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Fuzzy identity graph of finite Abelian groups

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Abstract

In this work, we attempt to link a fuzzy graph to a fuzzy subgroup. A fuzzy graph, with the vertices representing the fuzzy subgroup's elements, is used to represent a fuzzy subgroup. An identity graph is a specific type of graph that can be used to represent a finite group. We define fuzzy identity graph of finite abelian groups by combining the notions of fuzzy graph and identity graph of a group. For a fuzzy identity graph of a finite abelian group, we determine an upper bound on the number of fuzzy cut vertices and fuzzy bridges. The structure of the fuzzy identity graph of the direct product of abelian groups is also obtained.

Keywords: Fuzzy identity graph, fuzzy cut vertex, fuzzy bridge.

1 Introduction

The concept of fuzzy sets, which was introduced by Lotfi Zadeh in 1965 [33], offers a framework for depicting vagueness and uncertainty in data. Zadeh's fuzzy set theory permits elements to possess varying degrees of membership, as opposed to being entirely within or outside a set. This idea of fuzzy set theory established the basis for fuzzy graph theory. Kaufmann provided the initial definition of a fuzzy graph [19]. Subsequently, Rosenfeld [27] and Yeh and Bang [32] contributed to the foundational aspects of fuzzy graph theory. Rosenfeld presented fuzzy analogs for several fundamental graph-theoretic concepts, such as subgraphs, paths, connectedness, cliques, bridges, cut vertices, forests, and trees.

The generalization of a crisp graph is a fuzzy graph. The fuzzy vertex set $\mathcal{V}$ and the fuzzy edge set $\mathcal{E}$ make up a fuzzy graph, where $\mathcal{E}$ is a fuzzy relation on $\mathcal{V}$. Bhattacharya explained the concepts of eccentricity and center in fuzzy graphs and talked about some of their characteristics. Bhutani first proposed the idea of complete fuzzy graphs in 1989 [12]. The concepts of connectedness, fuzzy cycles, and fuzzy trees were defined differently by Delgado, Verdegay, and Vila [14] in 1985 than by Rosenfeld. Mordeson conducted research on fuzzy line graphs [23]. Bhutani and Rosenfeld first proposed the idea of strong edges in fuzzy graphs in 2002. Outer planar graphs in fuzzy graphs and its applications were studied in [17].

Molodtsov [22] introduced the idea of soft set theory, which is essentially an extension of fuzzy set theory. [3] discusses soft graphs and operations on soft graphs. Feng et al.[15, 16] created rough soft sets, soft rough sets, and soft-rough fuzzy sets by combining soft sets with rough and fuzzy sets. Rough soft graphs and rough soft hypergraphs were introduced by Sarwar et al.[29]. Among the many new ideas presented by Akram et al. [2, 4, 5, 6, 7, 8, 9, 10] were bipolar fuzzy graphs, certain ideas of m-polar fuzzy graphs, m-polar fuzzy competition graphs, fuzzy soft competition hypergraphs and m-polar fuzzy hypergraphs. [11] investigates the use of m-polar interval-valued fuzzy hypergraphs in decision-making problems. Decision making approach with Type-2 soft graphs was discussed in [30]. Some operations on complicated picture fuzzy graphs with applications are described in [31]. Energy of fuzzy soft set and its application is discussed in [21]. Domination in rough fuzzy digraphs was studied in [1]. In [28], a scientometric review of fuzzy graph research was carried out.

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In 1971, A. Rosenfeld introduced the notion of fuzzy subset of a set to introduce the notion of a fuzzy subgroup of a group [26]. This led to the development of fuzzy abstract algebra. The theoretical aspects of fuzzy subgroup of a group are covered in the book [24].

Several graph types can be used to represent a finite group. [13] provides a thorough analysis of graphs defined on groups. V Kandasamy and F Smarandache [18] introduce the identity graphs of groups and semigroups. Since the vertices in an identity graph of a group represent the group's elements, the group's order is equal to the number of vertices in the identity graph. The identity graph of a group shows all of the self-inversed and mutually inversed elements.

A finite group can be represented by a special graph called the identity graph. The motivation behind this work is to explore whether the fuzzy graph associated with the identity graph of a group can be related to the corresponding fuzzy subgroup. The objective of this work is to establish connections between the concepts of fuzzy subgroups, fuzzy graphs, and the identity graph of a group. Combining these concepts, we define fuzzy identity graph of a group. The fuzzy identity graph of a group allows us to readily identify fuzzy subgroups whose support contains self-inverse elements.

In this paper, we study fuzzy identity graph of finite abelian groups. The arrangement of the paper is as follows. In Section 2 we discuss essential preliminaries connected to our work. In Section 3 , we define fuzzy identity graph of a finite group and also determine an upper bound for fuzzy cut vertices and fuzzy bridges of Z_n. In Sections 4 and 5, we extend our work to direct products of finite Abelian groups. Section 6 explains applications of fuzzy identity graph and the conclusion is given in Section 7.

2 Preliminaries

This section includes the basic definitions and results used in the sequel.

2.1 Identity graph of a group

Definition 2.1. [18] *The identity graph of group G is a graph where the vertices are the elements of G satisfying the following properties:*

(i) *$a, b \in G$ are adjacent in the identity graph of G if $ab = e$.*

(ii) *Each element of G is connected with the identity element e.*

Remark 2.2. *The identity graph of G consists of lines and triangles originating from e. Lines indicate the number of self-inverse elements in the group, while triangles represent elements that are mutually inversed.*

Example 2.3. [18] *Let $G_1 = < g/g^8 = 1 >$ be the cyclic group of order 8. Then G_1 is isomorphic to Z_8. Figure 1 represents the identity graph of G_1.*

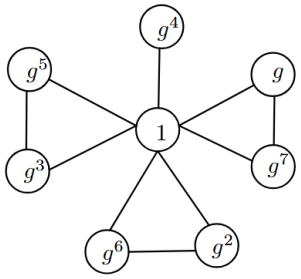

Figure 1: The Identity graph of a cyclic group of order 8

Theorem 2.4. [18] *If G is a cyclic group of order n, n an odd number, then the identity graph of G is formed with $\frac{n-1}{2}$ triangles.*

Theorem 2.5. [18] *If G is a cyclic group of order m, m an even number, then the identity graph of G has $\frac{m-2}{2}$ triangles and a line.*

2.2 Fuzzy graph

Definition 2.6. [20] *A fuzzy graph $\mathcal{G} = (V, E, \mathcal{V}, \mathcal{E})$ consists of a nonempty set V together with an edge set E of all unordered pairs of distinct vertices and a pair of functions $\mathcal{V} : V \to [0, 1]$ and $\mathcal{E} : E \to [0, 1]$ such that $\mathcal{V}(u) \neq 0$ for at least one u and $\mathcal{E}(uv) \leq \mathcal{V}(u) \wedge \mathcal{V}(v)$ for all $uv \in E$. The fuzzy set $\mathcal{V}$ is called the fuzzy vertex set and $\mathcal{E}$ is called the fuzzy edge set of $\mathcal{G}$. Also $\mathcal{E}$ is a fuzzy relation on $\mathcal{V}$.*

Notation 2.7. *We denote the fuzzy graph $\mathcal{G} = (V, E, \mathcal{V}, \mathcal{E})$ by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The support of $\mathcal{V}$ is denoted by $\mathcal{V}^*$ and is defined by $\mathcal{V}^* = \{u \in V \ \mathcal{V}(u) > 0\}$.*

2.3 Connectivity in fuzzy graphs

Definition 2.8. [20] *A path $\mathcal{P}$ in a fuzzy graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a sequence of distinct vertices x_0, x_1, ..., x_n such that $\mathcal{E}(x_{i-1}x_i) > 0, i = 1, 2, ..., n$. Here, n is called the length of the path. The diameter of $x, y \in V$ is the length of the longest path joining x to y. The strength of $\mathcal{P}$ is defined to be $\wedge_{i=1}^{n}\mathcal{E}(x_{i-1}x_i)$, $i = 1, 2, , ..., n$. The strength of connectedness between two vertices x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $\mathcal{E}^\infty(x, y)$ or $CONN_G(x, y)$. A strongest path joining any two vertices x, y has strength $\mathcal{E}^\infty(x, y)$.*

Definition 2.9. [20] *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a fuzzy graph. Let $\mathcal{G}'$ be the partial fuzzy subgraph of $\mathcal{G}$ obtained by deleting the edge xy. That is, $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ where $\mathcal{E}'(xy) = 0$ and $\mathcal{E}' = \mathcal{E}$ for all other pairs. An edge xy is said to be a fuzzy bridge in $\mathcal{G}$ if the deletion of the edge xy reduces the strength of connectedness between some pair of vertices in $\mathcal{G}$. xy is a fuzzy bridge in $\mathcal{G}$ if $\mathcal{E}'^\infty(u, v) < \mathcal{E}^\infty(u, v)$ for some $u, v \in \mathcal{V}^*$. Thus, xy is a fuzzy bridge if and only if there exists vertices u, v such that xy is an edge of every strongest path from u to v.*

Definition 2.10. [20] *A vertex w in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be a fuzzy cut vertex if deleting the vertex w reduces the strength of connectedness between some other pair of vertices. Thus, w is a fuzzy cut vertex if and only if there exists vertices u, v distinct from w such that w is on every strongest path from u to v.*

Theorem 2.11. [20] *Every fuzzy graph $\mathcal{G}$ has at least two vertices which are not cut vertices.*

Theorem 2.12. [20] *If w is a common vertex of at least two fuzzy bridges, then w is a fuzzy cut vertex.*

Theorem 2.13. [20] *If uv is a fuzzy bridge, then $\mathcal{E}^\infty(u, v) = \mathcal{E}(uv)$.*

2.4 Fuzzy subgroup

Definition 2.14. [26] *Let G be a group. A fuzzy subset $\mathcal{A}$ of a group G is called a fuzzy subgroup of the group G if*

(i) $\mathcal{A}(xy) \geqslant \wedge\{\mathcal{A}(x), \mathcal{A}(y)\}$ *for every $x, y \in G$ and*

(ii) $\mathcal{A}(x^{-1}) = \mathcal{A}(x)$ *for every $x \in G$.*

Remark 2.15. *From condition i, we have $\mathcal{A}(x) \leq \mathcal{A}(e), \forall x \in G$, where e is the identity element of the group G.*

Remark 2.16. *If G is a finite cyclic group, then the membership degree of a generator is the least among all the elements of G and all generators of G have the same membership degree.*

Example 2.17. [25] *Consider the group $G = \{1, -1, i, -i\}$ where the binary operation is ordinary multiplication. The identity element 1 is given the maximum membership degree. i and -i are inverse elements and so they should be given same membership degrees.*
$\mathcal{A}_1 =< \frac{1}{0.4}, \frac{-1}{0.3}, \frac{i}{0.2}, \frac{-i}{0.2} >$,
$\mathcal{A}_2 =< \frac{1}{0.4}, \frac{-1}{0.4}, \frac{i}{0.1}, \frac{-i}{0.1} >$,
$\mathcal{A}_3 =< \frac{1}{0.4}, \frac{-1}{0.3}, \frac{i}{0.1}, \frac{-i}{0.1} >$,
$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are fuzzy subgroups defined on G.

Any number of fuzzy subgroups can be defined on a finite group G, provided the conditions for fuzzy subgroup are satisfied.

3 Fuzzy identity graph

In this section, we define fuzzy identity graph of a finite group.

Definition 3.1. *Let G be a finite group with identity element e. A fuzzy graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a fuzzy identity graph of the group G if $\mathcal{G}$ is the identity graph of G and the fuzzy set $\mathcal{V}$ is a fuzzy subgroup of G. ie., A fuzzy graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a fuzzy identity graph of a group G if it satisfies the following conditions.*

(1) $\mathcal{E}(xy) \leq \mathcal{V}(x) \wedge \mathcal{V}(y)\ \forall x, y \in G$.

(2) $\mathcal{G}$ *is the identity graph of* G.

(3) $\mathcal{V}(xy) \geq \wedge\{\mathcal{V}(x), \mathcal{V}(y)\}$ *and* $\mathcal{V}(x^{-1}) = \mathcal{V}(x)\ \forall x, y \in G$.

Remark 3.2. *A fuzzy identity graph corresponding to a fuzzy subgroup of G is not unique.*

Remark 3.3. *The self inversed elements of a fuzzy subgroup of G can be easily determined from fuzzy identity graph of G. If $a[\alpha]$ is a vertex of $\mathcal{G}$ of degree one, then the fuzzy subgroup of G whose support contains a and e has the property that each element is its own inverse.*

Example 3.4. *Consider the Example 2.17. The fuzzy identity graphs of G corresponding to the fuzzy subgroups $\mathcal{A}_1$ and $\mathcal{A}_3$ are given in Figure 2 and are denoted by $\mathcal{G}_1$ and $\mathcal{G}_3$ respectively. A fuzzy identity graph of a fuzzy subgroup is not unique. By giving different membership degrees to fuzzy vertices and fuzzy edges we get different fuzzy identity graphs corresponding to a fuzzy subgroup.*

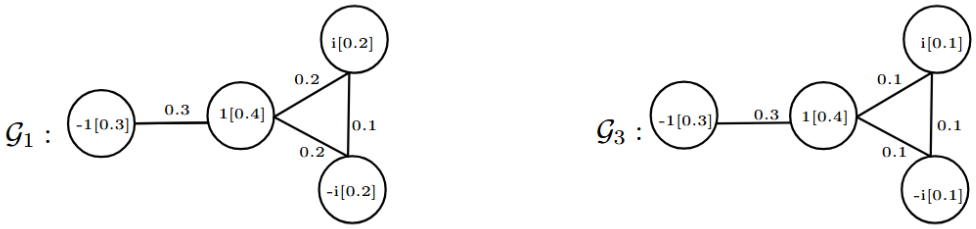

Figure 2: Fuzzy Identity graphs of G corresponding to $\mathcal{A}_1$ and $\mathcal{A}_3$

Remark 3.5. *From the definition, it is clear that fuzzy identity graph of a finite group is a connected graph and corresponding to each fuzzy subgroup of G, we get a fuzzy identity graph of G.*

Example 3.6. *Consider the group Z_5 under addition modulo 5. Figure 3 represents a fuzzy identity graph of Z_5.*

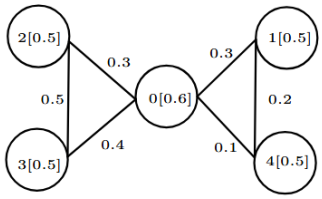

Figure 3: A fuzzy identity graph of Z_5

Theorem 3.7. *The fuzzy identity graph of Z_n, where $n > 3$, n odd, has at most $\frac{n-1}{2} + 1$ fuzzy cut vertices and at most $n - 1$ fuzzy bridges.*

Proof. When n is odd, the identity graph of Z_n has $\frac{n-1}{2}$ triangles.
Let $\mathcal{G}$ denote a fuzzy identity graph of Z_n.
We consider three cases.
Case 1: The membership of all edges in each triangle of $\mathcal{G}$ are distinct.

In each triangle, the edges corresponding to the larger two of the membership degrees will be fuzzy bridges. The number of fuzzy bridges in $\mathcal{G}$ is then $2 \times \frac{n-1}{2} = n-1$.
By Theorem 2.12, in each triangle, the vertex incident with the two fuzzy bridges will be a fuzzy cut vertex. Each triangle then contributes atmost one to the number of fuzzy cut vertices, other than identity. If $\mathcal{G}$ has two or more triangles, then at least one of the fuzzy bridges of each triangle is incident with the identity element of Z_n. Consequently, the identity element will be a fuzzy cut vertex. The number of fuzzy cut vertices is then atmost $\frac{n-1}{2} + 1$.

Case 2: Two edges of each triangle have same membership degree.
In a triangle, if the membership of two edges are equal and is greater than the membership of the third edge, then this case reduces to the case 1.
In a triangle, if the membership degree of two edges are equal and is smaller than the membership of the third edge, the edge with the largest membership grade will be a fuzzy bridge. Then there will be $\frac{n-1}{2}$ fuzzy bridges. The identity element of G is the only fuzzy cut vertex.

Case 3: All edges of each triangle have same membership degree.
In this case, there is no fuzzy bridge and the identity element is the only fuzzy cut vertex.
Hence, in Z_n, $n > 3$ and n is odd, the number of fuzzy cut vertices is at most $\frac{n-1}{2} + 1$ and the number of fuzzy bridges is at most $n-1$. □

Remark 3.8. *In the preceding theorem, if $n = 3$, the fuzzy identity graph of Z_3 has at most one fuzzy cut vertex and at most two fuzzy bridges.*

Example 3.9. *Consider the fuzzy identity graph of Z_5 in Example 3.6. Here, $n = 5$. In this fuzzy identity graph of Z_5, there are **three** fuzzy cut vertices given by 0,1 and 3. The edges 01, 14, 32 and 03 are the **four** fuzzy bridges.*

Example 3.10. *Figure 4 represents a fuzzy identity graph of the group Z_6 under addition modulo 6.*

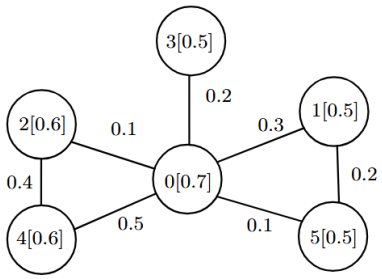

Figure 4: A fuzzy identity graph of Z_6

Theorem 3.11. *A fuzzy identity graph of Z_n, where $n > 2$ is even, has at most $\frac{n-2}{2} + 1$ fuzzy cut vertices and at most $n-1$ fuzzy bridges.*

Proof. When n is even, the fuzzy identity graph of Z_n consists of $\frac{n-2}{2}$ triangles and a line. Removal of the vertex of degree one does not effect the strength of connectedness between any other pair of vertices. i.e., the self inversed element other than the identity is not a fuzzy cut vertex. The line in the fuzzy identity graph of Z_n is a fuzzy bridge. Hence, as in Theorem 3.7, the number of fuzzy cut vertices is at most $\frac{n-2}{2} + 1$ and the number of fuzzy bridges is at most $2 \times \frac{n-2}{2} + 1 = n-1$. □

Remark 3.12. *In the preceding theorem, for $n = 2$, the fuzzy identity graph of Z_2 has no fuzzy cut vertex and exactly one fuzzy bridge.*

Example 3.13. *Consider the fuzzy identity graph in Example 3.10. Here, $n = 6$. In this fuzzy identity graph of Z_6, there are **three** fuzzy cut vertices given by 0,1 and 4. The edges 01, 15, 04, 42 and 03 are the **five** fuzzy bridges.*

Corollary 3.14. *For a finite group G of order n in which every element is its own inverse, each edge in fuzzy identity graph of G is a fuzzy bridge and the identity element is the only fuzzy cut vertex. Hence, fuzzy identity graph of G has exactly one fuzzy cut vertex and $n-1$ fuzzy bridges.*

Proof. A fuzzy identity graph of a group G in which every element is its own inverse, is a star graph with n vertices and $n-1$ edges. The identity element is the only fuzzy cut vertex because on the removal of the identity element there is no path connecting any other two vertices. Each edge in the fuzzy identity graph is a fuzzy bridge since the removal of any edge results in a disconnected graph and strength of connectivity from any vertex to the vertex deleted becomes zero. □

Example 3.15. *In $Z_2 \times Z_2$ each element is its own inverse and the elements of the group are $\{(0,0),(0,1),(1,0),(1,1)\}$. Figure 5 represents a fuzzy identity graph of $Z_2 \times Z_2$ and has exactly **one** fuzzy cut vertex which is the identity element $(0,0)$ and the edges $(0,0)(0,1)$, $(0,0)(1,0)$, $(0,0)(1,1)$ are the **three** fuzzy bridges.*

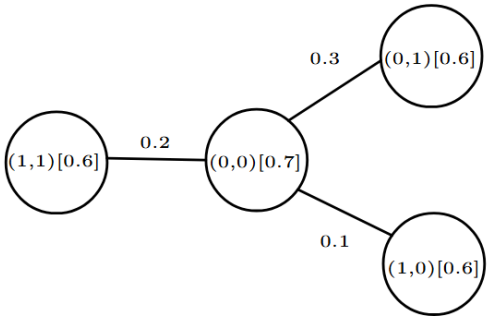

Figure 5: A fuzzy identity graph of $Z_2 \times Z_2$

4 Fuzzy identity graph of $Z_m \times Z_n$

Theorem 4.1. *The self inversed elements (other than the identity (0,0)) exist in $Z_m \times Z_n$ if and only if either m or n or both are even.*

Proof. Let $(a,b) \in Z_m \times Z_n$ such that $(a,b) = (a,b)^{-1}$.
$(a,b) = (a,b)^{-1} \iff (a,b) = (m-a, n-b) \iff m-a = a$ and $n-b = b \iff 2a = M(m)$ and $2b = M(n)$ $\iff a = \frac{rm}{2}$ in Z_m, $r = 0,1$ and $b = \frac{sn}{2}$ in Z_n, $s = 0,1 \iff a = 0$ or $\frac{m}{2}$ in Z_m and $b = 0$ or $\frac{n}{2}$ in Z_n. a takes the value $\frac{m}{2}$ in Z_m only when m is even and b takes the value $\frac{n}{2}$ in Z_n only when n is even. Depending on the values of m and n, the possible self inversed elements in $Z_m \times Z_n$ other than the identity are $(\frac{m}{2},0),(0,\frac{n}{2})$ and $(\frac{m}{2},\frac{n}{2})$. Hence, the self inversed elements other than the identity exist in $Z_m \times Z_n$ if and only if either m or n or both are even. □

Corollary 4.2. *If m and n are even, the self inversed elements in $Z_m \times Z_n$ are $(0,0),(\frac{m}{2},0),(0,\frac{n}{2})$ and $(\frac{m}{2},\frac{n}{2})$.*

Example 4.3. *In the Example 3.15, m and n are even and both equal to 2. So all the four elements of $Z_2 \times Z_2$ are self inversed elements.*

Corollary 4.4. *If m is even and n is odd, the self inversed elements in $Z_m \times Z_n$ are $(0,0)$and $(\frac{m}{2},0)$.*

Corollary 4.5. *If m and n are odd, the only self inversed element in $Z_m \times Z_n$ is $(0,0)$.*

4.1 Identity graph of $Z_m \times Z_n$

Remark 4.6. *The number of triangles and lines in the identity graph of $Z_m \times Z_n$ depends on the values of m and n.*

Case(i): *m and n even*
When m and n are even, by Corollary 4.2, there are four self inversed elements in $Z_m \times Z_n$. The identity graph of $Z_m \times Z_n$ has three lines and $\frac{mn-4}{2}$ triangles.

Case(ii): *m even and n odd*
When m is even and n is odd, by Corollary 4.4, there are two self inversed elements in $Z_m \times Z_n$. The identity graph of $Z_m \times Z_n$ has only one line and $\frac{mn-2}{2}$ triangles.

Case(iii): *m and n odd*
When m and n are odd, by Corollary 4.5, there is only one self inversed element in $Z_m \times Z_n$. The identity graph of $Z_m \times Z_n$ has only triangles, infact $\frac{mn-1}{2}$ triangles.

4.2 Fuzzy identity graph of $Z_m \times Z_n$

Theorem 4.7. *The fuzzy identity graph of $Z_m \times Z_n$, when m and n are even, has at most $\frac{mn-2}{2}$ fuzzy cut vertices and at most $mn-1$ fuzzy bridges.*

Proof. When m and n are even, the identity graph of $Z_m \times Z_n$ has three lines and $\frac{mn-4}{2}$ triangles. Hence the number of fuzzy cut vertices is at most $\frac{mn-4}{2}$. Since identity element is also a fuzzy cut vertex, the number of fuzzy cut vertices is at most $\frac{mn-4}{2}+1=\frac{mn-2}{2}$.
The three lines in the fuzzy identity graph of $Z_m \times Z_n$ are fuzzy bridges. Each triangle contributes at most two to the number of fuzzy bridges. The number of fuzzy bridges is at most $3+2\times\frac{mn-4}{2}=mn-1$. □

Example 4.8. *Consider a fuzzy identity graph of the group $Z_2 \times Z_4$ given in Figure 6. The graph has two triangles and three lines. There are **two** fuzzy cut vertices given by $(0,0)$ and $(0,3)$. The fuzzy bridges are the edges $(0,0)(0,2)$, $(0,0)(1,0)$, $(0,0)(1,2)$, $(0,1)(0,3)$ and $(0,0)(0,3)$.*

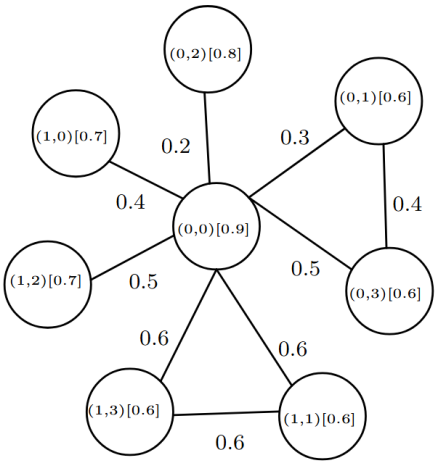

Figure 6: A fuzzy identity graph of $Z_2 \times Z_4$

Theorem 4.9. *If m is even and n is odd, the fuzzy identity graph of $Z_m \times Z_n$ has at most $\frac{mn}{2}$ fuzzy cut vertices and $mn-1$ fuzzy bridges.*

Proof. When m is even and n is odd, the identity graph of $Z_m \times Z_n$ has one line and $\frac{mn-2}{2}$ triangles. There will be at most $\frac{mn-2}{2}$ fuzzy cut vertices, from $\frac{mn-2}{2}$ triangles. Since identity element is also a fuzzy cut vertex, the number of fuzzy cut vertices is at most $\frac{mn-2}{2}+1=\frac{mn}{2}$.
Each triangle contributes at most two to the number of fuzzy bridges. The line in the fuzzy identity graph is a fuzzy bridge. Consequently, the number of fuzzy bridges is at most $2\times\frac{mn-2}{2}+1=mn-1$. □

Example 4.10. *A fuzzy identity graph of $Z_2 \times Z_3$ given in Figure 7 has two triangles and a line. The **three** fuzzy cut vertices are given by $(0,0)$, $(0,1)$ and $(1,1)$. The edges $(0,0)(0,1)$, $(0,1)(0,2)$, $(0,0)(1,0)$, $(0,0)(1,1)$ and $(1,1)(1,2)$ are the **five** fuzzy bridges.*

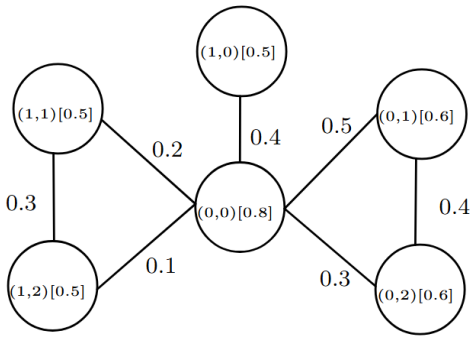

Figure 7: A fuzzy Identity graph of $Z_2 \times Z_3$

Theorem 4.11. *If m and n are odd, the fuzzy identity graph of $Z_m \times Z_n$ has at most $\frac{mn+1}{2}$ fuzzy cut vertices and $mn-1$ fuzzy bridges.*

Proof. When m and n are odd, the identity graph of $Z_m \times Z_n$ has $\frac{mn-1}{2}$ triangles and no lines. There will be at most $\frac{mn-1}{2}$ fuzzy cut vertices from $\frac{mn-1}{2}$ triangles. Together with the identity element, the number of fuzzy cut vertices is at most $\frac{mn-1}{2}+1=\frac{mn+1}{2}$.
As in Theorem 4.9, the number of fuzzy bridges is at most $2 \times \frac{mn-1}{2} = mn-1$. □

Example 4.12. *A fuzzy identity graph of $Z_3 \times Z_5$ is given in Figure 8. It has seven triangles. The* ***four*** *fuzzy cut vertices are $(0,0)$, $(0,3)$, $(2,1)$ and $(1,3)$. The edges $(0,0)(0,3)$, $(0,3)(0,2)$, $(0,0)(1,0)$, $(0,0)(2,0)$, $(0,0)(2,4)$, $(2,3)(1,2)$, $(0,0)(1,3)$, $(1,3)(2,2)$, $(0,0)(2,1)$ and $(2,1)(1,4)$ are the* ***ten*** *fuzzy bridges.*

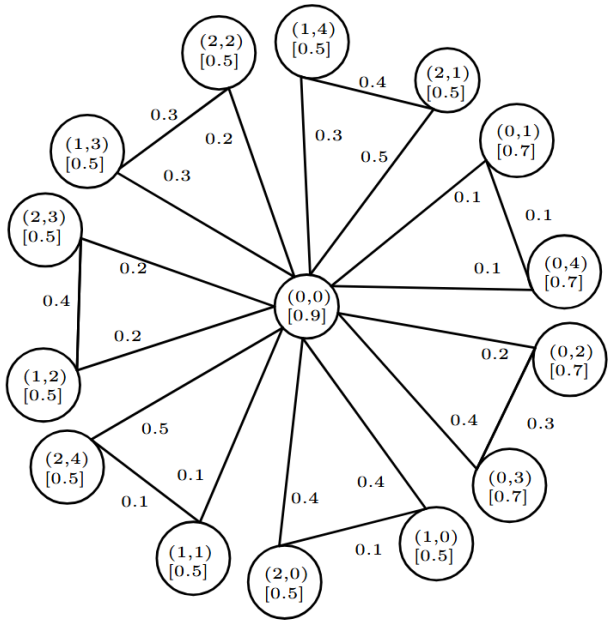

Figure 8: A fuzzy Identity graph of $Z_3 \times Z_5$

5 Fuzzy identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$

This is the generalisation of the case of $Z_m \times Z_n$. Let $t = m_1 \times m_2 \times m_3 \times ... \times m_n$ and $k = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n}$.
Case(i): If all $m_i's$ are even.
When all $m_i's$ are even, the number of self inversed elements in $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ is k.
In this case, the identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ has $k-1$ lines and $\frac{t-k}{2}$ triangles.

Case(ii): If r of $m_i's$ are even.
When r of $m_i's$ are even, the number of self inversed elements in $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ is $\binom{r}{0} + \binom{r}{1}+...+\binom{r}{r}$.
Let $s = \binom{r}{0} + \binom{r}{1}+...+\binom{r}{r}$.
The identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ has $s-1$ lines and $\frac{t-s}{2}$ triangles.
Case(iii): If all $m_i's$ are odd.
When all $m_i's$ are odd, the only self inversed elements in $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ is $(0,0,0,...0)$.
In this case, the identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3}...Z_{m_n}$ has only triangles and the number of triangles is $\frac{t-1}{2}$.

Theorem 5.1. *A fuzzy identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ has at most $\frac{t-k}{2}+1$ fuzzy cut vertices and $t-1$ fuzzy bridges, if all $m_i's$ are even.*

Proof. When all $m_i's$ are even, the identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ has $k-1$ lines and $\frac{t-k}{2}$ triangles. There will be at most $\frac{t-k}{2}$ fuzzy cut vertices from $\frac{t-k}{2}$ triangles. Together with the identity element, the number of fuzzy cut vertices is at most $\frac{t-k}{2}+1 = \frac{t-k+2}{2}$.
Each $k-1$ line in the fuzzy identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ is a fuzzy bridge. Each triangle contributes at most two to the number of fuzzy bridges. The number of fuzzy bridges is then at most $(k-1)+2\times\frac{t-k}{2} = t-1$. □

Example 5.2. *The fuzzy identity graph of $Z_2 \times Z_2 \times Z_4$ given in Figure 9 has seven lines and four triangles. The **five** fuzzy cut vertices are $(0,0,0)$, $(0,0,3)$, $(1,0,3)$, $(0,1,1)$, $(1,1,3)$. The **fifteen** fuzzy bridges are the edges $(0,0,0)(0,0,3)$, $(0,0,3)(0,0,1)$, $(0,0,0)(1,0,3)$, $(1,0,3)(1,0,1)$, $(0,0,0)(0,1,1)$, $(0,1,1)(0,1,3)$, $(0,0,0)(1,1,3)$, $(1,1,3)(1,1,1)$, $(0,0,0)(1,1,0)$, $(0,0,0)(0,0,2)$, $(0,0,0)(1,1,2)$, $(0,0,0)(1,0,0)$, $(0,0,0)(1,0,2)$, $(0,0,0)(0,1,0)$ and $(0,0,0)(0,1,2)$.*

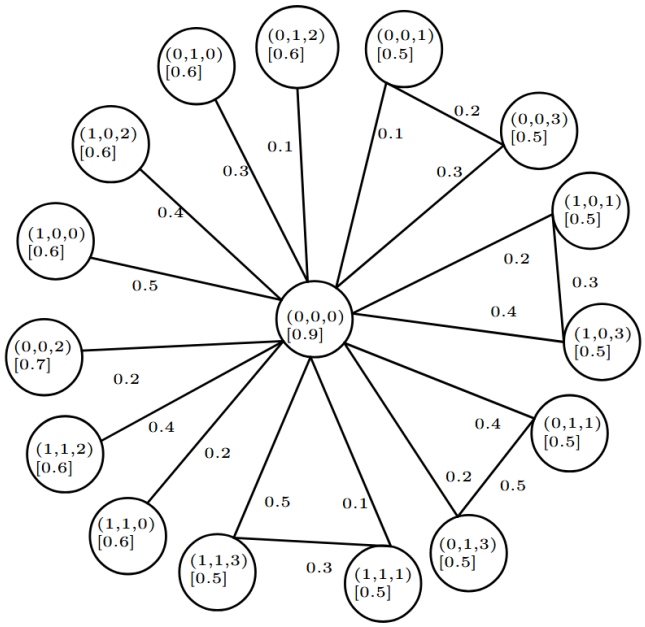

Figure 9: A fuzzy identity graph of $Z_2 \times Z_2 \times Z_4$

Theorem 5.3. *If r of $m_i's$ are even, then the fuzzy identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ has at most $\frac{t-s}{2}+1$ fuzzy cut vertices and $t-1$ fuzzy bridges,where $s = \binom{r}{0} + \binom{r}{1}+...+\binom{r}{r}$.*

Proof. When r of $m_i's$ are even, the identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ has $s-1$ lines and $\frac{t-s}{2}$ triangles. The number of fuzzy cut vertices from $\frac{t-s}{2}$ triangles, is at most $\frac{t-s}{2}$. Together with the identity element, the number of fuzzy cut vertices is at most $\frac{t-s}{2}+1 = \frac{t-s+2}{2}$.
Each $s-1$ line in the identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ is a bridge. Each triangle also contributes at most two to the number of fuzzy bridges. Thus the number of fuzzy bridges is at most $(s-1)+2\times\frac{t-s}{2} = t-1$. □

Example 5.4. *The fuzzy identity graph of $Z_2 \times Z_3 \times Z_4$ given in Figure 10 has three lines and ten triangles. The **nine** fuzzy cut vertices are $(0,0,0)$, $(0,0,1)$, $(0,2,0)$, $(0,1,3)$, $(1,0,3)$, $(1,1,0)$, $(1,1,1)$, $(1,2,2)$ and $(1,2,1)$. The*

twenty one *fuzzy bridges are the edges* $(0,0,0)(0,0,2)$, $(0,0,0)(1,0,0)$, $(0,0,0)(1,0,2)$, $(0,0,0)(0,0,1)$, $(0,0,1)(0,0,3)$, $(0,0,0)(0,2,0)$, $(0,2,0)(0,1,0)$, $(0,0,0)(0,1,1)$, $(0,0,0)(0,2,3)$, $(0,0,0)(0,1,3)$, $(0,2,1)(0,1,3)$, $(0,0,0)(1,0,3)$, $(1,0,3)(1,0,1)$, $(0,0,0)(1,1,0)$, $(1,1,0)(1,2,0)$, $(0,0,0)(1,1,1)$, $(1,2,3)(1,1,1)$, $(0,0,0)(1,2,2)$, $(1,2,2)(1,1,2)$, $(0,0,0)(1,2,1)$ *and* $(1,2,1)(1,1,3)$.

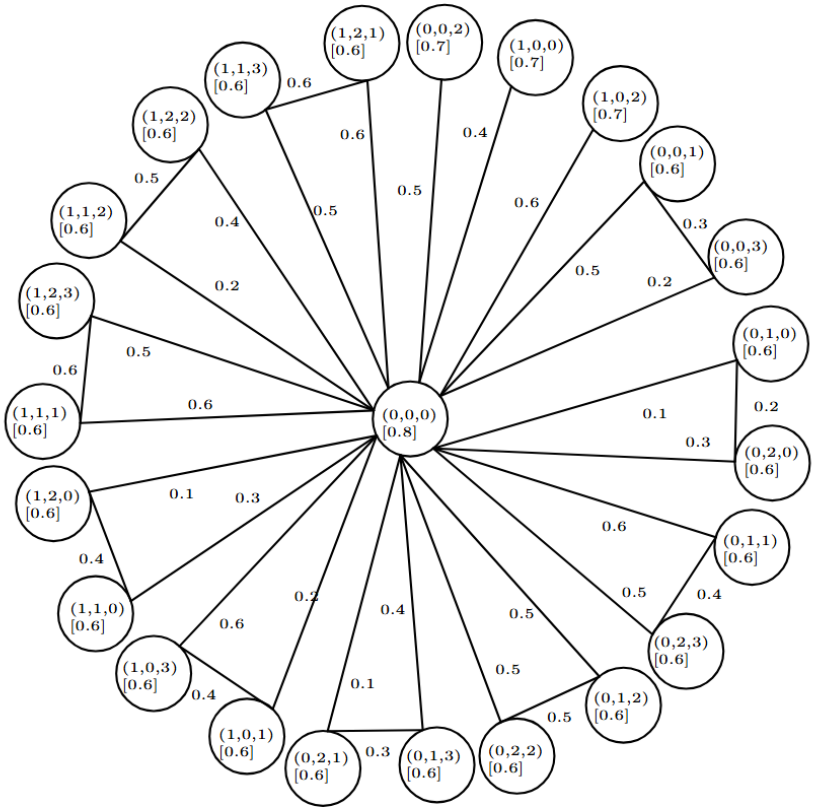

Figure 10: A fuzzy identity graph of $Z_2 \times Z_3 \times Z_4$

Theorem 5.5. *The fuzzy identity graph of* $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ *has at most* $\frac{t+1}{2}$ *fuzzy cut vertices and* $t-1$ *fuzzy bridges, if all* $m_i's$ *are odd.*

Proof. When all $m_i's$ are odd, the identity graph of $Z_{m_1} \times Z_{m_2} \times Z_{m_3} ... Z_{m_n}$ has only triangles, infact $\frac{t-1}{2}$ triangles.Thus there will be at most $\frac{t-1}{2}$ fuzzy cut vertices from $\frac{t-1}{2}$ triangles, together with the identity element. Thus the total number of fuzzy cut vertices is at most $\frac{t-1}{2} + 1 = \frac{t+1}{2}$.

Each triangle contributes at most two to the number of fuzzy bridges. Hence the number of fuzzy bridges is at most $2 \times \frac{t-1}{2} = t-1$. □

Example 5.6. *The fuzzy identity graph of* $Z_3 \times Z_3 \times Z_3$ *given in Figure 11 has only triangles and the number of triangles is thirteen. In this fuzzy identity graph, there are* ***eleven*** *fuzzy cut vertices and* ***twenty two*** *fuzzy bridges.*

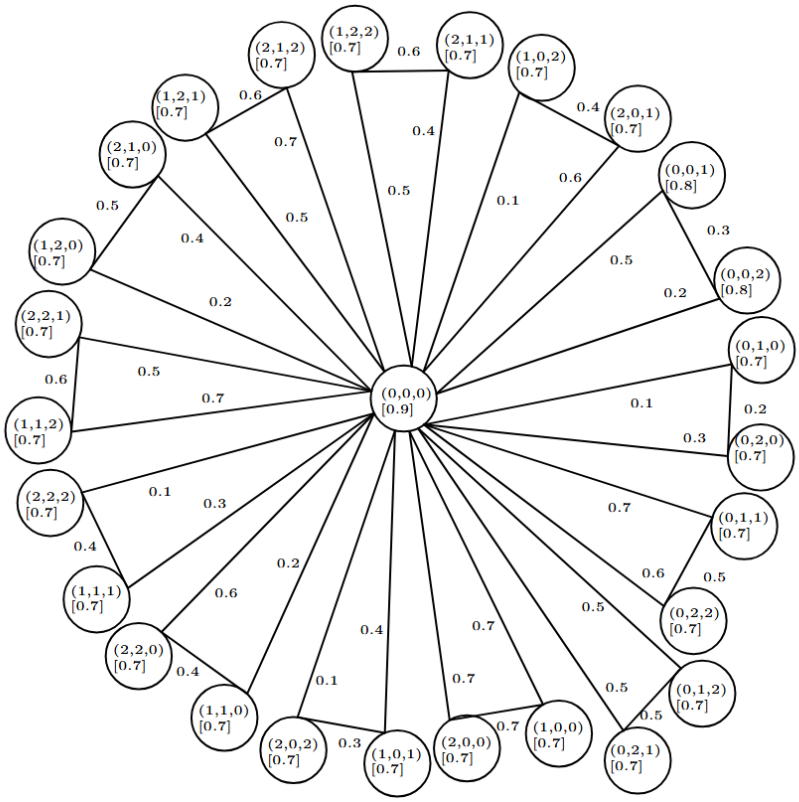

Figure 11: A fuzzy identity graph of $Z_3 \times Z_3 \times Z_3$

6 Application

The problem of installing cameras at the traffic junctions with minimum cost can be modeled by a fuzzy graph. If the transportation network mentioned above is represented by a crisp graph with vertices for junctions and edges for the routes that connect them, then selecting every vertex for camera installation could yield a significant amount of information but at a higher cost. If we model the aforementioned situation using a fuzzy graph, we can extract the most information at the lowest possible cost. In this instance, the number of edges that intersect a vertex determines the membership grade assigned to it. The density of the vehicles on the route determines which edges are included. It should be noted that the vertex and edge memberships meet the requirements for a fuzzy graph. The vertices are selected in the non-increasing order of their membership degrees in order to minimize installation costs and obtain the most traffic information.

If the network has a graphical representation isomorphic to that of a fuzzy identity graph, then the identity element has the maximum membership grade and is also a fuzzy cut vertex. The camera should be placed initially at the position of identity element. The other vertices are chosen in the non-increasing order of membership grades.

Consider a transportation network system given in Figure 12 that connects five junctions A,B,C D and I. In this network, we place camera initially at the position I. Next, choose one vertex from each triangle for installing cameras. In this way we get full information about the vehicles in each junction. A maximum information with minimum cost of installation can be obtained by placing camera at the I.

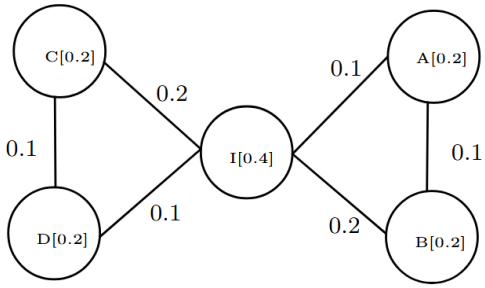

Figure 12:

In telecommunication networks, especially in mobile tower allocation, the application of fuzzy cut vertices and fuzzy bridges allows for better dynamic optimization, fault tolerance, and resource management in the face of uncertainties such as fluctuating demand, traffic load, and environmental conditions. These fuzzy models provide flexibility in decision-making, helping operators to ensure network stability, coverage, and performance while optimizing costs and improving service quality for users. An in-depth exploration of the applications of fuzzy identity graphs presents an area for further research.

7 Conclusion

In this paper we defined the fuzzy identity graph for a finite abelian group. The structure of identity graph of direct product of abelian groups is determined by finding the number of self inversed elements. From a fuzzy identity graph of abelian we can easily determine fuzzy subgroups whose support contains self inversed elements. The maximum number of fuzzy cut vertices and fuzzy bridges of fuzzy identity graphs of Z_n, $Z_m \times Z_n$ and $Z_{m_1} \times Z_{m_2} \times Z_{m_3} \times ... \times Z_{m_n}$ are also determined.

A fuzzy identity graph of a group in which the identity element is the only self inversed element consists of only triangles. Such a fuzzy identity graph is a fuzzy Euler graph. Hence a fuzzy identity graph can be applied in solving problems in which Euler graph is involved. We try to relate fuzzy subgroup with a fuzzy graph. A particular representation of a fuzzy subgroup by a fuzzy graph has been defined which helps in identifying the identity element and inverse of each element in a fuzzy subgroup.

A limitation of our work is that any relation between an element with elements other than the identity element and its inverse cannot be found out from a fuzzy identity graph of a group.

Future work could involve extending the notion of fuzzy identity graphs in soft set theory. Determining the structures of fuzzy identity graphs of non-abelian groups can also be considered.

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