership constraints:

Case A: If $(w_B^*/w_j^*) \leq m_{Bj}$, then the constraint becomes:

$$\frac{(w_B/w_j) - l_{Bj}}{m_{Bj} - l_{Bj}} \geq \lambda_{Bj} \geq \frac{(w_B^*/w_j^*) - l_{Bj}}{m_{Bj} - l_{Bj}}, \quad \text{with } w_B/w_j \leq m_{Bj}, \quad j = 1, 2, \ldots, n, \ (j \neq B). \tag{12}$$

Case B: If $(w_B^*/w_j^*) > m_{Bj}$, then:

$$\frac{u_{Bj} - (w_B/w_j)}{u_{Bj} - m_{Bj}} \geq \lambda_{Bj} \geq \frac{u_{Bj} - (w_B^*/w_j^*)}{u_{Bj} - m_{Bj}}, \quad \text{with } w_B/w_j > m_{Bj}, \quad j = 1, 2, \ldots, n, \ (j \neq B). \tag{13}$$

Case C: If $(w_j^*/w_W^*) \leq m_{jW}$, then:

$$\frac{(w_j/w_W) - l_{jW}}{m_{jW} - l_{jW}} \geq \lambda_{jW} \geq \frac{(w_j^*/w_W^*) - l_{jW}}{m_{jW} - l_{jW}}, \quad \text{with } w_j/w_W \leq m_{jW}, \quad j = 1, 2, \ldots, n, \ (j \neq B, W). \tag{14}$$

Case D: If $(w_j^*/w_W^*) > m_{jW}$, then:

$$\frac{u_{jW} - (w_j/w_W)}{u_{jW} - m_{jW}} \geq \lambda_{jW} \geq \frac{u_{jW} - (w_j^*/w_W^*)}{u_{jW} - m_{jW}}, \quad \text{with } w_j/w_W > m_{jW}, \quad j = 1, 2, \ldots, n, \ (j \neq B, W). \tag{15}$$

Model (16): Maximization of the mean membership degree

$$\bar{y}_W(W^{**}) = \max \frac{1}{2n-3} \left(\sum_{\substack{j=1 \\ j\neq B}}^{n} \lambda_{Bj} + \sum_{\substack{j=1 \\ j\neq B,W}}^{n} \lambda_{jW} \right)$$

$$\begin{aligned} \text{s.t.} \quad & \mu_{Bj}(w_B/w_j) \geq \lambda_{Bj} \geq \mu_{Bj}(w_B^*/w_j^*), \quad j = 1,2,\ldots,n, (j \neq B) \\ & \mu_{jW}(w_j/w_W) \geq \lambda_{jW} \geq \mu_{jW}(w_j^*/w_W^*), \quad j = 1,2,\ldots,n, (j \neq B,W) \\ & \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0, \quad j = 1,2,\ldots,n \\ & \lambda_{Bj},\ \lambda_{jW} \text{ free variables} \end{aligned} \tag{16}$$

Remark 3.1. *In Phase 1, let W^* denote the locally optimal weight vector obtained from Model (9), and let $W^* \in \mathcal{N}(W^*)$. Without loss of generality, assume $(w_B^*/w_j^*) \leq m_{Bj},\ (j \neq B)$ and $(w_j^*/w_W^*) \leq m_{jW},\ (j \neq B,W)$ (i.e., Cases A & C). Then for all $W \in \mathcal{N}(W^*)$, it holds that $(w_B/w_j) \leq m_{Bj}$ and $(w_j/w_W) \leq m_{jW}$. Since multiple locally optimal weight vectors may exist in $\mathcal{N}(W^*)$, Phase 2 refines W^* to W^{**} by maximizing the mean membership degree under the same constraints.*

To enforce these bounds, we rewrite the original membership constraints as:

$$\frac{(w_B/w_j) - l_{Bj}}{m_{Bj} - l_{Bj}} \geq \lambda_{Bj} \geq \frac{(w_B^*/w_j^*) - l_{Bj}}{m_{Bj} - l_{Bj}}, \quad \text{with } w_B/w_j \leq m_{Bj}$$

$$\frac{(w_j/w_W) - l_{jW}}{m_{jW} - l_{jW}} \geq \lambda_{jW} \geq \frac{(w_j^*/w_W^*) - l_{jW}}{m_{jW} - l_{jW}}, \quad \text{with } w_j/w_W \leq m_{jW}$$

This reformulation ensures that the refined vector W^{**} stays within the neighborhood $\mathcal{N}(W^*)$ and satisfies the consistency conditions. Mixed-case scenarios (A & B, A & D, B & C, B & D, C & D) are derived from Equations (12)-(15) and omitted here for brevity.

The overall framework is visually summarized in Figure 1.

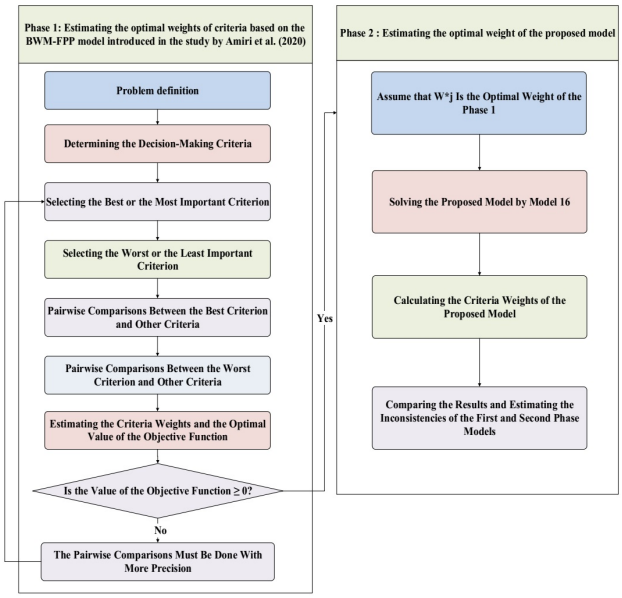


Figure 1: Outline of the proposed two-phase model.

3.4 Some key definitions and theorems for weight vector assessment

In the following, Definitions 3.2 to 3.6 provide a structured framework for evaluating and comparing weight vectors in decision-making scenarios, considering dominance, reliability, locally weakly efficient weight vector, locally efficient weight vector and mean membership degree. These definitions try to combine efficiency and robustness, making it well-suited for the BWM-FPP model.

Definition 3.2. *(dominate) Let* $V^* = (\mu_{Bj}\left(\frac{w_B^*}{w_j^*}\right), \mu_{jW}\left(\frac{w_j^*}{w_W^*}\right))$ *and* $V^{**} = (\mu_{Bj}\left(\frac{w_B^{**}}{w_j^{**}}\right), \mu_{jW}\left(\frac{w_j^{**}}{w_W^{**}}\right)), (j = 1, 2, \ldots, n)$ *be two criterion membership vectors in the objective space. Then, the vector* V^{**} *dominates the vector* V^* *if and only if* $V^{**} \geq V^*$ *with* $V^{**} \neq V^*$ *(i.e.,* $\mu_{Bj}\left(\frac{w_B^{**}}{w_j^{**}}\right) \geq \mu_{Bj}\left(\frac{w_B^*}{w_j^*}\right)$ *and* $\mu_{jW}\left(\frac{w_j^{**}}{w_W^{**}}\right) \geq \mu_{jW}\left(\frac{w_j^*}{w_W^*}\right), (j = 1, 2, \ldots, n)$*, with at least a strict inequality, for* $j = 1, 2, \ldots, n$*).*

Definition 3.3. *(reliable weight vector) The weight vector* $\mathbf{W}^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})$ *is more reliable than the weight vector* $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)$ *if and only if* $\mathbf{V}^{**} = (\mu_{Bj}(\frac{w_B^{**}}{w_j^{**}}), \mu_{jW}(\frac{w_j^{**}}{w_W^{**}}))$ *dominates* $\mathbf{V}^* = (\mu_{Bj}(\frac{w_B^*}{w_j^*}), \mu_{jW}(\frac{w_j^*}{w_W^*}))$ *for* $j = 1, 2, \ldots, n$*.*

Definition 3.4. *(locally weakly efficient weight vector) A weight vector* $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ *in Model (7) is called a locally weakly efficient weight vector if there exists a neighborhood* $\mathcal{N}(\mathbf{W}^*)$ *around* $\mathbf{W}^*$*, where:*

$$\mathcal{N}(\mathbf{W}^*) \subset \left\{ \mathbf{W} = (w_1, w_2, \ldots, w_n)^T : \sum_{j=1}^{n} w_j = 1,\ w_j \geq 0,\ j = 1, 2, \ldots, n \right\},$$

such that there does not exist another weight vector $\bar{\mathbf{W}} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^T \in \mathcal{N}(\mathbf{W}^*)$ *satisfying* $\bar{\mathbf{V}} > \mathbf{V}^*$*. In other words, there does not exist another weight vector* $\bar{\mathbf{W}} \in \mathcal{N}(\mathbf{W}^*)$ *such that:*

$$\mu_{Bj}(\bar{w}_B/\bar{w}_j) > \mu_{Bj}(w_B^*/w_j^*) \text{ for all } j = 1, 2, \ldots, n,\ (j \neq B),$$

$$\mu_{jW}(\bar{w}_j/\bar{w}_W) > \mu_{jW}(w_j^*/w_W^*) \text{ for all } j = 1, 2, \ldots, n,\ (j \neq B, W).$$

Definition 3.5. *(locally efficient weight vector) A weight vector* $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ *of Model (7) is called a locally efficient weight vector if there exists a neighborhood* $\mathcal{N}(\mathbf{W}^*)$ *around* $\mathbf{W}^*$*, where:*

$$\mathcal{N}(\mathbf{W}^*) \subset \left\{ \mathbf{W} = (w_1, w_2, \ldots, w_n)^T : \sum_{j=1}^{n} w_j = 1,\ w_j \geq 0,\ j = 1, 2, \ldots, n \right\},$$

such that there does not exist another weight vector $\bar{\mathbf{W}} \in \mathcal{N}(\mathbf{W}^*)$ *satisfying* $\bar{\mathbf{V}} \geq \mathbf{V}^*$ *with* $\bar{\mathbf{V}} \neq \mathbf{V}^*$*. In other words, there does not exist another weight vector* $\bar{\mathbf{W}} \in \mathcal{N}(\mathbf{W}^*)$ *such that:*

$$\mu_{Bj}(\bar{w}_B/\bar{w}_j) \geq \mu_{Bj}(w_B^*/w_j^*) \text{ for all } j = 1, 2, \ldots, n,\ (j \neq B),$$

$$\mu_{jW}(\bar{w}_j/\bar{w}_W) \geq \mu_{jW}(w_j^*/w_W^*) \text{ for all } j = 1, 2, \ldots, n,\ (j \neq B, W),$$

with at least one strict inequality holding for some j*.*

Definition 3.6. *(mean membership degree) The mean membership degree of the weight vector* $\mathbf{W} = (w_1, w_2, \ldots, w_n)^T$ *is defined as follows:*

$$\bar{y}_W(\mathbf{W}) = \frac{1}{2n-3}\left(\sum_{\substack{j=1 \\ j \neq B}}^{n} \mu_{Bj}(w_B/w_j) + \sum_{\substack{j=1 \\ j \neq B, W}}^{n} \mu_{jW}(w_j/w_W) \right).$$

In the following, we investigate the theoretical properties of the MOP-BWM-FPP model (Model (7)), the BWM-FPP model (Model (9)) and the proposed model (Model (16)), as well as the relationships among them.

Theorem 3.7. *Let* $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ *and* $\mathbf{W}^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})^T$ *denote the locally optimal weight vectors from Models (9) and (16), respectively. If the following equalities hold:*

$$\mu_{Bj}\left(\frac{w_B^*}{w_j^*}\right) = \mu_{Bj}\left(\frac{w_B^{**}}{w_j^{**}}\right) \quad \text{and} \quad \mu_{jW}\left(\frac{w_j^*}{w_W^*}\right) = \mu_{jW}\left(\frac{w_j^{**}}{w_W^{**}}\right), \quad \text{for all } j = 1, 2, \ldots, n,$$

then it follows that $\mathbf{W}^* = \mathbf{W}^{**}$*.*

Proof. Without loss of generality, suppose that $\frac{w_B^*}{w_j^*} \leq m_{Bj}$ and $\frac{w_j^*}{w_W^*} \leq m_{jW}$. According to Expressions (12) and (14), and given:

$$\mu_{Bj}\left(\frac{w_B^*}{w_j^*}\right) = \mu_{Bj}\left(\frac{w_B^{**}}{w_j^{**}}\right), \quad \mu_{jW}\left(\frac{w_j^*}{w_W^*}\right) = \mu_{jW}\left(\frac{w_j^{**}}{w_W^{**}}\right),$$

we obtain:

$$\frac{w_B^{**}}{w_j^{**}} = \frac{w_B^*}{w_j^*} \Rightarrow \frac{w_j^{**}}{w_B^{**}} = \frac{w_j^*}{w_B^*}.$$

Summing both sides over all $j = 1, 2, \ldots, n$, and using the normalization $\sum_{j=1}^n w_j^* = \sum_{j=1}^n w_j^{**} = 1$, we get:

$$\frac{1}{w_B^{**}} = \frac{1}{w_B^*} \Rightarrow w_B^* = w_B^{**}.$$

Substituting this into the previous ratio gives:

$$\frac{w_B^{**}}{w_j^{**}} = \frac{w_B^*}{w_j^*} \Rightarrow w_j^* = w_j^{**} \quad \text{for all } j = 1, 2, \ldots, n.$$

Similarly, for the second condition:

$$\frac{w_j^{**}}{w_W^{**}} = \frac{w_j^*}{w_W^*} \Rightarrow \frac{1}{w_W^{**}} = \frac{1}{w_W^*} \Rightarrow w_W^* = w_W^{**},$$

which implies $w_j^* = w_j^{**}$ for all $j = 1, 2, \ldots, n$. Therefore, $\mathbf{W}^* = \mathbf{W}^{**}$. □

Theorem 3.8. *Let $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ be a locally optimal weight vector within a neighborhood $\mathcal{N}(\mathbf{W}^*)$ around $\mathbf{W}^*$ for Model (9). Then, $\mathbf{W}^*$ is a locally weakly efficient weight vector within $\mathcal{N}(\mathbf{W}^*)$ for Model (7), but it is not necessarily a locally efficient weight vector for Model (7).*

Proof. We prove by contradiction. Suppose $\mathbf{W}^*$ is a locally optimal weight vector in $\mathcal{N}(\mathbf{W}^*)$ for Model (9), but not a locally weakly efficient weight vector for Model (7). Then, by Definition 3.4, there exists a feasible weight vector $\bar{\mathbf{W}} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^T \in \mathcal{N}(\mathbf{W}^*)$ such that:

$$\mu_{Bj}(\bar{w}_B/\bar{w}_j) > \mu_{Bj}(w_B^*/w_j^*) \geq \lambda \quad \text{for all } j = 1, 2, \ldots, n, \ (j \neq B),$$

$$\mu_{jW}(\bar{w}_j/\bar{w}_W) > \mu_{jW}(w_j^*/w_W^*) \geq \lambda \quad \text{for all } j = 1, 2, \ldots, n, \ (j \neq B, W).$$

These inequalities imply $\bar{\mathbf{W}}$ is feasible for Model (9), and also:

$$\lambda(\bar{\mathbf{W}}) = \min_j(\mu_{Bj}(\bar{w}_B/\bar{w}_j), \mu_{jW}(\bar{w}_j/\bar{w}_W)) > \min_j(\mu_{Bj}(w_B^*/w_j^*), \mu_{jW}(w_j^*/w_W^*)) = \lambda(\mathbf{W}^*),$$

which contradicts the assumption that $\mathbf{W}^*$ is locally optimal in $\mathcal{N}(\mathbf{W}^*)$ for Model (9). Therefore, $\mathbf{W}^*$ must be a locally weakly efficient weight vector for Model (7) within $\mathcal{N}(\mathbf{W}^*)$. Finally, as demonstrated in Examples 4.1-4.4, $\mathbf{W}^*$ is not necessarily locally efficient for Model (7), even though it is locally weakly efficient. □

Theorem 3.9. *Let $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ be a unique locally optimal weight vector in a neighborhood $\mathcal{N}(\mathbf{W}^*)$ for Model (9). Then, $\mathbf{W}^*$ is a locally efficient weight vector in $\mathcal{N}(\mathbf{W}^*)$ for Model (7).*

Proof. We prove this by contradiction. Let $\mathbf{W}^*$ be the unique locally optimal weight vector in $\mathcal{N}(\mathbf{W}^*)$ for Model (9), but not locally efficient for Model (7). Then, by Definition 3.5, there exists a feasible weight vector $\bar{\mathbf{W}} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^T \in \mathcal{N}(\mathbf{W}^*)$ such that:

$$\mu_{Bj}(\bar{w}_B/\bar{w}_j) \geq \mu_{Bj}(w_B^*/w_j^*) \geq \lambda, \quad \text{for all } j \neq B,$$

$$\mu_{jW}(\bar{w}_j/\bar{w}_W) \geq \mu_{jW}(w_j^*/w_W^*) \geq \lambda, \quad \text{for all } j \neq B, W,$$

with at least one strict inequality. These imply $\bar{\mathbf{W}}$ is feasible for Model (9). Moreover, the objective function value of Model (9) at $\bar{\mathbf{W}}$ is:

$$\lambda(\bar{\mathbf{W}}) = \min_j\{\mu_{Bj}(\bar{w}_B/\bar{w}_j), \mu_{jW}(\bar{w}_j/\bar{w}_W)\} = \lambda(\mathbf{W}^*),$$

since $\mathbf{W}^*$ is assumed to be locally optimal for Model (9). This contradicts the uniqueness of $\mathbf{W}^*$. Thus, $\mathbf{W}^*$ must be locally efficient in $\mathcal{N}(\mathbf{W}^*)$ for Model (7). □

Let $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ be an arbitrary feasible weight vector of Model (7). If $\mathbf{W}^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})^T$ is a locally optimal weight vector in the neighborhood $\mathcal{N}(\mathbf{W}^{**})$ for Model (16), then $\mathbf{W}^{**}$ is a locally efficient weight vector in $\mathcal{N}(\mathbf{W}^{**})$ for Model (7).

Proof. We prove this by contradiction. Assume $\mathbf{W}^{**}$ is locally optimal for Model (16), but not locally efficient for Model (7). Then, by Definition 3.5, there exists a feasible weight vector $\mathbf{W}' = (w_1', w_2', \ldots, w_n')^T \in \mathcal{N}(\mathbf{W}^{**})$ such that:

$$\mu_{Bj}(w_B'/w_j') \geq \mu_{Bj}(w_B^{**}/w_j^{**}), \quad \text{for all } j \neq B,$$
$$\mu_{jW}(w_j'/w_W') \geq \mu_{jW}(w_j^{**}/w_W^{**}), \quad \text{for all } j \neq B, W,$$

with at least one strict inequality. Moreover, since $\mathbf{W}^{**}$ is optimal in Model (16):

$$\mu_{Bj}(w_B^{**}/w_j^{**}) = \lambda_{Bj}^{**} \geq \mu_{Bj}(w_B^*/w_j^*), \quad \text{for all } j \neq B,$$
$$\mu_{jW}(w_j^{**}/w_W^{**}) = \lambda_{jW}^{**} \geq \mu_{jW}(w_j^*/w_W^*), \quad \text{for all } j \neq B, W.$$

Hence, $\mu_{Bj}(w_B'/w_j') \geq \mu_{Bj}(w_B^*/w_j^*)$ and $\mu_{jW}(w_j'/w_W') \geq \mu_{jW}(w_j^*/w_W^*)$ for all j, with at least one strict inequality. So $\mathbf{W}'$ is feasible in Model (16). Summing these yields:

$$y_{\bar{W}}(\mathbf{W}') > y_{\bar{W}}(\mathbf{W}^{**}),$$

contradicting the local optimality of $\mathbf{W}^{**}$ of Model (16). Thus, $\mathbf{W}^{**}$ is the locally efficient for Model (7). □

Theorem 3.10. *Let $\mathbf{W}^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})^T$ be a locally optimal weight vector for Model (9), and $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ be a locally efficient weight vector for Model (7). Then $\mathbf{W}^{**} = \mathbf{W}^*$.*

Proof. At optimality, Model (16) guarantees:

$$\mu_{Bj}(w_B^{**}/w_j^{**}) \geq \mu_{Bj}(w_B^*/w_j^*), \quad \text{for all } j \neq B,$$
$$\mu_{jW}(w_j^{**}/w_W^{**}) \geq \mu_{jW}(w_j^*/w_W^*), \quad \text{for all } j \neq B, W.$$

Since $\mathbf{W}^*$ is locally efficient, no other vector can improve without worsening others, implying equality:

$$\mu_{Bj}(w_B^{**}/w_j^{**}) = \mu_{Bj}(w_B^*/w_j^*), \quad \mu_{jW}(w_j^{**}/w_W^{**}) = \mu_{jW}(w_j^*/w_W^*).$$

By Theorem 3.7, this implies $\mathbf{W}^{**} = \mathbf{W}^*$. □

Corollary 3.11. *Let $\mathbf{W}^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})^T$ be a locally optimal weight vector for Model (16). If $\mathbf{W}^* \neq \mathbf{W}^{**}$, then $\mathbf{W}^*$ is not a locally efficient weight vector for Model (7).*

Theorem 3.12. *Let $\mathbf{W}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ and $\mathbf{W}^{**} = (w_1^{**}, w_2^{**}, \ldots, w_n^{**})^T$ be locally optimal weight vectors for Models (9) and (16), respectively. Then:*

$$\lambda(\mathbf{W}^{**}) \geq \lambda(\mathbf{W}^*), \quad \bar{y}_W(\mathbf{W}^{**}) \geq \bar{y}_W(\mathbf{W}^*).$$

Proof. In Model (9), constraints are:

$$\mu_{Bj}(w_B/w_j) \geq \lambda_{Bj} \quad \text{for all } j \neq B,$$
$$\mu_{jW}(w_j/w_W) \geq \lambda_{jW}, \quad \text{for all } j \neq B, W.$$

These ensure:

$$\bar{y}_W(\mathbf{W}) = \frac{1}{2n-3}\left(\sum_{j\neq B}\mu_{Bj}(w_B/w_j) + \sum_{j\neq B,W}\mu_{jW}(w_j/w_W)\right) \geq \frac{1}{2n-3}\left(\sum_{j\neq B}\lambda_{Bj} + \sum_{j\neq B,W}\lambda_{jW}\right).$$

Model (16) maximizes this expression. Thus, at optimality:

$$\bar{y}_W(\mathbf{W}^{**}) = \frac{1}{2n-3}\left(\sum_{j\neq B}\lambda_{Bj}^{**} + \sum_{j\neq B,W}\lambda_{jW}^{**}\right).$$

With constraints:

$$\mu_{Bj}(w_B^{**}/w_j^{**}) = \lambda_{Bj}^{**} \geq \mu_{Bj}(w_B^*/w_j^*), \quad \mu_{jW}(w_j^{**}/w_W^{**}) = \lambda_{jW}^{**} \geq \mu_{jW}(w_j^*/w_W^*).$$

Hence:

$$\lambda(\mathbf{W}^{**}) \geq \lambda(\mathbf{W}^*), \quad \bar{y}_W(\mathbf{W}^{**}) \geq \bar{y}_W(\mathbf{W}^*).$$

Hence, both the minimum and the mean membership degree achieved by w_j^{**} are at least as good as those obtained by w_j^*. These results confirm that Phase 2 (i.e., Model (16)) effectively refines the weight vector derived in Phase 1 (i.e., Model (9)), leading to improved reliability and stability in decision-making within the BWM-FPP framework. □

Table 5: Prioritization of criteria for Examples 4.1-4.4.

Example	No. of criteria	Best criterion	Worst criterion	C1	C2	C3	C4	C5	C6
E1	6	C1	–	(1,1,1)	(2,3,4)	(4,5,6)	(3,4,5)	(5,6,7)	(7,8,9)
		–	C6	(7,8,9)	(3,4,5)	(1,2,3)	(2,3,4)	(1,2,3)	(1,1,1)
E2	6	C2	–	(2,3,4)	(1,1,1)	(4,5,6)	(3,4,5)	(7,8,9)	(6,7,8)
		–	C5	(2,3,4)	(7,8,9)	(1,2,3)	(2,3,4)	(1,1,1)	(1,2,3)
E3	4	C1	–	(1,1,1)	(1,2,3)	(7,8,9)	(1,2,3)	–	–
		–	C5	(7,8,9)	(3,4,5)	(1,1,1)	(1,2,3)	–	–
E4	5	C2	–	(4,5,6)	(1,1,1)	(3,4,5)	(7,8,9)	(2,3,4)	–
		–	C4	(2,3,4)	(7,8,9)	(2,3,4)	(1,1,1)	(4,5,6)	–

4 Example validation

This section provides several numerical examples to validate the feasibility and effectiveness of the proposed two-phase model. Finally, we make a comparative analysis with the BWM-FPP model and discuss it to illustrate the convenience and benefits of the proposed two-phase model.

4.1 Numerical examples

This section presents four numerical examples to demonstrate the limitations of the BWM-FPP model—specifically Model (9) or its equivalent, Model (11)-in producing locally efficient weight vectors for the multi-objective BWM-FPP framework (Model (7)). These examples serve as counterarguments to the assumption that the BWM-FPP model always yields a robust and efficient solution.

Each example highlights instances where Model (9) fails to produce a weight vector that meets the local efficiency required by Model (7). The examples are constructed based on DMs' preferences, articulated using fuzzy pairwise comparisons in accordance with the linguistic scale of the fuzzy BWM (as detailed in Table 5).

These examples have been carefully selected to underscore the key shortcoming of the traditional BWM-FPP approach: while it may yield a locally weakly efficient solution, it does not necessarily guarantee local efficiency in the objective space. The proposed two-phase model addresses this gap by refining the initial weight vector through an additional optimization phase, thereby improving both membership degrees and consistency. Table 5 presents the fuzzy comparison values used in these four examples, capturing the decision-makers' judgments for both best-to-others and others-to-worst evaluations.

Example 4.1. *The six criteria C1 to C6 are evaluated by an expert. C1 is selected as the best criterion and C6 as the worst criterion.*

Example 4.2. *The six criteria C1 to C6 are evaluated by an expert. C2 is selected as the best criterion and C5 as the worst criterion.*

Example 4.3. *(Dehshiri et al.* [9]*). The four criteria C1 to C4 are evaluated by an expert. C1 is selected as the best criterion and C3 as the worst criterion.*

Example 4.4. *(Amiri et al.* [1]*). The five criteria C1 to C5 are evaluated by an expert. C2 is chosen as the best criterion and C4 as the worst criterion.*

Phase 1: Maximization of the minimum membership degree (Model (9))

In this phase, the goal is to determine a weight vector by maximizing the minimum membership degree, thereby ensuring a baseline level of consistency across all fuzzy comparisons. This process is demonstrated through the full modeling of Example **??**E1. The fuzzy constraints in Model (9), based on the comparisons from Example 4.1, are as follows:

$$
\begin{aligned}
&\max \lambda \\
&\text{s.t.} \\
&-w_1+2w_2+\lambda w_2 \le 0; w_1-4w_2+\lambda w_2 \le 0; -w_1+4w_3+\lambda w_3 \le 0; w_1-6w_3+\lambda w_3 \le 0; \\
&-w_1+3w_4+\lambda w_4 \le 0; w_1-5w_4+\lambda w_4 \le 0; -w_1+5w_5+\lambda w_5 \le 0; w_1-7w_5+\lambda w_5 \le 0; \\
&-w_1+7w_6+\lambda w_6 \le 0; w_1-9w_6+\lambda w_6 \le 0; -w_2+3w_6+\lambda w_6 \le 0; w_2-5w_6+\lambda w_6 \le 0; \\
&-w_3+1w_6+\lambda w_6 \le 0; w_3-3w_6+\lambda w_6 \le 0; -w_4+2w_6+\lambda w_6 \le 0; w_4-4w_6+\lambda w_6 \le 0; \\
&-w_5+1w_6+\lambda w_6 \le 0; w_5-3w_6+\lambda w_6 \le 0; \\
&\sum_{j=1}^{6} w_j = 1, \quad w_j \ge 0, \quad \lambda \text{ free variable}, \quad j=1,2,\ldots,6.
\end{aligned}
$$

Solving this model for each example produces the optimal weights by Lingo 11, as presented in Table 6.

Table 6: Optimal weights of criteria determined by Amiri's model (Model (9)) for Examples 4.1-4.4.

Example	λ	w_1^*	w_2^*	w_3^*	w_4^*	w_5^*	w_6^*
E1	0.46410	0.46215	0.18755	0.08348	0.13341	0.07927	0.05414
E2	0.18968	0.10327	0.45015	0.13402	0.05209	0.07079	0.18968
E3	0.29844	0.51044	0.23069	0.06994	0.18894	–	–
E4	0.14005	0.11707	0.48468	0.11707	0.05470	0.22648	–

These results represent the initial weight vectors derived by maximizing the minimum membership degree, serving as input for Phase 2. To transition into the second phase, the ratios w_B^*/w_j^* and w_j^*/w_W^* are computed and compared with the fuzzy thresholds m_{Bj} and m_{jW}, respectively. Based on these comparisons, appropriate constraints are selected from the predefined cases (Expressions 12–15) to formulate Model (16). This leads into Phase 2: the maximization of the mean membership degree, aimed at improving overall reliability while preserving the consistency of Phase 1.

Phase 2: Maximization of the mean membership degree (Model (16))

In this phase, the weight vector obtained from Phase 1 is further refined by maximizing the mean membership degree, denoted as $\bar{y}_W$. This optimization aims to improve the global consistency of the derived weights with the fuzzy preference relations. The constraints are selected based on the outcomes from Phase 1 and Expressions 12–15.

$$\max\ \bar{y}_W = \frac{\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{16}+\lambda_{26}+\lambda_{36}+\lambda_{46}+\lambda_{56}}{9}$$

s.t.

$$\frac{\left(\frac{w_1}{w_2}\right)-2}{1} \geq \lambda_{12} \geq \frac{\left(\frac{0.46215}{0.18755}\right)-2}{1}; w_1-3w_2 \leq 0$$

$$\frac{6-\left(\frac{w_1}{w_3}\right)}{1} \geq \lambda_{13} \geq \frac{6-\left(\frac{0.46215}{0.08348}\right)}{1}; w_1-5w_3 \geq 0$$

$$\frac{\left(\frac{w_1}{w_4}\right)-3}{1} \geq \lambda_{14} \geq \frac{\left(\frac{0.46215}{0.13341}\right)-3}{1}; w_1-4w_4 \leq 0$$

$$\frac{\left(\frac{w_1}{w_5}\right)-5}{1} \geq \lambda_{15} \geq \frac{\left(\frac{0.46215}{0.07927}\right)-5}{1}; w_1-6w_5 \leq 0$$

$$\frac{9-\left(\frac{w_1}{w_6}\right)}{1} \geq \lambda_{16} \geq \frac{9-\left(\frac{0.46215}{0.05414}\right)}{1}; w_1-8w_6 \geq 0$$

$$\frac{\left(\frac{w_2}{w_6}\right)-3}{1} \geq \lambda_{26} \geq \frac{\left(\frac{0.18755}{0.05414}\right)-3}{1}; w_2-4w_6 \leq 0$$

$$\frac{\left(\frac{w_3}{w_6}\right)-1}{1} \geq \lambda_{36} \geq \frac{\left(\frac{0.08348}{0.05414}\right)-1}{1}; w_3-2w_6 \leq 0$$

$$\frac{\left(\frac{w_4}{w_6}\right)-2}{1} \geq \lambda_{46} \geq \frac{\left(\frac{0.13341}{0.05414}\right)-2}{1}; w_4-3w_6 \leq 0$$

$$\frac{\left(\frac{w_5}{w_6}\right)-1}{1} \geq \lambda_{56} \geq \frac{\left(\frac{0.07927}{0.05414}\right)-1}{1}; w_5-2w_6 \leq 0$$

$$\sum_{j=1}^{6} w_j = 1, \quad w_j \geq 0, j=1,2,\cdots,n, \quad \lambda_{Bj} \text{ (for } j \neq B), \ \lambda_{jW} \text{ (for } j \neq B, W) \text{ free variables.}$$

Table 7: Optimal weights of criteria determined by the proposed model (Model (16)) for Examples 4.1-4.4.

Example 4.1		Example 4.2		Example 4.3		Example 4.4	
Variable	Value	Variable	Value	Variable	Value	Variable	Value
λ_{12}	0.46414	λ_{21}	0.88059	λ_{12}	1	λ_{21}	1
λ_{13}	1	λ_{23}	0.35896	λ_{13}	0.29826	λ_{23}	1
λ_{14}	0.46413	λ_{24}	0.35883	λ_{14}	0.29840	λ_{24}	0.13931
λ_{15}	0.83007	λ_{25}	0.35823	λ_{23}	0.64913	λ_{25}	0.14006
λ_{16}	0.46380	λ_{26}	0.35895	λ_{43}	0.29854	λ_{14}	0.21517
λ_{26}	0.46417	λ_{15}	1	–	–	λ_{34}	0.21517
λ_{36}	0.70724	λ_{35}	0.98253	–	–	λ_{54}	0.14040
λ_{46}	0.46417	λ_{45}	0.57285	–	–	–	–
λ_{56}	0.46417	λ_{65}	0.35899	–	–	–	–
w_1^{**}	0.45805	w_1^{**}	0.16168	w_1^{**}	0.49821	w_1^{**}	0.12018
w_2^{**}	0.18589	w_2^{**}	0.46570	w_2^{**}	0.24911	w_2^{**}	0.48074
w_3^{**}	0.13223	w_3^{**}	0.10684	w_3^{**}	0.06826	w_3^{**}	0.12018
w_4^{**}	0.07857	w_4^{**}	0.13865	w_4^{**}	0.18441	w_4^{**}	0.05426
w_5^{**}	0.09161	w_5^{**}	0.05389	–	–	w_5^{**}	0.22464
w_6^{**}	0.05366	w_6^{**}	0.07324	–	–	–	–

4.1.1 Evaluation of the quality of the derived weights

To evaluate the quality of the derived weights, membership degrees are computed based on how closely the weight ratios conform to the fuzzy reference comparisons. The membership degree for $\frac{w_B}{w_j}$ is defined as:

$$y_{Bj}(\frac{w_B}{w_j}) = \begin{cases} \frac{\frac{w_B}{w_j} - l_{Bj}}{m_{Bj} - l_{Bj}}, & \text{if } \frac{w_B}{w_j} \leq m_{Bj}, \quad j = 1, 2, \ldots, n, \ (j \neq B), \\ \\ \frac{u_{Bj} - \frac{w_B}{w_j}}{u_{Bj} - m_{Bj}}, & \text{if } \frac{w_B}{w_j} > m_{Bj}, \quad j = 1, 2, \ldots, n, \ (j \neq B), \end{cases} \tag{17}$$

Similarly, the membership degree for $\frac{w_j}{w_W}$ is:

$$y_{jW}(\frac{w_j}{w_W}) = \begin{cases} \frac{\frac{w_j}{w_W} - l_{jW}}{m_{jW} - l_{jW}}, & \text{if } \frac{w_j}{w_W} \leq m_{jW}, \quad j = 1, 2, \ldots, n, j \neq B, W), \\ \\ \frac{u_{jW} - \frac{w_j}{w_W}}{u_{jW} - m_{jW}}, & \text{if } \frac{w_j}{w_W} > m_{jW}, \quad j = 1, 2, \ldots, n, j \neq B, W). \end{cases} \tag{18}$$

The mean membership degree $\bar{y}_W$ is calculated by:

$$\bar{y}_W = \frac{1}{2n-3} \left(\sum_{\substack{j=1 \\ j \neq B}}^{n} \mu_{Bj} \left(\frac{w_B}{w_j} \right) + \sum_{\substack{j=1 \\ j \neq B, W}}^{n} \mu_{jW} \left(\frac{w_j}{w_W} \right) \right). \tag{19}$$

A higher $\bar{y}_W$ value reflects improved global consistency in satisfying the fuzzy preferences. To compare the performance of the BWM-FPP model (Model (9)) with the proposed model (Model (16)), we compute the relative change (RC) in weights and membership degrees as follows:

$$\text{RC}(w_j^{**}) = \frac{w_j^{**} - w_j^*}{w_j^*}, \tag{20}$$

$$\text{RC}(y_{Bj}^{**}) = \frac{y_{Bj}^{**} - y_{Bj}^*}{y_{Bj}^*}, \tag{21}$$

$$\text{RC}(y_{jW}^{**}) = \frac{y_{jW}^{**} - y_{jW}^*}{y_{jW}^*}, \tag{22}$$

$$\text{RC}(\bar{y}_W^{**}) = \frac{\bar{y}_W^{**} - \bar{y}_W^*}{\bar{y}_W^*}, \tag{23}$$

Table 8 presents the relative changes in the calculated weights based on Equation 20. The comparison between the weights from the BWM-FPP model (Model (9), Table 6) and the proposed model (Model (16), Table 7) highlights several key observations:

- **Notable differences in weights:** There are significant differences in the weights calculated by the two models, suggesting that the proposed two-phase model, which integrates the BWM and FPP approaches, produces more refined and potentially more accurate weight values.
- **Impact of the proposed model:** The proposed approach better adjusts weight distributions, reflecting the decision-making context more accurately.

In Example 4.1, the weights derived from the BWM-FPP model (Model (9)) are:

$$w_1^* = 0.46215, \quad w_2^* = 0.18755, \quad w_3^* = 0.08348, \quad w_4^* = 0.13341, \quad w_5^* = 0.07927, \quad w_6^* = 0.05414.$$

The proposed model (Model (16)) produces the following weights:

$$w_1^{**} = 0.45805, \quad w_2^{**} = 0.18589, \quad w_3^{**} = 0.13223, \quad w_4^{**} = 0.07857, \quad w_5^{**} = 0.09161, \quad w_6^{**} = 0.05366.$$

The corresponding relative changes in weights are:

$$\begin{aligned} &\text{RC}(w_1^{**}) = -0.00887, \quad \text{RC}(w_2^{**}) = -0.00885, \quad \text{RC}(w_3^{**}) = 0.58397, \\ &\text{RC}(w_4^{**}) = -0.41106, \quad \text{RC}(w_5^{**}) = 0.15567, \quad \text{RC}(w_6^{**}) = -0.00887. \end{aligned}$$

Table 8: Optimal weights, ranking of criteria and relative changes-Examples 4.1-4.4.

Example	Model	w_1	w_2	w_3	w_4	w_5	w_6	$\overline{\mathrm{RC}}(W^{**})$
E1	Amiri's model	0.46215	0.18755	0.08348	0.13341	0.07927	0.05414	
	Rank	1	2	4	3	5	6	
	Proposed model	0.45805	0.18589	0.13223	0.07857	0.09161	0.05366	
	Rank	1	2	3	5	4	6	
	$\mathrm{RC}(w_j^{**})$	-0.00887	-0.00885	0.58397	-0.41106	0.15567	-0.00887	0.19622
E2	Amiri's model	0.18968	0.45015	0.10327	0.13402	0.05209	0.07079	
	Rank	2	1	4	3	6	5	
	Proposed model	0.16168	0.46570	0.10684	0.13865	0.05389	0.07324	
	Rank	2	1	4	3	6	5	
	$\mathrm{RC}(w_j^{**})$	-0.14762	0.03454	0.03457	0.03455	0.03456	0.03461	0.05341
E3	Amiri's model	0.51044	0.23069	0.06994	0.18894	–	–	
	Rank	1	2	4	3	–	–	
	Proposed model	0.49821	0.24911	0.06826	0.18441	–	–	
	Rank	1	2	4	3	–	–	
	$\mathrm{RC}(w_j^{**})$	-0.02396	0.07985	-0.02402	-0.02398	–	–	0.03795
E4	Amiri's model	0.11707	0.48468	0.11707	0.05470	0.22648	–	
	Rank	3	1	3	5	2	–	
	Proposed model	0.12018	0.48074	0.12018	0.05426	0.22464	–	
	Rank	3	1	3	5	2	–	
	$\mathrm{RC}(w_j^{**})$	0.02657	-0.00813	0.02657	-0.00804	-0.00812	–	0.01549

The mean relative change is:

$$\overline{\mathrm{RC}}(W^{**}) = \frac{1}{6}\sum_{j=1}^{6}\left|\mathrm{RC}(w_j^{**})\right| = 0.19622.$$

Also,

- w_1 consistently ranks as the most important criterion in both models (0.46215 in BWM-FPP, 0.45805 in the proposed model).
- w_2 retains its second-place rank.
- w_3 rises from Rank 4 to Rank 3, with a weight increase from 0.08348 to 0.13223.
- w_4 drops from Rank 3 to Rank 5, decreasing from 0.13341 to 0.07857.
- w_5 moves from Rank 5 to Rank 4, increasing from 0.07927 to 0.09161.
- w_6 remains ranked 6th in both models.

These shifts in rankings and weight distributions demonstrate the effectiveness of the proposed model in better reflecting the decision-making context, reprioritizing the criteria to emphasize more critical factors while reducing the influence of less significant ones.

Table 8 also includes additional examples (E2, E3, and E4), each demonstrating how the proposed model adjusts weights and rankings relative to the BWM-FPP model. Noteworthy observations include:

- In Example 4.2, the RC values are relatively small, suggesting minimal changes in the weights, with a mean relative change of 0.05341.
- In Example 4.3, RC values indicate noticeable shifts, such as a -0.02396 change for w_1 and a 0.07985 increase for w_2, showing that the proposed model provides a finer granularity in adjusting weights.
- Example 4.4 highlights similar trends, with RC values showing relatively minor shifts, except for w_1 and w_2, which show modest changes of 0.02657 and -0.00813, respectively.

Table 9: Comparison of membership degrees and their relative changes for Example 4.1

Formula for calculating membership degrees	Amiri's model Model (9)	Proposed model Model (16)	Comparison	RC
$y_{12} = \left(\frac{w_1}{w_2} - 2\right)/1$	0.46414	0.46414	Similar	0
$y_{13} = \left(6 - \frac{w_1}{w_3}\right)/1$	0.46394	1.00000	Improved	1.15545
$y_{14} = \left(\frac{w_1}{w_4} - 3\right)/1$	0.46413	0.46413	Similar	0
$y_{15} = \left(\frac{w_1}{w_5} - 5\right)/1$	0.83007	0.83007	Similar	0
$y_{16} = \left(9 - \frac{w_1}{w_6}\right)/1$	0.46380	0.46380	Similar	0
$y_{26} = \left(\frac{w_2}{w_6} - 3\right)/1$	0.46417	0.46417	Similar	0
$y_{36} = \left(\frac{w_3}{w_6} - 1\right)/1$	0.54193	0.70724	Improved	0.30504
$y_{46} = \left(\frac{w_4}{w_6} - 2\right)/1$	0.46417	0.46417	Similar	0
$y_{56} = \left(\frac{w_5}{w_6} - 1\right)/1$	0.46417	0.46417	Similar	0
Mean ($\bar{y}_W$)	0.51339	0.59132	Improved	0.15179

Table 10: Comparison of membership degrees and their relative changes for Example 4.2

Formula for calculating membership degrees	Amiri's model Model (9)	Proposed model Model (16)	Comparison	RC
$y_{21} = \left(\frac{w_2}{w_1} - 2\right)/1$	0.37321	0.88059	Improved	1.35950
$y_{23} = \left(\frac{w_2}{w_3} - 4\right)/1$	0.35896	0.35896	Similar	0
$y_{24} = \left(\frac{w_2}{w_4} - 3\right)/1$	0.35883	0.35883	Similar	0
$y_{25} = \left(9 - \frac{w_2}{w_5}\right)/1$	0.35823	0.35823	Similar	0
$y_{26} = \left(\frac{w_2}{w_6} - 6\right)/1$	0.35895	0.35895	Similar	0
$y_{15} = \left(4 - \frac{w_1}{w_5}\right)/1$	0.35861	1.00000	Improved	1.78854
$y_{35} = \left(\frac{w_3}{w_5} - 1\right)/1$	0.98253	0.98253	Similar	0
$y_{45} = \left(\frac{w_4}{w_5} - 2\right)/1$	0.57285	0.57285	Similar	0
$y_{65} = \left(\frac{w_6}{w_5} - 1\right)/1$	0.35899	0.35899	Similar	0
Mean ($\bar{y}_W$)	0.45346	0.58110	Improved	0.28148

Tables 9 to 12 present a comparison of the results obtained from the proposed two-phase model and the BWM-FPP model for Examples 4.1 through 4.4. For both models, the membership degrees and the relative changes in y_{Bj}, y_{jW}, and $\bar{y}_W$ are calculated with a precision of up to five decimal places. The findings indicate that the proposed two-phase model consistently produces more efficient solutions, thereby outperforming the BWM-FPP model.

The relative change results for Example 1 are illustrated in Figure 7, based on the data provided in Tables 8 and 9. In these figures, positive RC values are highlighted in blue, while negative RC values are indicated in red. RC values equal to zero are omitted from the diagrams to enhance visual clarity. This color-coding scheme is consistently applied across all examples. Specifically, the RC outcomes for Example 4.2 are derived from Tables 8 and 10 and visualized in Figure 8. Similarly, Example 4.3 results are drawn from Tables 8 and 11, and shown in Figure 9, while the findings for Example 4.4 are presented in Tables 8 and 12, and depicted in Figure 10.

Table 11: Comparison of membership degrees and their relative changes for Example 4.3

Formula for calculating membership degrees	Amiri's model Model (9)	Proposed model Model (16)	Comparison	RC
$y_{12} = \left(3 - \frac{w_1}{w_2}\right)/1$	0.78733	1.00000	Improved	0.27012
$y_{13} = \left(\frac{w_1}{w_3} - 7\right)/1$	0.29826	0.29826	Similar	0
$y_{14} = \left(3 - \frac{w_1}{w_4}\right)/1$	0.29840	0.29840	Similar	0
$y_{23} = \left(\frac{w_2}{w_3} - 3\right)/1$	0.29840	0.64913	Improved	1.17537
$y_{43} = \left(3 - \frac{w_4}{w_3}\right)/1$	0.29854	0.29854	Similar	0
Mean $(\bar{y}_W)$	0.39619	0.50887	Improved	0.28441

Table 12: Comparison of membership degrees and their relative changes for Example 4.4.

Formula for calculating membership degrees	Amiri's model Model (9)	Proposed model Model (16)	Comparison	RC
$y_{21} = \left(5 - \frac{w_2}{w_1}\right)/1$	0.85991	1.00000	Improved	0.16291
$y_{23} = \left(5 - \frac{w_2}{w_3}\right)/1$	0.85991	1.00000	Improved	0.16291
$y_{24} = \left(9 - \frac{w_2}{w_4}\right)/1$	0.13931	0.13931	Similar	0
$y_{25} = \left(\frac{w_2}{w_5} - 2\right)/1$	0.14006	0.14006	Similar	0
$y_{14} = \left(\frac{w_1}{w_4} - 2\right)/1$	0.14022	0.21517	Improved	0.53452
$y_{34} = \left(\frac{w_3}{w_4} - 2\right)/1$	0.14022	0.21517	Improved	0.53452
$y_{54} = \left(\frac{w_5}{w_4} - 4\right)/1$	0.14040	0.14040	Similar	0
Mean $(\bar{y}_W)$	0.34572	0.40716	Improved	0.17772

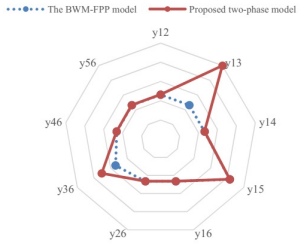


Figure 2: Comparing for Example 4.1.

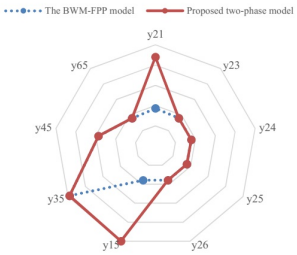


Figure 3: Comparing for Example 4.2.

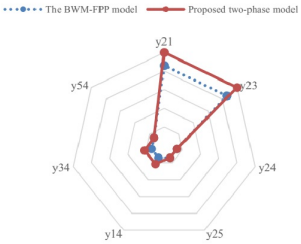


Figure 4: Comparing for Example 4.3.

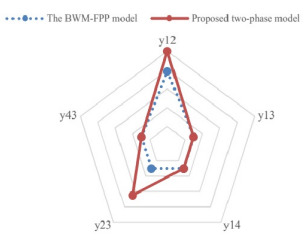


Figure 5: Comparing for Example 4.4.

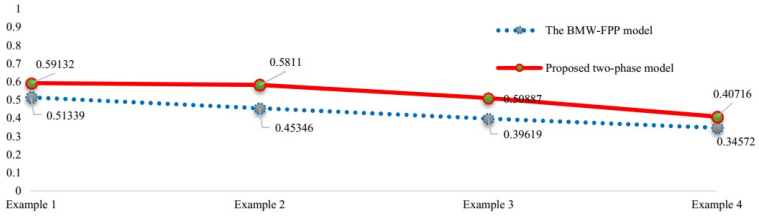


Figure 6: Comparing the mean membership degree-Examples 4.1-4.4.

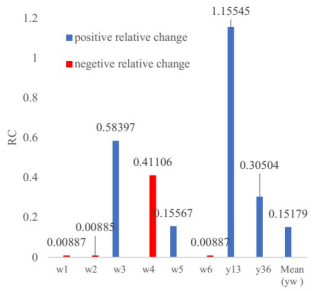


Figure 7: Relative changes for Example 4.1.

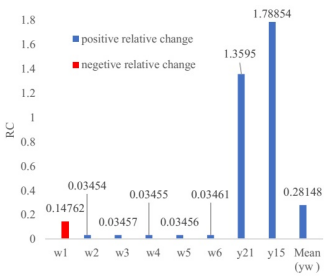


Figure 8: Relative changes for Example 4.2.

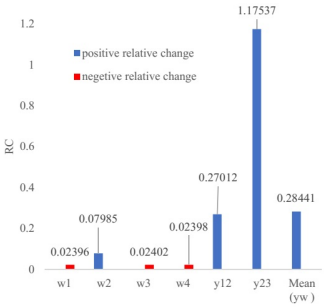


Figure 9: Relative changes for Example 4.3.

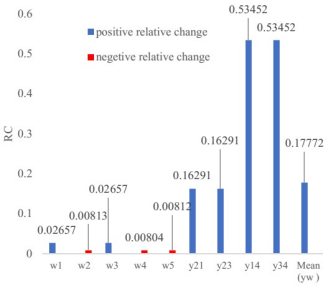


Figure 10: Relative changes for Example 4.4.

4.2 Theoretical interpretation and analysis of Examples 4.1–4.4

To reinforce the theoretical foundation and validity of the proposed two-phase fuzzy model, Examples 4.1 through 4.4 are analyzed with reference to Definitions 3.2–3.6 and Theorems 3.7–3.12. These definitions and theorems together provide a rigorous framework for evaluating the quality, efficiency, and consistency of weight vectors derived in fuzzy MADM environments.

In Example 4.1, the initial weight vector W^* derived from the BWM-FPP model (Model (9)) satisfies the condition of local weak efficiency as defined in Definition 3.4 and formally supported by Theorem 3.8. However, it is not locally efficient, as the refined vector W^{**}, obtained from Model (16), strictly improves at least one membership degree—specifically y_{13}, which increases from 0.46394 to 1.00000-without deteriorating any others. This dominance relationship satisfies Definition 3.2 and confirms that W^{**} is more reliable than W^* (Definition 3.3). Furthermore, W^{**} qualifies as a locally efficient weight vector under Definition 3.5, in agreement with Theorem 3.4. The increase in the mean membership degree from 0.51339 to 0.59132 (a relative change of 15.18%) aligns with Definition 3.6 and confirms the theoretical guarantee of improved consistency as stated in Theorem 3.12.

In Example 4.2, the initial solution W^* also meets the local weak efficiency condition but not local efficiency, as it is further refined by W^{**} through improvements in membership degrees such as y_{21} (from 0.37321 to 0.88059) and y_{15} (from 0.35861 to 1.00000). The strict improvement in these values, with no trade-offs, satisfies the dominance criteria (Definition 3.2) and establishes the reliability of the refined vector (Definition 3.3). As $W^* \neq W^{**}$, Corollary 3.11 derived from Theorem 3.10 implies the inefficiency of the original vector. The enhanced solution W^{**} thus satisfies local efficiency under Theorem 3.4, and the significant increase in the mean membership degree—from 0.45346 to 0.58110 (a relative change of 28.15%)—confirms the benefit predicted by Theorem 3.12 and supports Definition 3.6.

In Example 4.3, the initial vector W^* again qualifies as a locally weakly efficient solution (Definition 3.4, Theorem 3.8), but not as a locally efficient one. The refined vector W^{**} exhibits improvements such as in y_{24}, which rises from 0.29840 to 0.64913. These improvements are consistent with the dominance condition (Definition 3.2) and imply increased reliability (Definition 3.3). The relative change in the mean membership degree—from 0.39619 to 0.50887 (an increase of 28.44%)—aligns with the implications of Definition 3.6 and Theorem 3.12. Since $W^* \neq W^{**}$, Theorem 3.10 confirms the local inefficiency of the initial vector, while the refined one satisfies local efficiency according to Theorem 3.4.

In Example 4.4, the initial solution W^* is close to local efficiency, but measurable gains are still achieved through the proposed model. Membership degrees such as y_{14} and y_{34} are both improved by 0.53452 in the refined vector W^{**}. While the magnitude of improvement is more moderate compared to earlier examples, the dominance and reliability conditions (Definitions 3.2 and 3.3) are still met. The rise in the mean membership degree from 0.34572 to 0.40716 (a relative change of 17.77%) illustrates enhanced consistency as outlined in Definition 3.6 and Theorem 3.12. Moreover, since $W^* \neq W^{**}$, Theorem 3.10 confirms that the original solution is not locally efficient, whereas W^{**} satisfies the conditions of Theorem 3.4.

In summary, the theoretical interpretation and analysis of Examples 4.1-4.4 confirm that the proposed two-phase model consistently transforms locally weakly efficient weight vectors into locally efficient and more reliable ones. The empirical results observed in all four cases are in full agreement with the foundational definitions and theorems, thereby validating the robustness and consistency of the model. These findings underscore the model's effectiveness in improving the quality of decision-making outcomes under uncertainty in fuzzy MADM contexts.

4.3 Analysis of the difference between two weight vectors and their rank components

Beyond the numerical comparisons, this section strengthens the validation of the proposed two-phase model through statistical analysis employing three widely accepted evaluation metrics: Euclidean distance analysis, Spearman's rank correlation coefficient, and Garuti's compatibility index (G Index). These measures are used to assess the consistency, ranking similarity, and compatibility between the outcomes of the proposed model and those of the BWM-FPP model.

To demonstrate the application of these statistical tools, detailed computations are provided for Example 4.1, offering insight into the methodology and interpretation of results. Subsequently, the summarized outcomes for Examples 4.2 through 4.4 are presented in Table 14, reinforcing the model's robustness and superior performance across a broader set of decision scenarios.

4.3.1 Euclidean distance analysis

Euclidean distance (denoted as d_E) is a widely used metric in MCDM and related disciplines for measuring the dissimilarity between two vectors—typically, weight vectors generated by different decision-making models. As emphasized by Duleba and Szádoczki [12], this metric provides a direct and intuitive means of evaluating how closely the outputs of two models align.

Given two weight vectors: $W^{(1)} = (w_1^{(1)}, w_2^{(1)}, \ldots, w_n^{(1)})$ and $W^{(2)} = (w_1^{(2)}, w_2^{(2)}, \ldots, w_n^{(2)})$, the Euclidean distance is calculated using the following formula:

$$d_E\left(W^{(1)}, W^{(2)}\right) = \sqrt{\sum_{i=1}^{n}\left(w_i^{(1)} - w_i^{(2)}\right)^2}, \tag{24}$$

In Example 4.1, let the weight vectors for Model (9) (BWM-FPP) and Model (16) (proposed model) be:

$$W^{(1)} = (0.46215, 0.18755, 0.08348, 0.13341, 0.07927, 0.05414),$$

$$W^{(2)} = (0.45805, 0.18589, 0.13223, 0.07857, 0.09161, 0.05366),$$

$$d_E\left(W^{(1)}, W^{(2)}\right) = 0.07457.$$

This value indicates a moderate but meaningful difference in weight distributions between the two models.

4.3.2 Spearman's rank correlation coefficient

Spearman's rank correlation coefficient (ρ) quantifies the strength and direction of a monotonic relationship between two ranked variables. In this context, it evaluates the similarity between the rankings derived from the BWM-FPP model and the proposed model. Following Duleba and Szádoczki [12], the coefficient is computed as:

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n^3 - n}, \tag{25}$$

where d_i is the difference in rank for criterion i. The value $\rho = 1$ indicates perfect agreement, while $\rho = 0$ suggests no correlation. In Example 4.1, Spearman's rank correlation coefficient (ρ) is calculated as:

- Rank differences: $d = [0, 0, 1, -2, 1, 0]$,
- Squared differences: $d^2 = [0, 0, 1, 4, 1, 0]$,
- $\sum_{i=1}^{6} d_i^2 = 6$,
- $\rho = 1 - \frac{6\times 6}{6^3 - 6} = 1 - 0.1714 = 0.8286$.

This result reflects a strong, albeit not perfect, correlation between the rankings of the two models. The rankings of criteria are derived from the normalized simulated preference vectors presented in Table 8. For the BWM-FPP model, w_1 holds the 1st rank, and w_6 ranks last. Similarly, in the proposed model, w_1 also retains the 1st position, and w_6 remains the least important. However, some discrepancies are observed: for instance, w_4 drops from 3rd (the BWM-FPP model) to 5th (proposed model), while w_5 improves from 5th to 4th. These shifts highlight variations in the prioritization logic of the two approaches and emphasize the necessity of correlation measures such as Spearman's ρ to evaluate the overall alignment in rankings.

Table 13: Calculation of G index for Example 4.1.

Criterion	$\min(w_i^{(1)}, w_i^{(2)})$	$\max(w_i^{(1)}, w_i^{(2)})$	Ratio	$(w_i^{(1)} + w_i^{(2)})$	Product
C_1	0.45805	0.46215	0.9911	$0.46215 + 0.45805 = 0.9202$	$0.9911 \times 0.9202 = 0.9119$
C_2	0.18589	0.18755	0.9911	$0.18755 + 0.18589 = 0.37344$	$0.9911 \times 0.37344 = 0.3698$
C_3	0.08348	0.13223	0.6313	$0.08348 + 0.13223 = 0.21571$	$0.6313 \times 0.21571 = 0.1361$
C_4	0.07857	0.13341	0.5889	$0.13341 + 0.07857 = 0.21198$	$0.5889 \times 0.21198 = 0.1248$
C_5	0.07927	0.09161	0.8653	$0.07927 + 0.09161 = 0.17088$	$0.8653 \times 0.17088 = 0.1480$
C_6	0.05366	0.05414	0.9911	$0.05414 + 0.05366 = 0.10780$	$0.9911 \times 0.10780 = 0.1068$

4.3.3 Garuti's compatibility index

Garuti's compatibility index (G index) is a metric used to measure the compatibility between two vectors, taking into account both their values and rankings. As defined by Garuti [15] in his work on compatibility indices for evaluating the alignment of individual preferences with aggregated group preferences, the G index is applicable to two preference vectors and calculated using as:

$$G = \frac{1}{2} \sum_{i=1}^{n} \frac{\min\left(w_i^{(1)}, w_i^{(2)}\right)}{\max\left(w_i^{(1)}, w_i^{(2)}\right)} \cdot \left(w_i^{(1)} + w_i^{(2)}\right), \tag{26}$$

where:

- $w_i^{(1)}$: Weight of the i-th criterion in the BWM-FPP model (Model (9)) in Table 8.
- $w_i^{(2)}$: Weight of the i-th criterion in the proposed model (Model (16)) in Table 8.
- n: Number of criteria.

In Example 4.1, we calculate the G index for small-scale examples ($n = 6$) derived from normalized simulated preference vectors. The data is referenced from Table 8. We will calculate the G index for each of the four examples using the formula.

In Example 4.1, G index is calculated as:

- **Step 1.** Calculate $\min\left(w_i^{(1)}, w_i^{(2)}\right)$, $\max\left(w_i^{(1)}, w_i^{(2)}\right)$, the ratio $\min\left(w_i^{(1)}, w_i^{(2)}\right) / \max\left(w_i^{(1)}, w_i^{(2)}\right)$, the sum $\left(w_i^{(1)} + w_i^{(2)}\right)$, and the product of the ratio. For each criterion, the calculations are summarized in Table 13.
- **Step 2.** (Sum all products) Sum $= 0.9119 + 0.3698 + 0.1361 + 0.1248 + 0.1480 + 0.1068 = 1.7974$.
- **Step 3.** (Calculate G index) $G = \frac{1}{2} \times 1.7974 = 0.8987$.

To provide a clear and consolidated overview, the results for all three metrics-Euclidean distance (d_E), Spearman's rank correlation coefficient (ρ), and G index -are summarized in Table 14.

- **Euclidean distance analysis:** The smallest value (0.0062 in Example 4.4) indicates the highest similarity between the two models. A decreasing trend in distance values demonstrates increasing convergence between model outputs.
- **Spearman's rank correlation coefficient:** Perfect agreement is achieved in Examples 4.2-4.4, while Example 4.1 maintains a high correlation (0.8286), suggesting consistency in ranking logic with slight variations.
- **G index:** All values indicate high compatibility, with Example 4.3 showing the highest alignment (0.9807). Even the lowest observed G value (0.8812) denotes substantial agreement.

These findings confirm the robustness of the proposed two-phase model, which closely aligns with the established BWM-FPP model. The detailed example enhances methodological transparency, while the summarized statistics collectively underscore the proposed model's consistency, reliability, and validity across diverse scenarios.

Table 14: Results for Euclidean distance (d_E), Spearman's rank correlation coefficient (ρ), and G index.

Example	Euclidean distance (d_E)	Spearman's rank correlation coefficient (ρ)	G index
E1	0.07457	0.8286	0.8987
E2	0.03269	1.0000	0.8812
E3	0.02263	1.0000	0.9807
E4	0.00620	1.0000	0.9737

Table 15: Quality elements of the automotive after-sales services.

Main criteria (SERVQUAL dimensions)		Sub-criteria (Quality elements)	
C1	Tangibles	C11	Availability of information and solutions at service centers
		C12	Proximity of service centers
		C13	Modern equipment and fixtures
		C14	Convenient operating hours
C2	Reliability	C21	Provision of service as promised
		C22	Availability of spare parts during service calls
		C23	Availability of technical services staff
		C24	Consistency in service quality
		C25	Good customer service during the warranty period
C3	Responsiveness	C31	Immediate defect detection
		C32	Time spent on servicing
		C33	Time to resolution of complaints
		C34	Immediate service provision to customers by store personnel
		C35	Responsiveness to customer complaints
C4	Assurance	C41	Competency and experience of personnel
		C42	General attitude and behavior of technician
		C43	Customer handling
		C44	Professionalism of service personnel
		C45	Interpersonal behavior of service personnel
C5	Empathy	C51	Individuated interactions between service front liners and customers
		C52	Availability of service personnel

5 A case study in the automotive after-sales services

5.1 Problem context and criteria identification

To demonstrate the practical applicability and effectiveness of the proposed two-phase BWM-FPP model, this section presents a real-world case study in the context of the automotive after-sales service industry. The objective is to evaluate and prioritize key service quality elements that influence customer satisfaction, based on expert judgment and the SERVQUAL framework.

This study incorporates 21 quality elements previously identified by Shokouhyar et al. [35] as critical to customer satisfaction in after-sales services. These elements are systematically categorized under the five SERVQUAL dimensions: Tangibles, Reliability, Responsiveness, Assurance, and Empathy (see Table 15).

5.2 Data collection and expert evaluation

A panel of three experienced DMs, including the service manager, quality supervisor, and customer relations officer, participated in the evaluation process. Using linguistic terms from Table 4, the DMs provided pairwise comparisons between the best and worst criteria and the remaining criteria. These linguistic assessments are then converted into triangular fuzzy numbers.

Table 16: Optimal weights of main criteria determined by Model (9).

Main criteria		Optimal weights of criteria		
		DM1	DM2	DM3
C1	Tangibles	0.15928	0.06481	0.13964
C2	Reliability	0.50368	0.57415	0.57809
C3	Responsiveness	0.07367	0.08370	0.13964
C4	Assurance	0.15928	0.13868	0.07741
C5	Empathy	0.10412	0.13868	0.06525

Table 17: Optimal weights of sub-criteria determined by Model (9).

Main Criteria		Sub-criteria		Optimal weights of the sub-criteria		
				DM1	DM2	DM3
C1	Tangibles	C11	Availability of information and solutions at service centers	0.22105	0.04966	0.54895
		C12	Proximity of service centers	0.07790	0.43992	0.13260
		C13	Modern equipment and fixtures	0.16844	0.20557	0.06196
		C14	Convenient operating hours	0.53263	0.30487	0.25651
C2	Reliability	C21	Provision of service as promised	0.31426	0.15794	0.45097
		C22	Availability of spare parts during service calls	0.36525	0.52380	0.23585
		C23	Availability of technical services staff	0.05342	0.06818	0.13019
		C24	Consistency in service quality	0.11551	0.15794	0.05284
		C25	Good customer service during the warranty period	0.15159	0.09217	0.13019
C3	Responsiveness	C31	Immediate identification of defects	0.08655	0.06181	0.06181
		C32	Time spent on servicing	0.05276	0.17697	0.17697
		C33	Time to resolution of complaints	0.40531	0.47488	0.47488
		C34	The store employee gives prompt service to customers	0.17496	0.14318	0.14318
		C35	Responsiveness to customer complaints	0.28046	0.14318	0.14318
C4	Assurance	C41	Competency and experience of personnel	0.14532	0.13138	0.13138
		C42	General attitude and behavior of technician	0.14337	0.18809	0.18809
		C43	Customer handling	0.21036	0.18809	0.18809
		C44	Professionalism of service personnel	0.45017	0.43574	0.43574
		C45	Interpersonal behavior of service personnel	0.05081	0.05672	0.05672
C5	Empathy	C51	Individuated interactions between service front liners and customers	0.33333	0.33333	0.88889
		C52	Availability of service personnel	0.66667	0.66667	0.11111

5.3 Results of Phase 1: Initial weights by Model (9)

The initial step of the proposed two-phase model involves determining the weights of main and sub-criteria through Model (9), which maximizes the minimum membership degree. Table 16 displays the optimal weights of the five SERVQUAL main criteria-Tangibles, Reliability, Responsiveness, Assurance, and Empathy-as determined independently by each expert.

Table 16 shows that Reliability (C2) consistently received the highest weight across all three decision-makers (DM1: 0.50368, DM2: 0.57415, DM3: 0.57809), confirming its central importance in customer satisfaction. Conversely, criteria such as Empathy (C5) and Responsiveness (C3) received relatively lower weights, suggesting these may be perceived as secondary concerns when compared to reliability of service.

The corresponding sub-criteria weights are presented in Table 17. These results reveal expert-specific prioritizations within each main criterion. For instance, within Tangibles, DM1 emphasized C14: Convenient operating hours (0.53263), whereas DM2 prioritized C12: Proximity of service centers (0.43992), and DM3 rated C11: Availability of information and solutions highest (0.54895). Such differences underscore the subjective nature of expert judgment and reinforce the value of a flexible fuzzy approach to weight modeling.

5.4 Results of Phase 2: Refined weights by Model (16)

Tables 18-20 present the detailed outcomes of the proposed two-phase model, derived from solving Model (16) for each of the three DMs. These tables include the final weight vectors, as well as the associated membership degrees, for all main and sub-criteria across the SERVQUAL dimensions. For a detailed comparative analysis of the results for DMs 1-3, please refer to Appendix.

Table 18: Optimal weights of criteria for DM1 determined by Model (16).

Weights of main criteria		Weights of sub-criteria									
		Tangibles		Reliability		Responsiveness		Assurance		Empathy	
Variable	Value	Variable	Value	Variable	Value	Variable	Value	Variable	Value	Variable	Value
λ_{21}	1	λ_{21}	1	λ_{41}	0.59045	λ_{21}	0.16667	λ_{31}	1	λ_{41}	0.90222
λ_{23}	0.16302	λ_{23}	0.16302	λ_{42}	0.16264	λ_{23}	1	λ_{32}	0.31785	λ_{42}	0.13992
λ_{24}	0.16223	λ_{24}	0.16223	λ_{43}	0.16213	λ_{24}	0.2373	λ_{34}	0.31659	λ_{43}	0.14000
λ_{25}	1	λ_{25}	1	λ_{12}	0.16239	λ_{25}	1	λ_{35}	0.44516	λ_{45}	0.14013
λ_{13}	0.27899	λ_{13}	0.27899	λ_{32}	0.16226	λ_{13}	1	λ_{12}	0.92054	λ_{15}	0.13993
λ_{43}	0.16207	λ_{43}	0.16207	–	–	λ_{43}	0.1623	λ_{42}	0.31615	λ_{25}	0.82169
λ_{53}	0.70924	λ_{53}	0.70924	–	–	λ_{53}	0.66667	λ_{52}	0.31577	λ_{35}	0.14013
w_1	0.16293	w_1	0.16293	w_1	0.22105	w_1	0.32440	w_1	0.09985	w_2	0.33333
w_2	0.4888	w_2	0.48880	w_2	0.07790	w_2	0.37847	w_2	0.05199	w_2	0.66667
w_3	0.07149	w_3	0.07149	w_3	0.16844	w_3	0.05407	w_3	0.39939	–	–
w_4	0.15457	w_4	0.15457	w_4	0.53262	w_4	0.11691	w_4	0.17241	–	–
w_5	0.1222	w_5	0.12220	–	–	w_5	0.12616	w_5	0.27637	–	–

Table 19: Optimal weights of criteria for DM2 determined by Model (16).

Weights of main criteria		Weights of sub-criteria									
		Tangibles		Reliability		Responsiveness		Assurance		Empathy	
Variable	Value	Variable	Value	Variable	Value	Variable	Value	Variable	Value	Variable	Value
λ_{21}	0.14103	λ_{21}	0.14103	λ_{21}	0.14136	λ_{21}	0.31645	λ_{31}	0.31710	λ_{41}	0.31664
λ_{23}	1	λ_{23}	1	λ_{23}	0.14000	λ_{23}	0.31740	λ_{32}	0.43903	λ_{42}	1
λ_{24}	0.14011	λ_{24}	0.14011	λ_{24}	0.44298	λ_{24}	1	λ_{34}	0.31666	λ_{43}	0.31666
λ_{25}	1	λ_{25}	1	λ_{31}	0.13955	λ_{25}	1	λ_{35}	1	λ_{45}	0.31770
λ_{31}	0.4765	λ_{31}	0.47650	λ_{41}	0.13915	λ_{13}	0.31652	λ_{21}	1	λ_{15}	0.31629
λ_{41}	0.13979	λ_{41}	0.13979	–	–	λ_{43}	0.31652	λ_{41}	0.31645	λ_{25}	0.84115
λ_{51}	0.21474	λ_{51}	0.21474	–	–	λ_{53}	0.53652	λ_{51}	0.56097	λ_{35}	0.31611
w_1	0.06373	w_1	0.06373	w_1	0.04966	w_1	0.15345	w_1	0.06039	w_1	0.33333
w_2	0.56463	w_2	0.56463	w_2	0.43991	w_2	0.50890	w_2	0.18116	w_2	0.66667
w_3	0.0941	w_3	0.09410	w_3	0.20557	w_3	0.06624	w_3	0.46393	w_3	0.18265
w_4	0.13638	w_4	0.13638	w_4	0.30486	w_4	0.16963	w_4	0.13988	w_4	0.42313
w_5	0.14116	w_5	0.14116	–	–	w_5	0.10178	w_5	0.15464	w_5	0.05508

5.5 Expert weighting and aggregated global priority assessment

Following the application of the proposed two-phase model, the weights of all main and sub-criteria are calculated for each DM, as summarized in Tables 18-20. Among the SERVQUAL dimensions, Reliability consistently emerged as the most significant criterion for evaluating automotive after-sales services. However, variations are observed in the prioritization of other criteria across experts. To account for the differing expertise levels among the DMs, a weighting system is introduced based on their educational qualifications and professional experience. The evaluation framework for this purpose is presented in Tables 21 and 22, which assign a numerical grade to each DM's background. In Table 23, the final grades and normalized significance weights $\bar{v}_j$ are computed for each expert. These weights represent the relative influence of each DM's evaluations and are derived using the following expression:

$$\bar{v}_j = \frac{v_j^1 \times v_j^2}{\sum_{k=1}^{3} v_k^1 \times v_k^2}, \quad j = 1, 2, 3. \tag{27}$$

With the normalized significance weights established, local weights of the criteria and sub-criteria are aggregated to estimate global weights of sub-criteria, which reflect both the importance of each quality element and the credibility of the source expert. Table 24 displays the optimal weights of main and sub-criteria as determined individually by the experts, while Table 25 presents the final global weights and the resulting ranking of all sub-criteria across the SERVQUAL framework (see Figure 11).

Table 20: Optimal weights of criteria for DM3 determined by Model (16).

Weights of main criteria		Weights of sub-criteria									
		Tangibles		Reliability		Responsiveness		Assurance		Empathy	
Variable	Value	Variable	Value	Variable	Value	Variable	Value	Variable	Value	Variable	Value
λ_{21}	0.13986	λ_{12}	1	λ_{12}	0.91211	λ_{31}	0.14098	λ_{41}	1	λ_{12}	1
λ_{23}	0.13986	λ_{13}	0.14025	λ_{13}	0.46394	λ_{32}	1	λ_{42}	1	–	–
λ_{24}	1	λ_{14}	0.14007	λ_{14}	0.46537	λ_{34}	0.14006	λ_{43}	0.13994	–	–
λ_{25}	0.14038	λ_{23}	0.21494	λ_{15}	0.53606	λ_{35}	1	λ_{45}	0.14072	–	–
λ_{15}	0.14008	λ_{43}	0.13993	λ_{24}	0.46347	λ_{21}	0.26557	λ_{15}	0.21482	–	–
λ_{35}	0.14008	–	–	λ_{34}	0.46385	λ_{41}	0.13983	λ_{25}	0.26561	–	–
λ_{45}	0.26566	–	–	λ_{54}	0.46385	λ_{51}	0.4765	λ_{35}	0.13995	–	–
w_1	0.13892	w_1	0.54641	w_1	0.45095	w_1	0.06784	w_1	0.14308	w_1	0.88889
w_2	0.5751	w_2	0.1366	w_2	0.23584	w_2	0.08585	w_2	0.08176	w_2	0.11111
w_3	0.13892	w_3	0.06167	w_3	0.13018	w_3	0.60098	w_3	0.13824	–	–
w_4	0.08216	w_4	0.25532	w_4	0.05284	w_4	0.14516	w_4	0.57232	–	–
w_5	0.06491	–	–	w_5	0.13018	w_5	0.10016	w_5	0.0646	–	–

Table 21: Education grades.

Education	Grade
Diploma	1
Associate degree	2
Bachelor's degree	3
Master's degree	4
PhD and higher	5

Table 22: Work experience grades.

Work experience	Grade
$<$ 5 years	1
5–10 years	2
10–15 years	3
15–20 years	4
$>$ 20 years	5

Table 23: Grades and significance of experts' evaluations.

Expert	Position	Education grade (v_j^1)	Work experience grade (v_j^2)	Final grade ($v_j^1 \times v_j^2$)	$\bar{v}_j$
DM1	Technical expert/Instructor	Master's degree (4)	10–15 years (3)	12	0.25532
DM2	Warehouse manager	Associate degree (2)	>20 years (5)	10	0.21277
DM3	Branch manager/Instructor	PhD and higher (5)	>20 years (5)	25	0.53191

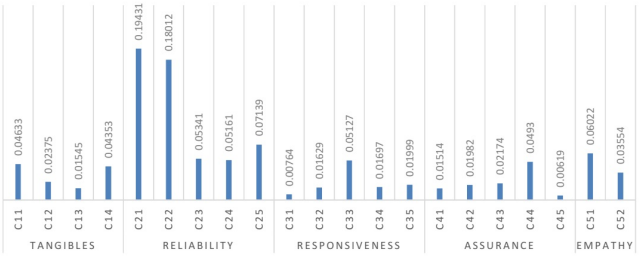


Figure 11: Global weights of sub-criteria.

5.6 Description of the results

Table 25 presents the global weights and ranks of the sub-criteria derived from the proposed two-phase model. The global weights were computed by multiplying the normalized weight of each main criterion by the normalized weight of each corresponding sub-criterion, aggregated across decision-makers. Figure 11 further illustrates the distribution of these global weights for enhanced visualization.

Among the five main criteria, Reliability (C2) holds the highest importance with a normalized weight of 0.55084, followed by Tangibles (C1) at 0.12905, Responsiveness (C3) at 0.11217, Assurance (C4) at 0.11218, and Empathy (C5) at 0.09576. These results reflect the experts' prioritization of reliable service delivery in the automotive after-sales services context.

Table 24: Weights of main criteria and sub-criteria determined by DMs.

	Main criteria	Optimal weights of the main criteria			Sub-criteria	Optimal weights of the sub-criteria		
		DM1	DM2	DM3		DM1	DM2	DM3
C1	Tangibles	0.16293	0.06373	0.13892	C11	0.22105	0.04966	0.54895
					C12	0.0779	0.43992	0.1326
					C13	0.16844	0.20557	0.06196
					C14	0.53263	0.30487	0.25651
C2	Reliability	0.4888	0.56463	0.5751	C21	0.31426	0.15345	0.45095
					C22	0.36525	0.5089	0.23584
					C23	0.05342	0.06624	0.13018
					C24	0.11551	0.16963	0.05284
					C25	0.15159	0.10178	0.13018
C3	Responsiveness	0.07149	0.0941	0.13892	C31	0.08655	0.06181	0.06181
					C32	0.05276	0.17697	0.17697
					C33	0.40531	0.47488	0.47488
					C34	0.17496	0.14318	0.14318
					C35	0.28046	0.14318	0.14318
C4	Assurance	0.15457	0.13638	0.08216	C41	0.14532	0.13138	0.13138
					C42	0.14337	0.18809	0.18809
					C43	0.21036	0.18809	0.18809
					C44	0.45017	0.43574	0.43574
					C45	0.05081	0.05672	0.05672
C5	Empathy	0.1222	0.14116	0.06491	C51	0.33333	0.33333	0.88889
					C52	0.66667	0.66667	0.11111

Table 25: Global weights and ranks of sub-criteria.

	Main criteria	(Optimal weights of the main criteria)$\times\bar{v}_j$			Normalized weights of	Sub-criteria	(Optimal weights of the sub-criteria) $\times\bar{v}_j$			Normalized weights of	Global weights of	Rank
		DM1	DM2	DM3	main-criteria		DM1	DM2	DM3	the sub-criteria	the sub-criteria	
C1	Tangibles	0.0416	0.01356	0.07389	0.12905	C11	0.05644	0.01057	0.29199	0.359	0.04633	9
						C12	0.01989	0.0936	0.07053	0.18402	0.02375	12
						C13	0.04301	0.04374	0.03296	0.11971	0.01545	18
						C14	0.13599	0.06487	0.13644	0.3373	0.04353	10
C2	Reliability	0.1248	0.12013	0.3059	0.55084	C21	0.08024	0.03265	0.23986	0.35275	0.19431	1
						C22	0.09326	0.10828	0.12545	0.32699	0.18012	2
						C23	0.01364	0.01409	0.06924	0.09697	0.05341	5
						C24	0.02949	0.03609	0.02811	0.09369	0.05161	6
						C25	0.0387	0.02166	0.06924	0.1296	0.07139	3
C3	Responsiveness	0.01825	0.02002	0.07389	0.11217	C31	0.0221	0.01315	0.03288	0.06813	0.00764	20
						C32	0.01347	0.03765	0.09413	0.14525	0.01629	17
						C33	0.10348	0.10104	0.25259	0.45711	0.05127	7
						C34	0.04467	0.03046	0.07616	0.15129	0.01697	16
						C35	0.07161	0.03046	0.07616	0.17823	0.01999	14
C4	Assurance	0.03946	0.02902	0.0437	0.11218	C41	0.0371	0.02795	0.06988	0.13493	0.01514	19
						C42	0.03661	0.04002	0.10005	0.17668	0.01982	15
						C43	0.05371	0.04002	0.10005	0.19378	0.02174	13
						C44	0.11494	0.09271	0.23177	0.43942	0.0493	8
						C45	0.01297	0.01207	0.03017	0.05521	0.00619	21
C5	Empathy	0.0312	0.03003	0.03453	0.09576	C51	0.08511	0.07092	0.47281	0.62884	0.06022	4
						C52	0.17021	0.14185	0.0591	0.37116	0.03554	11

Regarding the sub-criteria, the most significant sub-criterion is C21 (Service is performed accurately and dependably), which achieved the highest global weight of 0.19431 and ranked first. The next most important sub-criteria are C22 (Provision of service as promised) with a global weight of 0.18012 (Rank 2), and C25 (Willingness to solve customer problems) with a global weight of 0.07139 (Rank 3). These findings suggest that precision, dependability, and proactive problem-solving are the most critical factors influencing customer satisfaction.

In contrast, sub-criteria such as C45 (Security of transactions) and C31 (Promptness of service) received the lowest global weights (0.00619 and 0.00764, respectively), placing them at the bottom of the ranking (Ranks 21 and 20, respectively). This indicates that while important, these aspects are perceived as less critical compared to factors directly related to reliability and assurance.

Overall, the results reveal that in the automotive after-sales service domain, customers place the greatest value on service reliability and problem resolution capabilities. Elements related to physical tangibility and security, although necessary, are comparatively less influential in determining overall satisfaction. These insights can guide service

managers in resource allocation, emphasizing improvements in reliability and responsiveness to better meet customer expectations.

6 Theoretical discussions and managerial implications

6.1 Theoretical discussions

This study makes a significant contribution to the fuzzy MADM literature by addressing a key limitation in the BWM-FPP model-its inability to guarantee locally efficient weight vectors. In response, we developed a novel two-phase fuzzy model that systematically enhances both the minimum and mean membership degree. This two-phase structure not only improves the internal consistency of the model but also increases the reliability and interpretability of results in uncertain decision environments.

In the first phase, our model retains the core structure of the BWM-FPP model by maximizing the minimum membership degree. This step provides a solid foundation for decision reliability by ensuring a baseline level of consistency across judgments. The second phase of our approach refines the initial weight vector by maximizing the mean membership degree. This additional optimization ensures that the final weight vector is not only consistent but also locally efficient-leading to more robust and dependable prioritizations.

Our application of the model to the automotive after-sales service sector demonstrates its practical relevance. The results clearly indicate that Reliability, Tangibles, and Assurance are the most influential SERVQUAL dimensions for assessing service quality in this domain. These findings are consistent with prior studies using both SERVQUAL and fuzzy MADM models, thereby reinforcing the contextual validity of the proposed method. Furthermore, the improvement in membership degrees across all case studies confirms the method's theoretical and empirical robustness.

A comparative analysis with existing methods highlights the advantages of the two-phase model. In all test scenarios, it outperformed the traditional BWM-FPP method in terms of solution quality, offering higher membership degrees and ensuring local efficiency. This suggests its broader applicability in other complex decision-making contexts where reliability and interpretability are crucial.

Additionally, this research introduces important theoretical advancements–most notably, the formal definitions of locally efficient weight vectors and dominance in fuzzy membership spaces. These constructs provide a more precise lens for evaluating decision outputs in fuzzy environments and can form the basis for future research in fuzzy optimization and preference modeling.

In summary, the proposed two-phase model advances the methodological rigor and practical utility of MADM tools. It offers DMs a more reliable, interpretable, and theoretically grounded framework, particularly suited for complex, ambiguous environments such as automotive after-sales services.

6.2 Managerial implications

The findings of this study, particularly those presented in Table 27, offer valuable strategic insights for managers operating in the automotive after-sales service industry. The pronounced dominance of the Reliability dimension—especially sub-criteria C21 (Performing the promised service dependably and accurately) and C22 (Providing service as promised)—underscores the central importance of dependable service execution in shaping customer satisfaction.

To meet this expectation, managers should prioritize initiatives that strengthen process consistency. This includes the development and enforcement of standardized operating procedures, implementation of robust quality control mechanisms, and deployment of real-time performance monitoring tools. Such measures will help ensure that service commitments are consistently fulfilled, minimizing customer dissatisfaction due to unmet expectations.

Notably, the relatively high ranking of sub-criterion C51 (Understanding specific needs of customers) under the Empathy dimension highlights the value of personalized customer interaction. While operational excellence forms the foundation of service quality, the ability to address individual customer needs adds an important layer of differentiation in competitive markets. Managers should therefore invest in employee training programs aimed at enhancing communication skills, empathy, and situational awareness, fostering a customer-centric service culture.

In contrast, sub-criteria related to the Tangibles dimension—such as C11 (Modern equipment) and C14 (Professional appearance of staff)-received moderate weights. This indicates that while the physical environment and staff presentation contribute to perceived quality, they are not the primary drivers of customer satisfaction. Managers are advised to maintain acceptable standards in these areas without overinvesting, instead channeling resources toward more impactful factors like service reliability and responsiveness.

Interestingly, criteria associated with Responsiveness and Assurance, including C31 (Prompt service delivery) and C45 (Staff instilling confidence), were ranked lower in terms of global importance. This suggests that, from the

customer's perspective, timely service or symbolic displays of assurance hold less weight than the core requirement of reliable, accurate service fulfillment. As such, organizations should avoid overemphasizing speed or superficial assurance efforts at the expense of substantive service performance.

Overall, the prioritization patterns revealed by this study point to a clear managerial takeaway: strategic improvements in service reliability and customer-centric delivery will yield the greatest returns in customer satisfaction. By leveraging the structured insights generated through the proposed two-phase fuzzy model, managers can make more informed decisions regarding resource allocation, service design, and performance management in after-sales operations. This model thus serves as a valuable decision support tool, particularly in environments characterized by uncertainty and subjectivity in expert judgments.

7 Conclusions, limitations and future researches

7.1 Conclusions

This study introduced a novel two-phase model that significantly advances the BWM-FPP framework developed by Amiri et al. [1]. The primary enhancement lies in addressing the methodological limitation of deriving locally weakly efficient weight vectors by enabling the computation of locally efficient solutions. This dual-phase structure—first maximizing the minimum membership degree to ensure internal consistency, and then optimizing the mean membership degree to enhance interpretability-results in weight vectors that are both theoretically robust and practically reliable.

Application of the model in the context of automotive after-sales services confirms its value in real-world decision-making scenarios. The model effectively prioritized quality dimensions and sub-criteria under uncertainty, highlighting Reliability, Tangibles, and Assurance as the most critical drivers of customer satisfaction. These results not only align with established SERVQUAL frameworks but also validate the model's ability to capture nuanced expert judgments in complex service environments.

Moreover, the proposed method demonstrated superior performance in terms of consistency, membership degree optimization, and interpretability when compared with the traditional BWM-FPP model. The formal introduction of new analytical constructs-such as dominance in fuzzy membership spaces and locally efficient weight vectors—represents a meaningful theoretical contribution to the fuzzy MADM literature. These constructs improve the evaluation and comparison of decision-making outputs under ambiguity and offer a more refined analytical lens for future research.

In sum, this study advances both the methodological depth and practical applicability of MADM models. It offers a more reliable, consistent, and interpretable framework for prioritizing criteria in uncertain decision environments, supporting strategic decision-making with greater confidence.

7.2 Limitations

Despite the theoretical and practical advancements introduced in this study, several limitations should be acknowledged. First, the proposed two-phase BWM-FPP model was validated through numerical examples and a case study within the automotive after-sales service sector. While the findings demonstrate improved reliability and weight vector efficiency, the model's applicability to other domains remains to be empirically verified. Second, the two-phase nonlinear structure, while enhancing robustness, also increases computational complexity. This may pose challenges in large-scale or time-sensitive decision contexts, where computational efficiency is critical. Addressing these limitations will contribute to the broader applicability, usability, and robustness of the proposed approach.

7.3 Future researches

While the proposed two-phase model presents significant advancements, several promising avenues exist for further development and exploration:

- **Integration with other MADM techniques:** Combining the two-phase model with complementary fuzzy MADM methods—such as fuzzy TOPSIS, fuzzy VIKOR, or fuzzy DEMATEL–could enhance its utility in multi-layered or interdependent decision-making problems, offering more comprehensive assessments.
- **Cross-Domain application:** Although validated in the automotive after-sales service sector, the model is adaptable to diverse application areas such as healthcare, education, sustainable development, and supply chain management. Applying the model across these sectors can further evaluate its generalizability and practical robustness.

- **Software and tool development:** Developing a user-friendly decision-support tool or software platform based on this model could greatly facilitate its use by practitioners. This would enhance the model's accessibility and accelerate its adoption in industry.

In conclusion, the proposed model lays a strong foundation for future research in fuzzy decision analysis and practical applications. By bridging theoretical gaps and addressing real-world decision complexities, it offers both scholars and practitioners a robust tool for navigating uncertain, multi-criteria environments with clarity and confidence.

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A Appendix. (Comparative and visual analysis of the proposed two-phase model in the case study)

This appendix presents detailed visualizations (Figures A1–A18) and comparative analyses (Tables A1 and A2) to demonstrate the performance and advantages of the proposed two-phase model compared to existing approaches. The analyses focus on the results obtained from DMs in the context of the automotive after-sales service domain, highlighting the model's ability to produce locally efficient weights and more consistent rankings.

To quantify the performance improvement achieved through the proposed two-phase model over the BWM-FPP model, the relative change in weights was calculated for both main and sub-criteria across all decision-makers. These results

are summarized in Table A1. The RC metric, derived using Expression (22), measures the relative difference between the weights obtained from the proposed model (Model (16)) and those derived from the BWM-FPP model (Model (9)). A positive RC indicates an increased weight in the proposed model—suggesting improved representation or emphasis of that criterion-whereas a negative RC implies a reduction in its relative importance.

To further assess the effectiveness of the proposed model, Table A2 presents a comparative summary of the main and sub-criteria weights derived from both the BWM-FPP and the proposed two-phase model. These findings confirm that the two-phase approach enhances both the precision and consistency of decision-making outcomes. As a result, it offers a more robust and insightful tool for multi-criteria analysis, especially in complex service quality evaluations under uncertainty.

Moreover, the comparative analysis of main criteria, sub-criteria, and average membership degree weight ratios for DMs 1–3 is illustrated in Figures A1–A18. These visualizations provide detailed insight into how the proposed model adapts to varying decision contexts and refines the output weight vectors.

Table A1: Optimal weights and relative changes of criteria determined by DMs.

Quality element	Model	Optimal weights of the main criteria and sub criteria.		
		DM1	DM2	DM3
C1	Amiri's model	0.15928	0.06481	0.13964
	Proposed model	0.16293	0.06373	0.13892
	RC(w_j^{**})	0.02292	-0.01666	-0.00516
C2	Amiri's model	0.50368	0.57415	0.57809
	Proposed model	0.4888	0.56463	0.5751
	RC(w_j^{**})	-0.02954	-0.01658	-0.00517
C3	Amiri's model	0.07367	0.0837	0.13964
	Proposed model	0.07149	0.0941	0.13892
	RC(w_j^{**})	-0.02959	0.12425	-0.00516
C4	Amiri's model	0.15928	0.13868	0.07741
	Proposed model	0.15457	0.13638	0.08216
	RC(w_j^{**})	-0.02957	-0.01658	0.06136
C5	Amiri's model	0.10412	0.13868	0.06525
	Proposed model	0.1222	0.14116	0.06491
	RC(w_j^{**})	0.17365	0.01788	-0.00521
	Mean ($\overline{\text{RC}}(W^{**})$)	0.09509	0.06399	0.02735
C11	Amiri's model	0.22105	0.04966	0.54895
	Proposed model	0.22105	0.04966	0.54641
	RC(w_j^{**})	0	0	-0.00463
C12	Amiri's model	0.0779	0.43992	0.1326
	Proposed model	0.0779	0.43991	0.1366
	RC(w_j^{**})	0	-0.00002	0.03017
C13	Amiri's model	0.16844	0.20557	0.06196
	Proposed model	0.16844	0.20557	0.06167
	RC(w_j^{**})	0	0	-0.00468
C14	Amiri's model	0.53263	0.30487	0.25651
	Proposed model	0.53262	0.30486	0.25532
	RC(w_j^{**})	-0.00002	-0.00003	-0.00464
	Mean ($\overline{\text{RC}}(W^{**})$)	0.00001	0.00002	0.0147
C21	Amiri's model	0.31426	0.15794	0.45097
	Proposed model	0.3244	0.15345	0.45095
	RC(w_j^{**})	0.03227	-0.02843	-0.00004
C22	Amiri's model	0.36525	0.5238	0.23585
	Proposed model	0.37847	0.5089	0.23584
	RC(w_j^{**})	0.03619	-0.02845	-0.00004
C23	Amiri's model	0.05342	0.06818	0.13019
	Proposed model	0.05407	0.06624	0.13018
	RC(w_j^{**})	0.01217	-0.02845	-0.00008
C24	Amiri's model	0.11551	0.15794	0.05284
	Proposed model	0.11691	0.16963	0.05284
	RC(w_j^{**})	0.01212	0.07402	0
C25	Amiri's model	0.15159	0.09217	0.13019
	Proposed model	0.12616	0.10178	0.13018

Quality element	Model	Optimal weights of the main criteria and sub criteria.		
		DM1	DM2	DM3
	RC(w_j^{**})	-0.16776	0.10426	-0.00008
	Mean ($\overline{\text{RC}}(W^{**})$)	0.0521	0.05272	0.00005
C31	Amiri's model	0.08655	0.06181	0.06181
	Proposed model	0.09985	0.06039	0.06784
	RC(w_j^{**})	0.15367	-0.02297	0.09756
C32	Amiri's model	0.05276	0.17697	0.17697
	Proposed model	0.05199	0.18116	0.08585
	RC(w_j^{**})	-0.01459	0.02368	-0.51489
C33	Amiri's model	0.40531	0.47488	0.47488
	Proposed model	0.39939	0.46393	0.60098
	RC(w_j^{**})	-0.01461	-0.02306	0.26554
C34	Amiri's model	0.17496	0.14318	0.14318
	Proposed model	0.17241	0.13988	0.14516
	RC(w_j^{**})	-0.01457	-0.02305	0.01383
C35	Amiri's model	0.28046	0.14318	0.14318
	proposed	0.27637	0.15464	0.10016
	RC(w_j^{**})	-0.01458	0.08004	-0.30046
	Mean ($\overline{\text{RC}}(W^{**})$)	0.04241	0.03456	0.23846
C41	Amiri's model	0.14532	0.13138	0.13138
	Proposed model	0.14532	0.12758	0.14308
	RC(w_j^{**})	0	-0.02892	0.08905
C42	Amiri's model	0.14337	0.18809	0.18809
	Proposed model	0.14337	0.21157	0.08176
	RC(w_j^{**})	0	0.12483	-0.56531
C43	Amiri's model	0.21036	0.18809	0.18809
	Proposed model	0.21035	0.18265	0.13824
	RC(w_j^{**})	-0.00005	-0.02892	-0.26503
C44	Amiri's model	0.45017	0.43574	0.43574
	Proposed model	0.45016	0.42313	0.57232
	RC(w_j^{**})	-0.00002	-0.02894	0.31344
C45	Amiri's model	0.05081	0.05672	0.05672
	Proposed model	0.05081	0.05508	0.0646
	RC(w_j^{**})	0	-0.02891	0.13893
	Mean ($\overline{\text{RC}}(W^{**})$)	0.00001	0.04811	0.27435
C51	Amiri's model	0.33333	0.33333	0.88889
	Proposed model	0.33333	0.33333	0.88889
	RC(w_j^{**})	0	0	0
C52	Amiri's model	0.66667	0.66667	0.11111
	Proposed model	0.66667	0.66667	0.11111
	RC(w_j^{**})	0	0	0
	Mean ($\overline{\text{RC}}(W^{**})$)	0	0	0

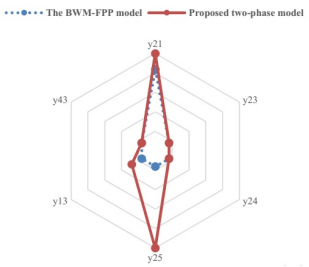


Figure A1:
Main criteria comparison– DM1.

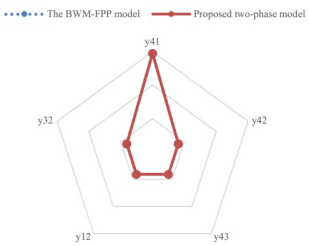


Figure A2:
Tangibles comparison– DM1.

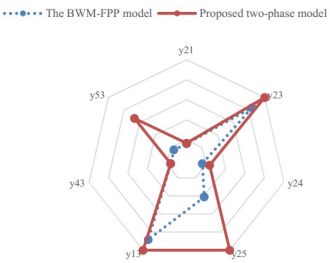


Figure A3:
Reliability comparison– DM1.

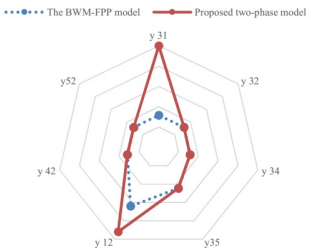


Figure A4:
Responsiveness comparison– DM1.

Table A2: Comparative analysis of the results for DMs using Amiri's model and the proposed two-phase model.

	Comparison of the results for DM1			Comparison of the results for DM2				Comparison of the results for DM3			
	Amiri's model	Proposed model	Comparison		Amiri's model	Proposed model	Comparison		Amiri's model	Proposed model	Comparison
Main criteria											
Y21	0.83777	1	Improved	Y21	0.14103	0.14103	Similar	Y21	0.13986	0.13986	Similar
Y23	0.16302	0.16302	Similar	Y23	0.14038	1	Improved	Y23	0.13986	0.13986	Similar
Y24	0.16223	0.16223	Similar	Y24	0.14011	0.14011	Similar	Y24	0.5321	1	Improved
Y25	0.1625	1	Improved	Y25	0.85989	1	Improved	Y25	0.14038	0.14038	Similar
Y13	0.16207	0.27899	Improved	Y31	0.29147	0.4765	Improved	Y15	0.14008	0.14008	Similar
Y43	0.16207	0.16207	Similar	Y41	0.13979	0.13979	Similar	Y35	0.14008	0.14008	Similar
Y53	0.27494	0.70924	Improved	Y51	0.13979	0.21474	Improved	Y45	0.18636	0.26566	Improved
Mean ($\bar{y}_W$)	0.27494	0.49651	Improved		0.26446	0.4446	Improved		0.20267	0.28085	Improved
Tangibles											
Y41	0.59045	0.59045	Similar	Y21	0.14136	0.14136	Similar	Y12	0.86011	1	Improved
Y42	0.16264	0.16264	Similar	Y23	0.14	0.14	Similar	Y13	0.14025	0.14025	Similar
Y43	0.16213	0.16213	Similar	Y24	0.44298	0.44298	Similar	Y14	0.14007	0.14007	Similar
Y12	0.16239	0.16239	Similar	Y31	0.13955	0.13955	Similar	Y23	0.14009	0.21494	Improved
Y32	0.16226	0.16226	Similar	Y41	0.13915	0.13915	Similar	Y43	0.13993	0.13993	Similar
Mean ($\bar{y}_W$)	0.247974	0.247974	Similar		0.20061	0.20061	Similar		0.28409	0.32704	Improved
Reliability											
Y21	0.16225	0.16667	Improved	Y21	0.31645	0.31645	Similar	Y12	0.91211	0.91211	Similar
Y23	0.83733	1	Improved	Y23	0.3174	0.3174	Similar	Y13	0.46394	0.46394	Similar
Y24	0.16206	0.2373	Improved	Y24	0.68355	1	Improved	Y14	0.46537	0.46537	Similar
Y25	0.40946	1	Improved	Y25	0.31702	1	Improved	Y15	0.53606	0.53606	Similar
Y13	0.88282	1	Improved	Y13	0.31652	0.31652	Similar	Y24	0.46347	0.46347	Similar
Y43	0.1623	0.1623	Similar	Y43	0.31652	0.31652	Similar	Y34	0.46385	0.46385	Similar
Y53	0.1623	0.66667	Improved	Y53	0.35186	0.53652	Improved	Y54	0.46385	0.46385	Similar
Mean ($\bar{y}_W$)	0.39693	0.60471	Improved		0.32742	0.54334	Improved		0.53838	0.53838	Similar
Responsiveness											
Y31	0.31704	1	Improved	Y31	0.3171	0.3171	Similar	Y31	0.14098	0.14098	Similar
Y32	0.31785	0.31785	Similar	Y32	0.31661	0.43903	Improved	Y32	0.22873	1	Improved
Y34	0.31659	0.31659	Similar	Y34	0.31666	0.31666	Similar	Y34	0.14006	0.14006	Similar
Y35	0.44516	0.44516	Similar	Y35	0.68334	1	Improved	Y35	0.14045	1	Improved
Y12	0.64045	0.92054	Improved	Y21	0.86313	1	Improved	Y21	0.13997	0.26557	Improved
Y42	0.31615	0.31615	Similar	Y41	0.31645	0.31645	Similar	Y41	0.13983	0.13983	Similar
Y52	0.31577	0.31577	Similar	Y51	0.31645	0.56097	Improved	Y51	0.29149	0.4765	Improved
Mean ($\bar{y}_W$)	0.38129	0.51887	Improved		0.44711	0.56432	Improved		0.1745	0.45185	Improved
Assurance											
Y41	0.90222	0.90222	Similar	Y41	0.31664	0.31664	Similar	Y41	0.86006	1	Improved
Y42	0.13992	0.13992	Similar	Y42	0.68334	1	Improved	Y42	0.22943	1	Improved
Y43	0.14	0.14	Similar	Y43	0.31666	0.31666	Similar	Y43	0.13994	0.13994	Similar
Y45	0.14013	0.14013	Similar	Y45	0.3177	0.3177	Similar	Y45	0.14072	0.14072	Similar
Y15	0.13993	0.13993	Similar	Y15	0.31629	0.31629	Similar	Y15	0.13995	0.21482	Improved
Y25	0.82169	0.82169	Similar	Y25	0.31611	0.84115	Improved	Y25	0.14011	0.26561	Improved
Y35	0.14013	0.14013	Similar	Y35	0.31611	0.31611	Similar	Y35	0.13995	0.13995	Similar
Mean ($\bar{y}_W$)	0.34629	0.34629	Similar		0.36898	0.48922	Improved		0.25574	0.41443	Improved

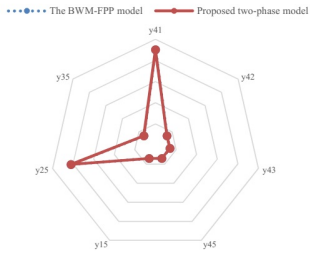


Figure A5:
Assurance comparison– DM1.

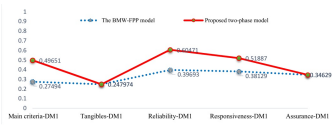


Figure A6:
Comparing the mean membership degree – DM1.

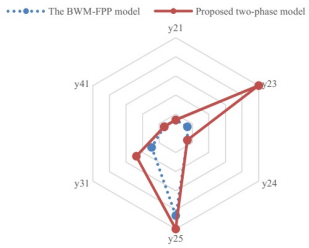


Figure A7:
Main criteria comparison– DM2.

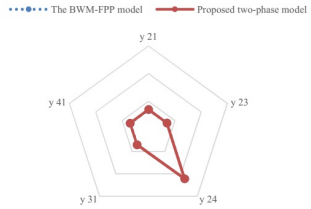


Figure A8:
Tangibles comparison– DM2.

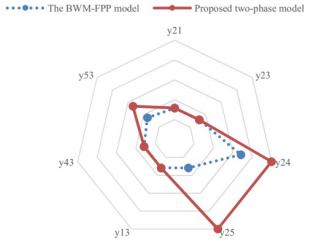


Figure A9:
Reliability comparison– DM2.

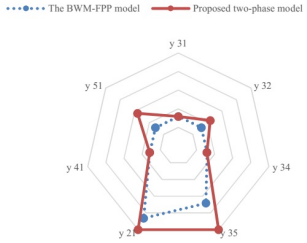


Figure A10:
Responsiveness comparison– DM2.

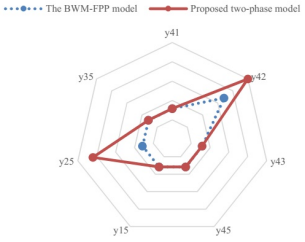


Figure A11:
Assurance comparison– DM2.

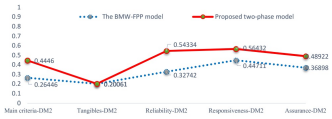


Figure A12:
Comparison of the mean membership degree – DM2.

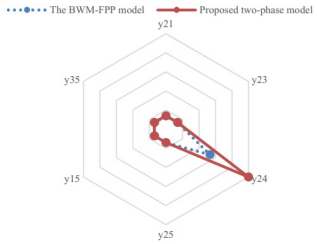


Figure A13:
Main criteria comparison– DM3.

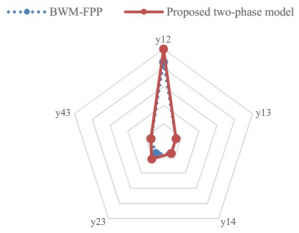


Figure A14:
Tangibles comparison– DM3.

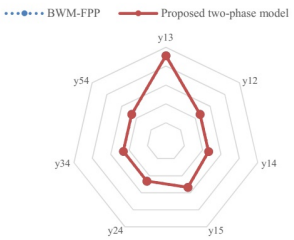


Figure A15:
Reliability comparison– DM3.

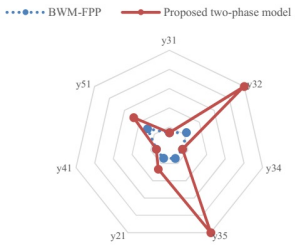


Figure A16:
Responsiveness comparison– DM3.

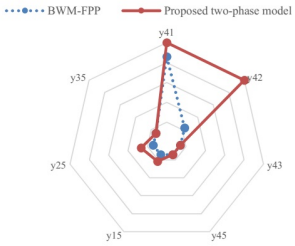


Figure A17:
Assurance comparison– DM3.

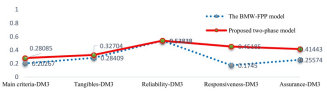


Figure A18:
Comparing the mean membership degree – DM3.