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Original paper

A new insight of the distributivity for S-uninorms

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Abstract

Although the scholars studied the distributivity for S-uninorms, at least, the underlying uninorm of one S-uninorm in the distributivity equations was assumed to be in $\mathcal{U}_{\min}$ except the distributivity for S-uninorms over t-(co)norms. In this paper, we further characterize the distributivity for S-uninorms, where the conjunctive underlying uninorms of the S-uninorms in the distributivity equations are not fixed in $\mathcal{U}_{\min}$ but arbitrary. Firstly, we discuss the distributivity between S-uninorms. Secondly, we analyze the distributivity for S-uninorms over T-uninorms. Thirdly, we investigate the distributivity for S-uninorms over disjunctive uninorms. Because S-uninorms in those distributivity equations are arbitrary, our results are extensions of the previous results on the distributivity for S-uninorms.

Keywords: Aggregation operators, distributivity, uninorms, S-uninorms, T-uninorms.

1 Introduction

Aczél [1] firstly discussed the distributivity equations from the perspective of functional equations. The distributivity equations involving aggregation operators [4, 7, 18, 24] are successfully applied in fuzzy sets and fuzzy logic, i.e., the distributivity for t-(co)norms [25] and the distributivity for fuzzy implications [2, 3, 35, 37, 41, 55]. Meanwhile, the distributivity equations play an important role in utility theory [14, 15, 18, 20] and integration theory [43].

Yager and Rybalov [53] introduced uninorms, which are special aggregation operators. Fodor et al. [17] discussed the structure of uninorms, which is a generalization of both t-norms and t-conorms. Moreover, uninorms have been applied in many topics, i.e., data mining, expert systems, fuzzy system modeling, image processing, neutral networks and so on [5, 6, 10, 50, 51, 52, 54]. Due to the great quantity of applications, different classes of uninorms have been characterized from a theoretical point of view, that is, uninorms in $\mathcal{U}_{\max}$ and $\mathcal{U}_{\min}$ [17], uninorms continuous in the open unit square [19, 40], locally internal uninorms [11, 12, 13], uninorms locally internal on the boundary [27], idempotent uninorms [9, 30, 42] and so on [26, 33, 34, 36]. Moreover, many authors studied the distributivity between uninorms [31, 38, 39, 44, 45, 48, 49]. In particular, the distributivity between uninorms are characterized by two methods to dates. Firstly, two uninorms in the distributivity equation are fixed, such as uninorms in $\mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$ [31, 38], and idempotent uninorms [39]. Secondly, one uninorm in the distributivity equation is fixed and the other one is assumed to be an arbitrary uninorm [44, 45, 48, 49].

S-uninorms and T-uninorms were introduced in [32] as special aggregation operators with annihilators, which have conjunctive and disjunctive uninorms as special cases, respectively. Due to the great quantity of applications of uninorms, Jočić and Štajner-Papuga [21, 22, 23] discussed the distributivity for S-uninorms (resp. T-uninorms) over uninorms. Meanwhile, Fang and Hu [16] investigated the distributivity for S-uninorms, which was further studied in [8, 47]. In particular, Chen et al. [8] analyzed the distributivity between two aggregation operators, where one of them is given as a t-(co)norm, a uninorm in $\mathcal{U}_{\max}$ and an S-uninorm with the underlying uninorm in $\mathcal{U}_{\min}$, and the other one is an arbitrary S-uninorm.

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In this paper, we intend to further discuss the distributivity for S-uninorms, where the conjunctive underlying uninorms of S-uninorms in the distributivity equations are not required to be any special classes of uninorms but arbitrary. Our motivation is driven by the following facts.

(1) Although the distributivity for uninorms were analyzed by many scholars, the underlying uninorm of one S-uninorm in the distributivity equations discussed in [8, 16, 22, 23, 47] is limited into $\mathcal{U}_{\min}$ except the distributivity for S-uninorms over t-(co)norms in [8]. Because there are still some other well-known classes of uninorms except the uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, the distributivity for S-uninorms should be studied more thoroughly in this context to remedy that defect, where the underlying uninorms of S-uninorms are not only in $\mathcal{U}_{\min}$, but also other types of conjunctive uninorms, such as uninorms continuous in the open unit, idempotent uninorms, locally internal uninorms, uninorms locally internal on the boundary and so on. That is the direct motivation for our paper.

(2) Because S-uninorms are partially constructed by conjunctive uninorms [32], there is a strong feeling that the distributivity between S-uninorms may be related to the distributivity between uninorms. For the similar reason, we choose to discuss the distributivity for S-uninorms over T-uninorms. As disjunctive uninorms can be viewed as a special case of T-uninorms, we investigate the distributivity for S-uninorms over disjunctive uninorms to fill the gap of the distributivity between S-uninorms and special T-uninorms in [8].

(3) As we intend to study the distributivity between S-uninorms, where the conjunctive underlying uninorms of those S-uninorms are not restricted to $\mathcal{U}_{\min}$, but can be arbitrary, we cannot fix any S-uninorms in the distributivity equation to be special. Otherwise, the discussion seems to be tedious and cannot completely solved the distributivity between S-uninorms. Different from the results in [8, 16, 22, 23, 47], we obtain that the distributivity between S-uninorms is equivalent to the distributivity between conjunctive uninorms. Although the distributivity between uninorms is not entirely resolved, the distributivity between S-uninorms is completely figured out to some extent, which fully depends on the distributivity between conjunctive uninorms. Similarly, we characterize the distributivity for S-uninorms over T-uninorms (resp. disjunctive uninorms). In particular, we do not fix any of the underlying uninorms of S-uninorms, T-uninorms or disjunctive uninorms to be special in those discussions mentioned above. That is our new insight of the distributivity property for S-uninorms.

(4) As the conjunctive underlying uninorms of the S-uninorms in our paper are arbitrary, i.e., uninorms in $\mathcal{U}_{\min}$, uninorms continuous in the open unit square, idempotent uninorms, locally internal uninorms and so on, our results have the results in [8, 16, 22, 23, 47] as special cases, which also can be viewed as a supplement to [8, 16, 22, 23, 47]. Finally, the conclusions in this paper, which show the distributivity for S-uninorms, may be applied in many topics, such as fuzzy logic, approximate reasoning, image processing, modeling some specific problems in the utility theory.

The content of this paper is organized as follows. In Section 2, we recall some basic definitions and facts about different aggregation operators and distributivity equations. Section 3 studies the distributivity between S-uninorms, where S-uninorms in the distributivity equation are arbitrary. In Section 4, we characterize the distributivity for S-uninorms over T-uninorms. Section 5 analyzes the distributivity for S-uninorms over disjunctive uninorms. In the final section, we present some conclusions of our research.

2 Preliminaries

In this paper, the readers are assumed to be familiar with basic definitions and properties of t-(co)norms. Please see [25] for more details on t-(co)norms. In this section, we only recall some basic definitions and facts applied in this paper.

Definition 2.1. [4, 7, 18, 24]*A binary aggregation operator is a function* $A : [0,1] \times [0,1] \to [0,1]$*, which is increasing in each variable and satisfies boundary conditions* $A(0,0) = 0$ *and* $A(1,1) = 1$.

Definition 2.2. [53]*A uninorm* $U : [0,1] \times [0,1] \to [0,1]$ *is a binary aggregation operator, which is commutative, associative and exists a neutral element* $e \in [0,1]$.

Obviously, a uninorm U degenerates into a t-norm, when the neutral element $e = 1$. Similarly, a uninorm U degenerates into a t-conorm, when the neutral element $e = 0$. A uninorm U always satisfies $U(0,1) = 0$ or $U(0,1) = 1$. U is called a *conjunctive* uninorm, if $U(0,1) = 0$. Meanwhile, U is called a *disjunctive* uninorm, if $U(0,1) = 1$. Furthermore, the characterizations of uninorms are presented as follows.

Proposition 2.3. [17] *Let U be a uninorm with a neutral element $e \in]0,1[$. Then U is given as*

$$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e}, \frac{y}{e}\right), & \text{if } (x,y) \in [0,e] \times [0,e]; \\ e + (1-e)S_U\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & \text{if } (x,y) \in [e,1] \times [e,1]; \\ C_U(x,y), & \text{otherwise;} \end{cases} \tag{1}$$

where T_U is a t-norm, S_U is a t-conorm and C_U satisfies $\min(x,y) \leqslant C_U(x,y) \leqslant \max(x,y)$ for all $(x,y) \in [0,e[\times]e,1]\cup]e,1]\times[0,e[$.

The t-norm T_U and the t-conorm S_U in Eq. (1) is called the *underlying t-norm* and the *underlying t-conorm* of U, respectively. A uninorm U with a underlying t-norm T_U, a underlying t-conorm S_U and a neutral element $e \in]0,1[$ is denoted as $U \equiv \langle T_U, e, S_U \rangle$. There are many classes of uninorms, which are mostly discussed. We recalled them as follows.

Definition 2.4. [9, 11, 12, 13, 17, 19, 30, 42] *Let a uninorm $U \equiv \langle T_U, e, S_U \rangle$. Then U is called*

(i) *a uninorm in $\mathcal{U}_{\min}$, if $C_U(x,y) = \min(x,y)$ for all $(x,y) \in [0,e[\times]e,1]\cup]e,1]\times[0,e[$;*

(ii) *a uninorm in $\mathcal{U}_{\max}$, if $C_U(x,y) = \max(x,y)$ for all $(x,y) \in [0,e[\times]e,1]\cup]e,1]\times[0,e[$;*

(iii) *a representable uninorm, if there exists $h : [0,1] \to [0,+\infty]$, which is continuous strictly increasing with $h(0) = 0$, $h(e) = 1$ and $h(1) = +\infty$ such that*

$$U(x,y) = \begin{cases} 0 \text{ or } 1, & \text{if } (x,y) \in \{(1,0),\ (0,1)\}; \\ h^{-1}\big(h(x)\cdot h(y)\big), & \text{otherwise;} \end{cases} \tag{2}$$

(iv) *a uninorm continuous in the open unit square, if U is continuous in the open unit square;*

(v) *a locally internal uninorm, if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,e[\times]e,1]\cup]e,1]\times[0,e[$;*

(vi) *an idempotent uninorm, if $U(x,x) = x$ for all $x \in [0,1]$.*

Notice that there are still some other types of uninorms, such as uninorms with (non)continuous underlying operators [26, 33, 34], uninorms locally internal on the boundary [27] and so on [36]. Meanwhile, a representable uninorm is also continuous in the open unit square. The structures of uninorms continuous in the open unit square were provided in [19, 40].

Proposition 2.5. [19, 40] *Let a uninorm $U \equiv \langle T_U, e, S_U \rangle$ be continuous in the open unit square. Then either one of the following cases is satisfied.*

(i) *There exist $\mu \in [0,e[$, $\lambda \in [0,\mu]$, two continuous t-norms T and T' and a representable uninorm R such that U is given as*

$$U(x,y) = \begin{cases} \lambda T\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right), & \text{if } (x,y) \in [0,\lambda] \times [0,\lambda]; \\ \lambda + (\mu-\lambda)T'\left(\frac{x-\lambda}{\mu-\lambda}, \frac{y-\lambda}{\mu-\lambda}\right), & \text{if } (x,y) \in [\lambda,\mu] \times [\lambda,\mu]; \\ \mu + (1-\mu)R\left(\frac{x-\mu}{1-\mu}, \frac{y-\mu}{1-\mu}\right), & \text{if } (x,y) \in]\mu,1[\times]\mu,1[; \\ 1, & \text{if } (x,y) \in]\lambda,1] \times \{1\} \cup \{1\}\times]\lambda,1]; \\ 1 \text{ or } \lambda, & \text{if } (x,y) \in \{(\lambda,1),\ (1,\lambda)\}; \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{3}$$

(ii) *There exist $\nu \in]e,1]$, $\omega \in [\nu,1]$, two continuous t-conorms S and S' and a representable uninorm R such that U is given as*

$$U(x,y) = \begin{cases} \nu R\left(\frac{x}{\nu}, \frac{y}{\nu}\right), & \text{if } (x,y) \in]0,\nu[\times]0,\nu[; \\ \nu + (\omega-\nu)S\left(\frac{x-\nu}{\omega-\nu}, \frac{y-\nu}{\omega-\nu}\right), & \text{if } (x,y) \in [\nu,\omega] \times [\nu,\omega]; \\ \omega + (1-\omega)S'\left(\frac{x-\omega}{1-\omega}, \frac{y-\omega}{1-\omega}\right), & \text{if } (x,y) \in [\omega,1] \times [\omega,1]; \\ 0, & \text{if } (x,y) \in [0,\omega[\times\{0\} \cup \{0\} \times [0,\omega[; \\ 0 \text{ or } \omega, & \text{if } (x,y) \in \{(\omega,0),\ (0,\omega)\}; \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{4}$$

The symbol $\mathcal{U}_{\cos,\min}$ is applied to denote the family of all uninorms given by Eq. (3) and the symbol $\mathcal{U}_{\cos,\max}$ is applied to denote the family of all uninorms given by Eq. (4). Moreover, a uninorm U given by Eq. (3) is denoted as $U \equiv \langle T, \lambda, T', \mu, (R,e) \rangle_{\cos,\min}$. Similarly, a uninorm given by Eq. (4) is denoted as $U \equiv \langle S, \nu, S', \omega, (R,e) \rangle_{\cos,\max}$.

Remark 2.6. *Consider a conjunctive uninorm U be continuous in the open unit square. Then one of the following items hold.*

(i) $U \equiv \langle T, \lambda, T', \mu, (R, e)\rangle_{\cos,\min}$ *and* $\lambda > 0$.

(ii) $U \equiv \langle T, \lambda, T', \mu, (R, e)\rangle_{\cos,\min}$, $\lambda = 0$ *and* $U(1, \lambda) = \lambda$.

(iii) $U \equiv \langle S, \nu, S', \omega, (R, e)\rangle_{\cos,\max}$, $\omega = 1$ *and* $U(0, \omega) = 0$.

Similarly, we easily obtain the characterizations of a disjunctive uninorm continuous in the open unit square.

The scholars [30, 42] studied the relationship between the values of a uninorm $U \equiv \langle T_U, e, S_U\rangle$ in the set $[0, e[\times]e, 1]\cup]e, 1]\times [0, e[$ and the idempotent property of that uninorm.

Proposition 2.7. [30, 42] *Let a uninorm* $U \equiv \langle T_U, e, S_U\rangle$ *be idempotent. Then* $U(x, y) \in \{x, y\}$ *for all* $(x, y) \in [0, e[\times]e, 1]\cup]e, 1] \times [0, e[$.

Definition 2.8. [32] *Let an aggregation operator* $A : [0, 1]^2 \to [0, 1]$ *be commutative and associative. Then A is called*

(i) *an S-uninorm, if the following hold,*

- *there exists an annihilator* $a \in [0, 1[$ *for A, that is,* $A(x, a) = a$ *for all* $x \in [0, 1]$*;*
- $A(0, x)$ *is continuous and* $A(1, x)$ *is not;*
- *there is an IFC element* [47] $c \in]0, 1[$ *such that* $A(c, c) = c$, $A(c, 1) = 1$ *and* $A(c, x)$ *is continuous.*

(ii) *a T-uninorm, if the following hold,*

- *there exists an annihilator* $a \in]0, 1]$ *for A;*
- $A(1, x)$ *is continuous and* $A(0, x)$ *is not;*
- *there is an IFC element* [23, 47] $c \in]0, 1[$ *such that* $A(c, c) = c$, $A(c, 0) = 0$ *and* $A(c, x)$ *is continuous.*

Mas et al. [32] obtained the equivalent characterizations of S-uninorms and T-uninorms as follows.

Proposition 2.9. [32] *Let A be an aggregation operator.*

(i) *A is an S-uninorm if and only if there is* $a \in [0, 1[$, *a t-conorm S_A and a conjunctive uninorm U_A with the neutral element* $e' \in]0, 1[$ *such that A is given as*

$$A(x, y) = \begin{cases} aS_A\left(\frac{x}{a}, \frac{y}{a}\right), & \text{if } (x, y) \in [0, a] \times [0, a]; \\ a + (1 - a)U_A\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right), & \text{if } (x, y) \in [a, 1] \times [a, 1]; \\ a, & \text{otherwise.} \end{cases} \tag{5}$$

(ii) *A is a T-uninorm if and only if there is* $a \in]0, 1]$, *a t-norm T_A and a disjunctive uninorm U_A with the neutral element* $e' \in]0, 1[$ *such that A is given as*

$$A(x, y) = \begin{cases} aU_A\left(\frac{x}{a}, \frac{y}{a}\right), & \text{if } (x, y) \in [0, a] \times [0, a]; \\ a + (1 - a)T_A\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right), & \text{if } (x, y) \in [a, 1] \times [a, 1]; \\ a, & \text{otherwise.} \end{cases} \tag{6}$$

The conjunctive uninorm U_A and t-conorm S_A in Eq. (5) are called the *underlying uninorm* and the *underlying t-conorm* of the S-uninorm A, respectively. The IFC element of an S-uninorm A is denoted as e and given as $e = a + (1 - a)e'$. Moreover, an S-uninorm given by Eq. (5) is denoted as $A \equiv \langle S_A, a, U_A, e\rangle$. Similarly, the disjunctive uninorm U_A and t-norm T_A in Eq. (6) are called the *underlying uninorm* and the *underlying t-norm* of the T-uninorm A, respectively. The IFC element of a T-uninorm is also denoted as e and given as $e = ae'$. A T-uninorm A given by Eq. (6) is denoted as $A \equiv \langle U_A, e, a, T_A\rangle$. In particular, an S-uninorm (resp. a T-uninorm) degenerates into a conjunctive (resp. disjunctive) uninorm, if the annihilator is equal to 0 (resp. 1).

Definition 2.10. [1]*Let* $F, G : [0, 1] \times [0, 1] \to [0, 1]$ *be two binary operators. Then F is distributive over G if and only if the following equations hold for all* $x, y, z \in [0, 1]$,

$$F\big(x, G(y, z)\big) = G\big(F(x, y), F(x, z)\big) \text{ and } F\big(G(x, y), z\big) = G\big(F(x, z), F(y, z)\big).$$

Figure 1: Structure of A (left) and B (right) in Proposition 3.3.

3 Distributivity between S-uninorms

Although the distributivity between S-uninorms had been studied in [8, 16, 47], at least, one of the S-uninorms in the distributivity equation was always assumed to be the S-uninorm with the underlying uninorm in $\mathcal{U}_{\min}$. In fact, the uninorms in the distributivity equation between uninorms [44, 45, 48, 49] are not only in $\mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$ but also other types of uninorms, i.e., uninorms continuous in the open unit square, idempotent uninorms and locally internal uninorms. Inspired by the results on the distributivity between uninorms mentioned above, we discuss the distributivity between S-uninorms regardless of their detail forms of underlying uninorms in this section.

Lemma 3.1. [8] *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over an S-uninorm $B \equiv \langle S_B, b, U_B, e_2\rangle$. Then the following hold,*

(i) $a < e_2$ *and* $b < e_1$.

(ii) $A(x, b) = B(x, a)$ *for all* $x \in [0, 1]$.

Lemma 3.2. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over an S-uninorm $B \equiv \langle S_B, b, U_B, e_2\rangle$. Then $B(x, x) = x$ for all $x \in [0, a]$.*

Proof. We obtain $B(x, x) = B\big(A(x, 0), A(x, 0)\big) = A\big(x, B(0, 0)\big) = A(x, 0) = x$ for all $x \in [0, a]$ by Proposition 2.9(i) and the distributivity equation. Thus $B(x, x) = x$ holds for all $x \in [0, a]$. □

Firstly, we study the distributivity between S-uninorms with identical annihilators as follows.

Proposition 3.3. *Let $A \equiv \langle S_A, a, U_A, e_1\rangle$ and $B \equiv \langle S_B, b, U_B, e_2\rangle$ be two S-uninorms. If $a = b$, then A is distributive over B if and only if A is given by Eq. (5) and B is given as*

$$B(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,b]\times[0,b];\\ b + (1-b)U_B\big(\frac{x-b}{1-b}, \frac{y-b}{1-b}\big), & \text{if } (x,y) \in [b,1]\times[b,1];\\ b, & \text{otherwise;}\end{cases} \tag{7}$$

where the conjunctive uninorm U_A is distributive over the conjunctive uninorm U_B.

Proof. **Necessity.** By Lemma 3.2 and $a = b$, we have $B(x, y) = \max(x, y)$ for all $x, y \in [0, b]$. Due to the distributivity equation and $a = b$, the restriction of A to the set $[a, 1] \times [a, 1]$ is distributive over the restriction of B to the set $[b, 1] \times [b, 1]$, that is, the conjunctive uninorm U_A is distributive over the conjunctive uninorm U_B.

Sufficiency. It can be proven by a simple computation. □

Example 3.4. *Consider an S-uninorm A be given as*

$$A(x,y) = \begin{cases} x + y - 5xy, & \text{if } (x,y) \in [0,0.2]\times[0,0.2];\\ \max(x+y-0.4, 0.2), & \text{if } (x,y) \in [0.2,0.4]\times[0.2,0.4];\\ 10xy - 4x - 4y + 2, & \text{if } (x,y) \in [0.4,0.5]\times[0.4,0.5];\\ 10xy - 5x - 5y + 3, & \text{if } (x,y) \in [0.5,0.6]\times[0.5,0.6];\\ 0.6 + 0.4R(2.5x - 1.5, 2.5y - 1.5), & \text{if } (x,y) \in]0.6,1[\times]0.6,1[;\\ 0.2, & \text{if } (x,y) \in [0,0.2]\times[0.2,1] \cup [0.2,1]\times[0,0.2];\\ 1, & \text{if } (x,y) \in [0.4,1]\times\{1\} \cup \{1\}\times[0.4,1];\\ \min(x,y), & \text{otherwise;}\end{cases}$$

where R is a representable uninorm with the neutral element $e' = 0.5$. Then we have $A \equiv \langle S_P, 0.2, U_A, 0.8\rangle$ and $U_A \equiv \langle T_{LK}, 0.25, T', 0.5, (R, 0.75)\rangle_{\cos,\min}$ with $T' = (\langle 0, 0.5, T_P\rangle, \langle 0.5, 1, T_P\rangle)$ and $U_A(0.25, 1) = 1$, where S_P, T_P and T_{LK} are given as for all $x, y \in [0, 1]$, respectively,

$$S_P(x, y) = x + y - xy,\ T_P(x, y) = xy \text{ and } T_{LK}(x, y) = \max(x + y - 1, 0).$$

Consider an S-uninorm B be given as

$$B(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,0.2]\times[0,0.2] \cup [0.5,1[\times[0.5,1[; \\ 0.2, & \text{if } (x,y) \in [0,0.2]\times[0.2,1] \cup [0.2,1]\times[0,0.2]; \\ 1, & \text{if } (x,y) \in [0.4,1]\times\{1\} \cup \{1\}\times[0.4,1]; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then $B \equiv \langle \max, 0.2, U_B, 0.5\rangle$ holds and U_B is given as

$$U_B(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0.375,1[\times[0.375,1[; \\ 1, & \text{if } (x,y) \in [0.25,1]\times\{1\} \cup \{1\}\times[0.25,1]; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

By Proposition 3.3 and Theorem 9(ii) in [45], we have that A is distributive over B. However, the underlying uninorms of S-uninorms A and B are not in $\mathcal{U}_{\min}$.

Remark 3.5. *In contrast to Proposition 3.3, the distributivity between S-uninorms in [8] was only discussed the situation that the underlying uninorm of one S-uninorm in the distributivity equation was assumed to be in $\mathcal{U}_{\min}$ (see Section 3 in [8]). In fact, when the annihilators of S-uninorms are identical with each others, Proposition 3.3 does not only have the underlying uninorm of one S-uninorm in $\mathcal{U}_{\min}$ as a special case, but also discusses the underlying uninorm of one S-uninorm in other types of uninorms, i.e., uninorms continuous in the open unit square, idempotent uninorms and so on. Proposition 3.3 and Example 3.4 show that the distributivity between S-uninorms with identical annihilators turns out to be the distributivity between conjunctive uninorms. Thus the conclusions on the distributivity between S-uninorms with identical annihilators and special conjunctive underlying uninorms of S-uninorms are easily obtained based on the distributivity between conjunctive uninorms in [44, 45, 48, 49]. We do not list them here.*

Secondly, we characterize the distributivity between S-uninorms with different annihilators.

Lemma 3.6. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over an S-uninorm $B \equiv \langle S_B, b, U_B, e_2\rangle$ with $a < b$. Then the following hold,*

(i) $A(x, b) = B(x, a) = x$ *for all* $x \in [a, b]$.

(ii) $B(x, x) = x$ *for all* $x \in [0, b]$.

(iii) $A(x, y) = \min(x, y)$ *for all* $(x, y) \in [a, b] \times [b, 1] \cup [b, 1] \times [a, b]$.

Proof. (i) By Proposition 2.9(i) and Lemma 3.1, we obtain $x = \max(x, a) \leqslant B(x, a) = A(x, b) \leqslant \min(x, b) = x$ for all $x \in [a, b]$. Thus $A(x, b) = B(x, a) = x$ holds for all $x \in [a, b]$.
(ii) Due to item (i) and the distributivity equation, we have $B(x, x) = B\big(A(x, b), A(x, b)\big) = A\big(x, B(b, b)\big) = A(x, b) = x$ for all $x \in [a, b]$. Associating with Lemma 3.2, we get $B(x, x) = x$ for all $x \in [0, b]$.
(iii) By item (i), Proposition 2.9(i) and Lemma 3.1(ii), we have

$$A(x,b) = B(x,a) = \begin{cases} x, & \text{if } x \in [a,b]; \\ b, & \text{if } x \in [b,1]. \end{cases}$$

Because of Lemma 3.1(i), $b < e_1$ holds, so we obtain $x = A(x, b) \leqslant A(x, y) \leqslant A(x, e_1) = x$ for all $x \in [a, b]$ and $y \in [b, e_1]$. Hence, we get $A(x, y) = x = \min(x, y)$ for all $(x, y) \in [a, b] \times [b, e_1]$.

Assume that there is a point $(x_0, y_0) \in [a, b]\times]e_1, 1]$ such that $A(x_0, y_0) > x_0$, then $A\big(x_0, B(y_0, z_0)\big) = A(x_0, b) = x_0$ holds for all $z_0 \in [x_0, b]$. Meanwhile, by Lemma 3.1, we obtain for all $z_0 \in [x_0, b]$,

$$a \leqslant x_0 < A(x_0, y_0) \leqslant A(b, 1) = B(a, 1) = b \text{ and } a = A(a, a) \leqslant A(x_0, z_0) \leqslant \min(x_0, z_0) = x_0 \leqslant b,$$

so we have $B\big(A(x_0, y_0), A(x_0, z_0)\big) = \max\big(A(x_0, y_0), A(x_0, z_0)\big) = A(x_0, y_0) > x_0$ by item (ii) and the increasing property of A. Thus we obtain a contradiction to the distributivity equation. By Proposition 2.3, $A(x, y) = x = \min(x, y)$ holds for all $(x, y) \in [a, b]\times]e_1, 1]$.

Associating with the commutativity of the S-uninorm A, we directly get the conclusion. □

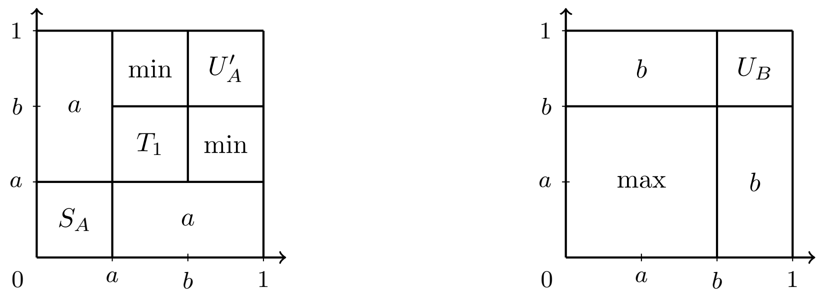


Figure 2: Structure of A (left) and B (right) in Proposition 3.7.

Proposition 3.7. *Let $A \equiv \langle S_A, a, U_A, e_1\rangle$ and $B \equiv \langle S_B, b, U_B, e_2\rangle$ be two S-uninorms. If $a < b$, then A is distributive over B if and only if A is given as*

$$A(x,y) = \begin{cases} aS_A\left(\frac{x}{a}, \frac{y}{a}\right), & \text{if } (x,y) \in [0,a] \times [0,a]; \\ a + (b-a)T_1\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right), & \text{if } (x,y) \in [a,b] \times [a,b]; \\ b + (1-b)U'_A\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right), & \text{if } (x,y) \in [b,1] \times [b,1]; \\ \min(x,y), & \text{if } (x,y) \in [a,b] \times [b,1] \cup [b,1] \times [a,b]; \\ a, & \text{otherwise;} \end{cases} \tag{8}$$

and B is given by Eq. (7), where the conjunctive uninorm U'_A is distributive over the conjunctive uninorm U_B.

Proof. **Necessity.** Due to $A(1,b) = B(1,a) = b$, we obtain that U'_A is conjunctive. The distributivity for U'_A over U_B follows immediately from the distributivity equation. Moreover, by Lemma 3.6, the S-uninorms A and B satisfy Eqs. (8) and (7), respectively.

Sufficiency. It can be proven by a simple computation. □

Remark 3.8. *Consider the conjunctive uninorm U'_A in Eq. (8) be continuous in the open unit square. Then we easily obtain the conclusions on the distributivity between S-uninorms by Remark 2.6 and the results on the distributivity between conjunctive uninorms in [44, 45, 48, 49]. If the underlying uninorm U_A of the S-uninorm A is continuous in the open unit square, then we have the following results.*

Lemma 3.9. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over an S-uninorm $B \equiv \langle S_B, b, U_B, e_2\rangle$ with $a < b$. If U_A is continuous in the open unit square, then $U_A \equiv \langle T, \lambda, T', \mu, (R, e)\rangle_{\cos,\min}$ with $\lambda \geqslant \frac{b-a}{1-a}$.*

Proof. Because U_A is continuous in the open unit square, we have $U_A \equiv \langle T, \lambda, T', \mu, (R,e)\rangle_{\cos,\min}$ by Proposition 2.5, Remark 2.6 and Lemma 3.6(iii). Meanwhile, associating with $A(x,1) = x$ for all $x \in [a, a+(1-a)\lambda[$ and $A(x,1) = 1$ for all $x \in]a+(1-a)\lambda, 1]$, we obtain $a + (1-a)\lambda \geqslant b$. Thus the conclusion holds. □

Corollary 3.10. *Let $A \equiv \langle S_A, a, U_A, e_1\rangle$ and $B \equiv \langle S_B, b, U_B, e_2\rangle$ be two S-uninorms. If $a < b$ and U_A is continuous in the open unit square, then A is distributive over B if and only if either of the following item holds,*

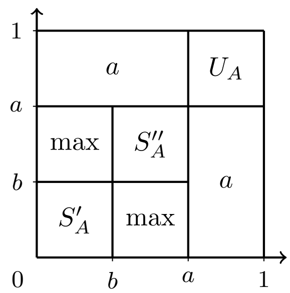
(i) *If $U_A \equiv \langle T, \lambda, T', \mu, (R,e)\rangle_{\cos,\min}$ with $\lambda > \frac{b-a}{1-a}$, then A and B are given by Eqs. (8) and (7), respectively, where T is given as $T = \left(\left\langle 0, \frac{b-a}{\lambda(1-a)}, T_1\right\rangle, \left\langle \frac{b-a}{\lambda(1-a)}, 1, T_2\right\rangle\right)$ and the conjunctive uninorm U'_A is given as $U'_A \equiv \left\langle T_2, \frac{\lambda(1-a)+a-b}{1-b}, T', \frac{\mu(1-a)+a-b}{1-b}, \left(R, \frac{e(1-a)+a-b}{1-b}\right)\right\rangle_{\cos,\min}$ distributive over the conjunctive uninorm U_B.*

(ii) *If $U_A \equiv \langle T, \lambda, T', \mu, (R,e)\rangle_{\cos,\min}$ with $\lambda = \frac{b-a}{1-a}$, then A and B are given by Eqs. (8) and (7), respectively, where the conjunctive uninorm $U'_A \equiv \left\langle T, 0, T', \frac{\mu(1-a)+a-b}{1-b}, \left(R, \frac{e(1-a)+a-b}{1-b}\right)\right\rangle_{\cos,\min}$ is distributive over the conjunctive uninorm U_B.*

Proof. It follows immediately from Proposition 3.7 and Lemma 3.9. □

Notice that the detail form of U_B in Corollary 3.10 can be easily obtained on the basis of the conclusions in [45, 48, 49] related to uninorms continuous in the open unit square. We do not list them here.

Lemma 3.11. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over an S-uninorm $B \equiv \langle S_B, b, U_B, e_2\rangle$ with $a > b$. Then the following hold.*



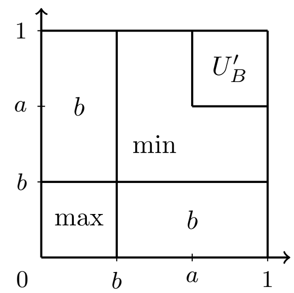


Figure 3: Structure of A (left) and B (right) in Proposition 3.12.

(i) $A(x,y) = \max(x,y)$ *for all* $(x,y) \in [0,b] \times [b,a] \cup [b,a] \times [0,b]$.

(ii) S_A *is an ordinal sum t-conorm, i.e.,* $S_A = \big(\langle 0, \frac{b}{a}, S'_A\rangle, \langle \frac{b}{a}, 1, S''_A\rangle\big)$.

(iii) $B(x,y) = \min(x,y)$ *for all* $(x,y) \in [b,a] \times [b,1] \cup [b,1] \times [b,a]$.

Proof. (i) By Lemma 3.2, we directly have $B(x,y) = \min(x,y)$ for all $x, y \in [b,a]$. Associating with Lemma 3.1(ii), we obtain

$$A(x,b) = B(x,a) = \begin{cases} b, & \text{if } x \in [0,b]; \\ \min(x,a), & \text{if } x \in [b,a]; \end{cases} = \begin{cases} b, & \text{if } x \in [0,b]; \\ x, & \text{if } x \in [b,a]. \end{cases}$$

Hence we have $x = A(x,0) \leqslant A(x,y) \leqslant A(x,b) = x$ for all $(x,y) \in [b,a] \times [0,b]$. Thus $A(x,y) = x = \max(x,y)$ holds for all $(x,y) \in [b,a] \times [0,b]$. Associating with the commutativity of A, we get the conclusion.
(ii) It follows immediately from item (i).
(iii) Firstly, due to item (i), we have $x = A(x,b) = B(x,a) \leqslant B(x,y) \leqslant B(x,e_2) = x$ for all $(x,y) \in [b,a] \times [a,e_2]$. Hence, $B(x,y) = x = \min(x,y)$ holds for all $(x,y) \in [b,a] \times [a,e_2]$.

Secondly, assume that there are $x_0 \in [b,a]$ and $y_0 \in]e_2,1]$ such that $B(x_0,y_0) > x_0$, then we obtain $b = B(b,e_2) \leqslant B(x_0,y_0) \leqslant B(a,1) = A(b,1) = a$ by Lemma 3.1(ii). Associating with item (i) and Lemma 3.1, we obtain for all $z_0 \in [0,b]$,

$$A\big(z_0, B(x_0,y_0)\big) = \max\big(z_0, B(x_0,y_0)\big) = B(x_0,y_0),$$
$$B\big(A(z_0,x_0), A(z_0,y_0)\big) = B(x_0,a) = A(x_0,b) = x_0,$$

which imply a contradiction to the distributivity equation. By Proposition 2.3, we have $B(x,y) = x = \min(x,y)$ for all $(x,y) \in [b,a] \times]e_2,1]$.

Associating with Lemma 3.2 and the commutativity of the S-uninorm A, we obtain the conclusion. □

Proposition 3.12. *Let* $A \equiv \langle S_A, a, U_A, e_1\rangle$ *and* $B \equiv \langle S_B, b, U_B, e_2\rangle$ *be two* S*-uninorms. If* $a > b$*, then* A *is distributive* B *if and only if* A *is given as*

$$A(x,y) = \begin{cases} bS'_A\big(\frac{x}{b}, \frac{y}{b}\big), & \text{if } (x,y) \in [0,b] \times [0,b]; \\ b + (a-b)S''_A\big(\frac{x-b}{a-b}, \frac{y-b}{a-b}\big), & \text{if } (x,y) \in [b,a] \times [b,a]; \\ a + (1-a)U_A\big(\frac{x-a}{1-a}, \frac{y-a}{1-a}\big), & \text{if } (x,y) \in [a,1] \times [a,1]; \\ \max(x,y), & \text{if } (x,y) \in [0,b] \times [b,a] \cup [b,a] \times [0,b]; \\ a, & \text{otherwise}; \end{cases} \tag{9}$$

and B *is given as*

$$B(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,b] \times [0,b]; \\ a + (1-a)U'_B\big(\frac{x-a}{1-a}, \frac{y-a}{1-a}\big), & \text{if } (x,y) \in [a,1] \times [a,1]; \\ \min(x,y), & \text{if } (x,y) \in [b,a] \times [b,1] \cup [b,1] \times [b,a]; \\ b, & \text{otherwise}; \end{cases} \tag{10}$$

where the conjunctive uninorm U_A *is distributive over the conjunctive uninorm* U'_B.

Proof. **Necessity.** It follows immediately from Proposition 2.9(i), Lemmas 3.2 and 3.11.

Sufficiency. It can be proven by a simple computation. □

Remark 3.13. *According to Propositions 3.3, 3.7 and 3.12, the distributivity between S-uninorms is equivalent to the distributivity between conjunctive uninorms regardless of the detail forms of the underlying uninorms of S-uninorms. Thus the results in Section 3 of the reference* [8] *turn out to be the special case of Propositions 3.3, 3.7 and 3.12. Moreover, the distributivity between T-uninorms can be easily obtained by duality. We do not list them here.*

4 Distributivity for S-uninorms over T-uninorms

The scholars [16, 47] only analyzed the distributivity between S-uninorms and T-uninorms, where the underlying uninorms of S-uninorms are in $\mathcal{U}_{\min}$ and the underlying uninorms of T-uninorms are in $\mathcal{U}_{\max}$. Meanwhile, the distributivity between S-uninorms and T-uninorms was not investigated in [8, 22]. In this section, we further study the distributivity for S-uninorms over T-uninorms neglecting their detail forms of underlying uninorms. By duality, we easily characterize the distributivity for T-uninorms over S-uninorms. Thus we do not list them here.

Lemma 4.1. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over a T-uninorm $B \equiv \langle U_B, e_2, b, T_B\rangle$ with $a, b \in]0,1[$. Then the following hold,*

(i) $a > e_2$ *and* $b < e_1$.

(ii) $A(x, b) = B(x, a)$ *for all* $x \in [0,1]$.

(iii) $B(x, x) = x$ *for all* $x \in [0, a]$.

Proof. (i) By Proposition 2.9 and the distributivity equation, we have $A(0, b) = A\big(0, B(0, 1)\big) = B\big(A(0, 0), A(0, 1)\big) = B(0, a)$. Assume that $a \leqslant e_2$ holds, then we obtain $B(0, a) \leqslant B(0, e_2) = 0$, which contradict to $A(0, b) = a > 0$, because of $a \leqslant e_2 < b$. Thus $a > e_2$ holds.

Similarly, we get $A(1, b) = A\big(1, B(0, 1)\big) = B\big(A(1, 0), A(1, 1)\big) = B(a, 1)$. Assume that $b \geqslant e_1$ holds, then due to $a < e_1 \leqslant b$, we obtain $A(1, b) \geqslant A(1, e_1) = 1$ and $B(a, 1) = b < 1$, which imply a contradiction. Thus $b < e_1$ holds.

(ii) According to item (i), we obtain $B(e_1, 0) = b$, so the following holds for all $x \in [0,1]$,

$$A(x, b) = A\big(x, B(e_1, 0)\big) = B\big(A(x, e_1), A(x, 0)\big) = \begin{cases} B(a, x), & \text{if } x \in [0, a]; \\ B(x, a), & \text{if } x \in [a, 1]; \end{cases} = B(x, a).$$

(iii) Due to the distributivity equation, we have $B(x, x) = B\big(A(x, 0), A(x, 0)\big) = A\big(x, B(0, 0)\big) = A(x, 0) = x$ for all $x \in [0, a]$. Thus $B(x, x) = x$ holds for all $x \in [0, a]$. □

Remark 4.2. *The condition $a, b \in]0,1[$ is indispensable in Lemma 4.1. Otherwise, either $a = 0$ or $b = 1$ may hold. Meanwhile, according to the proof of Lemma 4.1(i), Lemma 4.1(i) may not hold.*

Lemma 4.3. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over a T-uninorm $B \equiv \langle U_B, e_2, b, T_B\rangle$ with $a = b$. Then the following hold,*

(i) $B(x, y) = \min(x, y)$ *for all* $(x, y) \in [0, e_2] \times [0, e_2]$ *and* $B(x, y) = \max(x, y)$ *for all* $(x, y) \in [e_2, b] \times [e_2, b]$.

(ii) S_A *is an ordinal sum t-conorm, i.e.,* $S_A = \big(\langle 0, \frac{e_2}{a}, S'_A\rangle, \langle \frac{e_2}{a}, 1, S''_A\rangle\big)$.

(iii) $B(x, y) = \max(x, y)$ *for all* $(x, y) \in [0, e_2[\times]e_2, b] \cup]e_2, b] \times [0, e_2[$.

Proof. (i) It follows immediately from Lemma 4.1(iii).

(ii) Consider $x \in [e_2, a]$. Then we have $e_2 \leqslant x = \max(x, e_2) \leqslant A(x, e_2) \leqslant A(a, a) = a$. By item (i), we obtain for all $x \in [e_2, a]$,

$$A\big(x, B(e_2, 0)\big) = A(x, 0) = x,$$
$$B\big(A(x, e_2), A(x, 0)\big) = B\big(A(x, e_2), x\big) = \max\big(A(x, e_2), x\big) = A(x, e_2).$$

Hence, $A(x, e_2) = x$ holds for all $x \in [e_2, a]$. Moreover, we have $e_2 = A(0, e_2) \leqslant A(x, e_2) \leqslant A(e_2, e_2) = e_2$ for all $x \in [0, e_2]$. Thus we obtain the conclusion.

(iii) Due to item (i) and Proposition 2.7, we obtain $B(y, z) \in \{y, z\}$ for all $y \in [0, e_2[$ and $z \in]e_2, b]$. Assume that there is a point $(y_0, z_0) \in [0, e_2[\times]e_2, b]$ such that $B(y_0, z_0) = y_0$, then we get $e_2 < x_0 < z_0 \leqslant A(x_0, z_0) \leqslant A(b, b) = A(a, a) = a = b$ for all $x_0 \in]e_2, z_0[$. By item (ii), we have for all $x_0 \in]e_2, z_0[$,

$$A\big(x_0, B(y_0, z_0)\big) = A(x_0, y_0) = \max(x_0, y_0) = x_0,$$
$$B\big(A(x_0, y_0), A(x_0, z_0)\big) = B\big(x_0, A(x_0, z_0)\big) = \max\big(x_0, A(x_0, z_0)\big) = A(x_0, z_0) > x_0.$$

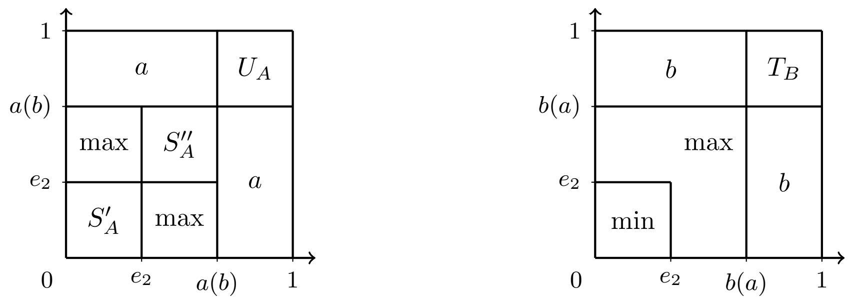


Figure 4: Structure of A (left) and B (right) in Proposition 4.4.

Hence, we obtain a contradiction to the distributivity equation. Thus $B(y,z)=z$ holds for all $(y,z)\in[0,e_2[\times]e_2,b]$. Associating with the commutativity of A, we have the conclusion. □

Notice that the auxiliary condition $a=b$ directly implies $a,b\in]0,1[$, because of $a\in[0,1[$ and $b\in]0,1]$.

Proposition 4.4. *Let $A\equiv\langle S_A,a,U_A,e_1\rangle$ be an S-uninorm and $B\equiv\langle U_B,e_2,b,T_B\rangle$ be a T-uninorm. If $a=b$, then A is distributive over B if and only if A is given as*

$$A(x,y)=\begin{cases} e_2S'_A\left(\frac{x}{e_2},\frac{y}{e_2}\right), & \text{if } (x,y)\in[0,e_2]\times[0,e_2];\\ e_2+(a-e_2)S''_A\left(\frac{x-e_2}{a-e_2},\frac{y-e_2}{a-e_2}\right), & \text{if } (x,y)\in[e_2,a]\times[e_2,a];\\ a+(1-a)U_A\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right), & \text{if } (x,y)\in[a,1]\times[a,1];\\ \max(x,y), & \text{if } (x,y)\in[0,e_2]\times[e_2,a]\cup[e_2,a]\times[0,e_2];\\ a, & \text{otherwise;}\end{cases}\tag{11}$$

and B is given as

$$B(x,y)=\begin{cases} \min(x,y), & \text{if } (x,y)\in[0,e_2]\times[0,e_2];\\ b+(1-b)T_B\left(\frac{x-b}{1-b},\frac{y-b}{1-b}\right), & \text{if } (x,y)\in[b,1]\times[b,1];\\ \max(x,y), & \text{if } (x,y)\in[0,b]\times]e_2,b]\cup]e_2,b]\times[0,b];\\ b, & \text{otherwise;}\end{cases}\tag{12}$$

where the conjunctive uninorm U_A is distributive over the t-norm T_B.

Proof. **Necessity.** It follows immediately from Proposition 2.9 and Lemma 4.3.
Sufficiency. It can be proven by a simple computation. □

Example 4.5. *Consider an S-uninorm A and a T-uninorm B be given as, respectively,*

$$A(x,y)=\begin{cases} x+y-2.5xy, & \text{if } (x,y)\in[0,0.4]\times[0,0.4];\\ \min(x+y-0.4,0.5), & \text{if } (x,y)\in[0.4,0.5]\times[0.4,0.5];\\ 0.5, & \text{if } (x,y)\in[0,0.5]\times[0.5,1]\cup[0.5,1]\times[0,0.5];\\ \min(x,y), & \text{if } 0.5\leqslant x\leqslant 1.5-y \text{ and } y\in[0.5,1];\\ \max(x,y), & \text{otherwise;}\end{cases}$$

and

$$B(x,y)=\begin{cases} \min(x,y), & \text{if } (x,y)\in[0,0.4]\times[0,0.4]\cup[0.5,1]\times[0.5,1];\\ 0.5, & \text{if } (x,y)\in[0,0.5]\times[0.5,1]\cup[0.5,1]\times[0,0.5];\\ \max(x,y), & \text{otherwise.}\end{cases}$$

Then $A\equiv\langle S_A,0.5,U_A,0.75\rangle$ and U_A is an idempotent uninorm. Because an arbitrary uninorm is distributive over min, *we obtain that A is distributive over B by Proposition 4.4.*

Remark 4.6. *Different from the results in* [16, 47], *Proposition 4.4 and Example 4.5 show that the distributivity for S-uninorms over T-uninorms with identical annihilators is equivalent to the distributivity for conjunctive uninorms over t-norms, where the underlying uninorms of S-uninorms are not limited in $\mathcal{U}_{\min}$ but arbitrary conjunctive uninorms. In particular, Chen et al.* [8] *did not discussed the distributivity for S-uninorms over T-uninorms.*

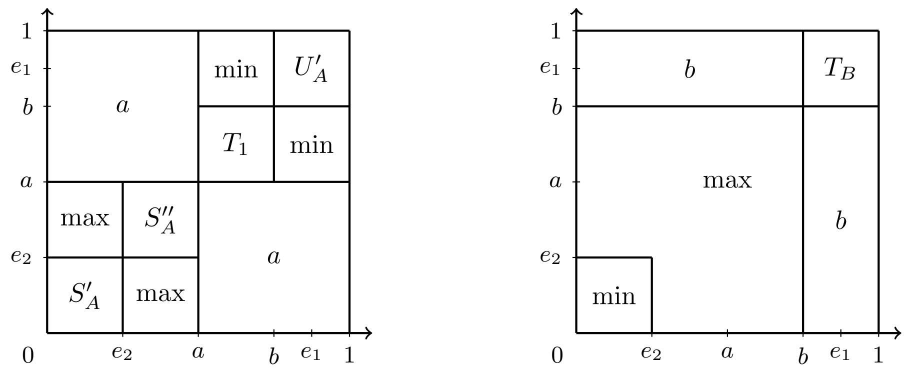


Figure 5: Structure of A (left) and B (right) in Proposition 4.8.

Lemma 4.7. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1\rangle$ be distributive over a T-uninorm $B \equiv \langle U_B, e_2, b, T_B\rangle$ with $0 < a < b < 1$. Then the following hold,*

(i) $A(x,b) = B(x,a) = x$ *for all* $x \in [a,b]$.

(ii) $B(x,x) = x$ *for all* $x \in [0,b]$.

(iii) S_A *is an ordinal sum t-conorm, i.e.,* $S_A = \big(\langle 0, \frac{e_2}{a}, S'_A\rangle, \langle \frac{e_2}{a}, 1, S''_A\rangle\big)$.

(iv) $B(x,y) = \min(x,y)$ *for all* $(x,y) \in [0,e_2]\times[0,e_2]$ *and* $B(x,y) = \max(x,y)$ *for all* $(x,y) \in [0,b]\times]e_2,b]\cup]e_2,b]\times[0,b]$.

(v) $A(x,y) = \min(x,y)$ *for all* $(x,y) \in [a,b]\times[b,1]\cup[b,1]\times[a,b]$.

Proof. (i) By Lemma 4.1(i), (ii) and $0 < a < b < 1$, we have $x = \max(x,a) \leqslant B(x,a) = A(x,b) \leqslant \min(x,b) = x$ for all $x \in [a,b]$. Thus $A(x,b) = B(x,a) = x$ holds for all $x \in [a,b]$.
(ii) According to item (i) and the distributivity equation, we obtain $B(x,x) = B\big(A(x,b), A(x,b)\big) = A\big(x, B(b,b)\big) = A(x,b) = x$ for all $x \in [a,b]$. Associating with Lemma 4.1(iii), $B(x,x) = x$ holds for all $x \in [0,b]$.
(iii) It can be proven in a similar way as for Lemma 4.3(ii).
(iv) Firstly, by item (ii), we directly have $B(x,y) = \min(x,y)$ for all $x,y \in [0,e_2]$ and $B(x,y) = \max(x,y)$ for all $x,y \in [e_2,b]$.

Secondly, according to item (ii) and Proposition 2.7, we obtain $B(y,z) \in \{y,z\}$ for all $(y,z) \in [0,e_2[\times]e_2,a]$. Assume that there are $y_0 \in [0,e_2[$ and $z_0 \in]e_2,a]$ such that $B(y_0,z_0) = y_0$, then by item (iii), we have $A\big(x_0, B(y_0,z_0)\big) = A(x_0,y_0) = \max(x_0,y_0) = x_0$ for all $x_0 \in]e_2,z_0[$. Meanwhile, we obtain $e_2 < x_0 < z_0 = \max(x_0,z_0) \leqslant A(x_0,z_0) \leqslant A(a,a) = a$ for all $x_0 \in]e_2,z_0[$, so $B\big(A(x_0,y_0), A(x_0,z_0)\big) = B\big(x_0, A(x_0,z_0)\big) = \max\big(x_0, A(x_0,z_0)\big) = A(x_0,z_0) > x_0$. Hence we get a contradiction to the distributivity equation. Thus $B(y,z) = z = \max(y,z)$ holds for all $(y,z) \in [0,e_2[\times]e_2,a]$.

Thirdly, due to item (ii) and Proposition 2.7, $B(y,z) \in \{y,z\}$ holds for all $(y,z) \in [0,e_2[\times]a,b]$. Meanwhile, by Lemma 4.1(ii) and the increasing property of A, we get $B(y,z) \geqslant B(0,a) = A(0,b) = a$ for all $(y,z) \in [0,e_2[\times]a,b]$. Thus we obtain $B(y,z) = z$ for all $(y,z) \in [0,e_2[\times]a,b]$.

Associating with the commutativity of B, we have the conclusion.
(v) Assume that there is a point $(x_0,y_0) \in [a,b]\times[b,1]$ such that $A(x_0,y_0) > x_0$, then we have $A\big(x_0, B(y_0,z_0)\big) = A(x_0,b) = x_0$ for all $z_0 \in [0,a]$ by item (i). Due to Lemma 4.1(ii), we get $A(1,b) = B(1,a) = b$, so $a \leqslant x_0 < A(x_0,y_0) \leqslant A(b,1) = b$. By item (i), we obtain $B\big(A(x_0,y_0), A(x_0,z_0)\big) = B\big(A(x_0,y_0), a\big) = A(x_0,y_0) > x_0$. Thus we have a contradiction to the distributivity equation. According to Proposition 2.3, we directly have $x = \min(x,y) \leqslant A(x,y) \leqslant \max(x,y) = y$ for all $(x,y) \in [a,b]\times[b,1]$. Thus $A(x,y) = x$ holds for all $(x,y) \in [a,b]\times[b,1]$. Associating with the commutativity of A, we obtain the conclusion. □

By Lemma 4.7, we directly obtain the following result.

Proposition 4.8. *Let $A \equiv \langle S_A, a, U_A, e_1\rangle$ be an S-uninorm and $B \equiv \langle U_B, e_2, b, T_B\rangle$ be a T-uninorm. If $0 < a < b < 1$,* *then A is distributive over B if and only if A is given as*

$$A(x,y)=\begin{cases} e_2S'_A\big(\frac{x}{e_2},\frac{y}{e_2}\big), & \text{if } (x,y)\in[0,e_2]\times[0,e_2];\\ e_2+(a-e_2)S''_A\big(\frac{x-e_2}{a-e_2},\frac{y-e_2}{a-e_2}\big), & \text{if } (x,y)\in[e_2,a]\times[e_2,a];\\ a+(b-a)T_1\big(\frac{x-a}{b-a},\frac{y-a}{b-a}\big), & \text{if } (x,y)\in[a,b]\times[a,b];\\ b+(1-b)U'_A\big(\frac{x-b}{1-b},\frac{y-b}{1-b}\big), & \text{if } (x,y)\in[b,1]\times[b,1];\\ \max(x,y), & \text{if } (x,y)\in[0,e_2]\times[e_2,a]\cup[e_2,a]\times[0,e_2];\\ \min(x,y), & \text{if } (x,y)\in[a,b]\times[b,1]\cup[b,1]\times[a,b];\\ a, & \text{otherwise;}\end{cases} \tag{13}$$

and B is given by Eq. (12), where the conjunctive uninorm U'_A is distributive over the t-norm T_B.

Consider the underlying uninorm of the S-uninorm A be continuous in the open unit square. Then we have the following conclusion.

Corollary 4.9. *Let $A\equiv\langle S_A,a,U_A,e_1\rangle$ be an S-uninorm and $B\equiv\langle S_B,b,U_B,e_2\rangle$ be a T-uninorms. If $0<a<b<1$ and U_A is continuous in the open unit square, then A is distributive over B if and only if either of the following item holds,*

(i) *If $U_A\equiv\langle T,\lambda,T',\mu,(R,e)\rangle_{\cos,\min}$ with $\lambda>\frac{b-a}{1-a}$, then A and B are given by Eqs. (13) and (12), respectively, where T is given as $T=\big(\langle 0,\frac{b-a}{\lambda(1-a)},T_1\rangle,\langle\frac{b-a}{\lambda(1-a)},1,T_2\rangle\big)$ and the conjunctive uninorm U'_A is given as $U'_A\equiv\big\langle T_2,\frac{\lambda(1-a)+a-b}{1-b},T',\frac{\mu(1-a)+a-b}{1-b},\big(R,\frac{e(1-a)+a-b}{1-b}\big)\big\rangle_{\cos,\min}$ distributive over t-norm T_B.*

(ii) *If $U_A\equiv\langle T,\lambda,T',\mu,(R,e)\rangle_{\cos,\min}$ with $\lambda=\frac{b-a}{1-a}$, then A and B are given by Eqs. (13) and (12), respectively, where the conjunctive uninorm $U'_A\equiv\big\langle T,0,T',\frac{\mu(1-a)+a-b}{1-b},\big(R,\frac{e(1-a)+a-b}{1-b}\big)\big\rangle_{\cos,\min}$ is distributive over t-norm T_B.*

Proof. Similarly to Lemma 3.9, we have $U_A\equiv\langle T,\lambda,T',\mu,(R,e)\rangle_{\cos,\min}$ and $\lambda\geqslant\frac{b-a}{1-a}$. Associating with Propositions 2.5 and 4.8, we obtain the conclusion. □

Lemma 4.10. *Let an S-uninorm $A\equiv\langle S_A,a,U_A,e_1\rangle$ be distributive over a T-uninorm $B\equiv\langle U_B,e_2,b,T_B\rangle$ with $a>b$. Then the following hold,*

(i) *$A(x,b)=B(x,a)=x$ for all $x\in[b,a]$.*

(ii) *S_A is an ordinal sum t-conorm, i.e., $S_A=\big(\langle 0,\frac{e_2}{a},S'_A\rangle,\langle\frac{e_2}{a},\frac{b}{a},S''_A\rangle,\langle\frac{b}{a},1,S'''_A\rangle\big)$.*

(iii) *$B(x,y)=\min(x,y)$ for all $(x,y)\in[0,e_2]\times[0,e_2]\cup[b,a]\times[b,1]\cup[b,1]\times[b,a]$.*

(iv) *$B(x,y)=\max(x,y)$ for all $(x,y)\in[0,b]\times]e_2,b]\cup]e_2,b]\times[0,b]$.*

Proof. (i) By Lemma 4.1(ii), we have $x=\max(x,b)\leqslant A(x,b)=B(x,a)\leqslant\min(x,a)=x$ for all $x\in[b,a]$. Thus $A(x,b)=B(x,a)=x$ holds for all $x\in[b,a]$.
(ii) Similarly to Lemma 4.3(ii), we get

$$A(x,e_2)=\begin{cases} e_2, & \text{if } x\in[0,e_2];\\ x, & \text{if } x\in[e_2,a].\end{cases}$$

According to item (i) and Lemma 4.1(ii), we obtain

$$A(x,b)=\begin{cases} B(x,a), & \text{if } x\in[0,b];\\ x, & \text{if } x\in[b,a];\end{cases}=\begin{cases} b, & \text{if } x\in[0,b];\\ x, & \text{if } x\in[b,a].\end{cases}$$

Thus the points e_2 and b are the points of division of A. Therefore, we obtain the conclusion.
(iii) By Lemma 4.1(iii), we directly have $B(x,y)=\min(x,y)$ for all $(x,y)\in[0,e_2]\times[0,e_2]\cup[b,a]\times[b,a]$. Assume that there is a point $(x_0,y_0)\in[b,a]\times]a,1]$ such that $B(x_0,y_0)<x_0$, then due to the increasing property of A, we have $b=B(b,b)\leqslant B(x_0,y_0)<x_0\leqslant a$. Hence, according to items (i) and (ii), we get for all $z_0\in[0,b]$,

$$A\big(z_0,B(x_0,y_0)\big)=\max\big(z_0,B(x_0,y_0)\big)=B(x_0,y_0)<x_0,$$
$$B\big(A(z_0,x_0),A(z_0,y_0)\big)=B\big(\max(z_0,x_0),a\big)=B(x_0,a)=x_0,$$

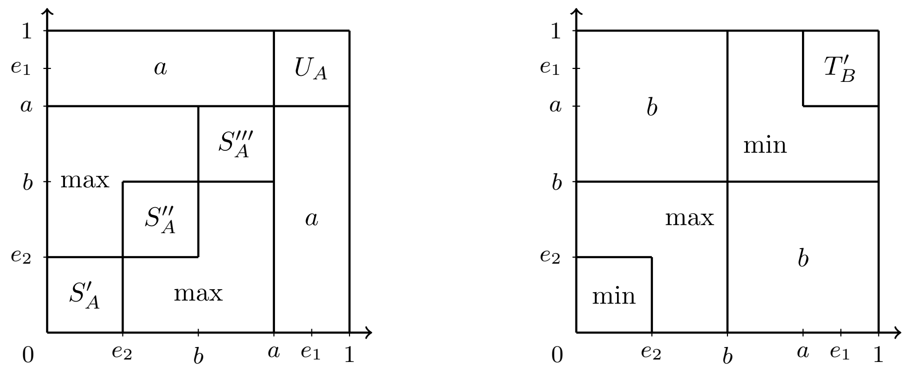


Figure 6: Structure of A (left) and B (right) in Proposition 4.11.

which imply a contradiction to the distributivity equation. Because the restriction of the T-uninorm B to the set $[b,1]\times[b,1]$ is isomorphism to a t-norm T_B, we have $B(x,y)=x$ for all $(x,y)\in[b,a]\times]a,1]$. Associating with the commutativity of B, we obtain the conclusion.
(iv) According to Lemma 4.1(iii), we obtain $B(x,y)=\max(x,y)$ for all $(x,y)\in[e_2,b]\times[e_2,b]$. By Proposition 2.7 and Lemma 4.1(iii), we have $B(x,y)\in\{x,y\}$ for all $(x,y)\in[0,e_2[\times]e_2,b]$. Assume that there are $x_0\in[0,e_2[$ and $y_0\in]e_2,b]$ such that $B(x_0,y_0)=x_0$, then we have $e_2<z_0<y_0=\max(z_0,y_0)\leqslant A(z_0,y_0)\leqslant A(b,b)=b$ from item (ii) for all $z_0\in]e_2,y_0[$. Applying item (ii) again, we obtain

$$A\big(z_0,B(x_0,y_0)\big)=A(z_0,x_0)=\max(z_0,x_0)=z_0,$$
$$B\big(A(z_0,x_0),A(z_0,y_0)\big)=B\big(z_0,A(z_0,y_0)\big)=\max\big(z_0,A(z_0,y_0)\big)=A(z_0,y_0)>z_0,$$

which imply a contradiction to the distributivity equation. Thus $B(x,y)=y=\max(x,y)$ holds for all $(x,y)\in[0,e_2[\times]e_2,b]$. Associating with the commutativity of B, we obtain the conclusion. □

Notice that we directly obtain $a,b\in]0,1[$ in the case $a>b$, due to $a<e_1$ and $b>e_2$. Moreover, by Proposition 2.9 and Lemma 4.10, we easily obtain the following conclusion.

Proposition 4.11. *Let $A\equiv\langle S_A,a,U_A,e_1\rangle$ be an S-uninorm and $B\equiv\langle U_B,e_2,b,T_B\rangle$ be a T-uninorm. If $a>b$, then A is distributive over B if and only if A is given as*

$$A(x,y)=\begin{cases} e_2S'_A\big(\frac{x}{e_2},\frac{y}{e_2}\big), & \text{if } (x,y)\in[0,e_2]\times[0,e_2];\\ e_2+(b-e_2)S''_A\big(\frac{x-e_2}{b-e_2},\frac{y-e_2}{b-e_2}\big), & \text{if } (x,y)\in[e_2,b]\times[e_2,b];\\ b+(a-b)S'''_A\big(\frac{x-b}{a-b},\frac{y-b}{a-b}\big), & \text{if } (x,y)\in[b,a]\times[b,a];\\ a+(1-a)U_A\big(\frac{x-a}{1-a},\frac{y-a}{1-a}\big), & \text{if } (x,y)\in[a,1]\times[a,1];\\ a, & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a];\\ \max(x,y), & \text{otherwise;}\end{cases}\tag{14}$$

and B is given as

$$B(x,y)=\begin{cases} \min(x,y), & \text{if } (x,y)\in[0,e_2]\times[0,e_2];\\ a+(1-a)T'_B\big(\frac{x-a}{1-a},\frac{y-a}{1-a}\big), & \text{if } (x,y)\in[a,1]\times[a,1];\\ \max(x,y), & \text{if } (x,y)\in[0,b]\times]e_2,b]\cup]e_2,b]\times[0,b];\\ \min(x,y), & \text{if } (x,y)\in[b,a]\times[b,1]\cup[b,1]\times[b,a];\\ b, & \text{otherwise;}\end{cases}\tag{15}$$

where the conjunctive uninorm U_A is distributive over the t-norm T'_B.

Remark 4.12. *Due to Propositions 4.4, 4.8 and 4.11, the distributivity for S-uninorms over T-uninorms with their annihilators in the open unit is identical with the distributivity for conjunctive uninorms over t-norms* [26, 28, 29, 40, 46, 56]*. Thus we easily obtain them by discussing the forms of either conjunctive uninorms or t-norms. Here we do not list them. Meanwhile, we easily obtain that the distributivity for T-uninorms over S-uninorms is equivalent to the distributivity for disjunctive uninorms over t-conorms* [26, 28, 29, 40, 46, 56] *by duality.*

5 Distributivity for S-uninorms over disjunctive uninorms

In this section, we further study the distributivity between an S-uninorm $A \equiv \langle S_A, a, U_A, e_1 \rangle$ and a T-uninorm $B \equiv \langle U_B, e_2, b, T_B \rangle$. Consider $a = 0$ or $b = 1$ by Remark 4.2. Then we have the following cases.

Firstly, if $a = 0$ and $b = 1$, then that distributivity equation degenerates into the distributivity between uninorms, which had been discussed in [44, 45, 48, 49].

Secondly, if $a \in]0, 1[$ and $b = 1$, then we need study the distributivity between an S-uninorm A and a disjunctive uninorm U.

Thirdly, if $a = 0$ and $b \in]0, 1[$, then that distributivity equation degenerates into the distributivity between conjunctive uninorms and T-uninorms, which can be easily characterized by duality.

Because Chen et al. [8] discussed the distributivity between S-uninorms and uninorms in $\mathcal{U}_{\max}$, this section further analyzes the distributivity for S-uninorms over disjunctive uninorms with the annihilators of the S-uninorms in the open unit.

Lemma 5.1. *Let an S-uninorm $A \equiv \langle S_A, a, U_A, e_1 \rangle$ be distributive over a disjunctive uninorm $U \equiv \langle T_U, e_2, S_U \rangle$ with $a \in]0, 1[$. Then the following hold,*

(i) $a > e_2$.

(ii) $U(x, x) = x$ *for all* $x \in [0, a]$.

(iii) S_A *is an ordinal sum t-conorm, i.e.,* $S_A = \big(\langle 0, \frac{e_2}{a}, S'_A \rangle, \langle \frac{e_2}{a}, 1, S''_A \rangle\big)$.

(iv) $U(x, y) = \max(x, y)$ *for all* $(x, y) \in [e_2, a] \times [e_2, 1] \cup [e_2, 1] \times [e_2, a]$.

(v) $U(x, y) = \max(x, y)$ *for all* $(x, y) \in [0, e_2[\times]e_2, a] \cup]e_2, a] \times [0, e_2[$.

(vi) $U(x, y) = \max(x, y)$ *for all* $(x, y) \in [0, e_2[\times]a, 1] \cup]a, 1] \times [0, e_2[$.

Proof. (i) Because U is disjunctive, we have $U(0, a) = U\big(A(0, 0), A(0, 1)\big) = A\big(0, U(0, 1)\big) = A(0, 1) = a$. If $a \leqslant e_2$, then $U(0, a) \leqslant U(0, e_2) = 0$ holds, which implies a contradiction to $a \in]0, 1[$. Thus $a > e_2$ holds.
(ii) By Proposition 2.9(i), we obtain $U(x, x) = U\big(A(x, 0), A(x, 0)\big) = A\big(x, U(0, 0)\big) = A(x, 0) = x$ for all $x \in [0, a]$. Thus $U(x, x) = x$ holds for all $x \in [0, a]$.
(iii) By items (i) and (ii), we get $e_2 = A(e_2, 0) = A\big(e_2, U(e_2, 0)\big) = U\big(A(e_2, e_2), A(e_2, 0)\big) = A(e_2, e_2)$. Hence, we have $e_2 = \max(x, e_2) \leqslant A(x, e_2) \leqslant A(e_2, e_2) = e_2$ for all $x \in [0, e_2]$. According to item (i), we obtain $e_2 \leqslant x = \max(x, e_2) \leqslant A(x, e_2) \leqslant A(a, e_2) = a$ for all $x \in [e_2, a]$. Thus it follows from item (ii) and the distributivity equation that the following hold for all $x \in [e_2, a]$,

$$A\big(x, U(e_2, 0)\big) = A(x, 0) = x,$$
$$U\big(A(x, e_2), A(x, 0)\big) = U\big(A(x, e_2), x\big) = \max\big(A(x, e_2), x\big) = A(x, e_2).$$

We have $A(x, e_2) = x$ for all $x \in [e_2, a]$. Therefore, the conclusion holds.
(iv) According to the distributivity equation, we obtain $x = A(x, e_1) = A\big(x, U(e_1, e_2)\big) = U\big(A(x, e_1), A(x, e_2)\big) = U(x, a)$ for all $x \in [a, 1]$. Consider $x \in [a, 1]$ and $y \in [e_2, a]$. Then we have $x = U(x, e_2) \leqslant U(x, y) \leqslant U(x, a) = x$. Thus $U(x, y) = x$ holds for $(x, y) \in [a, 1] \times [e_2, a]$. Associating with item (ii) and the commutativity of U, we obtain the conclusion.
(v) By item (ii), $U(a, a) = a$ holds. Hence the restriction of the disjunctive uninorm U to the set $[0, a] \times [0, a]$ is an idempotent uninorm. According to Proposition 2.7, we have $U(x, y) \in \{x, y\}$ for all $(x, y) \in [0, e_2[\times]e_2, a]$. Assume that there are $x_0 \in [0, e_2[$ and $y_0 \in]e_2, a]$ such that $U(x_0, y_0) = x_0$, then it follows from items (iii) and (iv) that the following hold for all $z_0 \in]e_2, y_0[$,

$$A\big(z_0, U(x_0, y_0)\big) = A(z_0, x_0) = \max(z_0, x_0) = z_0,$$
$$U\big(A(z_0, x_0), A(z_0, y_0)\big) = U\big(z_0, A(z_0, y_0)\big) = \max\big(z_0, A(z_0, y_0)\big) = A(z_0, y_0) \geqslant \max(y_0, z_0) > z_0,$$

which imply a contradiction to the distributivity equation. Thus $U(x, y) = y$ holds for all $(x, y) \in [0, e_2[\times]e_2, a]$. Associating with the commutativity of U, we obtain the conclusion.
(vi) Assume that there is a point $(x_0, y_0) \in [0, e_2[\times]a, 1]$ such that $U(x_0, y_0) < y_0$, then by item (v), we have $U(x_0, y_0) \geqslant U(0, a) = a$, so $A\big(e_1, U(x_0, y_0)\big) = U(x_0, y_0) < y_0$. According to item (iv), we obtain $U\big(A(e_1, x_0), A(e_1, y_0)\big) = U(a, y_0) = \max(a, y_0) = y_0$. Thus we get a contradiction to the distributivity equation. Associating with Proposition 2.3 and the commutativity of U, the conclusion holds. □

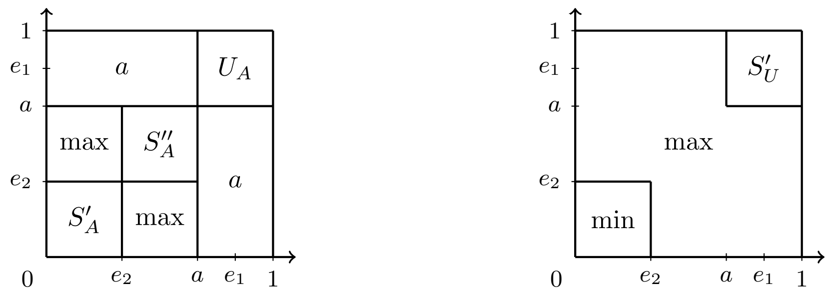


Figure 7: Structure of A (left) and U (right) in Proposition 5.2.

Proposition 5.2. *Let $A \equiv \langle S_A, a, U_A, e_1\rangle$ be an S-uninorm and $U \equiv \langle T_U, e_2, S_U\rangle$ be a disjunctive uninorm with $a \in]0,1[$. Then A is distributive over U if and only if A is given as*

$$A(x,y) = \begin{cases} e_2 S'_A\big(\frac{x}{e_2}, \frac{y}{e_2}\big), & \text{if } (x,y) \in [0,e_2]\times[0,e_2];\\ e_2 + (a-e_2)S''_A\big(\frac{x-e_2}{a-e_2}, \frac{y-e_2}{a-e_2}\big), & \text{if } (x,y) \in [e_2,a]\times[e_2,a];\\ a + (1-a)U_A\big(\frac{x-a}{1-a}, \frac{y-a}{1-a}\big), & \text{if } (x,y) \in [a,1]\times[a,1];\\ a, & \text{if } (x,y) \in [0,a]\times[a,1]\cup[a,1]\times[0,a];\\ \max(x,y), & \text{otherwise;} \end{cases} \tag{16}$$

and U is given as

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,e_2]\times[0,e_2];\\ a + (1-a)S'_U\big(\frac{x-a}{1-a}, \frac{y-a}{1-a}\big), & \text{if } (x,y) \in [a,1]\times[a,1];\\ \max(x,y), & \text{otherwise;} \end{cases} \tag{17}$$

where the conjunctive uninorm U_A is distributive over the t-conorm S'_U.

Proof. **Necessity.** It follows immediately from Lemma 5.1.

Sufficiency. It can be proven by a simple computation. □

Example 5.3. *Consider an S-uninorm A be given as*

$$A(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,0.2]\times[0,0.2];\\ \max(x+y-0.4, 0.2), & \text{if } (x,y) \in [0.2,0.4]\times[0.2,0.4];\\ 5xy-2x-2y+1.2, & \text{if } (x,y) \in [0.4,0.6]\times[0.4,0.6];\\ 0.6+0.4R(2.5x-1.5, 2.5y-1.5), & \text{if } (x,y) \in]0.6,1[\times]0.6,1[;\\ 0.2, & \text{if } (x,y) \in [0,0.2]\times[0.2,1]\cup[0.2,1]\times[0,0.2];\\ 1, & \text{if } (x,y) \in [0.4,1]\times\{1\}\cup\{1\}\times[0.4,1];\\ \min(x,y), & \text{otherwise;} \end{cases}$$

and a disjunctive uninorm U be given as

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,0.1]\times[0,0.1];\\ 1, & \text{if } (x,y) \in [0.4,1]\times[0.4,1];\\ \max(x,y), & \text{otherwise,} \end{cases}$$

where R is a representable uninorm with the neutral element $e' = 0.5$. Then $A \equiv \langle \max, 0.2, U_A, 0.8\rangle$ and $U_A \equiv \langle T_{LK}, 0.25, T_P, 0.5, (R, 0.75)\rangle_{\cos,\min}$ with $U_A(0.25,1) = 1$. Meanwhile it follows from Propositions 4.3(ii) in [56] *and 5.2 that A is distributive over U. In particular, U_A is not in $\mathcal{U}_{\min}$ but continuous in the open unit square.*

Due to Proposition 5.2, the distributivity for S-uninorms over disjunctive uninorms with the annihilators in the open unit turns out to be the distributivity for conjunctive uninorms over t-conorms, which had been studied in [26, 28, 29, 40, 46, 56]. Thus we do not list the details of them here.

Remark 5.4. *Because the underlying uninorms of S-uninorms in Proposition 5.2 are not limited in $\mathcal{U}_{\min}$ but arbitrary conjunctive uninorms, Theorem 5.2 in* [8] *is a special case of Proposition 5.2, where the underlying uninorms of S-uninorms were required to be in $\mathcal{U}_{\min}$. However, there are some problems on the distributivity for disjunctive uninorms over S-uninorms, that is, we cannot obtain a conclusion different from Theorem 5.1 in* [8] *until now.*

6 Conclusions

This paper further characterizes the distributivity between S-uninorms, the distributivity for S-uninorms over T-uninorms, and the distributivity for S-uninorms over disjunctive uninorms as the distributivity between conjunctive uninorms, the distributivity for conjunctive uninorms over t-norms and the distributivity for conjunctive uninorms over t-conorms, respetively. Although the distributivity between uninorms was studied in [31, 38, 39, 44, 45, 48, 49] and the distributivity for uninorms over t-(co)norms was discussed in [26, 28, 29, 40, 46, 56], those distributivity equations are not fully solved. The reasons are listed as follows. Firstly, we usually fix one uninorm (resp. the uninorm) in the distributivity between uninorms (resp. the distributivity for uninorms over t-(co)norms) as a special uninorm. However, there are some special uninorms without the completely clear expressions. For example, some cases of uninorms locally internal on the boundary were partially characterized in [27]. Meanwhile, a uninorm not locally internal on the boundary is not discussed in [36], when its boundary functions have infinite points of noncontinuity. Secondly, in contrast to the continuous t-(co)norms, there does not exist a conclusion representing all noncontinuous t-(co)norms. Hence, the scholars [56] partially analyzed the distributivity for uninorms over noncontinuous t-(co)norms with some auxiliary condition(s) to show the noncontinuity of t-(co)norms.

In the forthcoming work, we continue to further discuss the distributivity for uninorms over noncontinuous t-(co)norms.

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