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Study of fuzzy fractional dynamic equations on time scales for $\mathcal{S}$-correlated fuzzy functions

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Abstract

In this study, we introduce a concept of fuzzy fractional integrability and differentiability on time scales for $\mathcal{S}$-correlated fuzzy functions and propose their fundamental properties. Using these concepts, we study fuzzy fractional dynamic equations on time scales for $\mathcal{S}$-correlated fuzzy functions, establishing existence and uniqueness results for their fuzzy solutions. Additionally, we study non-homogeneous first-order linear fuzzy fractional dynamic equations with both real and fuzzy coefficients and establish their general solution forms on time scales. Furthermore, we provide illustrative examples to illustrate our results.

Keywords: Time scales, fuzzy fractional dynamic equations, $\mathcal{S}$-correlated fuzzy functions.

1 Introduction

Fractional calculus, which generalizes classical calculus by allowing derivatives and integrals of non-integer order have a deep-rooted history, dating back to the works of Leibniz, Liouville, and Riemann. Over the years, it has found extensive applications in various fields, including signal processing [10], viscoelastic material [15], and so on. Agarwal et al. [1] developed a framework for fuzzy fractional calculus and explored solutions for fractional differential equations under uncertainty, incorporating Hukuhara fractional Riemann–Liouville differentiability. Moreover, the existence and uniqueness of solutions to fuzzy fractional differential equations are investigated, as shown in [2, 3, 4, 16].

Hilger [13] introduced the formulation of dynamic equations on time scales to establish a unified framework for analyzing both difference and differential equations. Due to their significance in applications, extensive research has been conducted on dynamic equations on time scales within diverse fields, including control theory, economics, and so on [5, 14]. Building on this foundation, the development of fuzzy dynamic equations on time scales was pioneered by Fard and Bidgoli [21] to incorporate uncertainty into this framework, enabling the modeling of real-world systems with imprecise or vague data. This theory has been applied to investigating the uniqueness and existence of solutions to fuzzy dynamic equations on time scales, as exemplified in [12, 17, 22].

The fuzzy conformable derivative (FCD) offers a promising alternative to both classical fuzzy derivatives (Hukuhara-based) and classical fractional derivatives. In particular, the use of $\mathcal{S}$-correlated fuzzy functions-some classes of fuzzy functions introduced by [19]-further enhances FCD by providing a rigorous mathematical structure, improving stability, and making computations more efficient. Although classical approaches to fuzzy fractional calculus have been explored, existing methods often fail to simultaneously capture the memory effects of fractional systems and the structured uncertainty inherent in fuzzy functions. By incorporating $\mathcal{S}$-correlated fuzzy functions within the framework of time scales, we develop a more flexible and mathematically consistent approach. This framework allows for improved modeling of uncertain dynamical systems, with applications in fields such as control engineering, economics, and biomedicine.

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Although the combination of fractional theory with fuzzy dynamic equations on time scales has significant potential, it has received little attention. Notably, Shahidi and Khastan [20] introduced fuzzy fractional calculus on time scales, extended the framework proposed in [8], and applied it to solve fuzzy fractional dynamic equations (FFDEs) under Hukuhara differentiability on time scales. Subsequently, Shahidi and Esmi [18] further extended the concept of fuzzy fractional differentiability introduced in [20] by incorporating the gH-difference [7] and establishing its properties. Indeed, in the analysis of fuzzy fractional derivatives on time scales, ensuring the existence of fuzzy differences is essential, while the emergence of the switching point for the gH-derivative introduces additional challenges. To address these challenges, we introduce a notion of fractional differentiability and integrability for $\mathcal{S}$-correlated fuzzy functions on time scales. Within this framework, we investigate the existence and uniqueness of solutions to fuzzy fractional dynamic equations on time scales. The main contributions of this paper are summarized as follows:

- We introduce a notion of fractional differentiability and integrability for $\mathcal{S}$-correlated fuzzy functions on arbitrary time scales.
- We establish fundamental properties of these operators, showing their consistency with existing frameworks.
- We investigate the existence and uniqueness of solutions to fuzzy fractional dynamic equations on time scales.
- We provide a detailed analysis of first-order linear fuzzy fractional dynamic equations with both real and fuzzy coefficients, an area not previously explored in the literature.

The paper is structured as follows. In Section 2, we introduce the concept of fuzzy fractional integral and derivative for $\mathcal{S}$-correlated fuzzy functions on time scales and present some of their properties. Section 3 investigates the existence and uniqueness of solutions to fuzzy fractional differential equations on time scales. In Section 4, we study first-order linear fuzzy fractional dynamic equations with both real and fuzzy coefficients and present their fuzzy general form solutions on time scales.

2 Fuzzy fractional calculus on time scales for $\mathcal{S}$-correlated fuzzy functions

In this section, we introduce the concepts of fuzzy fractional derivatives and integrals and present their properties for $\mathcal{S}$-correlated fuzzy functions on time scales. For this purpose, we recall several relevant concepts.

Definition 2.1. [8] *Let $g : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}$. For $t > 0$, we define $\delta^\alpha g(t)$ to be the number (if it exists) with the feature that for any $\epsilon > 0$, there exists a neighbourhood $\mathcal{N}_\mathbb{T}(t, \delta) := (t - \delta, t + \delta) \cap \mathbb{T}$ of t, for some $\delta > 0$, such that*

$$\left|(g(\sigma(t)) - g(s))\, t^{1-\alpha} - \delta^\alpha g(t)\,(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|\,,$$

for all $s \in \mathcal{N}_\mathbb{T}(t, \delta)$. The function $\delta^\alpha g(t)$ is called the conformable fractional derivative of g of order $\alpha \in (0, 1]$ at t.

Remark 2.1. [9] *A function $q : \mathbb{T} \to \mathbb{R}$ is referred to as regressive if it satisfies the inequality*

$$1 + \mu(t)q(t) \neq 0,\ \forall t \in \mathbb{T}.$$

The set of all rd-continuous and regressive functions is represented by $\mathcal{R}$.

Definition 2.2. [9] *The exponential function for $q \in \mathcal{R}$ is given by*

$$e_q(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(q(\tau))\delta\tau\right), \quad \text{for} \quad s, t \in \mathbb{T},$$

where $\xi_h(z)$ is defined in [9].

Theorem 2.1. [9] *Let $q \in \mathcal{R}$, then the following holds:*

(i) $e_0(t, \tau) = 1$ *and* $e_q(t, t) = 1$.

(ii) $e_q\big(\sigma(t), \tau\big) = [1 + \mu(t)q(t)]e_q(t, \tau)$.

(iii) $e_q(t, r)e_q(r, \tau) = e_q(t, \tau)$.

(iv) $e_q(t, \tau) = \frac{1}{e_q(\tau, t)} = e_{\ominus q}(\tau, t)$, *where* $(\ominus q)(t) = -\frac{q(t)}{1+\mu(t)q(t)}$.

Remark 2.2. [9] *It is worth noting that, in a special case when $\mathbb{T} = h\mathbb{Z}$, with $h > 0$, the exponential function for $q \in \mathcal{R}$ is expressed by $e_q(t, t_0) = (1 + q(t)h)^{\frac{t-t_0}{h}}$. Additionally, when $\mathbb{T} = 2^{\mathbb{N}_0}$, it is stated by $e_q(t, t_0) = \prod_{\tau=t_0}^{\frac{t}{2}} (1 + \tau q(\tau))$, for $t > t_0$.*

Definition 2.3. *The fractional dynamic equation*

$$\delta^\alpha y(t) = t^{1-\alpha} (q(t)y(t)) + w(t), \tag{1}$$

is said to be regressive if $w : \mathbb{T} \to \mathbb{R}$ is rd-continuous and $q \in \mathcal{R}$.

Theorem 2.2. *Let $x_1 \in \mathbb{R}$ and $t_1 \in \mathbb{T}$. If Equation (1) is regressive, then the solution of the following fractional dynamic equation*

$$\begin{cases} \delta^\alpha x(t) = -t^{1-\alpha} (q(t)x^\sigma(t)) + w(t), \\ x(t_1) = x_1, \end{cases} \tag{2}$$

is given by

$$x(t) = e_{\ominus q}(t, t_1) \left(x_1 + \int_{t_1}^{t} w(\tau) e_q(\tau, t_1) \delta^\alpha \tau \right).$$

Proof. This proof is analogous to Theorem 2.74 in [9]. □

Let us now revisit some fundamental concepts related to fuzzy numbers. A fuzzy subset of real numbers is a function that maps each real number to a value in the interval $[0, 1]$. For $r \in (0, 1]$, the r-level set of A is defined by $[A]_r = \{t \in \mathbb{R} | A(t) \geq r\}$ and for $r = 0$ by the closure of the support $[A]_0 = cl\{t \in \mathbb{R} | A(t) > 0\}$. A fuzzy subset of $\mathbb{R}$ such that its r-level are non-empty, bounded, and closed intervals of $\mathbb{R}$ is called a fuzzy number [6]. The set of all fuzzy numbers is represented by $\mathbb{R}_F$. Thus, the r-levels of the fuzzy number A can be described as $[A]_r = [A_r^-, A_r^+]$, for each $r \in [0, 1]$. Moreover, fuzzy numbers can be classified into two types: symmetric and non-symmetric, a distinction that is essential for understanding strongly linearly independent (SLI).

Definition 2.4. [11] *A fuzzy number $A \in \mathbb{R}_F$ is symmetric with respect to $t \in \mathbb{R}$ if it satisfies $A(t - s) = A(t + s)$, for all $s \in \mathbb{R}$, and we represent it by $(A|t)$. If there exists no t for which A is symmetric, then A is called non-symmetric or asymmetric.*

Definition 2.5. [11] *Let $A_1, \ldots, A_n \in \mathbb{R}_F$. We define the subset $\mathcal{S}(A_1, \ldots, A_n)$ of $\mathbb{R}_F$ as the set given by*

$$\mathcal{S}(A_1, \ldots, A_n) = \{p_1 A_1 + \cdots + p_n A_n \mid p_1, \ldots, p_n \in \mathbb{R}\},$$

where $p_i A_i$ and $+$ represent the scalar multiplication and usual addition in $\mathbb{R}_F$,, respectively. Moreover, the set $\{A_1, \ldots, A_n\} \subseteq \mathbb{R}_F$ is said to be SLI if for all $A \in \mathcal{S}(A_1, \ldots, A_n)$, then we have

$$(A = p_1 A_1 + \cdots + p_n A_n \mid 0) \Rightarrow p_1 = p_2 = \cdots = p_n = 0.$$

Theorem 2.3. [11] *Let $\psi : \mathbb{R}^n \to \mathcal{S}(A_1, \ldots, A_n)$ given by $\psi(p_1, \ldots, p_n) = p_1 A_1 + \cdots + p_n A_n$. Then, the function ψ is a bijection iff the set $\{A_1, \ldots, A_n\}$ is SLI.*

Remark 2.3. [11] *Let $\{A_1, \ldots, A_n\}$ be SLI. Theorem 2.3 indicates that $(\mathcal{S}(A_1, \ldots, A_n), \oplus_\psi, \cdot_\psi, \|\cdot\|_\psi)$ is a Banach space where the operations of scalar multiplication $\cdot_\psi$, addition $\oplus_\psi$, and norm $\|\cdot\|_\psi$ are given as follows:*

$$\begin{aligned} c \cdot_\psi A &= \psi(c \cdot \psi^{-1}(A)), \\ A \oplus_\psi B &= \psi(\psi^{-1}(A) + \psi^{-1}(B)), \\ \|A\|_\psi &= \left\|\psi^{-1}(A)\right\|_\infty, \end{aligned}$$

for all $A, B \in \mathcal{S}(A_1, \ldots, A_n)$ and $c \in \mathbb{R}$, where $+$ and $.$ are respectively the usual vector addition and scalar multiplication in $\mathbb{R}^n$. In fact, for all $c \in \mathbb{R}$ and $A = p_1 A_1 +, \cdots + p_n A_n \in \mathcal{S}(A_1, \ldots, A_n)$, we have

$$c \cdot_\psi A = \psi(c \cdot \psi^{-1}(A)) = c(p_1 A_1 + \ldots + p_n A_n) = cA.$$

Thus, to simplify notation, we denote cA by $c \cdot_\psi A$.
Moreover, for $A, B \in \mathcal{S}(A_1, \ldots, A_n)$, the ψ-difference can be observed by

$$A \ominus_\psi B = A \oplus_\psi (-1) \cdot_\psi B.$$

Let $A = p_1A_1+, \cdots + p_nA_n$ *and* $B = b_1A_1 + \cdots + b_nA_n$*, then we have*

$$\begin{aligned}\mathcal{D}_\psi(A,B) &= \|A \ominus_\psi B\|_\psi \\ &= \|\psi(p_1,\ldots,p_n) \ominus_\psi \psi(b_1,\ldots,b_n)\|_\psi \\ &= \|\psi(p_1-b_1,\ldots,p_n-b_n)\|_\psi \\ &= \|(p_1-b_1,\ldots,p_n-b_n)\|_\infty \\ &= \max_{i=1,\ldots,n} |p_i-b_i|\,.\end{aligned}$$

In the following, we revisit some properties of $\mathcal{D}_\psi$.

Theorem 2.4. [19] *Let* $A, B, C, E \in \mathcal{S}(A_1,\ldots,A_n)$*, then we have*

(i) $\mathcal{D}_\psi(A \oplus_\psi E, B \oplus_\psi E) = \mathcal{D}_\psi(A,B)$.

(ii) $\mathcal{D}_\psi(A \oplus_\psi B, C \oplus_\psi E) \leq \mathcal{D}_\psi(A,C) + \mathcal{D}_\psi(B,E)$.

Definition 2.6. *Let* $\odot_P$ *be a distributive product on* $\mathcal{S}(A_1,\ldots,A_n)$ *and let* $P_{i,j} = A_i \odot_P A_j$ *for all* $i,j = 1,\ldots,n$*. For every* $A, B \in \mathcal{S}(A_1,\ldots,A_n)$ *such that* $A = p_1A_1 + \ldots + p_nA_n$ *and* $B = b_1A_1 + \ldots + b_nA_n$*, we have that*

$$A \odot_P B = \sum_{i=1}^{n}\sum_{j=1}^{n} p_i b_j P_{i,j}. \tag{3}$$

Example 2.1. *Let* A *be an asymmetric triangular fuzzy number and the SLI set* $\{1, A, A^2, \ldots, A^n\}$*. Setting*

$$P_{i,j} = A^i \odot_P A^j = \begin{cases} A^{i+j}, & \text{if } i+j \leq n, \\ 0, & \text{if } i+j > n. \end{cases} \tag{4}$$

for $i,j = 0,1,\ldots,n$*. For* $B, C \in S(1, A, \ldots, A^n)$*, we have*

$$\begin{aligned} B \odot_P C &= \begin{bmatrix} b_0 & \ldots & b_n \end{bmatrix} \begin{bmatrix} 1 & A & \ldots & A^n \\ A & A^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^n & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} b_i c_j A^{i+j}, \end{aligned} \tag{5}$$

where $B = b_0 + b_1A + \ldots + b_nA^n$ *and* $C = c_0 + c_1A + \ldots + c_nA^n$.

For a given SLI set $\{A_1,\ldots,A_n\}$, an $\mathcal{S}$-correlated fuzzy function is a fuzzy function $g : \mathbb{T} \to \mathcal{S}(A_1,\ldots,A_n)$ represented by

$$g(t) = g_1(t)A_1 + \cdots + g_n(t)A_n, \ \forall t \in \mathbb{T},$$

with $g_i : \mathbb{T} \to \mathbb{R}, i = 1,\ldots,n$. In the rest of the paper, $\{A_1,\ldots,A_n\}$ is assumed to be an SLI set.

Remark 2.4. [19] *A fuzzy function* $g : \mathbb{T} \to \mathcal{S}(A_1,\ldots,A_n)$ *is referred to as rd-continuous, provided its left-sided limits exist at all left-dense points in* $\mathbb{T}$ *and it is continuous at all right-dense points in* $\mathbb{T}$*. The space of fuzzy rd-continuous functions on* $\mathbb{T}$ *is represented by* $\mathcal{C}_{frd}(\mathbb{T})$.

Theorem 2.5. [19] *Let* $g : \mathbb{T} \to \mathcal{S}(A_1,\ldots,A_n)$ *with* $g(t) = g_1(t)A_1 + \cdots + g_n(t)A_n$*. Then, the fuzzy function* g *is rd-continuous at* t *if and only if* g_i *is rd-continuous at* t*, for all* $i = 1,\ldots,n$.

Definition 2.7. *A function* $g : \mathbb{T} \to \mathcal{S}(A_1,\ldots,A_n)$ *expressed as* $g(t) = g_1(t)A_1 + \cdots + g_n(t)A_n$ *is called* $\mathcal{S}$*-fuzzy regressive if the real function* $g_i(t)$ *is regressive for all* $i = 1,\ldots,n$*. The set of all fuzzy regressive and rd-continuous functions is represented by* $\mathcal{R}_\mathcal{S}$.

We are now prepared to introduce the fuzzy fractional derivative on time scales for $\mathcal{S}$-correlated fuzzy functions. For the reminder of the paper, the time scale $\mathbb{T}^+ = (0,\infty) \cap \mathbb{T}$ is employed.

Definition 2.8. *Let $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$. The fuzzy function g is said to be α_ψ-fuzzy fractional differentiable (α_ψ-differentiable, briefly) of order $\alpha \in (0, 1]$ at t if there exists a $\delta^\alpha_\psi g(t) \in \mathcal{S}(A_1, \ldots, A_n)$ such that for all $\epsilon > 0$ there exists a $\delta > 0$ and an open ball $\mathcal{N}_{\mathbb{T}^+}(t, \delta)$ of t with $t \mp h \in \mathcal{N}_{\mathbb{T}^+}$ and*

$$\mathcal{D}_\psi \left((g(t+h) \ominus_\psi g(\sigma(t)))\, t^{1-\alpha}, \delta^\alpha_\psi g(t)(h - \mu(t)) \right) \leq \epsilon\, |h - \mu(t)|\,,$$

$$\mathcal{D}_\psi \left((g(\sigma(t)) \ominus_\psi g(t-h))\, t^{1-\alpha}, \delta^\alpha_\psi g(t)(h + \mu(t)) \right) \leq \epsilon(h + \mu(t)).$$

If $D \subseteq \mathbb{T}^+$, we say that g is α_ψ-differentiable on D if it is α_ψ-differentiable at any point of the set D.

Theorem 2.6. *Let $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ with $g(t) = g_1(t)A_1 + \cdots + g_n(t)A_n$ be α_ψ-differentiable at $t \in \mathbb{T}^+$ if and only if the function $g_i : \mathbb{T}^+ \to \mathbb{R}$ is conformable differentiable at t, for all $i = 1, \ldots, n$. Moreover, we have*

$$\delta^\alpha_\psi g(t) = \delta^\alpha g_1(t)A_1 + \cdots + \delta^\alpha g_n(t)A_n.$$

Proof. The proof follows similarly to Theorem 3.6 in [19]. □

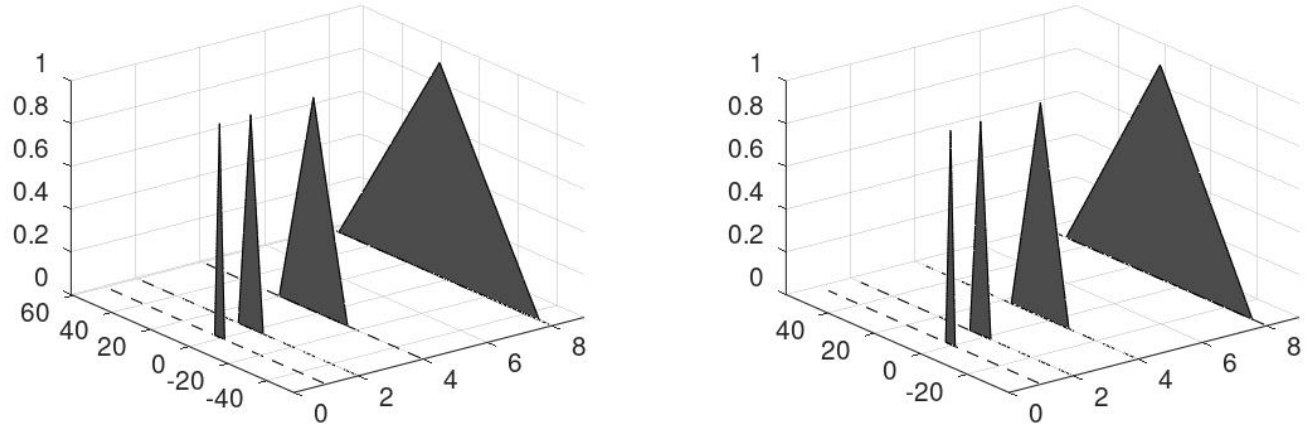

Figure 1: Plot of the function and its α_ψ-derivative in Example 2.2

Example 2.2. *Let $g : 2^{\mathbb{N}} \to \mathcal{S}(A_1, A_2, A_3)$ given by*

$$g(t) = tA_1 + (t+1)^2 A_2 + \frac{1}{t+1} A_3,$$

where $\{A_1, A_2, A_3\}$ is SLI. According to Theorem 2.6, the fuzzy function g is α_ψ-differentiable at t and

$$\delta^{\frac{1}{3}}_\psi g(t) = t^{\frac{2}{3}} A_1 + \left(3t^{\frac{5}{3}} + 2t^{\frac{2}{3}} \right) A_2 - \frac{t^{\frac{2}{3}}}{(2t+1)\,(t+1)} A_3.$$

Figure 1 illustrates the fuzzy function and its α-derivative, where the function is shown in the upper plot and its α-derivative in the lower plot.

In what follows, we establish the key properties of α-differentiability.

Theorem 2.7. *Let $f, g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ be α_ψ-differentiable in $t \in \mathbb{T}^+$ and $c \in \mathbb{R}$. Then*

(1) *$f + g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ is α_ψ-differentiable at t and*

$$\delta^\alpha_\psi (f + g)(t) = \delta^\alpha_\psi f(t) + \delta^\alpha_\psi g(t).$$

(2) *$c \cdot g : \mathbb{T} \to \mathcal{S}(A_1, \ldots, A_n)$ is α_ψ-differentiable at t and*

$$\delta^\alpha_\psi (c \cdot g)(t) = c \cdot \delta^\alpha_\psi g(t).$$

Proof. This proof is directly deduced from Theorem 2.6 and Theorem 15 in [8]. □

Theorem 2.8. *Suppose that $f : \mathbb{T}^+ \to \mathbb{R}$ is conformable differentiable at $t \in \mathbb{T}^+$, and $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ is α_ψ-differentiable at t. Then, fg is α_ψ-differentiable at t and*

$$\begin{aligned}\delta^\alpha_\psi(fg)(t) &= \delta^\alpha f(t)g(t) + f(\sigma(t))\delta^\alpha_\psi g(t)\\ &= f(t)\delta^\alpha_\psi g(t) + \delta^\alpha f(t)g(\sigma(t)).\end{aligned}$$

Proof. It is derived directly from Theorem 2.6 and Theorem 15 in [8]. □

Theorem 2.9. *If $f, g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ are α_ψ-differentiable at t, then $f \odot_P g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ is α_ψ-differentiable at t, and we get*

$$\delta^\alpha_\psi(f \odot_P g)(t) = \delta^\alpha_\psi f(t) \odot_P g(t) \oplus_\psi f(\sigma(t)) \odot_P \delta^\alpha_\psi g(t).$$

Proof. It is a direct result of Definition 2.6 and Theorem 2.6. □

Definition 2.9. *Let $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ and $t \in \mathbb{T}^+$. The α_ψ-Fuzzy fractional integral (α_ψ-integral, briefly) on $[a, b]_{\mathbb{T}^+}$ is given by*

$$\int_a^b g(t)\delta^\alpha_\psi t = \int_a^b t^{\alpha-1} \cdot_\psi g(t)\delta_\psi t,$$

in which the integral on the right is the Riemann δ_ψ-fuzzy integral presented in [19].

The following result is a straightforward consequence of Definition 2.9 and [19].

Theorem 2.10. *Let $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ given by $g(t) = g_1(t)A_1 + \cdots + g_n(t)A_n$ for all $t \in \mathbb{T}^+$. Then, the function g is α_ψ-integrable on $[a, b]_{\mathbb{T}^+}$ if and only if the real function g_i is the Cauchy α-fractional integrable, introduced in* [8], *on $[a, b]_{\mathbb{T}^+}$ for any $i = 1, \ldots, n$. Moreover, we have*

$$\int_a^b g(t)\delta^\alpha_\psi t = \left(\int_a^b g_1(t)\delta^\alpha t\right) A_1 + \cdots + \left(\int_a^b g_n(t)\delta^\alpha t\right) A_n.$$

Example 2.3. *Let $\mathbb{T}^+ = 2^{\mathbb{N}_0}$ and $g : \mathbb{T}^+ \to \mathcal{S}(A_1, A_2, A_3)$ be defined by*

$$g(t) = tA_1 + t^2A_2 + 5t^3A_3,$$

for all $t \in \mathbb{T}^+$. Therefore, by Theorem 2.10, we have

$$\begin{aligned}&\int_1^4 \left(tA_1 + t^2A_2 + 5t^3A_3\right)\delta^{\frac{1}{3}}_\psi t\\ &= \left(\int_1^4 t\delta^{\frac{1}{3}} t\right) A_1 + \left(\int_1^4 t^2\delta^{\frac{1}{3}} t\right) A_2 + \left(\int_1^4 5t^3\delta^{\frac{1}{3}} t\right) A_3\\ &= \left(\int_1^4 t^{\frac{1}{3}}\delta t\right) A_1 + \left(\int_1^4 t^{\frac{4}{3}}\delta t\right) A_2 + \left(\int_1^4 5t^{\frac{7}{3}}\delta t\right) A_3\\ &= \left(\frac{1}{2^{\frac{4}{3}} - 1}\left(4^{\frac{4}{3}} - 1\right)\right) A_1 + \left(\frac{1}{2^{\frac{7}{3}} - 1}\left(4^{\frac{7}{3}} - 1\right)\right) A_2 + \left(\frac{5}{2^{\frac{10}{3}} - 1}\left(4^{\frac{10}{3}} - 1\right)\right) A_3.\end{aligned}$$

Several basic properties of the α_ψ-integral are as follows.

Theorem 2.11. *Let $a, b, c \in \mathbb{T}^+, \lambda \in \mathbb{R}$ and $f, g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ be rd-continuous. Then*

(1) $\int_a^b (\lambda f(t) + g(t))\delta^\alpha_\psi t = \lambda \int_a^b f(t)\delta^\alpha_\psi t + \int_a^b g(t)\delta^\alpha_\psi t.$

(2) $\int_a^b g(t)\delta^\alpha_\psi t = \int_a^c g(t)\delta^\alpha_\psi t + \int_c^b g(t)\delta^\alpha_\psi t,\ c \in [a, b]_{\mathbb{T}^+}.$

Theorem 2.12. *Let $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ with $g(t) = g_1(t)A_1 + \cdots + g_n(t)A_n$ for all $t \in \mathbb{T}^+$. If g is rd-continuous on $[a, b]_{\mathbb{T}^+}$, then $G(t) = \int_a^t g(s)\delta_\psi^\alpha s$ is α_ψ-differentiable and $\delta_\psi^\alpha G(t) = g(t)$, for all $t \in [a, b]_{\mathbb{T}^+}$.*

Proof. By Theorem 2.10, we have

$$\begin{aligned}\delta_\psi^\alpha G(t) &= \delta_\psi^\alpha \left(\int_a^t g(s)\delta_\psi^\alpha s \right) \\ &= \delta^\alpha \left(\int_a^t g_1(s)\delta^\alpha s \right) A_1 + \cdots + \delta^\alpha \left(\int_a^t g_n(s)\delta^\alpha s \right) A_n \\ &= g_1(t)A_1 + \cdots + g_n(t)A_n.\end{aligned}$$

□

Theorem 2.13. *Let $g : \mathbb{T}^+ \to \mathcal{S}(A_1, \ldots, A_n)$ be α_ψ-differentiable on $[a, b]_{\mathbb{T}^+}$. If g is α_ψ-integrable on $[a, b]_{\mathbb{T}^+}$, then*

$$\int_a^t \delta_\psi^\alpha g(s)\delta_\psi^\alpha s = g(t) \ominus_\psi g(a),$$

for all $t \in [a, b]_{\mathbb{T}^+}$.

Proof. It is a straightforward consequence of Definition 8 in [8] and Theorem 2.10. □

3 Existence and uniqueness of solutions for FFDEs on time scales

Let $\xi > 0$. For $v, w \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+})$, we present

$$d_\xi(v, w) = \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \frac{\mathcal{D}_\psi(v(t), w(t))}{e_\xi(t, t_1)}.$$

Moreover, for $\xi = 0$, the metric is given by

$$d_0(v, w) = \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \mathcal{D}_\psi(v(t), w(t)),$$

because, for all $t, s \in \mathbb{T}$, by Theorem 2.1, $e_0(t, s) = 1$.

Theorem 3.1. *The metric space $(\mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+}), d_\xi)$ is complete .*

Proof. Let $\{v_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+}), d_\xi)$. Take $\epsilon > 0$ arbitrarily. Then, for any $m, n \geq N$, $m, n \in \mathbb{N}$ there exists an $N = N(\epsilon) > 0$ such that

$$\begin{aligned} d_\xi(v_m, v_n) &= \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \frac{\mathcal{D}_\psi(v_m(t), v_n(t))}{e_\xi(t_2, t_1)} \\ &\leq \frac{\epsilon}{e_\xi(t_2, t_1)}. \end{aligned}$$

Therefore, it yields that

$$\mathcal{D}_\psi(v_m(t), v_n(t)) \leq \epsilon,$$

for all $t \in [t_1, t_2]_{\mathbb{T}^+}$. As a result of the completeness of the metric space $(\mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+}), d_0)$, there exists a $v \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+})$ so that, for $t \in [t_1, t_2]_{\mathbb{T}^+}$, we observe

$$\lim_{m \to \infty} \mathcal{D}_\psi(v_m(t), v(t)) = 0.$$

As a result, it implies that

$$\lim_{m \to \infty} \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \mathcal{D}_\psi(v_m(t), v(t)) = 0.$$

In consequence, we observe

$$\begin{aligned} \lim_{m \to \infty} d_\xi(v_m, v) &\leq \frac{1}{e_\xi(t_2, t_1)} \lim_{m \to \infty} \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \mathcal{D}_\psi(v(t), v_m(t)) \\ &= 0, \end{aligned}$$

which implies $\lim_{m \to \infty} d_\xi(v_m, v) = 0$, yielding the desired result. □

Consider the following FFDE

$$\begin{cases} \delta_{\psi}^{\alpha} w(t) = g(t, w^{\sigma}(t)), \quad t \in [t_1, t_2]_{\mathbb{T}^+}, \\ w(t_1) = w_1, \end{cases} \tag{6}$$

where $g \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+} \times \mathcal{S}(A_1, \ldots, A_n))$ and $w_1 \in \mathcal{S}(A_1, \ldots, A_n)$.

Theorem 3.2. *Assume that there exists a constant $L > 0$ with $\frac{L}{\xi} < 1$ and*

$$(1 + \mu(t)\xi)\mathcal{D}_{\psi}\left(t^{\alpha-1} g(t, v), t^{\alpha-1} g(t, w)\right) \leq L\mathcal{D}_{\psi}(v, w),$$

for any $(t, v), (t, w) \in [t_1, t_2]_{\mathbb{T}^+} \times \mathcal{S}(A_1, \ldots, A_n)$, then the FFDE (6) has a unique solution.

Proof. For $w \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+})$, consider the operator $\mathcal{B}$ given by

$$\mathcal{B}w(t) = w_1 \oplus_{\psi} \int_{t_1}^{t} g(s, w^{\sigma}(s)) \delta_{\psi}^{\alpha} s, \quad t \in [t_1, t_2]_{\mathbb{T}^+}.$$

Let an arbitrary $\zeta \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+})$. As g is bounded, it follows that

$$\mathcal{D}_{\psi}\left(t^{\alpha-1} g(t, \zeta^{\sigma}(t)), \widetilde{0}\right) \leq M, \quad t \in [t_1, t_2]_{\mathbb{T}^+},$$

for a constant $M > 0$. Let $t_3, t \in [t_1, t_2]_{\mathbb{T}^+}$ with $t_3 < t$, then we get

$$\begin{aligned} \mathcal{D}_{\psi}(\mathcal{B}\zeta(t), \mathcal{B}\zeta(t_3)) &= \mathcal{D}_{\psi}\left(\int_{t_1}^{t} g(s, \zeta^{\sigma}(s)) \delta_{\psi}^{\alpha} s, \int_{t_1}^{t_3} g(s, \zeta^{\sigma}(s)) \delta_{\psi}^{\alpha} s\right) \\ &= \mathcal{D}_{\psi}\left(\widetilde{0}, \int_{t_3}^{t} g(s, \zeta^{\sigma}(s)) \delta_{\psi}^{\alpha} s\right) \\ &= \mathcal{D}_{\psi}\left(\widetilde{0}, \int_{t_3}^{t} s^{\alpha-1} g(s, \zeta^{\sigma}(s)) \delta_{\psi} s\right) \\ &\leq \int_{t_3}^{t} \mathcal{D}_{\psi}\left(\widetilde{0}, s^{\alpha-1} g(s, \zeta^{\sigma}(s))\right) \delta_{\psi} s \\ &\leq M(t - t_3). \end{aligned}$$

Hence, $\mathcal{B}\zeta \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+})$. Next, for $v, w \in \mathcal{C}_{frd}([t_1, t_2]_{\mathbb{T}^+})$, we obtain

$$\begin{aligned} d_{\xi}(\mathcal{B}v, \mathcal{B}w) &= \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \frac{\mathcal{D}_{\psi}(\mathcal{B}v(t), \mathcal{B}w(t))}{e_{\xi}(t, t_1)} \\ &= \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \frac{1}{e_{\xi}(t, t_1)} \mathcal{D}_{\psi}\left(\int_{t_1}^{t} g(s, v^{\sigma}(s)) \delta_{\psi}^{\alpha} s, \int_{t_1}^{t} g(s, w^{\sigma}(s)) \delta_{\psi}^{\alpha} s\right) \\ &= \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \frac{1}{e_{\xi}(t, t_1)} \mathcal{D}_{\psi}\left(\int_{t_1}^{t} s^{\alpha-1} g(s, v^{\sigma}(s)) \delta_{\psi} s, \int_{t_1}^{t} s^{\alpha-1} g(s, w^{\sigma}(s)) \delta_{\psi} s\right) \\ &\leq \sup_{t \in [t_1, t_2]_{\mathbb{T}^+}} \frac{1}{e_{\xi}(t, t_1)} \int_{t_1}^{t} \mathcal{D}_{\psi}\left(s^{\alpha-1} g(s, v^{\sigma}(s)), s^{\alpha-1} g(s, w^{\sigma}(s))\right) \delta_{\psi} s \end{aligned}$$

$$
\begin{aligned}
&= \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{1}{e_\xi(t,t_1)} \int_{t_1}^{t} \frac{1+\mu(s)\xi}{1+\mu(s)\xi} \mathcal{D}_\psi \left(s^{\alpha-1} g(s, v^\sigma(s)), s^{\alpha-1} g(s, w^\sigma(s))\right) \delta_\psi s \\
&\leq L \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{1}{e_\xi(t,t_1)} \int_{t_1}^{t} \frac{\mathcal{D}_\psi \left(v^\sigma(s), w^\sigma(s)\right)}{1+\mu(s)\xi} \delta_\psi s \\
&= L \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{1}{e_\xi(t,t_1)} \int_{t_1}^{t} \frac{e_\xi(s,t_1)}{e_\xi(\sigma(s),t_1)} \mathcal{D}_\psi \left(v^\sigma(s), w^\sigma(s)\right) \delta_\psi s \\
&\leq L d_\xi(v,w) \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{1}{e_\xi(t,t_1)} \int_{t_1}^{t} e_\xi(s,t_1) \delta_\psi s \\
&= \frac{L}{\xi} d_\xi(v,w) \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{1}{e_\xi(t,t_1)} \int_{t_1}^{t} \xi e_\xi(s,t_1) \delta_\psi s \\
&= \frac{L}{\xi} d_\xi(v,w) \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{e_\xi(s,t_1)}{e_\xi(t,t_1)}\Big|_{s=t_1}^{s=t} \\
&= \frac{L}{\xi} d_\xi(v,w) \sup_{t\in[t_1,t_2]_{\mathbb{T}^+}} \frac{e_\xi(t,t_1)-1}{e_\xi(t,t_1)} \\
&\leq \frac{L}{\xi} d_\xi(v,w).
\end{aligned}
$$

Thus, the operator $\mathcal{B} : \mathcal{C}_{frd}([t_1,t_2]_{\mathbb{T}^+}) \to \mathcal{C}_{frd}([t_1,t_2]_{\mathbb{T}^+})$ has a unique fixed point, concluding the proof. □

4 Study of first-order linear fuzzy fractional dynamic equations on time scales

In this section, we study non-homogeneous first-order fuzzy fractional dynamic equations with both real and fuzzy coefficients. For this purpose, we present the following definition.

Definition 4.1. *The fuzzy fractional dynamic equation*

$$\delta_\psi^\alpha u(t) = t^{1-\alpha} \left(l(t)u(t)\right) + w(t), \tag{7}$$

is called fuzzy regressive if $w : [t_1,t_2]_{\mathbb{T}^+} \to \mathcal{S}(A_1,\ldots,A_n)$ is rd-continuous and $l(t) \in \mathcal{R}$.

Consider the subsequent first-order linear fuzzy fractional dynamic equation

$$\begin{cases} \delta_\psi^\alpha v(t) = -t^{1-\alpha} \left(l(t)v^\sigma(t)\right) \oplus_\psi w(t), \\ v(t_1) = v_1, \end{cases} \tag{8}$$

where $v_1 \in \mathcal{S}(A_1,\ldots,A_n)$, $l : [t_1,t_2]_{\mathbb{T}^+} \to \mathbb{R}$ and $w, v : [t_1,t_2]_{\mathbb{T}^+} \to \mathcal{S}(A_1,\ldots,A_n)$ are *rd*-continuous.

Theorem 4.1. *If Equation* (7) *is regressive, then the solution to Equation (8) is expressed by*

$$v(t) = e_{\ominus l}(t,t_1) \left(v_1 \oplus_\psi \int_{t_1}^{t} w(\tau) e_l(\tau,t_1) \delta_\psi^\alpha \tau\right). \tag{9}$$

Proof. To establish that $v(t)$ satisfies Equation (8), we proceed, as follows. It is evident that $v(t_1) = v_1$. Furthermore, let us consider $f(t) = e_{\ominus l}(t,t_1)$ and $g(t) = v_1 \oplus_\psi \int_{t_1}^{t} w(\tau) e_l(\tau,t_1) \delta^\alpha \tau$. Then, by Theorems 2.7 and 2.8, we achieve

$$\delta_\psi^\alpha v(t) = \left((\ominus l)(t) e_{\ominus_l}(t,t_1) t^{1-\alpha}\right) \left(v_1 \oplus_\psi \int_{t_1}^{\sigma(t)} w(\tau) e_l(\tau,t_1) \delta_\psi^\alpha \tau\right) \oplus_\psi w(t).$$

Furthermore, due to Theorem 2.1, we achieve

$$\begin{aligned}
&\left(-l(t)t^{1-\alpha}\right)v^{\sigma}(t)\oplus_{\psi}w(t) = -l(t)t^{1-\alpha}\left(e_{\ominus l}(\sigma(t),t_1)\left(v_1\oplus_{\psi}\int_{t_1}^{\sigma(t)}w(\tau)e_l(\tau,t_1)\delta_{\psi}^{\alpha}\tau\right)\right)\\
&\oplus_{\psi}w(t) = \frac{-l(t)}{e_l(\sigma(t),t_1)}t^{1-\alpha}\left(v_1\oplus_{\psi}\int_{t_1}^{\sigma(t)}w(\tau)e_l(\tau,t_1)\delta_{\psi}^{\alpha}\tau\right)\oplus_{\psi}w(t)\\
&= \frac{-l(t)}{e_l(\sigma(t),t_1)}e_l(t,t_1)e_{\ominus l}(t,t_1)t^{1-\alpha}\left(v_1\oplus_{\psi}\int_{t_1}^{\sigma(t)}w(\tau)e_l(\tau,t_1)\delta_{\psi}^{\alpha}\tau\right)\oplus_{\psi}w(t)\\
&= \frac{-l(t)}{1+\mu(t)l(t)}e_{\ominus l}(t,t_1)t^{1-\alpha}\left(v_1\oplus_{\psi}\int_{t_1}^{\sigma(t)}w(\tau)e_l(\tau,t_1)\delta_{\psi}^{\alpha}\tau\right)\oplus_{\psi}w(t)\\
&= (\ominus l)(t)e_{\ominus_l}(t,t_1)t^{1-\alpha}\left(v_1\oplus_{\psi}\int_{t_1}^{\sigma(t)}w(\tau)e_l(\tau,t_1)\delta_{\psi}^{\alpha}\tau\right)\oplus_{\psi}w(t)\\
&= \delta_{\psi}^{\alpha}v(t),
\end{aligned}$$

which is the required result. □

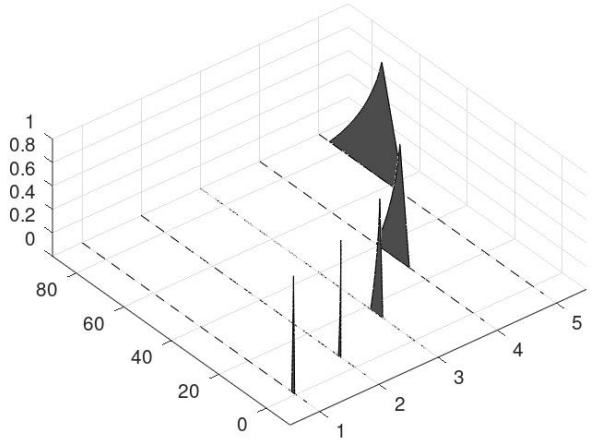

Figure 2: Fuzzy solution to Equation (10) in Example 4.1 for $\{1, A, A^2\}$ with $A = (-0.49, 0, 0.05)$.

Example 4.1. *Let $v : [1,20]_{\mathbb{N}} \to \mathcal{S}(1, A, A^2)$. Consider the following fuzzy fractional dynamic equation*

$$\begin{cases}\delta_{\psi}^{\frac{1}{3}}v(t) = \frac{t^{\frac{2}{3}}}{2}v^{\sigma}(t)\oplus_{\psi}\left(2 - tA + tA^2\right),\\ v(1) = 1 + A + A^2.\end{cases} \tag{10}$$

Let $l(t) = -\frac{1}{2}$ and $1+\mu(t)l(t) = 1-\frac{1}{2} \neq 0$. So, by Theorem 4.1, we get

$$\begin{aligned}
v(t) &= e_{\ominus(-\frac{1}{2})}(t,1)\left(\left(1+A+A^2\right)\oplus_{\psi}\int_1^t\left(2-\tau A+\tau A^2\right)e_{-\frac{1}{2}}(\tau,1)\delta_{\psi}^{\frac{1}{3}}\tau\right)\\
&= 2^{t-1}\left(\left(1+A+A^2\right)\oplus_{\psi}\left(\int_1^t\left(\left(2^{2-\tau}\right)-\left(\tau 2^{1-\tau}\right)A+\left(\tau 2^{1-\tau}\right)A^2\right)\delta_{\psi}^{\frac{1}{3}}\tau\right)\right)\\
&= 2^{t-1}\left(\left(1+\sum_{\tau=1}^{t-1}2^{2-\tau}\tau^{\frac{-2}{3}}\right)+\left(1-\sum_{\tau=1}^{t-1}2^{1-\tau}\tau^{\frac{1}{3}}\right)A+\left(1+\sum_{\tau=1}^{t-1}2^{1-\tau}\tau^{\frac{1}{3}}\right)A^2\right),
\end{aligned}$$

which is illustrated in Figure 2.

Next, we study non-homogeneous first-order fuzzy fractional dynamic equations with fuzzy coefficients. To this end, the definition is given as follows. For convenience in calculations and without loss of generality, we assume the given SLI set $\{1, A, A^2\}$ and A is an asymmetric triangular fuzzy number for the remainder of the paper.

Definition 4.2. *The fuzzy fractional dynamic equation*

$$\delta_{\psi}^{\alpha} u(t) = t^{1-\alpha} \left(q \odot_P u(t)\right) \oplus_{\psi} w(t), \tag{11}$$

is called $\mathcal{S}$-fuzzy regressive if $w : [t_1, t_2]_{\mathbb{T}^+} \to \mathcal{S}(1, A, A^2)$ is rd-continuous and $q \in \mathcal{R}_{\mathcal{S}}$ as given in Definition 2.7.

Consider the subsequent first-order linear fuzzy fractional dynamic equation

$$\begin{cases} \delta_{\psi}^{\alpha} v(t) = -t^{1-\alpha} \left(q \odot_P v^{\sigma}(t)\right) \oplus_{\psi} w(t), \\ v(t_1) = v_1, \end{cases} \tag{12}$$

where $v_1, q \in \mathcal{S}(1, A, A^2)$ and $w, v : [t_1, t_2]_{\mathbb{T}^+} \to \mathcal{S}(1, A, A^2)$ are rd-continuous.

Theorem 4.2. *Let $q = q_0 + q_1 A + q_2 A^2$, $v_1 = v_{1,0} + v_{1,1} A + v_{1,2} A^2$ and $w : [t_1, t_2]_{\mathbb{T}^+} \to \mathcal{S}(1, A, A^2)$ given by $w(t) = w_0(t) + w_1(t)A + w_2(t)A^2$. If Equation (11) is $\mathcal{S}$-fuzzy regressive, then the solution to Equation (12) is given by $v(t) = v_0(t) + v_1(t)A + v_2(t)A^2$ where*

$$v_0(t) = e_{\ominus q_0}(t, t_1) \left(v_{1,0} + \int_{t_1}^{t} w_0(\tau) e_{q_0}(\tau, t_1) \delta^{\alpha} \tau \right),$$

$$v_1(t) = e_{\ominus q_0}(t, t_1) \left(v_{1,1} + \int_{t_1}^{t} \left(\left(-\tau^{1-\alpha}\right) q_1 v_0^{\sigma}(\tau) + w_1(\tau) \right) e_{q_0}(\tau, t_1) \delta^{\alpha} \tau \right),$$

and

$$v_2(t) = e_{\ominus q_0}(t, t_1) \Big(v_{1,2} + \int_{t_1}^{t} \Big(\left(-\tau^{1-\alpha}\right) \left(q_1 v_1^{\sigma}(\tau) + q_2 v_0^{\sigma}(\tau)\right) + w_2(\tau) \Big) e_{q_0}(\tau, t_1) \delta^{\alpha} \tau \Big),$$

for all $t \in [t_1, t_2]_{\mathbb{T}^+}$.

Proof. From Definition 2.6, we have

$$q \odot_P v^{\sigma}(t) = q_0 v_0^{\sigma}(t) + \left(q_0 v_1^{\sigma}(t) + q_1 v_0^{\sigma}(t)\right) A + \left(q_0 v_2^{\sigma}(t) + q_1 v_1^{\sigma}(t) + q_2 v_0^{\sigma}(t)\right) A^2. \tag{13}$$

By substituting Equation (13) into (12) and by Theorem 2.6, we have

$$\begin{aligned} \delta^{\alpha} v_0(t) + \delta^{\alpha} v_1(t) A + \delta^{\alpha} v_2(t) A^2 = -t^{1-\alpha} \Big(& \left(q_0 v_0^{\sigma}(t)\right) + \left(q_0 v_1^{\sigma}(t) + q_1 v_0^{\sigma}(t)\right) A \\ & + \left(q_0 v_2^{\sigma}(t) + q_1 v_1^{\sigma}(t) + q_2 v_0^{\sigma}(t)\right) A^2 \Big) \oplus_{\psi} \left(w_0(t) + w_1(t) A + w_2(t) A^2\right), \end{aligned}$$

and subject to the initial value condition

$$v_0(t_1) + v_1(t_1) A + v_2(t_1) A^2 = v_{1,0} + v_{1,1} A + v_{1,2} A^2.$$

Therefore, by Theorem 2.3, we obtain the following system of classical fractional dynamic equations:

$$\begin{cases} \delta^{\alpha} v_0(t) = -t^{1-\alpha} q_0 v_0^{\sigma}(t) + w_0(t), \\ \delta^{\alpha} v_1(t) = -t^{1-\alpha} \left(q_0 v_1^{\sigma}(t) + q_1 v_0^{\sigma}(t)\right) + w_1(t), \\ \delta^{\alpha} v_2(t) = -t^{1-\alpha} \left(q_0 v_2^{\sigma}(t) + q_1 v_1^{\sigma}(t) + q_2 v_0^{\sigma}(t)\right) + w_2(t), \\ v_0(t_1) = v_{1,0}, \\ v_1(t_1) = v_{1,1}, \\ v_2(t_1) = v_{1,2}. \end{cases} \tag{14}$$

Therefore, by Theorem 2.2, we obtain $v_0(t), v_1(t)$ and $v_2(t)$. Consequently, the fuzzy solution $v(t)$ follows. This concludes the demonstration. □

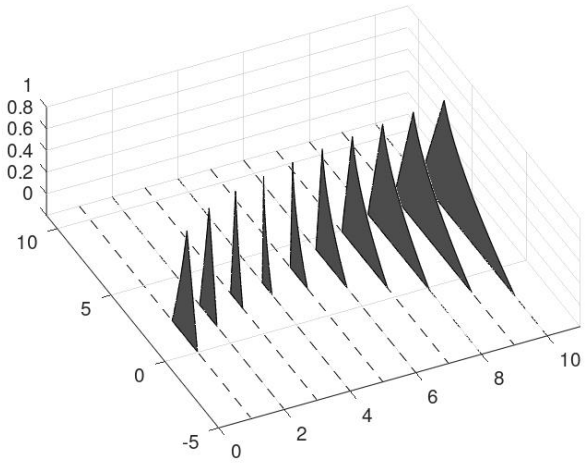

Figure 3: Fuzzy solution to Equation (17) in Example 4.2 for $\{1, A, A^2\}$ with $A = (-0.49, 0, 0.05)$.

Example 4.2. *Let $v : [1, 10]_{\mathbb{N}} \to \mathcal{S}(1, A, A^2)$. Consider the following dynamic equation*

$$\begin{cases} \delta_{\psi}^{\frac{1}{4}} v(t) = -t^{\frac{3}{4}} \left(A^2 \odot_P v^{\sigma}(t) \right) \oplus_{\psi} \left(1 + \frac{t}{2} A + t A^2 \right), \\ v(1) = 1 + 2A + \frac{3}{2} A^2. \end{cases} \tag{15}$$

Let $v(t) = v_0(t) + v_1(t)A + v_2(t)A^2$. It is straightforward to verify that the assumptions of Theorem 4.2 are satisfied. Moreover, by Remark 2.2, we have $e_{q_0}(t, 1) = e_0(t, 1) = 1$ and $e_{\ominus 0}(t, 1) = 1$. Hence, by Theorem 4.2, we observe

$$v_0(t) = 1 + \int_1^t \delta^{\frac{1}{4}} \tau = 1 + \sum_{\tau=1}^{t-1} \tau^{\frac{-3}{4}}.$$

$$v_1(t) = 2 + \int_1^t \frac{\tau}{2} \delta^{\frac{1}{4}} \tau = 2 + \frac{1}{2} \int_1^t \tau^{\frac{1}{4}} \delta\tau = 2 + \frac{1}{2} \sum_{\tau=1}^{t-1} \tau^{\frac{1}{4}}.$$

and

$$\begin{aligned} v_2(t) &= \frac{3}{2} + \int_1^t \left(-\tau^{\frac{3}{4}} \left(1 + \sum_{s=1}^{\tau} s^{\frac{-3}{4}} \right) + \tau \right) \delta^{\frac{1}{4}} \tau \\ &= \frac{3}{2} + \int_1^t \left(-\tau^{\frac{3}{4}} \left(1 + \sum_{s=1}^{\tau} s^{\frac{-3}{4}} \right) + \tau \right) \tau^{\frac{-3}{4}} \delta\tau \\ &= \frac{3}{2} - \int_1^t \left(1 - \tau^{\frac{1}{4}} + \sum_{s=1}^{\tau} s^{\frac{-3}{4}} \right) \delta\tau \\ &= \frac{3}{2} - \sum_{\tau=1}^{t-1} \left(1 - \tau^{\frac{1}{4}} + \sum_{s=1}^{\tau} s^{\frac{-3}{4}} \right) \\ &= \left(\frac{3}{2} - t + 1 \right) + \sum_{\tau=1}^{t-1} \tau^{\frac{1}{4}} - \sum_{\tau=1}^{t-1} \sum_{s=1}^{\tau} s^{\frac{-3}{4}}, \end{aligned}$$

which is illustrated in Figure 3.

Example 4.3. *Let $v : [1, 10]_{2^{\mathbb{N}_0}} \to \mathcal{S}(1, A, A^2)$ given by $v(t) = v_0(t) + v_1(t)A + v_2(t)A^2$. Consider the following equation*

$$\begin{cases} \delta_{\psi}^{\frac{2}{3}} v(t) = -t^{\frac{1}{3}} \left(\left(1 + A^2 \right) \odot_P v^{\sigma}(t) \right) \oplus_{\psi} \left(tA + A^2 \right), \\ v(1) = 1 + 5A + \frac{1}{2} A^2. \end{cases} \tag{16}$$

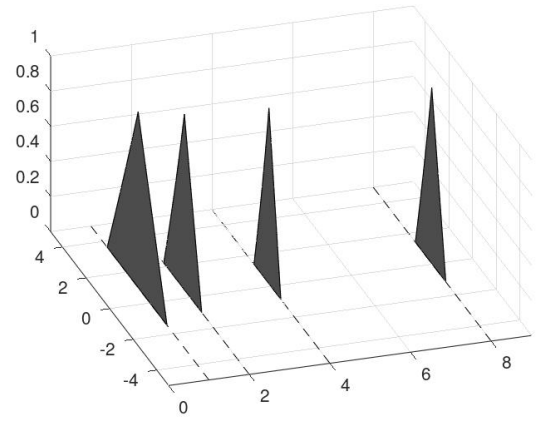

Figure 4: Fuzzy solution to Equation (16) in Example 4.3 for $\{1, A, A^2\}$ with $A = (-0.49, 0, 0.05)$.

By Remark 2.2, we have $e_{q_0}(t,1) = e_1(t,1) = \prod_{s=1}^{\frac{t}{2}}(1+s)$ *and* $e_{\ominus 1}(t,1) = \frac{1}{\prod_{s=1}^{\frac{t}{2}}(1+s)}$. *According to Theorem 4.2, we have* $v_0(t) = \frac{1}{\prod_{s=1}^{\frac{t}{2}}(1+s)}$ *and*

$$v_1(t) = \frac{1}{\prod_{s=1}^{t/2}(1+s)}\left(5 + \int_1^t \tau \prod_{s=1}^{\tau/2}(1+s)\,\delta^{\frac{2}{3}}\tau\right) = \frac{1}{\prod_{s=1}^{t/2}(1+s)}\left(5 + \int_1^t \tau^{\frac{2}{3}} \prod_{s=1}^{\tau/2}(1+s)\,\delta\tau\right)$$
$$= \frac{1}{\prod_{s=1}^{t/2}(1+s)}\left(5 + \sum_{\tau=1}^{t/2} \tau^{\frac{5}{3}} \prod_{s=1}^{\tau/2}(1+s)\right).$$

$$v_2(t) = \frac{1}{\prod_{s=1}^{t/2}(1+s)}\left(\frac{1}{2} + \int_1^t \left(-\tau^{\frac{1}{3}} \frac{\prod_{s=1}^{\tau/2}(1+s)}{\prod_{s=1}^{\tau}(1+s)} + \prod_{s=1}^{\tau/2}(1+s)\right)\delta^{\frac{2}{3}}\tau\right)$$
$$= \frac{1}{\prod_{s=1}^{t/2}(1+s)}\left(\frac{1}{2} + \int_1^t \left(-\frac{\prod_{s=1}^{\tau/2}(1+s)}{\prod_{s=1}^{\tau}(1+s)} + \tau^{\frac{-1}{3}} \prod_{s=1}^{\tau/2}(1+s)\right)\delta\tau\right)$$
$$= \frac{1}{\prod_{s=1}^{t/2}(1+s)}\left(\frac{1}{2} + \sum_{\tau=1}^{t/2}\left(\frac{-\tau \prod_{s=1}^{\tau/2}(1+s)}{\prod_{s=1}^{\tau}(1+s)} + \tau^{\frac{2}{3}} \prod_{s=1}^{\tau/2}(1+s)\right)\right),$$

which is illustrated in Figure 4.

Example 4.4. *We consider a fuzzy population model in which the population size $v(t)$ evolves under the effect of both intrinsic decay and an external fuzzy input. Due to uncertainties in the growth environment and limited biological information, the model includes fuzzy-valued coefficients and initial conditions. The evolution is described on $[1,10]_{\mathbb{N}}$ using the following fuzzy dynamic system:*

$$\begin{cases} \delta_{\psi}^{\frac{1}{2}} v(t) = -t^{\frac{1}{2}} \left(A \odot_P v^{\sigma}(t)\right) \oplus_{\psi} (1 + tA), \\ v(1) = 1 + 5A. \end{cases} \tag{17}$$

where $v : [1,10]_{\mathbb{N}} \to \mathcal{S}(1,A)$ *given by* $v(t) = v_0(t) + v_1(t)A$ *represents the fuzzy population size at time* $t \in [1,10]_{\mathbb{N}}$*, and* A *is a fuzzy parameter capturing the uncertainty in the interaction intensity or decay rate within the population. The term* $-t^{1/2}\,(A \odot_P v^{\sigma}(t))$ *models the natural vital dynamics of population at some instant* t*, while the term* $1 + tA$ *represents an external fuzzy source (e.g. migration) that influences the population. It is straightforward to verify that the assumptions of Theorem 4.2 are satisfied. Moreover, by Remark 2.2, we have* $e_{q_0}(t,1) = e_0(t,1) = 1$ *and* $e_{\ominus 0}(t,1) = 1$*. Hence, by Theorem 4.2, we observe*

$$v_0(t) = 1 + \int_1^t \delta^{\frac{1}{2}}\tau = 1 + \sum_{\tau=1}^{t-1} \tau^{\frac{-1}{2}}.$$

and

$$v_1(t) = 5 + \int_1^t \left(-\tau^{\frac{1}{2}}\left(1 + \sum_{s=1}^{\tau} s^{\frac{-1}{2}}\right) + \tau\right)\delta^{\frac{1}{2}}\tau = 5 - \sum_{\tau=1}^{t-1}\left(1 + \sum_{s=1}^{\tau} s^{\frac{-1}{2}}\right) + \sum_{\tau=1}^{t-1}\tau^{\frac{1}{2}}.$$

5 Conclusions

In this work, we have established the notions of fractional differentiability and integrability on time scales for $\mathcal{S}$-correlated fuzzy functions. This allowed us to formulate and analyze fuzzy fractional dynamic equations within this setting. We established existence and uniqueness results for the $\mathcal{S}$-correlated fuzzy functions. Moreover, we have investigated first-order linear fuzzy fractional dynamic equations with both real and fuzzy coefficients, as well as presented their general form solutions on time scales. Our findings lay the groundwork for further research in fuzzy fractional calculus. Potential directions include extending these results to higher-order equations, investigating stability properties, and developing numerical schemes for approximating solutions. Additionally, applications in control theory and real-world modeling could further demonstrate the practical relevance of this framework.

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