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Original paper

Deferred pointwise f-statistical convergence of sequences of fuzzy mappings of order α

E. Bayram [1], M. Et [2] and A. Aydın [3]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Tekirdağ Namık Kemal University, Tekirdağ, Türkiye*
[2] *Department of Mathematics, Faculty of Sciences, Fırat University, Elazığ, Türkiye*
[3] *Department of Mathematics, Faculty of Science and Literature, Muş Alparslan University, Muş, Türkiye*

ebayram@nku.edu.tr, met@firat.edu.tr, a.aydin@alparslan.edu.tr

Abstract

This research paper introduces and investigates two new concepts, namely pointwise deferred f-statistical convergence of order α and strong pointwise deferred f-summability of order α, within the context of sequences of fuzzy mappings. The study explores the relationships between these newly proposed concepts and establishes several inclusion theorems. These findings contribute valuable insights into the properties of the introduced concepts, shedding light on their characteristics and applications in the realm of fuzzy mappings.

Keywords: Fuzzy number, fuzzy mappings, deferred statistical convergence, Cesàro summability, pointwise statistical convergence.

1 Introduction and background information

The concept of statistical convergence, first studied by Steinhaus [31] and Fast [19], has been widely investigated across various mathematical fields. Schoenberg [30] extended this notion, leading to applications in Banach spaces, number theory, Fourier analysis, ergodic theory, and measure theory. Further developments have been made by Fridy [20], Et et al. [16, 17, 18, 32, 33], and others [3, 8, 11, 12, 21, 22, 23, 29], particularly within sequence spaces and summability theory.

Statistical convergence is based on the natural density of $K \subseteq \mathbb{N}$, defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),$$

where χ_K is the characteristic function of K. A sequence (x_k) is said to be statistically convergent to x (shortly, $x_k \xrightarrow{\text{stat}} x$, or $x_n \xrightarrow{st} x$ or $\text{S}-\lim_{k\to\infty} x_k = x$) if, for every $\varepsilon > 0$,

$$\delta\big(\{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}\big) = 0.$$

Agnew [1] introduced Cesàro means with deferred summation:

$$(D_{r,s}x)_n = \frac{1}{(s_n - r_n)} \sum_{k=r_n+1}^{s_n} x_k \ , \ n \in \mathbb{N}^+ \tag{1}$$

Corresponding Author: E. Bayram

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where $r = (r_n)$ and $s = (s_n)$ are sequences of non-negative integers satisfying $r_n < s_n$ and $s_n \to \infty$. This leads to the concept of deferred density $K \subseteq \mathbb{N}$:

$$\delta_{r,s}(K) = \lim_{n\to\infty} \frac{|\{k \in K : r_n < k \leq s_n\}|}{s_n - r_n},$$

As a result, *the deferred statistical convergence* of the sequence $x = (x_k)$ to x (denoting $S_{r,s}-\lim x_k = L$) is given by, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(s_n - r_n)} |\{r_n < k \leq s_n : |x_k - L| \geq \varepsilon\}| = 0.$$

Deferred statistical convergence extends statistical convergence by replacing natural density with deferred density. When $s_n = n$ and $r_n = 0$, it can be easily observed that the usual statistical convergence and deferred statistical convergence coincide [24].

The notion of modulus functions f, introduced by Nakano [26], plays a key role in defining f-statistical convergence. A modulus $f : [0,\infty) \to [0,\infty)$ is a non-negative, increasing function satisfying $f(0) = 0$, right continuity at 0, and subadditivity. It is evident that f is continuous everywhere on $[0,\infty)$ and we easily obtain the inequality $f(nx) \leq nf(x)$ and consequently, $f(n) \leq nf(1)$ for every positive integer n and real $x \geq 0$. Throughout the article, the set of all unbounded modulus functions is denoted by $\mathcal{M}^{ub}$. Aizpuru et al. [2] introduced f-density using unbounded modulus functions, later extended by Bhardwaj-Dhawan [7] to strong Cesàro summability and f-statistical convergence of order α.

The concept of statistical convergence for sequences of real and complex numbers has been extended to fuzzy numbers and fuzzy mappings [4, 5, 6, 9, 10, 14, 15, 27, 34, 35]. Fuzzy sets are investigated within the framework of a nonempty base set X comprising elements of interest. As articulated by Zadeh [36], a fuzzy subset of X is defined as a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$, and is represented by a membership function $u : X \to [0, 1]$. Specifically, $u(x) = 0$ signifies nonmembership, $0 < u(x) < 1$ indicates partial membership, and $u(x) = 1$ denotes full membership.

The set $L(\mathbb{R}^n)$ is denoted the space of fuzzy numbers, consisting of functions $\phi : \mathbb{R}^n \to [0, 1]$ satisfying:

i. Normality: $\exists x_0 \in \mathbb{R}^n$ with $\phi(x_0) = 1$.

ii. Fuzzy convexity: $\min\{\phi(x), \phi(y)\} \leq \phi(\lambda x + (1-\lambda)y)$ for all $x, y \in \mathbb{R}^n$, $\lambda \in [0, 1]$.

iii. Upper semicontinuity.

iv. Compact support: $[\phi]^0 = \{x \in \mathbb{R}^n : \phi(x) > 0\}$ has compact closure.

In addition, the γ-level set $[\phi]^\gamma$ is defined for $0 < \gamma \leq 1$ as follows:

$$[\phi]^\gamma = \{x \in \mathbb{R}^n : \phi(x) \geq \gamma\}.$$

For any $Y, Z \in C(\mathbb{R}^n)$, the family of all nonempty, convex, and compact subsets of $\mathbb{R}^n$, the distance between Y and Z with respect to the Hausdorff metric is defined by

$$\delta_\infty(Y, Z) = \max\Big\{\sup_{z\in Z} \inf_{y\in Y} \|y - z\|,\ \sup_{y\in Y} \inf_{z\in Z} \|y - z\|\Big\}.$$

It is well known that the metric space $(C(\mathbb{R}^n), \delta_\infty)$ is complete [13]. Hence, since all level sets belong to $C(\mathbb{R}^n)$, the space of fuzzy numbers can be equipped with the metric using the Hausdorff metric δ_∞, and further, for $1 \leq q < \infty$, by

$$d_q(\phi, \psi) = \left(\int_0^1 [\delta_\infty([\phi]^\gamma, [\psi]^\gamma)]^q d\gamma\right)^{1/q},$$

where $[\phi]^\gamma$ denotes the γ-level set. Obviously, $d_q \leq d_s$ when $q \leq s$ [13]. Throughout the paper, d denotes d_q.

A fuzzy mapping is a function $A : U \to L(\mathbb{R}^n)$, where $U \subset \mathbb{R}^n$. A sequence of fuzzy mappings (A_k) converges pointwise to A on U if, for each $u \in U$, the sequence $A_k(u)$ converges to $A(u)$ in the space of fuzzy numbers [25].

2 Pointwise deferred f-statistical convergence of order α

In this section, we introduce the notion of pointwise deferred f-statistical convergence of order α for sequences of fuzzy mappings. We systematically investigate inclusion relations between classes of sequences associated with different pairs (r,s) and (p,q). To establish these relations rigorously, we first present the following fundamental definition.

Throughout this paper, we consider sequences $r=(r_n)$ and $s=(s_n)$ of natural numbers satisfying $r_n < s_n$ for all $n \in \mathbb{N}$, with both $s_n \to \infty$ and $(s_n - r_n) \to \infty$ as $n \to \infty$. We denote the set of all such pairs by Γ.

Definition 2.1. *Consider $(r,s) \in \Gamma$, a fixed real number $\alpha \in (0,1]$ and $f \in \mathcal{M}^{ub}$. A fuzzy mappings sequence (A_k) is defined as pointwise deferred f-statistically convergent of order α (or pointwise $S^{\alpha}_{r,s}(f)$-statistically convergent) to X on a set U if we have*

$$\lim_{n\to\infty} \frac{1}{f\left((s_n - r_n)^{\alpha}\right)} f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k(u), A(u)\right) \geq \varepsilon\right\}\right|\right) = 0,$$

for every $\varepsilon > 0$ and for every $u \in U$.

In such instances, we express the limit as $S^{\alpha}_{r,s}(f)$ of the sequence $A_k(u)$ as equal to $A(u)$ on U. The ensemble encompassing all fuzzy mappings sequences that are pointwise $S^{\alpha}_{r,s}(f)$-statistically convergent will be labeled as $S^{\alpha}_{r,s}(f)$.

If the parameters are chosen specifically, the notation $S^{\alpha}_{r,s}(f)$ will be used differently as:

(i) $S_{r,s}(f)$, whenever $\alpha = 1$,

(ii) $S^{\alpha}(f)$, whenever $r_n = 0$ and $s_n = n$,

(iii) $S(f)$, whenever $r_n = 0$ and $s_n = n$ and $\alpha = 1$,

(iv) S, whenever $r_n = 0$ and $s_n = n$, $\alpha = 1$ and $f(x) = x$,

If $A = \bar{0}$, we will use $S^{\alpha}_{0r,s}(f)$ instead of $S^{\alpha}_{r,s}(f)$, where

$$\bar{0}(u) = \begin{cases} 1, & \text{for } u = (0,0,0,...,0) \\ 0, & \text{otherwise} \end{cases}$$

The proof of the subsequent theorem is straightforward, hence we present it without further elaboration.

Theorem 2.2. *Let $(r,s) \in \Gamma$, $\alpha \in (0,1]$ be a fixed real number and $f \in \mathcal{M}^{ub}$. Then, for the fuzzy mappings sequences (A_k) and (B_k), the following assertions hold:*

(i) *If $S^{\alpha}_{r,s}(f) - \lim A_k(u) = A_0(u)$ and $S^{\alpha}_{r,s}(f) - \lim B_k(u) = B_0(u)$, then*

$$S^{\alpha}_{r,s}(f) - \lim\left(A_k(u) + B_k(u)\right) = A_0(u) + B_0(u),$$

(ii) *If $S^{\alpha}_{r,s}(f) - \lim A_k(u) = A_0(u)$ and $c \in \mathbb{R}$, then*

$$S^{\alpha}_{r,s}(f) - \lim(cA_k(u)) = cA_0(u).$$

Theorem 2.3. *Consider $(r,s) \in \Gamma$, a real number $\alpha \in (0,1]$, and $f \in \mathcal{M}^{ub}$. In this context, it is affirmed that every fuzzy mappings sequence that converges pointwise is also pointwise deferred f-statistically convergent of order α.*

Proof. Assume that $(A_k(u))$ is an arbitrary pointwise convergent fuzzy mappings sequence to $A(u)$ for every $u \in U$. Then, for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : d(A_k(u), A(u)) \geq \varepsilon, \forall u \in U\}$ is finite. Say

$$\left|\left\{k \in \mathbb{N} : d\left(A_k(u), A(u)\right) \geq \varepsilon, \forall u \in U\right\}\right| = K.$$

Since the inclusion

$$\left\{k \in (r_n, s_n] : d\left(A_k(u), A(u)\right) \geq \varepsilon, \forall u \in U\right\} \subset \left\{k \in \mathbb{N} : d\left(A_k(u), A(u)\right) \geq \varepsilon, \forall u \in U\right\},$$

holds and f is increasing, we have

$$f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k(u), A(u)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \leq f\left(\left|\left\{k \in \mathbb{N} : d\left(A_k(u), A(u)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) = f(K),$$

and hence

$$\frac{f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon\ \right\}\right|\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} \leq \frac{f\left(K\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)}.$$

Since $f\left(K\right)$ is a constant, taking limit as $n \to \infty$ on the both sides we get

$$\lim_{n\to\infty} \frac{f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon\ \right\}\right|\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} = 0.$$

This means that (A_k) is pointwise deferred f-statistically convergent of order α. □

By giving special values to the pair of (r, s), α and f, we obtain the following results.

Corollary 2.4. (i) *For a real number $\alpha \in (0, 1]$ and $f \in \mathcal{M}^{ub}$, every pointwise convergent fuzzy mappings sequence is pointwise f-statistically convergent of order α.*

(ii) *For $(r, s) \in \Gamma$ and $f \in \mathcal{M}^{ub}$, every pointwise convergent fuzzy mappings sequence is pointwise deferred f-statistically convergent.*

(iii) *For $(r, s) \in \Gamma$ and a real number $\alpha \in (0, 1]$, every pointwise convergent fuzzy mappings sequence is pointwise deferred statistically convergent of order α.*

(iv) *Every pointwise convergent fuzzy mappings sequence is pointwise statistically convergent.*

Theorem 2.5. *Let $(r, s) \in \Gamma$, $\alpha \in (0, 1]$ be a real number and $f \in \mathcal{M}^{ub}$. If $\left(\frac{f\left(s_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)}\right)$ is bounded, then every pointwise f-statistically convergent sequence of order α is pointwise deferred f-statistically convergent of order α.*

Proof. First, we observe the inequality as follows:

$$\begin{aligned}
&\frac{1}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \\
&\qquad \leq \frac{1}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} f\left(\left|\left\{k \in [1, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \\
&\qquad + \frac{1}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} f\left(\left|\left\{k \in [1, r_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right).
\end{aligned}$$

By the assumption, if $\left(\frac{f\left(s_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)}\right)$ is bounded, then $\left(\frac{f\left(r_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)}\right)$ is bounded as well. Then, there exists $K_1, K_2 \in \mathbb{R}^+$ such that $\frac{f\left(s_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} \leq K_1$ and $\frac{f\left(r_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} \leq K_2$. Thus, by using that f is increasing, one can obtain

$$\begin{aligned}
&\frac{1}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \\
&\qquad \leq \frac{f\left(s_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} \frac{1}{f\left(s_n^{\alpha}\right)} f\left(\left|\left\{k \in [1, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \\
&\qquad + \frac{f\left(r_n^{\alpha}\right)}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} \frac{1}{f\left(r_n^{\alpha}\right)} f\left(\left|\left\{k \in [1, r_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \\
&\qquad \leq \frac{K_1}{f\left(s_n^{\alpha}\right)} f\left(\left|\left\{k \in [1, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) \\
&\qquad + \frac{K_2}{f\left(r_n^{\alpha}\right)} f\left(\left|\left\{k \in [1, r_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right).
\end{aligned}$$

Consequently, taking limit as n tends to ∞ on the both sides, we get

$$\lim_{n\to\infty} \frac{1}{f\left(\left(s_n - r_n\right)^{\alpha}\right)} f\left(\left|\left\{k \in (r_n, s_n] : d\left(A_k\left(u\right), A\left(u\right)\right) \geq \varepsilon, \forall u \in U\right\}\right|\right) = 0.$$

This means that (A_k) is pointwise deferred f-statistically convergent of order α. □

Corollary 2.6. (i) *Let $(r,s)\in\Gamma$, $\alpha\in(0,1]$ be a real number. If $\left(\dfrac{s_n^{\alpha}}{(s_n-r_n)^{\alpha}}\right)$ is bounded, then every fuzzy mappings sequence which is pointwise statistically convergent of order α is pointwise deferred statistically convergent of order α.*

(ii) *Let $(r,s)\in\Gamma$ and $f\in\mathcal{M}^{ub}$. If $\left(\dfrac{f(s_n)}{f(s_n-r_n)}\right)$ is bounded, then every fuzzy mappings sequence which is pointwise f-statistically convergent is pointwise deferred f-statistically convergent.*

In the next theorem, we modify the conditions on the sequences $r=(r_n)$ and $s=(s_n)$ to establish an analogous relation as in Corollary 2.6(i).

Theorem 2.7. *Suppose that $(r,s)\in\Gamma$ and $\alpha\in(0,1]$ is a real number. If $\liminf_n \dfrac{s_n}{r_n}>1$, then every fuzzy mappings sequence which is pointwise statistically convergent of order α is pointwise deferred statistically convergent of order α.*

Proof. If the inequality $\liminf_n \dfrac{s_n}{r_n}>1$ is satisfied, then one can find a positive number v such that $\dfrac{s_n}{r_n}>1+v$ for sufficiently large values of n. This implies that

$$\frac{s_n-r_n}{s_n}\geq\frac{v}{1+v}\Longrightarrow\left(\frac{s_n-r_n}{s_n}\right)^{\alpha}\geq\left(\frac{v}{1+v}\right)^{\alpha}\Longrightarrow\frac{1}{{s_n}^{\alpha}}\geq\frac{v^{\alpha}}{(1+v)^{\alpha}}\frac{1}{(s_n-r_n)^{\alpha}}.$$

If the fuzzy mappings sequence (A_k) is pointwise statistically convergent of order α then we have

$$\begin{aligned}\frac{1}{{s_n}^{\alpha}}\left|\{k\in[1,s_n]:\varepsilon\leq d(A_k(u),A(u)),\forall u\in U\}\right| &\geq\frac{1}{{s_n}^{\alpha}}\left|\{k\in(r_n,s_n]:\varepsilon\leq d(A_k(u),A(u)),\forall u\in U\}\right|\\ &\geq\frac{v^{\alpha}}{(1+v)^{\alpha}}\frac{1}{(s_n-r_n)^{\alpha}}\left|\{k\in(r_n,s_n]:\varepsilon\leq d(A_k(u),A(u)),\forall u\in U\}\right|.\end{aligned}$$

Therefore, (A_k) is pointwise deferred statistically convergent of order α. □

Corollary 2.8. *If $(r,s)\in\Gamma$ and $\liminf_n \dfrac{s_n}{r_n}>1$, then every fuzzy mappings sequence that is pointwise statistically convergent is also pointwise deferred statistically convergent.*

In the upcoming results, we will compare fuzzy mappings sequences that are pointwise $S^{\beta}_{r,s}(f)$-statistically convergent and fuzzy mappings sequences that are pointwise $S^{\alpha}_{p,q}(f)$-statistically convergent under the following condition:

$$r_n\leq p_n<q_n\leq s_n,\ \forall n\in\mathbb{N}, \tag{3}$$

where (r,s) and $(p,q)\in\Gamma$.

Theorem 2.9. *Suppose that α and β are two fixed real numbers such that $0<\alpha\leq\beta\leq1$, $f\in\mathcal{M}^{ub}$ and $(r,s),(p,q)\in\Gamma$. If*

$$\lim_{n\to\infty}\frac{f\left((q_n-p_n)^{\alpha}\right)}{f\left((s_n-r_n)^{\beta}\right)}=C>0, \tag{4}$$

holds then pointwise $S^{\beta}_{r,s}(f)$-statistically convergent fuzzy mappings sequences is pointwise $S^{\alpha}_{p,q}(f)$-statistically convergent.

Proof. Let (A_k) be a pointwise $S^{\beta}_{r,s}(f)$-statistically convergent sequence, then we have

$$\frac{1}{f\left((s_n-r_n)^{\beta}\right)}f\left(\left|\{k\in(r_n,s_n]:\varepsilon\leq d(A_k(u),A(u)),\forall u\in U\}\right|\right)=0.$$

From (3), for any $\varepsilon>0$ we have

$$\{k\in(r_n,s_n]:\varepsilon\leq d(A_k(u),A(u)),\forall u\in U\}\supset\{k\in(p_n,q_n]:\varepsilon\leq d(A_k(u),A(u)),\forall u\in U\},$$

and hence we can observe the following inequality:

$$\begin{aligned}&\frac{1}{f\left((s_n-r_n)^{\beta}\right)}f\left(\left|\left\{k\in(r_n,s_n]:\varepsilon\le d\left(A_k\left(u\right),A\left(u\right)\right),\forall u\in U\right\}\right|\right)\\&\qquad\ge\frac{1}{f\left((s_n-r_n)^{\beta}\right)}f\left(\left|\left\{k\in(p_n,q_n]:\varepsilon\le d\left(A_k\left(u\right),A\left(u\right)\right),\forall u\in U\right\}\right|\right)\\&\qquad=\frac{f\left((q_n-p_n)^{\alpha}\right)}{f\left((s_n-r_n)^{\beta}\right)}\frac{1}{f\left((q_n-p_n)^{\alpha}\right)}f\left(\left|\left\{k\in(p_n,q_n]:\varepsilon\le d\left(A_k\left(u\right),A\left(u\right)\right),\forall u\in U\right\}\right|\right).\end{aligned}$$

Taking limit as n tends to ∞ on the both sides, we obtain

$$\frac{1}{f\left((q_n-p_n)^{\alpha}\right)}f\left(\left|\left\{k\in(p_n,q_n]:\varepsilon\le d\left(A_k\left(u\right),A\left(u\right)\right),\forall u\in U\right\}\right|\right)=0.$$

This means that (A_k) is pointwise $S^{\alpha}_{p,q}(f)$-statistically convergent. □

From Theorem 2.9 we get the following results.

Corollary 2.10. (i) *Let $(r,s),(p,q)\in\Gamma$, $\alpha\in(0,1]$ be a fixed real number and $f\in\mathcal{M}^{ub}$. If (3) and*

$$\lim_{n\to\infty}\frac{f\left((q_n-p_n)^{\alpha}\right)}{f\left((s_n-r_n)^{\alpha}\right)}=C>0, \tag{5}$$

hold, then pointwise $S^{\alpha}_{r,s}(f)$-statistically convergent fuzzy mappings sequences is pointwise $S^{\alpha}_{p,q}(f)$-statistically convergent.

(ii) *Let $(r,s),(p,q)\in\Gamma$ and $f\in\mathcal{M}^{ub}$. If (3) and*

$$\lim_{n\to\infty}\frac{f\left(q_n-p_n\right)}{f\left(s_n-r_n\right)}=C>0, \tag{6}$$

hold, then pointwise $S_{r,s}(f)$-statistically convergent sequences of fuzzy mappings is pointwise $S_{p,q}(f)$-statistically convergent.

(iii) *Let $(r,s),(p,q)\in\Gamma$, $\alpha\in(0,1]$ be a fixed real number. If (3) and*

$$\lim_{n\to\infty}\frac{(q_n-p_n)^{\alpha}}{(s_n-r_n)^{\alpha}}=C>0, \tag{7}$$

hold, then pointwise $S^{\alpha}_{r,s}$-statistically convergent sequences of fuzzy mappings is pointwise $S^{\alpha}_{p,q}$-statistically convergent.

(iv) *Let $(r,s),(p,q)\in\Gamma$. If (3) and*

$$\lim_{n\to\infty}\frac{(q_n-p_n)}{(s_n-r_n)}=C>0, \tag{8}$$

hold, then pointwise $S_{r,s}$-statistically convergent fuzzy mappings sequences is pointwise $S_{p,q}$-statistically convergent.

3 Strong pointwise deferred f-summability of order α

The concept of strong pointwise deferred f-summability of order α of fuzzy mapping sequences is introduced in this section.

Definition 3.1. *Let $(r,s)\in\Gamma$, $\alpha\in(0,1]$ be a fixed real number and $f\in\mathcal{M}^{ub}$. A fuzzy mappings sequence (A_k) is referred to as strongly pointwise deferred f-summable of order α (or strongly pointwise $w^{\alpha}_{r,s}(f)$-summable) to A on a set U provided that*

$$\lim_{n\to\infty}\frac{1}{f\left((s_n-r_n)^{\alpha}\right)}\sum_{r_n+1}^{s_n}f\left(d\left(A_k\left(u\right),A\left(u\right)\right)\right)=0,$$

holds for every $\varepsilon>0$.

We express $w_{r,s}^{\alpha}(f) - \lim A_k(u) = A(u)$ on U. The collection of all strongly pointwise $w_{r,s}^{\alpha}(f)$-summable fuzzy mappings sequences will be denoted by $w_{r,s}^{\alpha}(f)$.

If the parameters are chosen specifically, the notation $w_{r,s}^{\alpha}(f)$ will be used differently as:

i) $w_{r,s}(f)$, whenever $\alpha = 1$,

ii) $w^{\alpha}(f)$, whenever $r_n = 0$ and $s_n = n$,

iii) $w(f)$, whenever $r_n = 0$, $s_n = n$ and $\alpha = 1$,

iv) w, whenever $\alpha = 1$, $r_n = 0$, $s_n = n$ and $f(x) = x$.

Theorem 3.2. *Suppose that $(r,s),(p,q) \in \Gamma$ such that (3) holds, α and β are two fixed real numbers within the range $0 < \alpha \leq \beta \leq 1$ and $f \in \mathcal{M}^{ub}$. If the condition (4) is satisfied, then $w_{r,s}^{\beta}(f) \subset w_{p,q}^{\alpha}(f)$.*

Proof. Let (A_k) be a strongly pointwise $w_{r,s}^{\beta}(f)$-summable sequence, then we have

$$\lim_{n\to\infty} \frac{1}{f\left((s_n - r_n)^{\beta}\right)} \sum_{r_n+1}^{s_n} f\left(d\left(A_k(u), A(u)\right)\right) = 0.$$

Since the following inequality has been satisfied

$$\sum_{r_n+1}^{s_n} f\left(d\left(A_k(u), A(u)\right)\right) \geq \sum_{p_n+1}^{q_n} f\left(d\left(A_k(u), A(u)\right)\right),$$

we have

$$\begin{aligned} \frac{1}{f\left((s_n - r_n)^{\beta}\right)} \sum_{r_n+1}^{s_n} f\left(d\left(A_k(u), A(u)\right)\right) &\geq \frac{1}{f\left((s_n - r_n)^{\beta}\right)} \sum_{p_n+1}^{q_n} f\left(d\left(A_k(u), A(u)\right)\right) \\ &= \frac{f\left((q_n - p_n)^{\alpha}\right)}{f\left((s_n - r_n)^{\beta}\right)} \frac{1}{f\left((q_n - p_n)^{\alpha}\right)} \sum_{p_n+1}^{q_n} f\left(d\left(A_k(u), A(u)\right)\right). \end{aligned}$$

As a results, taking limit as $n \to \infty$, on the both sides, we get

$$\lim_{n\to\infty} \frac{1}{f\left((q_n - p_n)^{\alpha}\right)} \sum_{p_n+1}^{q_n} f\left(d\left(A_k(u), A(u)\right)\right) = 0.$$

That is, (A_k) is a strongly pointwise $w_{p,q}^{\alpha}(f)$-summable sequence.

□

Corollary 3.3. (i) *Let $(r,s),(p,q) \in \Gamma$, α be a fixed real numbers such that $0 < \alpha \leq 1$ and $f \in \mathcal{M}^{ub}$. If (3) and (5) hold, then $w_{r,s}^{\alpha}(f) \subset w_{p,q}^{\alpha}(f)$.*

(ii) *Let $(r,s),(p,q) \in \Gamma$ and $f \in \mathcal{M}^{ub}$. If (3) and (6) hold, then $w_{r,s}(f) \subset w_{p,q}(f)$.*

(iii) *Let $(r,s),(p,q) \in \Gamma$, α be a fixed real numbers such that $0 < \alpha \leq 1$. If (3) and (7) hold, then $w_{r,s}^{\alpha} \subset w_{p,q}^{\alpha}$.*

(iv) *Let $(r,s),(p,q) \in \Gamma$. If (3) and (8) hold, then $w_{r,s}^{\alpha} \subset w_{p,q}^{\alpha}$.*

Lemma 3.4. [28] *Consider a modulus function f and a scalar $0 < \delta < 1$. Then, the inequality $f(x) \leq 2f(1)\delta^{-1}x$ holds for each $x \geq \delta$.*

Theorem 3.5. *Let $(r,s) \in \Gamma$, α be a fixed real number such that $0 < \alpha \leq 1$ and $f \in \mathcal{M}^{ub}$. If $\lim_{u\to\infty} \frac{f(u)}{u} > 0$ holds, then $w_{r,s}^{\alpha}(f) = w_{r,s}^{\alpha}[f]$, where $w_{r,s}^{\alpha}[f]$ defined by*

$$w_{r,s}^{\alpha}[f] = \left\{ A = (A_k) : \lim_{n\to\infty} \frac{1}{f\left((s_n - r_n)^{\alpha}\right)} \sum_{r_n+1}^{s_n} d\left(A_k(u), A(u)\right) = 0 \right\}.$$

Proof. Let (A_k) be strongly pointwise $w_{r,s}^{\alpha}(f)$-summable sequence. Under assumptions, we observe that $\underset{u\in(0,\infty)}{inf}\frac{f(u)}{u}=\lim_{u\longrightarrow\infty}\frac{f(u)}{u}$. Suppose that $m=\underset{u\in(0,\infty)}{inf}\frac{f(u)}{u}>0$. Hence, for every $u\in(0,\infty)$, the inequality $\frac{f(u)}{u}>m$ holds and so that $mu\leq f(u)$. Clearly,

$$\frac{1}{f((s_n-r_n)^{\alpha})}\sum_{r_n+1}^{s_n}f(d(A_k(u),A(u)))\geq\frac{m}{f((s_n-r_n)^{\alpha})}\sum_{r_n+1}^{s_n}[d(A_k(u),A(u))]=0,$$

and this follows that (A_k) be a strongly pointwise $w_{r,s}^{\alpha}[f]$-summable sequence.

If (A_k) be a strongly pointwise $w_{r,s}^{\alpha}[f]$-summable sequence then we have

$$\lim_{n\to\infty}\frac{1}{f((s_n-r_n)^{\alpha})}\sum_{r_n+1}^{s_n}d(A_k(u),A(u))=0.$$

Let $0<\delta<1$. From Lemma 3.4, it follows that

$$\begin{aligned}\frac{1}{f((s_n-r_n)^{\alpha})}\sum_{r_n+1}^{s_n}d(A_k(u),A(u))&\geq\frac{1}{f((s_n-r_n)^{\alpha})}\sum_{\substack{r_n+1\\ d(A_k(u),A(u))\geq\delta}}^{s_n}d(A_k(u),A(u))\\&\geq\frac{1}{f((s_n-r_n)^{\alpha})}\sum_{\substack{r_n+1\\ d(A_k(u),A(u))\geq\delta}}^{s_n}\frac{f(d(A_k(u),A(u)))}{2f(1)\delta^{-1}}\\&\geq\frac{1}{2f(1)\delta^{-1}}\frac{1}{f((s_n-r_n)^{\alpha})}\sum_{\substack{r_n+1\\ d(A_k(u),A(u))\geq\delta}}^{s_n}f(d(A_k(u),A(u))).\end{aligned}$$

Therefore, (A_k) is strongly pointwise $w_{r,s}^{\alpha}(f)$-summable sequence. □

Corollary 3.6. *Let $(r,s)\in\Gamma$ and $f\in\mathcal{M}^{ub}$. If $\lim_{u\to\infty}\frac{f(u)}{u}>0$ holds, then $w_{r,s}(f)=w_{r,s}[f]$.*

Theorem 3.7. *Let $(r,s)\in\Gamma$, α and β be two fixed real numbers such that $0<\alpha\leq\beta\leq1$ and $f\in\mathcal{M}^{ub}$ such that $\lim_{u\to\infty}\frac{f(u)}{u}>0$. Then, strongly pointwise $w_{r,s}^{\alpha}(f)$-summable fuzzy mappings sequence is pointwise $S_{r,s}^{\beta}(f)$-statistically convergent.*

Proof. Since $m=\underset{u\in(0,\infty)}{inf}\frac{f(u)}{u}>0$, the inequality $mu\leq f(u)$ holds for every $u\in(0,\infty)$. Now let (A_k) be a strongly pointwise $w_{r,s}^{\alpha}(f)$-summable sequence. Since f is modulus and

$$|\{k\in(r_n,s_n]:d(A_k(u),A(u))\geq\varepsilon,\forall u\in U\}|,$$

is a positive integer, the inequality

$$f(|\{k\in(r_n,s_n]:\varepsilon\leq d(A_k(u),A(u))\}|)\leq|\{k\in(r_n,s_n]:\varepsilon\leq d(A_k(u),A(u))\}|\,f(1),$$

is satisfied. Hence, for every $u\in U$ and any $\varepsilon>0$, we observe the following inequalities

$$\begin{aligned}\frac{1}{(s_n-r_n)^{\alpha}}\sum_{r_n+1}^{s_n}f(d(A_k(u),A(u)))&\geq\frac{m}{(s_n-r_n)^{\alpha}}\sum_{r_n+1}^{s_n}d(A_k(u),A(u))\\&\geq\frac{m}{(s_n-r_n)^{\beta}}\sum_{\substack{k\in(r_n,s_n]\\ d(A_k(u),A(u))\geq\varepsilon}}d(A_k(u),A(u))\\&\geq\frac{m}{(s_n-r_n)^{\beta}}(|\{k\in(r_n,s_n]:d(A_k(u),A(u))\geq\varepsilon\}|)\,\varepsilon\\&\geq\frac{f(|\{k\in(r_n,s_n]:d(A_k(u),A(u))\geq\varepsilon\}|)}{f\left((s_n-r_n)^{\beta}\right)}\frac{f\left((s_n-r_n)^{\beta}\right)}{(s_n-r_n)^{\beta}}\frac{m}{f(1)}\varepsilon.\end{aligned}$$

By taking limit of both sides of the above inequality as n tends to ∞, it can be seen that (A_k) is pointwise $S^{\beta}_{r,s}(f)$-statistically convergent while (A_k) is strongly pointwise $w^{\alpha}_{r,s}(f)$-summable. □

Considering special cases of $(r,s) \in \Gamma$, $0 < \alpha \leq \beta \leq 1$ and $f \in \mathcal{M}^{ub}$ in Theorem 3.7, we get the next results.

Corollary 3.8.

(i) *When* $(r,s) \in \Gamma$, $\alpha \in (0,1]$ *is a fixed real number and* $f \in \mathcal{M}^{ub}$ *with* $\lim_{u\to\infty} \dfrac{f(u)}{u} > 0$, *a fuzzy mappings sequence demonstrating strong pointwise* $w^{\alpha}_{r,s}(f)$*-summability implies pointwise* $S^{\alpha}_{r,s}(f)$*-statistical convergence.*

(ii) *For* $(r,s) \in \Gamma$ *and* $f \in \mathcal{M}^{ub}$ *with* $\lim_{u\to\infty} \dfrac{f(u)}{u} > 0$, *a fuzzy mappings sequence exhibiting strong pointwise* $w_{r,s}(f)$*-summability implies pointwise* $S_{r,s}(f)$*-statistical convergence.*

(iii) *For fixed real numbers* α *and* β *such that* $0 < \alpha \leq \beta \leq 1$, *and* $f \in \mathcal{M}^{ub}$ *with* $\lim_{u\to\infty} \dfrac{f(u)}{u} > 0$, *a fuzzy mappings sequence manifesting strong pointwise* $w^{\alpha}(f)$*-summability implies pointwise* $S^{\beta}(f)$*-statistical convergence.*

(iv) *For* $(r,s) \in \Gamma$, *and fixed real numbers* α *and* β *such that* $0 < \alpha \leq \beta \leq 1$, *a fuzzy mappings sequence displaying strong pointwise* $w^{\alpha}_{r,s}$*-summability implies pointwise* $S^{\beta}_{r,s}$*-statistical convergence.*

(v) *For* $f \in \mathcal{M}^{ub}$ *with* $\lim_{u\to\infty} \frac{f(u)}{u} > 0$, *a fuzzy mappings sequence with strong pointwise* $w(f)$*-summability implies pointwise* $S(f)$*-statistical convergence.*

4 Conclusion

Deferred statistical convergence and strong deferred Cesàro summability for real sequences were initially introduced and investigated by Küçükaslan and Yılmaztürk [24] in 2016, with subsequent refinements by Et et al. [16, 32]. This paper delves into the ideas of pointwise deferred f-statistical convergence of order α and strong pointwise deferred f-summability of order α for fuzzy mappings sequence. The results obtained in this study surpass those achieved by previous researchers. To derive these comprehensive results, we introduce extensive classes of sequences of real numbers utilizing two sequences of non-negative integers, denoted as $r = (r_n)$ and $s = (s_n)$, where the condition $r_n < s_n$ holds, and $\lim_{n\to\infty} s_n = \infty$. Researchers in this field can explore the concepts of uniformly deferred f-statistical convergence of order α for fuzzy number sequences with the constraint $0 < \alpha \leq 1$, and examine the relations between uniformly deferred f-statistical convergence and pointwise deferred f-statistical convergence.

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