Iranian Journal of Fuzzy Systems
Volume 22, Number 4, (2025), pp. 1-15
Original paper

Möbius representation of the bipolar decomposition integrals and its applications

J. Abbas [1], R. Mesiar [2] and R. Halaš [3]

[1] *Department of Applied Sciences, University of Technology, Al Sinaá Street, P. O. Box (19006), Baghdad, Iraq*
[2] *Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovakia*
[2,3] *Department of Algebra and Geometry, Palacký University of Olomouc, 17. listopadu 12, 779 00 Olomouc, Czech Republic.*

jabbar.a.ghafil@uotechnology.edu.iq, mesiar@math.sk, radomir.halas@upol.cz

Abstract

The bipolar decomposition integral, recently introduced in [5], is a general framework for handling integrals related to aggregation on unipolar and bipolar scales. The Möbius representation is related to the notion of k-additivity of a monotone set function and allows to derive simple expressions of some nonlinear integrals. In this paper, we propose Möbius representation for the bipolar decomposition integral, which includes the Möbius representation of each of the bipolar Choquet integral, bipolar Shilkret integral, and bipolar Pan integral. Then, we introduce the expressions for computing bipolar decomposition integrals concerning a 2-additive bi-capacity. Lastly, a practical numerical example is provided to illustrate the applicability of the proposed results in dealing with aggregation on bipolar scales, and simplicity of calculating the 2-additive bipolar decomposition integrals.

Keywords: 2-additive bi-capacity, aggregation functions, bipolar decomposition integrals, Möbius transform, multi-criteria decision-making.

1 Introduction

Fuzzy integrals [16] based on non-additive measures have been used as utility aggregation tools for representing interaction among multiple criteria, and combining a set of values for different criteria into an overall representative value to rank the alternatives. Fuzzy integrals have been extensively applied in various fields of real-world applications, including face recognition, decision-making problems, and interactive information fusion (see, e. g., [19, 21, 23, 30]). Bipolar fuzzy integrals are an advanced generalization of fuzzy integrals concerning aggregation on bipolar scales. These have been introduced in numerous studies (see, e.g., [1, 3, 15, 18, 20]).

The idea of decomposing the integrand function on unipolar scales was used in [10, 28] for introducing decomposition integral as a common framework for known integrals, such as the Choquet integral [9], the Shilkret integral [32], the PAN integral [34], and the concave integral [24]. The bipolar decomposition integral, recently introduced in [5], is a general framework for handling integrals related to aggregation on unipolar and bipolar scales. The Möbius representation [31] of a monotone set function is a fundamental concept permitting the derivation of simple expressions of nonlinear integrals, and its generalizations have been discussed by various researchers (see, e.g. [4, 11, 14]). Recently, based on Lovász idea of an extension of pseudo-boolean functions (leading to an alternative representation of the discrete Choquet integral), the Möbius representation of decomposition integrals has been introduced in [26] for the decomposition systems to get different types of decomposition integrals on unipolar scales.

In this paper, we propose Möbius representation for the bipolar decomposition integral, which includes the Möbius representation of each of the bipolar Choquet integral, bipolar Shilkret integral, and bipolar Pan integral. Furthermore, the computational complexity for bipolar decomposition integral by means of the Möbius representation is simpler when applying k-additive measures with small $k \in \{2, 3\}$ than that one based on bi-capacities.

Corresponding Author: J. Abbas
Received: July 2025; Accepted: Invited paper.
https://doi.org/10.22111/ijfs.2025.9334

Consequently, we then introduce the expressions for computing bipolar decomposition integrals concerning a 2-additive bi-capacity, and apply them with a practical example to demonstrate the efficiency and simplicity of calculating the 2-additive bipolar decomposition integrals.

The rest of the paper is arranged as follows: Section 2 recalls the preliminaries required for this work. According to the approach used in previous literature [5], Section 3 introduces the Möbius representation and k-additivity of bi-capacity. Section 4 describes the bipolar decomposition integral as a general way of selection and global utility analysis. In section 5, we propose the general main results. In Section 6, we perform special cases of bipolar decomposition integrals based on the 2-additive bi-capacities. Section 7 provides a practical numerical example to illustrate the applicability of the proposed results. Concluding remarks and future research direction are given in Section 8.

Throughout the paper, N denotes the universal set of n elements (players, criteria, states of nature, etc.), $\mathcal{P}(N)$ denotes the power set of N. We consider $[0,1]$ or $[0,\infty]$ to be prototypical unipolar scale, while $[-1,1]$ or $\mathbb{R}$ with 0 as a neutral level will be considered as a prototypical bipolar scale. To avoid heavy notations, we will often omit brackets and commas to denote sets. For example, $\{i\},\{1,2\},\{1,2,3\}$ are respectively denoted by $i, 12, 123$.

2 Preliminaries

In this section, we will review some basic concepts and definitions of bi-capacities that have been adopted in the literature [5, 13, 14, 17], which are relevant to the results obtained in this paper.

For any $s,t\in[-1,1]$, the symmetric maximum $\curlyvee$ has been introduced in [12] as follows:

$$s \curlyvee t = \operatorname{sign}(s+t)\cdot(|s|\vee|t|),$$

where sign is the sign-function. Observe that the symmetric maximum has a neutral element 0 but it is not associative and thus it opens the doors for several extension into related n-ary operations on $[-1,1]$, $n>2$.

Many extensions of the symmetric maximum have been proposed by various researchers [12, 13, 17, 27]. One of these extensions is the bipolar maximum, here denoted by $\check{\curlyvee}$, which is based on the splitting rule applied to the supremum and to the infimum, and given by:

$$\check{\curlyvee}_{s_i\in I} = (\bigvee_{s_i\geq 0} s_i) \curlyvee (\bigwedge_{s_i<0} s_i), \tag{1}$$

for any non-empty set (index set) I and a system $(s_i),\ i\in I$ of values $s_i\in[-1,1]$.

In the context of game theory, Bilbao et al. [8] have introduced the bi-cooperative game as an extension of a cooperative game.

Definition 2.1. [8] *Let $\mathcal{Q}(N) := \{(A,B)\in\mathcal{P}(N)\times\mathcal{P}(N)|\ A\cap B=\emptyset\}$ be the set of all disjoint pairs of sets. A bi-cooperative game is a function $\nu:\mathcal{Q}(N)\to\mathbb{R}$ with $\nu(\emptyset,\emptyset)=0$, where $\nu(A,B)$ is the worth when players in A are positive contributors, players in B are negative contributors, and the remaining players do not participate.*

In the field of multicriteria decision making, Grabisch and Labreuche have presented the following partial order $\sqsubseteq$ on $\mathcal{Q}(N)$ for bi-capacities [14]:

$$(A_1,B_1)\sqsubseteq(A_2,B_2) \text{ if and only if } A_1\subseteq A_2 \text{ and } B_1\supseteq B_2. \tag{2}$$

The bi-capacity is a monotonic bi-cooperative game, as specified by the following definition.

Definition 2.2. [14] *A function $\nu:\mathcal{Q}(N)\to[-1,1]$ is a bi-capacity on $\mathcal{Q}(N)$ if it satisfies:*
(i) $\nu(\emptyset,\emptyset)=0,\ \nu(N,\emptyset)=1,\ \nu(\emptyset,N)=-1$,
(ii) $\nu(A_1,B_1)\leq\nu(A_2,B_2)$ *for any* $(A_1,B_1),(A_2,B_2)\in\mathcal{Q}(N)$ *such that*
$(A_1,B_1)\sqsubseteq(A_2,B_2)$.

There are several order relations on the structure $\mathcal{Q}(N)$ that have been introduced by Grabisch and Labreuche [14], and Bilbao et al. [8]. The partially ordered set $(\mathcal{Q}(N),\sqsubseteq)$ is a lattice which has no center, but only the top $(N,\emptyset)$ and the bottom $(\emptyset,N)$.

On the other hand, the bipolar fuzzy integral is constructed by a symmetry around the $(\emptyset,\emptyset)$ point, which is the central point in $\mathcal{Q}(N)$. This leads to consider a more natural structure on $\mathcal{Q}(N)$ with replacement each element (A,B) by (A,B^c), introduced in [14] and denoted by $(\mathcal{Q}(N)^*,\sqsubseteq^*)$. The set $\mathcal{Q}(N)^*$ consists of all pairs

(A,B) such that $A\subseteq B$, and $\mathcal{Q}(N)^*$ is a lattice with respect to the product order $\sqsubseteq^*$ on $\mathcal{P}(N)^2$: for each $(A_1,B_1),(A_2,B_2)\in\mathcal{Q}(N)^*$,

$$(A_1,B_1)\sqsubseteq^*(A_2,B_2) \text{ if and only if } A_1\subseteq A_2,\ B_1\subseteq B_2. \tag{3}$$

Its lattice operations of infimum and supremum are given by formulas

$$(A_1,B_1)\sqcap^*(A_2,B_2):=(A\cap A_1,B\cap B_1),$$

and

$$(A_1,B_1)\sqcup^*(A_2,B_2):=(A\cup A_1,B\cup B_1).$$

For each element $(A_1,B_1)\in\mathcal{Q}(N)^*$, the characteristic function (denoted by $\mathbf{1}_{(A_1,B_1)}$) is defined by

$$\mathbf{1}_{(A_1,B_1)}(i)=\begin{cases}\ \ 1 & \text{iff } i\in A_1,\\ -1 & \text{iff } i\in B_1^c,\\ \ \ 0 & \text{iff } i\notin A_1\cup B_1^c\end{cases}$$

for $i\in N$.

According to order relation $\sqsubseteq^*$ on $\mathcal{Q}(N)^*$ and the lattice operations of infimum $\sqcap^*$ and supremum $\sqcup^*$, we can directly have the following definitions.

Definition 2.3. [5] *We say that (A_1,B_1) and (A_2,B_2) are nested if either $(A_1,B_1)\sqsubseteq^*(A_2,B_2)$ or $(A_2,B_2)\sqsubseteq^*(A_1,B_1)$, for any $(A_1,B_1),(A_2,B_2)\in\mathcal{Q}(N)^*$. A collection $\mathcal{C}$ is called a chain if any two different pairs of elements $(A_1,B_1),(A_2,B_2)\in\mathcal{C}$ are nested.*

Definition 2.4. [5] *For any element $(A,B)\in\mathcal{Q}(N)^*$, $(A,B)\neq(\emptyset,\emptyset)$, a collection $\mathcal{P}=\{(A_1,B_1),\ldots,(A_n,B_n)\}$ from $\mathcal{Q}(N)^*$ is called a partition of (A,B) if for all $(A_j,B_j),(A_k,B_k)\in\mathcal{P}$, $(A_j,B_j)\sqcap^*(A_k,B_k)=(\emptyset,\emptyset)$, and $\sqcup^*\{(A_i,B_i)\}_{i=1}^n=(A,B)$.*

3 Möbius representation of bi-capacities

In the sense of our approach, we introduce a Möbius representation of a bi-capacity ν on the partially ordered set $(\mathcal{Q}(N)^*,\sqsubseteq^*)$, which is consistent as generalization of the Möbius representation for capacities.

Definition 3.1. *The function $M^*:\mathcal{Q}(N)^*\longrightarrow\mathbb{R}$ is said to be Möbius transform of ν defined by*

$$\begin{aligned}M^*(A_1,B_1)&=\sum_{(A_2,B_2)\sqsubseteq^*(A_1,B_1^c)}(-1)^{card(A_1\setminus A_2)+card(B_1^c\setminus B_2)}\ \nu(A_2,B_2^c)\\&=\sum_{\substack{A_2\subseteq A_1\\B_2\subseteq B_1^c}}(-1)^{card(A_1\setminus A_2)+card(B_1^c\setminus B_2)}\ \nu(A_2,B_2^c),\end{aligned} \tag{4}$$

for all $(A_1,B_1)\in\mathcal{Q}(N)^*$.

Conversely, the following proposition proves a one-to-one correspondence between the bi-capacity ν and the Möbius transform of ν.

Proposition 3.2. *Consider a bi-capacity $\nu:\mathcal{Q}(N)^*\longrightarrow\mathbb{R}$ and let $M^*:\mathcal{Q}(N)^*\longrightarrow\mathbb{R}$ be the Möbius transform of ν. Then,*

$$\nu(A_1,B_1)=\sum_{(A_2,B_2)\sqsubseteq^*(A_1,B_1^c)}M^*(A_2,B_2^c)=\sum_{\substack{A_2\subseteq A_1\\B_2\subseteq B_1^c}}M^*(A_2,B_2^c), \tag{5}$$

for all $(A_1,B_1)\in\mathcal{Q}(N)^*$.

Proof. By Definition 3.1, for all $(A_2, B_2^c) \in \mathcal{Q}(N)^*$ we have

$$M^*(A_2, B_2^c) = \sum_{\substack{A_3 \subseteq A_2 \\ B_3 \subseteq B_2}} (-1)^{card(A_2 \setminus A_3)+card(B_2 \setminus B_3)} \ \nu(A_3, B_3^c).$$

Then,

$$\begin{aligned}
\sum_{\substack{A_2 \subseteq A_1 \\ B_2 \subseteq B_1^c}} M^*(A_2, B_2^c) &= \sum_{\substack{A_2 \subseteq A_1 \\ B_2 \subseteq B_1^c}} \sum_{\substack{A_3 \subseteq A_2 \\ B_3 \subseteq B_2}} (-1)^{card(A_2 \setminus A_3)+card(B_2 \setminus B_3)} \ \nu(A_3, B_3^c) \\
&= \sum_{\substack{A_3 \subseteq A_1 \\ B_3 \subseteq B_1^c}} (-1)^{card(A_3)+card(B_3)} \ \nu(A_3, B_3^c) \Big(\sum_{\substack{A_2 : A_3 \subseteq A_2 \subseteq A_1 \\ B_2 : B_3 \subseteq B_2 \subseteq B_1^c}} (-1)^{card(A_2)+card(B_2)} \Big) \\
&= \sum_{\substack{A_3 = A_1 \\ B_3 = B_1^c}} (-1)^{card(A_3)+card(B_3)} \ \nu(A_3, B_3^c) \Big(\sum_{\substack{A_2 : A_3 \subseteq A_2 \subseteq A_1 \\ B_2 : B_3 \subseteq B_2 \subseteq B_1^c}} (-1)^{card(A_2)+card(B_2)} \Big) \\
&\quad + \sum_{\substack{A_3 \subset A_1 \\ B_3 \subset B_1^c}} (-1)^{card(A_3)+card(B_3)} \ \nu(A_3, B_3^c) \Big(\sum_{\substack{A_2 : A_3 \subseteq A_2 \subseteq A_1 \\ B_2 : B_3 \subseteq B_2 \subseteq B_1^c}} (-1)^{card(A_2)+card(B_2)} \Big) \\
&= (-1)^{card(A_1)+card(B_1)} \ \nu(A_1, B_1) \ \big((-1)^{card(A_1)+card(B_1)} \big) \\
&\quad + \sum_{\substack{A_3 \subset A_1 \\ B_3 \subset B_1^c}} (-1)^{card(A_3)+card(B_3)} \ \nu(A_3, B_3^c) \ (0) \\
&= \nu(A_1, B_1).
\end{aligned}$$

□

Möbius representation is related to the concept of *k-additivity*. The fundamental notion of *k-additivity* proposed by Grabisch and Labreuche ([14]) enables to reduce the number of bi-capacity coefficients. Also, Fujimoto et al. [11] have proposed the characterization of k-additivity of bi-capacities by using the bipolar Möbius transform. In this section, we define the k-additivity of bi-capacities based on the partially ordered set $(\mathcal{Q}(N)^*, \sqsubseteq^*)$.

Definition 3.3. *Let $k \in \{1, \ldots, n-1\}$. A bi-capacity ν is said to be k-additive if and only if it's Möbius transform $M^*(A, B) = 0$ whenever $card(A) + card(B^c) > k$, and there exists some $(A, B) \in \mathcal{Q}(N)^*$, such that $card(A) + card(B^c) = k$ and $M^*(A, B) \neq 0$.*

Proposition 3.4. *Let ν be a 2-additive bi-capacity based on the partially ordered set $(\mathcal{Q}(N)^*, \sqsubseteq^*)$ and M^* its Möbius transform. For all $i, j \in \{1, \cdots, n\}$, $i \neq j$, and any $(A, B) \in \mathcal{Q}(N)^*$ we have:*

$$\begin{aligned}
\nu(A, B) = \sum_{i \in A} M^*(i, N) + \sum_{\{i,j\} \subseteq A} M^*(\{i, j\}, N) + \sum_{\substack{i \in A \\ j \in B}} M^*(i, j^c) \\
+ \sum_{j \in B} M^*(\emptyset, j^c) + \sum_{\{i,j\} \subseteq B} M^*(\emptyset, \{i, j\}^c). \qquad (6)
\end{aligned}$$

Proof. By Proposition 3.2, for all $(A_1, B_1) \in \mathcal{Q}(N)^*$ we have

$$\nu(A_1, B_1) = \sum_{(A_2, B_2) \sqsubseteq^* (A_1, B_1^c)} M^*(A_2, B_2^c).$$

Since ν is 2-additive bi-capacity, the equality (6) can be deduced by using the relation between bi-capacity ν and its Möbius transform M^*. □

4 Bipolar decomposition integral as a general way of selection and global utility analysis

Although there are several approaches for representing preferences, multi-attribute utility theory [22] is possibly the most widely used model in multi-criteria decision making. Consider a decision-making problem that depends on n

criteria (or attributes) described by the alternatives $g_1, \cdots, g_n$ and a set of criteria $N = \{1, \cdots, n\}$. An alternative is characterized by a value with respect to each criterion and it is thus identified with a point in the Cartesian product g of the criteria, i.e., $g = g_1 \times \cdots \times g_n$ is the set of potential alternatives. The preference relation of the Decision Maker (DM) over alternatives is denoted by $\succeq$. For any $x, y \in g$, "$x \succeq y$" means that the DM prefers alternative x before y. To conduct a qualitative analysis of preferences, decision makers will attempt to model $\succeq$ by means of an utility function $u : g \to \mathbb{R}$ such that

$$\forall\, x, y \in g,\ x \succeq y \Leftrightarrow u(x) \geq u(y).$$

Clearly u is a n-dimensional function. Probably the easiest manner to construct u is to consider one-dimensional utility function u_i on each criterion and then to aggregate them by a suitable operator: for all $x \in g$, we put

$$u(x) := u(x_1, \ldots, x_2) = F(u_1(x_1), \ldots, u_n(x_n)),$$

where, F is called an aggregation function.

The choice of utility function is subjective and related to the behavior of the decision maker behavior toward risk and uncertainty. Hence, each decision maker is asked to choose a utility function to express his preferences among criteria of g. Typically, utility function is constructed first, and then an appropriate aggregation function is choosed to model the decision-making process.

Recently, the bipolar decomposition integral has been introduced in [5] as an aggregation function to address aggregation-related integrals on unipolar and bipolar scales. Therefore, we can use the bipolar decomposition integral as a general way for aggregating criteria and analyzing optimal utility under specific constraints in decision-making.

We recall below the basic foundations of the bipolar decomposition integral, and present a new integral in terms of decomposition of the integrand function to be suitable for bipolar scales, which covers families of the decomposition systems for the bipolar Choquet, bipolar Shilkret, and the bipolar PAN Integrals.

For the real-valued function $g = (x_1, \ldots, x_i, \ldots, x_n)$, $x_i \in \mathbb{R}$ with $i \in N$, we denote by (A_g^*, B_g^{*c}) with $A_g^* = \{i \in N | x_i \geq 0\}$ and $B_g^* = \{i \in N | x_i < 0\}$ an element of $\mathcal{Q}(N)^*$ corresponding to the real-valued function g on N.

A sub-decomposition of the real-valued function g is a finite summation $\sum_{i=1}^k \alpha_i \mathbf{1}_{(A_i, B_i)}$ such that:

(i) $\alpha_i \geqslant 0$ and $\sum_{i=1}^k \alpha_i \mathbf{1}_{(A_i, B_i)} \leqslant g$,

(ii) $(A_i, B_i) \sqsubseteq^* (A_g^*, B_g^{*c})$ for every $i = 1, \ldots, k$.

If $\sum_{i=1}^k \alpha_i \mathbf{1}_{(A_i, B_i)} = g$, then $\sum_{i=1}^k \alpha_i \mathbf{1}_{(A_i, B_i)}$ is a decomposition of the real-valued function g, and the value of a decomposition with respect to bi-capacity ν is $\sum_{i=1}^k \alpha_i\, \nu(A_i, B_i)$.

Hereafter, a non-empty subfamily $\mathcal{D}^*$ of $(A_g^*, B_g^{*c}) \setminus \{(\emptyset, \emptyset)\}$ is called a bipolar collection. The class of all bipolar collections will be denoted by $\mathbb{D}^*$. A non-empty family $\mathcal{H}^*$ of bipolar collections on $(A_g^*, B_g^{*c}) \setminus \{(\emptyset, \emptyset)\}$ is called a bipolar decomposition system, and the class of all bipolar decomposition systems will be denoted by $\mathbb{H}^*$. Also, we shall denote the class of all bi-capacities on $\mathcal{Q}(N)^*$ by $\mathcal{M}^*$, and the class of all real-valued functions will be denoted by $\mathcal{X}^*$.

The following definition gives the decomposition integral in terms of decompositions of the integrand function when underlying scales are bipolar.

Definition 4.1. *Let $\nu \in \mathcal{M}^*$ be a bi-capacity and $\mathcal{H}^*$ be a decomposition system. The functional $F_{\mathcal{H}^*} : \mathcal{M}^* \times \mathcal{X}^* \to \mathbb{R}$ given by*

$$F_{\mathcal{H}^*}(\nu, g) = \check{\curlyvee}_{\mathcal{D}^* \in \mathcal{H}^*} \left\{ \sum_{i=1}^k \alpha_i\ \nu(A_i, B_i) \mid \sum_{i=1}^k \alpha_i \mathbf{1}_{(A_i, B_i)} \leqslant g,\ \alpha_i \geqslant 0 \right\}, \tag{7}$$

where $\mathcal{D}^* = \{(A_1, B_1), \ldots, (A_k, B_k)\}$, is called a bipolar decomposition integral with respect to the bipolar decomposition system $\mathcal{H}^*$.

Based on the choice of $\mathcal{H}^*$ in Eq. (7), we get several specific bipolar decomposition integrals based on some common decomposition systems. Here, we introduce some distinguished systems from $\mathbb{H}^*$ and the related integrals:

- Let $\mathcal{H}_1^* = \{\mathcal{D}^* | \mathcal{D}^*$ is a maximal chain with respect to the order relation $\sqsubseteq^*$ on $(A_g^*, B_g^{*c})\}$. Then the bipolar decomposition integral $F_{\mathcal{H}_1^*}(\nu, g)$ is the bipolar Choquet integral, i.e., for any $\nu \in \mathcal{M}^*$ and $g \in \mathcal{X}^*$:

$$BCh(\nu, g) = \check{\curlyvee}_{\mathcal{D}^* \in \mathcal{H}_1^*} \left\{ \sum_{i=1}^k \alpha_i\ \nu(A_i, B_i) \mid \sum_{i=1}^k \alpha_i \mathbf{1}_{(A_i, B_i)} \leqslant g,\ \alpha_i \geqslant 0 \right\}. \tag{8}$$

- Let $\mathcal{H}_2^* = \{\{(A,B)\} | \ (A,B) \sqsubseteq^* (A_g^*, B_g^{*c}),\ (A,B) \neq (\emptyset,\emptyset)\}$. Then the bipolar decomposition integral $F_{\mathcal{H}_2^*}(\nu, g)$ is the bipolar Shilkret integral $BSh(\nu, g)$ ([6, 18]), i.e., for any $\nu \in \mathcal{M}^*$ and $g \in \mathcal{X}^*$:

$$F_{\mathcal{H}_2^*}(\nu, g) = BSh(\nu, g) = \check{\curlyvee}_{(A,B)\in\mathcal{H}_2^*} \left\{ \alpha\ \nu(A,B) \mid \alpha\ \mathbf{1}_{(A,B)} \leqslant g, \alpha \geqslant 0 \right\}. \tag{9}$$

- Let $\mathcal{H}_3^* = \{\{(A_i, B_i)\}_{i\in J} | \ \{(A_i, B_i)\}_{i\in J}$ be a partition of $(A_g^*, B_g^{*c})\}$. Then the bipolar decomposition integral $F_{\mathcal{H}_3^*}(\nu, g)$ is the bipolar PAN integral ($BPan(\nu, g)$) (based on the standard arithmetic operations $+$ and $\cdot$), i.e.,

$$BPan(\nu, g) = \check{\curlyvee}_{\mathcal{D}^*\in\mathcal{H}_3^*} \left\{ \sum_{i\in J} \alpha_i\ \nu(A_i, B_i) \mid \sum_{i\in J} \alpha_i \mathbf{1}_{(A_i,B_i)} \leqslant g,\ \alpha_i \geqslant 0 \right\}. \tag{10}$$

- For the decomposition system $\mathcal{H}_4^* = \{\mathcal{D}^* | \ \forall\ (A_i, B_i), (A_j, B_j) \in \mathcal{D}^*$, $(A_i, B_i) \sqcap^* (A_j, B_j) \in \{(A_i, B_i), (A_j, B_j), (\emptyset, \emptyset)\}\}$, we can define a new corresponding decomposition integral in terms of decomposition of the integrand function to be suitable for bipolar scales, which is a generalization of the PC-integral introduced by Stupňanová [33]. We call this integral a bipolar PC-integral ($BPC(\nu, g)$), and define it by

$$BPC(\nu, g) = \check{\curlyvee}_{\mathcal{D}^*\in\mathcal{H}_4^*} \left\{ \sum_{i\in J} \alpha_i\ \nu(A_i, B_i) \mid \sum_{i\in J} \alpha_i \mathbf{1}_{(A_i,B_i)} \leqslant g,\ \alpha_i \geqslant 0 \right\}. \tag{11}$$

  Notice that the formula of bipolar PC-integral ($BPC(\nu, g)$) covers families of the decomposition systems for the bipolar Choquet, bipolar Shilkret, and the bipolar PAN Integrals.

5 Möbius representation of the bipolar decomposition integrals

In [26] we have established a formula for unifying a family of aggregation functions in terms of the Möbius transform of capacity and minimum operator in a way similar to the Lovász extension of pseudo-Boolean function.

In this section, we provide a general formula to deal with aggregation functions concerning aggregation on unipolar and bipolar scales, which encompasses the Möbius representation of each of the bipolar Choquet integral, bipolar Shilkret integral, and bipolar Pan integral.

The following theorem is the main general result of this paper, which satisfies several families of the bipolar decomposition systems for bipolar concave integral in terms of the Möbius transform of bi-capacities.

Theorem 5.1. *Let $\mathcal{H}^*$ be a bipolar decomposition system and $\mathcal{H}^* \in \{\mathcal{H}_1^*, \mathcal{H}_2^*, \mathcal{H}_3^*, \mathcal{H}_4^*\}$. The bipolar decomposition-integral $F_{\mathcal{H}^*}(M^*, g)$ in terms of Möbius transform M^* of bi-capacity ν is given by* $F_{\mathcal{H}^*}(M^*, g) =$

$$\check{\curlyvee}_{\mathcal{D}^*\in\mathcal{H}^*} \left\{ \sum_{(A_i,B_i)\in\mathcal{D}^*} \left(\sum_{\substack{(C,D)\in\mathcal{P}^*(A_i,B_i^c)\setminus\sqcup^*\mathcal{P}^*(A_j,B_j^c) \\ \forall (A_j,B_j)\in\mathcal{D}^*, j\neq i, (A_i,B_i^c)\sqcap^*(A_j,B_j^c)\in\{(A_j,B_j^c),(\emptyset,\emptyset)\}}} M^*(C, D^c) \right) \bigwedge_{x\in A_i\cup B_i^c} |g(x)| \right\}, \tag{12}$$

where $\mathcal{P}^*(A_i, B_i^c)$ is a set of all sub-coalitions in (A_i, B_i^c) and $\mathcal{P}^*(A_j, B_j^c)$ is a set of all sub-coalitions in (A_j, B_j^c).

Notice that, for any element $(A, B^c) \in \mathcal{Q}(N)^*, (A, B^c) \neq (\emptyset, \emptyset)$, the set of all sub-coalitions in (A, B^c) is denoted by $\mathcal{P}^*(A, B^c)$ and expressed as: $\mathcal{P}^*(A, B^c) = \{(C, D) | \ (C, D) \sqsubseteq^* (A, B^c)\}$.

Proof. The proof is based on the choice of $\mathcal{H}^*$ in the bipolar decomposition system, and the internal summation of the formula (12), i.e.,

$$\sum_{\substack{(C,D)\in\mathcal{P}^*(A_i,B_i^c)\setminus\sqcup^*\mathcal{P}^*(A_j,B_j^c) \\ \forall (A_j,B_j)\in\mathcal{D}^*, j\neq i, (A_i,B_i^c)\sqcap^*(A_j,B_j^c)\in\{(A_j,B_j^c),(\emptyset,\emptyset)\}}} M^*(C, D^c).$$

- Case 1 (Bipolar Choquet integral).
  Let us first consider the case $\mathcal{H}^* = \mathcal{H}_1^* := \{\mathcal{D}^* | \mathcal{D}^*$ is a maximal chain with respect to the order relation $\sqsubseteq^*$ on $(A_g^*, B_g^{*c})\}$. Without loss of generality, we assume that for any two elements $(A_i, B_i), (A_j, B_j) \in$

$\mathcal{D}^*$, $(A_i,B_i)\sqsupset^*(A_j,B_j)$ are nested (that is, $(A_i,B_i)\sqcap^*(A_j,B_j)=(A_j,B_j)$. Then the internal summation reduces to

$$\sum_{(C,D)\in\mathcal{P}^*(A_i,B_i^c)\setminus\mathcal{P}^*(A_j,B_j^c)} M^*(C,D^c)\bigwedge_{x\in A_i\cup B_i^c}|g(x)|,$$

and the right-hand side of (12) becomes

$$\curlyvee_{\mathcal{D}\in\mathcal{H}_1^*}\left\{\sum_{(A_i,B_i)\in\mathcal{D}^*}\left(\sum_{(C,D)\in\mathcal{P}^*(A_i,B_i^c)\setminus\mathcal{P}^*(A_j,B_j^c)} M^*(C,D^c)\bigwedge_{x\in A_i\cup B_i^c}|g(x)|\right)\right\}. \tag{13}$$

By Definition 3.1, we know that

$$M^*(C,D^c)=\sum_{\substack{E\subseteq C\\ F\subseteq D}}(-1)^{card(C\setminus E)+card(D\setminus F)}\ \nu(E,F^c).$$

Hence, we get

$$\curlyvee_{\mathcal{D}\in\mathcal{H}_1^*}\left\{\sum_{(A_i,B_i)\in\mathcal{D}^*}\left(\sum_{(C,D)\in\mathcal{P}^*(A_i,B_i^c)\setminus\mathcal{P}^*(A_j,B_j^c)}\sum_{\substack{E\subseteq C\\ F\subseteq D}}(-1)^{card(C\setminus E)+card(D\setminus F)}\ \nu(E,F^c)\bigwedge_{x\in A_i\cup B_i^c}|g(x)|\right)\right\}.$$

Since $(A_i,B_i)\sqsupset^*(A_j,B_j)$, and by taking $|g(x_{\pi(i)})|=\bigwedge_{x\in\{C\cup D\}\subseteq\{A_i\cup B_i^c\}}|g(x)|$, where π is a permutation on $(g(x_1),\cdots,g(x_n))$ making $|g(x)|$ non-decreasing, i.e., $|g(x_{\pi(1)})|\leq\cdots\leq|g(x_{\pi(n)})|$, we obtain

$$=\curlyvee_{\mathcal{D}\in\mathcal{H}_1^*}\{\ |g(x_{\pi(1)})|\sum_{x_{\pi(1)}\in\{C\cup D\}\subseteq\{A_1\cup B_1^c\}}\ \sum_{\substack{E\subseteq C\\ F\subseteq D}}(-1)^{card(C\setminus E)+card(D\setminus E)}\ \nu(E,F^c)$$

$$+|g(x_{\pi(2)})|\sum_{x_{\pi(2)}\in\{C\cup D\}\subseteq\{A_1\cup B_1^c\}\setminus\{x_{\pi(1)}\}}\ \sum_{\substack{E\subseteq C\\ F\subseteq D}}(-1)^{card(C\setminus E)+card(D\setminus E)}\ \nu(E,F^c)\ +\dots$$

$$+|g(x_{\pi(n-1)})|\sum_{x_{\pi(n-1)}\in\{C\cup D\}\subseteq\{x_{\pi(n-1)},\ x_{\pi(n)}\}}\ \sum_{\substack{E\subseteq C\\ F\subseteq D}}(-1)^{card(C\setminus E)+card(D\setminus E)}\ \nu(E,F^c)$$

$$+|g(x_{\pi(n)})|\sum_{x_{\pi(n)}\{C\cup D\}\subseteq\{x_{\pi(n)}\}}\ \sum_{\substack{E\subseteq C\\ F\subseteq D}}(-1)^{card(C\setminus E)+card(D\setminus E)}\ \nu(E,F^c)\ \}$$

$$=\curlyvee_{\mathcal{D}\in\mathcal{H}_1^*}\{\ |g(x_{\pi(1)})|\ [\nu(A_{\pi(1)},B_{\pi(1)}^c)-\nu(A_{\pi(2)},B_{\pi(2)}^c)]$$

$$+|g(x_{\pi(2)})|\ [\nu(A_{\pi(2)},B_{\pi(2)}^c)-\nu(A_{\pi(3)},B_{\pi(3)}^c)]\ +\dots$$

$$+|g(x_{\pi(n-1)})|\ [\nu(A_{\pi(n-1)},B_{\pi(n-1)}^c)-\nu(A_{\pi(n)},B_{\pi(n)}^c)]$$

$$+|g(x_{\pi(n)})|\ [\nu(A_{\pi(n)},B_{\pi(n)}^c)-\nu(A_{\pi(n+1)},B_{\pi(n+1)}^c)]\ \}$$

(with the convention $(A_{\pi(n+1)},B_{\pi(n+1)}^c)=(\emptyset,\emptyset)$).

$$=\curlyvee_{\mathcal{D}\in\mathcal{H}_1^*}\{\sum_{i=1}^{n}|g(x_{\pi(i)})|\ [\nu(A_{\pi(i)},B_{\pi(i)}^c)-\nu(A_{\pi(i+1)},B_{\pi(i+1)}^c)]\}=BCh(\nu,g),$$

which is a bipolar Choquet integral of g with respect to bi-capacity ν given in [5].

- Case 2 (Bipolar Shilkret integral).
If we consider $\mathcal{H}^* = \mathcal{H}_2^* := \{\{(A,B)\}|\ (A,B) \sqsubseteq^* (A_g^*, B_g^{*c}),\ (A,B) \neq (\emptyset,\emptyset)\}$ is the set of all singleton element systems covered by (A_g^*, B_g^{*c}), then $\{(A_i,B_i)\} = \mathcal{D}^*$ and there is no element (A_j,B_j). Hence, the internal summation reduces to
$$\sum_{(C,D)\in\mathcal{P}^*(A,B^c)} M^*(C,D^c) \bigwedge_{x\in A\cup B^c} |g(x)|,$$
and the right-hand side of (12) becomes
$$\curlyvee_{\{(A,B)\}\in\mathcal{H}_2^*} \left\{ \sum_{(C,D)\in\mathcal{P}^*(A,B^c)} M^*(C,D^c) \bigwedge_{x\in A\cup B^c} |g(x)| \right\}. \tag{14}$$
Therefore, the proof of this case can be completed in a similar way as the proof of the bipolar Choquet integral (Case 1).

- Case 3 (Bipolar Pan integral).
If we consider $\mathcal{H}^* = \mathcal{H}_3^* := \{\{(A_i,B_i)\}_{i\in J}|\ \{(A_i,B_i)\}_{i\in J}$ is a partition of $(A_g^*, B_g^{*c})\}$ to be a finite measurable partition of (A_g^*, B_g^{*c}), then the elements $(A_i,B_i),(A_j,B_j) \in \mathcal{D}^*$ are disjoint whenever i differs from j (i.e., $(A_i,B_i)\sqcap^*(A_j,B_j) = (\emptyset,\emptyset)$). Hence, the internal summation precisely equals to
$$\sum_{(C,D)\in\mathcal{P}^*(A_i,B_i^c)} M^*(C,D^c) \bigwedge_{x\in A_i\cup B_i^c} |g(x)|,$$
and the right-hand side of (12) becomes
$$\curlyvee_{\mathcal{D}\in\mathcal{H}_3^*} \left\{ \sum_{(A_i,B_i)\in\mathcal{D}^*} \left(\sum_{(C,D)\in\mathcal{P}^*(A_i,B_i^c)} M^*(C,D^c) \bigwedge_{x\in A_i\cup B_i^c} |g(x)| \right) \right\}. \tag{15}$$
Therefore, the proof of this case can be completed in a similar way as the proof of the bipolar Choquet integral (Case 1).

- Case 4 (Bipolar PC- integral).
Finaly, consider $\mathcal{H}^* = \mathcal{H}_4^* := \{\mathcal{D}^* \mid \forall\ (A_i,B_i),(A_j,B_j) \in \mathcal{D}^*,$
$(A_i,B_i)\sqcap^*(A_j,B_j) \in \{(A_i,B_i),(A_j,B_j),(\emptyset,\emptyset)\}$. Without loss of generality, we assume that for any two elements $(A_i,B_i),(A_j,B_j) \in \mathcal{D}^*, j \neq i, (A_i,B_i)\sqcap^*(A_j,B_j) \in \{(A_i,B_i),(A_j,B_j),(\emptyset,\emptyset)\}$. So, we have a bipolar collection $\mathcal{D}^*$ which contains nested elements, disjoint pairs of elements, or both.
Therefore, when the elements $(A_i,B_i)\sqsupseteq^*(A_j,B_j)$ are nested (that is, $(A_i,B_i)\sqcap^*(A_j,B_j) = (A_j,B_j)$, this leads to Case 1. Further, if the elements $(A_i,B_i),(A_j,B_j) \in \mathcal{D}^*$ are disjoint whenever i differs from j (i.e., $(A_i,B_i)\sqcap^*(A_j,B_j) = (\emptyset,\emptyset)$), this leads to Case 3. Consequently, the formula (12) satisfies the application of the bipolar decomposition system $\mathcal{H}^* = \mathcal{H}_4^*$ when constructing a bipolar decomposition integral, which completes the proof of the theorem.

□

As a particular case of the above theorem, the following corollary shows that the bipolar decomposition integral in terms of Möbius transform encompasses the decomposition integral in terms of Möbius transform with respect to a capacity.

Corollary 5.2. *Let $\nu \in \mathcal{M}$ be a capacity, M the Möbius transform of ν, and $\mathcal{H}$ a decomposition system. Then, for a positive real-valued function $g = (x_1,\ldots,x_i,\ldots,x_n),\ x_i \in [0,\infty]$, the bipolar decomposition integral $F_{\mathcal{H}^*}(M^*,g)$ in terms of the Möbius transform M^* reduces to the decomposition integral $F_{\mathcal{H}}(M,g)$ in terms of the Möbius transform M:*
$$F_{\mathcal{H}^*}(M^*,g) = F_{\mathcal{H}}(M,g) = \bigvee_{\mathcal{D}\in\mathcal{H}} \left\{ \sum_{A_i\in\mathcal{D}} \left(\sum_{\substack{B\in\mathcal{P}(A_i)\setminus\bigcup\mathcal{P}(A_j) \\ \forall A_j\in\mathcal{D}, j\neq i, A_i\cap A_j\in\{A_j,\emptyset\}}} M(B) \right) \bigwedge_{x\in A_i} g(x) \right\}, \tag{16}$$
where $\mathcal{P}(A_i)$ is a set of all sub-coalitions in A_i and $\mathcal{P}(A_j)$ is a set of all sub-coalitions in A_j.

Proof. For any capacity $\nu \in \mathcal{M}$, where $\mathcal{M}$ the class of all capacities on N, decomposition system $\mathcal{H}$ with respect to a capacity, and any $[0,\infty]$-valued function $g \in \mathcal{X}$, the bipolar maximum $\check{\Upsilon}$ coincides with $\bigvee$ on $[0,\infty]$. Hence, it is easy to verify from this fact that the bipolar decomposition integral $F_{\mathcal{H}^*}(M^*, g)$ in terms of the Möbius transform M^* with respect to the decomposition system $\mathcal{H}^* = \{\mathcal{D}^* \backslash \mathcal{D} \in \mathcal{H}\}$, where, for $\mathcal{D} = \{A_i\}_{i=1}^{j}$, we denote by $\mathcal{D}^*$ a related bipolar collection given by $\{(A_i, N)\}_{i=1}^{j}$, reduces to the decomposition integral $F_{\mathcal{H}}(M, g)$ in terms of the Möbius transform M with respect to the decomposition system $\mathcal{H}$. □

Applying Theorem 5.1, in the following result we propose a simple equivalent formula of the bipolar Choquet integral in terms of the Möbius transform M^* of the bi-capacity ν.

Corollary 5.3. *Let $\mathcal{H}_1^* = \{\mathcal{D}^* | \mathcal{D}^*$ is a maximal chain w. r. t. the order relation $\sqsubseteq^*$ on $(A_g^*, B_g^{*c})\}$ be a bipolar decomposition system. The bipolar Choquet integral $(BCh(M^*, g))$ in terms of the Möbius transform M^* is given by*

$$BCh(M^*, g) = \sum_{(C,D) \sqsubseteq^* (A_g^*, B_g^*)} M^*(C, D^c) \bigwedge_{x \in C \cup D} |g(x)|. \tag{17}$$

Proof. The bipolar decomposition system of a bipolar Choquet integral has a maximal chain in (A_g^*, B_g^{*c}), which is a bipolar collection $\{(A_i, B_i)\}_{i=1}^{k}$ such that $(A_i, B_i) \sqsupset^* (A_{i+1}, B_{i+1})$ for every $i = 1, \ldots, k-1$, , where $k = \operatorname{card}(A_g^*) + \operatorname{card}(B_g^{*c})$. Hence, we can rewrite equivalently the bipolar Choquet integral (formula (13)) in terms of the Möbius transform as follows: $BCh(M^*, g) =$

$$\check{\Upsilon}_{\mathcal{D}^* \in \mathcal{H}_1^*} \left\{ \sum_{(A_i, B_i) \in \mathcal{D}^*} \left(\sum_{(C,D) \in \mathcal{P}^*(A_i, B_i^c) \backslash \mathcal{P}^*(A_{i+1}, B_{i+1}^c)} M^*(C, D^c) \bigwedge_{x \in A_i \cup B_i^c} |g(x)| \right) \right\}, \tag{18}$$

where $\mathcal{P}^*(A_i, B_i^c)$ is a set of all sub-coalitions in (A_i, B_i^c), and $\mathcal{P}^*(A_{i+1}, B_{i+1}^c)$ is a set of all sub-coalitions in (A_{i+1}, B_{i+1}^c) with the convention $\mathcal{P}^*(A_{n+1}, B_{n+1}^c) = (\emptyset, \emptyset)$.

Then for any $\mathcal{D}^* \in \mathcal{H}_1^*$, $\mathcal{D}^* = \{(A_i, B_i)\}_{i=1}^{k}$, with $(A_i, B_i) \sqsupset^* (A_{i+1}, B_{i+1})$, it holds

$$\overset{*}{\bigsqcup_{(C,D) \in \mathcal{P}^*(A_i, B_i^c) \backslash \mathcal{P}^*(A_{i+1}, B_{i+1}^c)}} \{(C, D)\} = (A_g^*, B_g^*).$$

Thus, the right-hand side of the formula (18) becomes

$$BCh(M^*, g) = \sum_{(C,D) \sqsubseteq^* (A_g^*, B_g^*)} M^*(C, D^c) \bigwedge_{x \in C \cup D} |g(x)|.$$

□

6 The 2-additive bipolar decomposition integrals

In multicriteria decision making problems, the bipolar Choquet integral with respect to a bi-capacity typically requires the decision maker to set $3^n - 3$ values. However, when the number of parameters n is not small, this significantly complicates the problem and requires simplification to reduce the number of parameters. The use of bipolar Choquet integral with respect to a 2-additive bi-capacity has revealed to be useful in order to reduce the number of parameters from $3^n - 3$ to $2n^2$ (this is effective when $n > 2$). The bipolar Choquet integral with respect to a 2-additive bi-capacity was used in several applications, and it has been taken much attention in the literature, e. g. [2, 7, 25].

In this section, we propose special types of recognized bipolar decomposition integrals with respect to a 2-additive bi-capacity, to obtain more simple expressions those given by general formulas that deal with the general bipolar decomposition integrals.

Considering the case $\mathcal{H}^* = \mathcal{H}_1^*$ is a maximal chain with respect to the order relation $\sqsubseteq^*$ on (A_g^*, B_g^{*c}) (i.e., using Corollary 2), the bipolar Choquet integral with respect to a 2-additive bi-capacity, we will call it for short the 2-additive bipolar Choquet integral. It is given by

$$BCh(M^*, g) = \sum_{k \in A_g^*} M^*(k, N)\ |g(k)| + \sum_{\{k,l\} \subseteq A_g^*} M^*(\{k,l\}, N)\ (|g(k)| \wedge |g(l)|)$$

$$+\sum_{\substack{k\in A_g^*\\ l\in B_g^*}} M^*(k,\{l^c\})\ (|g(k)|\wedge|g(l)|)+\sum_{l\in B_g^*} M^*(\emptyset,\{l^c\})\ |g(l)|$$

$$+\sum_{\{k,l\}\subseteq B_g^*} M^*(\emptyset,\{k,l\}^c)\ (|g(k)|\wedge|g(l)|). \quad (19)$$

In the case ($\mathcal{H}^*=\mathcal{H}_2^*$) of the bipolar Shilkret integral ($BSh(M^*,g)$) with respect to a 2-additive bi-capacity, we will call it for short the 2-additive bipolar Shilkret integral, and it is given by

$$BSh(M^*,g)=\check{\curlyvee}_{\{(A,B)\}\in\mathcal{H}_2^*}\{(\sum_{l\in B^c} M^*(\emptyset,\{l^c\})+\sum_{\substack{k\in A\\ l\in B^c}} M^*(k,\{l^c\})+$$

$$\sum_{\{k,l\}\subseteq B^c} M^*(\emptyset,\{k,l\}^c)+\sum_{k\in A} M^*(k,N)+\sum_{\{k,l\}\subseteq A} M^*(\{k,l\},N))\bigwedge_{x\in A\cup B^c}|g(x)|\}. \quad (20)$$

For the Pan integral ($\mathcal{H}^*=\mathcal{H}_3^*$) with respect to a 2-additive bi-capacity, we will call it for short the 2-additive bipolar Pan integral, and it is given by

$$BPan(M^*,g)=\check{\curlyvee}_{\mathcal{D}^*\in\mathcal{H}_3^*}\{\sum_{(A_i,B_i)\in\mathcal{D}^*}(\sum_{l\in B_i^c} M^*(\emptyset,\{l^c\})+\sum_{\substack{k\in A_i\\ l\in B_i^c}} M^*(k,\{l^c\})+$$

$$\sum_{\{k,l\}\subseteq B_i^c} M^*(\emptyset,\{k,l\}^c)+\sum_{k\in A_i} M^*(k,N)+\sum_{\{k,l\}\subseteq A_i} M^*(\{k,l\},N))\bigwedge_{x\in A_i\cup B_i^c}|g(x)|\}. \quad (21)$$

7 Practical example

In this section, we provide a practical numerical example to illustrate the applicability of the suggested results when dealing with aggregation on bipolar scales, and present relative simplicity of calculating the 2-additive bipolar decomposition integrals. The first part of this example addresses how to evaluate and select the best medical treatment among treatment protocols for a patient with a chronic medical case. In the second part of this example, to achieve maximum recovery under specific constraints, we will extend the idea inspired by the literature [29] to include bipolar scales for organizing a treatment plan according to the preferred protocol.

7.1 Selecting the best treatment protocol

Suppose there are two treatment protocols (treatment pathways), Protocol P_1 and Protocol P_2 , and the treating physician selects between them (Protocol P_1 and Protocol P_2) based on three criteria: side effects (SE) as a negative criterion, treatment effectiveness (TE) as a positive criterion, and treatment cost (TC) as a negative criterion, with values ranging from -1 (very bad) to 1 (very good), i.e., on the bipolar scale [-1,1]. Let us assign values to the evaluation of both treatment protocols based on the criteria (SE, TE, and TC) as shown in Table 1.

Treatment Protocol	**SE**	**TE**	**TC**
Protocol P_1	- 0.5	0.7	0.4
Protocol P_2	- 0.3	0.4	0.6

Table 1: Treatment protocols with their respective TE, SE, and TC values

For the treatment protocol P_1, i.e., $g=(-0.5,0.7,0.4)$, the bipolar decomposition system is related to the g-based element $(A_g^*,B_g^{*c})=(23,23)$. That is, $\{(23,23),(2,23),(3,23),(2,123),(3,123),(23,123),(\emptyset,23),(\emptyset,123)\}$.

We express the effects (negative and positive) between individual and combined treatments as bi-capacity values, with values ranging from -1 (very bad) to 1 (very good), i.e., on the bipolar scale [-1,1]. Assume bi-capacity values and the Möbius transform M^* of a bi-capacity ν are given in Table 2.

(A, B)	(23,23)	(2,23)	(3,23)	(2,123)	(3,123)	(23,123)	(∅,23)	(∅,123)
$\nu(A,B)$	0.3	-0.1	0	0.2	0.2	0.6	-0.2	0
$M^*(A,B)$	0	-0.1	0	0.2	0.2	0.2	-0.2	0

Table 2: Bi-capacity values and the Möbius transform M^* of bi-capacity ν.

The bipolar Choquet integral in terms of decompositions (formula (8)) of the treatment protocol P_1 (i.e., the integrated function $g = (-0.5, 0.7, 0.4)$) is obtained at the chain of $\mathcal{D}^* = \{(23, 23), (2, 23), (2, 123)\}$, and g has a $\mathcal{D}^*-$decomposition:

$$\begin{aligned}(-0.5, 0.7, 0.4) &= 0.4\ (-1, 1, 1) + 0.1\ (-1, 1, 0) + 0.2\ (0, 1, 0)\\ &= 0.4\ \mathbf{1}_{(23,23)} + 0.1\ \mathbf{1}_{(2,23)} + 0.2\ \mathbf{1}_{(2,123)}\\ &= 0.4\ \nu(23, 23) + 0.1\ \nu(2, 23) + 0.2\ \nu(2, 123).\end{aligned}$$

Thus,

$$BCh(\nu, g) = 0.15.$$

Using the 2-additive bipolar Choquet integral (formula (19)) of the treatment protocol P_1, we obtain

$$\begin{aligned}BCh(M^*, g) &= 0.4\ [M^*(3, 23) + M^*(3, 123) + M^*(23, 123)]\\ &\quad + 0.5\ [M^*(2, 23) + M^*(\emptyset, 23)] + 0.7\ M^*(2, 123) = 0.15.\end{aligned}$$

The bipolar Shilkret integral in terms of decompositions (formula (9)) of the treatment protocol P_1 is obtained at $\mathcal{D}^* = \{(2, 23)\}$. That is,

$$(-0.5, 0.7, 0.4) = 0.5\ (-1, 1, 0) = 0.5\ \mathbf{1}_{(2,23)} = 0.5\ \nu(2, 23).$$

Thus,

$$BSh(\nu, g) = -0.05.$$

Using the 2-additive bipolar Shilkret integral (formula (20)) of the treatment protocol P_1, we obtain

$$BSh(M^*, g) = 0.5\ [M^*(2, 23) + M^*(2, 123) + M^*(\emptyset, 23)] = -0.05.$$

For the bipolar Pan integral in terms of decompositions (formula (10)) of the treatment protocol P_1 is obtained at $\mathcal{D}^* = \{(23, 123), (\emptyset, 23)\}$. That is,

$$\begin{aligned}(-0.5, 0.7, 0.4) &= 0.4\ (0, 1, 1) + 0.5\ (-1, 0, 0)\\ &= 0.4\ \mathbf{1}_{(23,123)} + 0.5\ \mathbf{1}_{(\emptyset,23)}\\ &= 0.4\ \nu(23, 123) + 0.5\ \nu(\emptyset, 23).\end{aligned}$$

Thus,

$$BPan(\nu, g) = 0.14.$$

Using the 2-additive bipolar Pan integral (formula (21)) of the treatment protocol P_1, we obtain

$$BPan(M^*, g) = 0.4\ [M^*(23, 123) + M^*(2, 123) + M^*(3, 123)] + 0.5\ M^*(\emptyset, 23)] = 0.14.$$

Similarly, we can evaluate the treatment protocol P_2 (i.e., the integrated function $g = (-0.3, 0.4, 0.6)$) using the above bipolar decomposition integrals. The calculation of bipolar decomposition integrals for both treatment protocols is shown in Table 3,

Treatment Protocol	BCh	BSh	BPan
Protocol P_1	0.15	-0.05	0.14
Protocol P_2	0.19	0.09	0.18

Table 3: Bipolar decomposition integrals for each treatment protocol

Under the assumption of the preference relationship $\succeq$ mentioned in Section 3 for the values in Table 3, the treating physician concludes that treatment protocol P_2 has the greater potential to cure the patient and would be a better choice than treatment protocol P_1.

7.2 Optimal medical plan for preferred treatment protocol

The preferred treatment protocol is evaluated after a period of treatment administration using clinical indicators (such as blood sugar measurements), functional indicators (such as walking ability), and other measurable indicators.

Let's assume that the preferred treatment protocol includes three treatments: T_1, T_2, and T_3. Suppose the preferred treatment protocol is evaluated based on measuring the effectiveness of the three treatments over a 20-day period (i.e., on [0,20]), such that the effectiveness of the first treatment is evident in a maximum of 16 days [or we can say that the maximum time period for administering the first treatment is 16 days] (i.e., $g(T_1) = 16$), the effectiveness of the second treatment is evident in a maximum of 5 days (i.e., $g(T_2) = 5$), and the effectiveness of the third treatment is evident in a maximum of 18 days (i.e., $g(T_3) = 18$). For a ratio scale, we can convert the regular values given in the interval [0, 20] to bipolar scale on [-10, 10], and this gives $g(T_1) = 6$, $g(T_2) = -5$, and $g(T_3) = 8$. Hence, the bipolar decomposition system is related to the g-based element $(A^*_g, B^{*c}_g) = (13, 13)$. That is, $\{(13, 13), (1, 13), (3, 13), (13, 123), (1, 123), (3, 123), (\emptyset, 13), (\emptyset, 123)\}$.

We express the effects (negative and positive) between individual and combined treatments as bi-capacity values, with values ranging from -1 (very bad) to 1 (very good), i.e., on the bipolar scale [-1,1]. Assume bi-capacity values and the Möbius transform M^* of bi-capacity ν are given in Table 4.

(A, B)	**(13,13)**	**(1,13)**	**(3,13))**	**(13,123)**	**1,123)**	**(3,123)**	**(∅,13)**	**(∅,123)**
$\nu(A, B)$	0.2	0	0	0.4	0.2	0.2	-0.2	0
$M^*(A, B)$	0	0	0	0	0.2	0.2	-0.2	0

Table 4: Bi-capacity values and the Möbius transform M^* of bi-capacity ν.

We aim to develop an optimal medical plan for a preferred treatment protocol to maximize a patient's recovery under the following condition:

A single treatment combination (includes three treatments) is administered to a patient, and then the treatment cannot be resumed after its administration period has ended.

In this case, since the treatment cannot be resumed after its administration period has ended, a combination of treatments must be developed that maximizes the patient's recovery before any treatment is discontinued. To ensure this plan is achieved, we consider a single combination that includes administering all treatments $\{T_1, T_2, T_3\}$ simultaneously until the minimum available treatment reaches its maximum administration period. Here, the minimum available treatment is T_2, where $|g(T_2)| = 5$ days. Then the treatment T_2 is stopped and the combination of treatments $\{T_1, T_3\}$ is kept together for $|g(T_1)| - |g(T_2)| = 1$ day. Finally, treatment T_1 is stopped and T_3 alone is kept $|g(T_3)| - |g(T_1)| = 2$ days. Then the maximum recovery of the patient under this plan is consistent with the result of the bipolar Choquet integral calculation.

The bipolar Choquet integral in terms of decompositions (formula (8)) of the preferred treatment protocol (i.e., the integrated function $g = (6, -5, 8)$) is obtained at the chain of $\mathcal{D}^* = \{(13, 13), (13, 123), (3, 123)\}$, and g has a $\mathcal{D}^*-$decomposition:

$$\begin{aligned}(6, -5, 8) &= 5\ (1, -1, 1) +\ (1, 0, 1) + 2\ (0, 0, 1)\\ &= 5\ \mathbf{1}_{(13,13)} +\ \mathbf{1}_{(13,123)} + 2\ \mathbf{1}_{(3,123)}\\ &= 5\ \nu(13, 13) +\ \nu(13, 123) + 2\ \nu(3, 123).\end{aligned}$$

Thus,

$$BCh(\nu, g) = 1.8.$$

Using the 2-additive bipolar Choquet integral (formula (19)) of the preferred treatment protocol (i.e., the integrated function $g = (6, -5, 8)$), we obtain

$$\begin{aligned}BCh(M^*, g) = 6\ &[M^*(1, 13) + M^*(1, 123) + M^*(13, 123)]\\ &+ 8\ [M^*(3, 13) + M^*(3, 123)] + 5\ M^*(\emptyset, 13)) = 1.8.\end{aligned}$$

By following this medical plan for the preferred treatment protocol, maximum recovery can be achieved for the patient within the constraint specified above.

8 Conclusions

In this work, we have presented a general framework for dealing with the Möbius representation for the bipolar decomposition integrals, which includes the Möbius representation of each of the bipolar Choquet integral, bipolar

Shilkret integral, and bipolar Pan integral. Then, we have introduced the expressions of computing bipolar decomposition integrals with respect to a 2-additive bi-capacity. Furthermore, we have provided a practical example to demonstrate the efficiency of the proposed results and their applicability in the medical sector. In the second part of the practical example (subsection 7.2), we aimed to describe how to implement the optimal medical plan for the preferred treatment protocol. As with the case of bipolar Choquet integral, we can also discuss other treatment plans consistent with the calculations of other bipolar decomposition integrals (such as the bipolar Shilkret, bipolar Pan, and bipolar concave integral).

Similarly to the discussion of the Möbius representation for the bipolar decomposition integrals, we can obtain the corresponding results via a dual way of bipolar decomposition integrals based on the Möbius representation (i.e., an extension of the Theorem 5 in the literature [26] to be suitable for bipolar scales). The study of the dual way for bipolar decomposition integrals based on the Möbius representation and its applications is an interesting area for the future research. In addition, this approach can also address optimization problems in other sectors.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The second author kindly acknowledges the support of the grant VEGA 1/0036/23. The third author was supported by the IGA grant of Palacky University Olomouc PrF2025011.

References

[1] J. Abbas, *The bipolar Choquet integrals based on ternary-element sets*, Journal of Artificial Intelligence and Soft Computing Research, **6**(1) (2016), 13-21. https://doi.org/10.1515/jaiscr-2016-0002

[2] J. Abbas, *The 2-additive Choquet integral of bi-capacities*, in Artificial Intelligence and Soft Computing. Lecture Notes in Computer Science, L. Rutkowski, R. Scherer, M. Korytkowski, W. Pedrycz, R. Tadeusiewicz and J. Zurada (eds.), Springer, 2019. https://doi.org/10.1007/978-3-030-20912-4_27

[3] J. Abbas, *The balancing bipolar Choquet integrals*, International Journal of Innovative Computing, Information and Control, **17**(3) (2021), 949-957. https://doi.org/10.24507/ijicic.17.03.949

[4] J. Abbas, *The Banzhaf interaction index for bicooperative game*, International Journal of General Systems, **50**(5) (2021), 486-500. https://doi.org/10.1080/03081079.2021.1924166

[5] J. Abbas, R. Mesiar, R. Halaš, *Bipolar decomposition integrals*, International Journal of Approximate Reasoning, **183** (2025), 109439. https://doi.org/10.1016/j.ijar.2025.109439

[6] J. Abbas Ghafil, *A new approach to the bipolar Shilkret integral*, Mathematics and Computational Sciences, **3**(4) (2022), 46-54. https://doi.org/10.30511/mcs.2022.1971146.1092

[7] J. Ah-Pine, B. Mayag, A. Rolland, *Elicitation of 2-additive bi-capacity parameters*, EURO Journal on Decision Processes, **3**(1-2) (2015), 5-28. https://doi.org/10.1007/s40070-015-0043-3

[8] J. M. Bilbao, J. R. Fernandez, A. Jimenez Losada, E. Lebron, *Bicooperative games*, First World Congress of the Game Theory Society (Games 2000) July 24-28, Bilbao, Spain, 2000.

[9] G. Choquet, *Theory of capacities*, Annales de L'institut Fourier, **5** (1953), 131-295. https://eudml.org/doc/73714

[10] Y. Even, E. Lehrer, *Decomposition-integral: Unifying Choquet and the concave integrals*, Econ Theory, **56** (2014), 33-58. https://doi.org/10.1007/s00199-013-0780-0

[11] K. Fujimoto, T. Murofushi, M. Sugeno, *k-Additivity and C-decomposability of bi-capacities and its integral*, Fuzzy Sets and Systems, **158**(15) (2007), 1698-1712. https://doi.org/10.1016/j.fss.2007.03.002

[12] M. Grabisch, *The symmetric Sugeno integral*, Fuzzy Sets and Systems, **139**(3) (2003), 473-490. https://doi.org/10.1016/S0165-0114(02)00499-2

[13] M. Grabisch, B. de Baets, J. Fodor, *The quest for rings on bipolar scales*, International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems, **12** (2004), 499-512. https://doi.org/10.48550/arXiv.0804.1270

[14] M. Grabisch, C. Labreuche, *Bi-capacities I: Definition, Möbius transform and interaction*, Fuzzy Sets and Systems, **151**(2) (2005), 211-236.

[15] M. Grabisch, C. Labreuche, *Bi-capacities, Part II: the Choquet integral*, Fuzzy Sets and Systems, **151**(2) (2005), 237-259. https://doi.org/10.1016/j.fss.2004.08.013

[16] M. Grabisch, T. Murofushi, M. Sugeno, *Fuzzy measures and integrals: Theory and applications*, Physica Verlag, Heidelberg, 2000. https://doi.org/10.1007/978-94-015-8449-4_6

[17] S. Greco, R. Mesiar, F. Rindone, *Discrete bipolar universal integrals*, Fuzzy Sets and Systems, **252** (2014), 55-65. https://doi.org/10.1016/j.fss.2014.02.002

[18] S. Greco, F. Rindone, *Bipolar fuzzy integrals*, Fuzzy Sets and Systems, **220** (2013), 21-33. https://doi.org/10.1016/j.fss.2012.11.021

[19] M. Hesham, J. Abbas, *Multi-criteria decision making on the optimal drug for rheumatoid arthritis disease*, Iraqi Journal of Science, **62**(5) (2021), 1659-1665. https://doi.org/10.24996/ijs.2021.62.5.28

[20] M. Kalina, B. Mihailović, M. Štrboja, *Bipolar ordered weighted quasi-averages and induced bipolar ordered weighted averages: BIGOWA and IBIOWA operators*, Iranian Journal of Fuzzy Systems, **22**(3) (2025), 39-54. https://doi.org/10.22111/ijfs.2025.50487.8917

[21] P. Karczmarek, A. Kiersztyn, W. Pedrycz, *Generalized Choquet integral for face recognition*, International Journal of Fuzzy Systems, **20**(3) (2017), 1047-1055. https://doi.org/10.1007/s40815-017-0355-5

[22] R. L. Keeney, H. Raiffa, *Decision with multiple objectives*, Cambridge University Press, 1976. https://doi.org/10.1017/CBO9781139174084

[23] D. Kim, H. Kim, L. C. Jang, *Some inequalities for generalized Choquet integrals of triangular fuzzy number-valued functions and its application*, Iranian Journal of Fuzzy Systems, **21**(6) (2024), 83-99. https://doi.org/10.22111/ijfs.2024.48347.8504

[24] E. Lehrer, *A new integral for capacities*, Econ Theory, **39** (2009), 157-176.

[25] Y. Liang, Y. Ju, X. Zeng, H. Li, P. Dong, T. Ju, *A user-generated content-based social network large-scale group decision-making approach in healthcare service: Case study of general practitioners selection in UK*, Expert Systems with Applications, **261** (2025), 125542. https://doi.org/10.1016/j.eswa.2024.125542

[26] R. Mesiar, J. Abbas, J. Li, *The 2-additive decomposition integrals and their applications*, Fuzzy Sets and Systems, **507** (2025), 109316. https://doi.org/10.1016/j.fss.2025.109316

[27] R. Mesiar, A. Mesiarová-Zemánková, K. Ahmad, *Discrete Choquet integral and some of its symmetric extensions*, Fuzzy Sets and Systems, **184**(1) (2011), 148-155. https://doi.org/10.1016/j.fss.2010.11.013

[28] R. Mesiar, A. Stupňanová, *Decomposition integrals*, International Journal of Approximate Reasoning, **54**(8) (2013), 1252-1259. https://doi.org/10.1016/j.ijar.2013.02.001

[29] R. Mesiar, A. Stupňanová, *Integral sums and integrals*, In Non-Additive Measures; V. Torra; Y. Narukawa; M. Sugeno, Eds., Springer, (2014), 63-78. https://doi.org/10.1007/978-3-319-03155-2_3

[30] A. Naeem, J. Abbas, *A computational model for multi-criteria decision making in traffic jam problem*, Journal of Automation Mobile Robotics and Intelligent Systems, **15**(2) (2022), 39-43. https://doi.org/10.14313/JAMRIS/2-2021/12

[31] G. C. Rota, *On the foundations of combinatorial theory I. Theory of Möbius functions*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **2** (1964), 340-368.

[32] N. Shilkret, *Maxitive measure and integration*, Indagationes Mathematicae, **33** (1971), 109-116. https://doi.org/10.1016/S1385-7258(71)80017-3

[33] A. Stupňanová, *A note on decomposition integrals*, Communications in Computer and Information Science, Advances in Computational Intelligence, Part IV 300, (2012), 542-548. https://doi.org/10.1007/978-3-642-31724-8_57

[34] Z. Wang, G. J. Klir, *Generalized measure theory*, Springer, Boston, 2009.