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Maximum product fuzzy relational inequalities with fuzzy constraints: An exact algorithm

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Abstract

Fuzzy relational inequalities with fuzzy constraints (FRI-FC) represent a generalized form of fuzzy relational inequalities (FRI), in which fuzzy inequality replaces ordinary inequality in the constraints. With fuzzy constraints, we can obtain optimal points (called super-optima) that provide better solutions than those obtained by addressing similar problems with ordinary inequality constraints. In this paper, a linear objective function optimization problem with respect to a max-product FRI-FC problem is considered. Several optimization problems are equivalent to the primal problem, as it has been proven. The main problem is converted into a more simplified problem by means of some simplification operations based on the algebraic structure of the primary problem and its equivalent forms. As a result of a few mathematical manipulations, the main problem has been transformed into a linear optimization model. A super-optimum (which is better than the ordinary feasible optimal solution) is not only found by applying the proposed linearization method, but the best super-optimum is also identified for the main problem. A comparison is made between the current approach and our previous work, in addition to some well-known heuristic algorithms, by applying them to random test problems of different sizes and seeing how they compare to each other. Results demonstrate that the proposed method produces optimal solutions with admissible infeasibilities, while the linearization algorithm produces better solutions than other heuristic algorithms. In addition, the results demonstrate that heuristic algorithms could not escape from poor solutions in most cases.

Keywords: Fuzzy relational inequalities, fuzzy relational equations, fuzzy constraints, product t-norm, linear optimization.

1 Introduction

Fuzzy relational equations (FREs) were initially outlined by Sanchez as an extended version of Boolean relational equations and have been used in medical diagnosis [40]. In the view of Pedrycz, two different ways can be taken into account to extend FRE, depending on the sets and various operations [36].

An essential aspect of FRE problems is the identification of solvability and determining the set of solutions. It has been shown by Di Nola et al. that a solution set for FRE, defined by a continuous max-t-norm composition if it is not empty, is highly likely to be a non-convex set. The solution to this non-convex set has only one maximum solution and a finite number of minimal solutions that determine it entirely [5]. Among the two bottlenecks, non-convexity is one of them, and it can lead to a significant increase in the complexity of FRE-related problems. Moreover, it should be noted that the detection of minimal solutions to FREs is another bottleneck. There have been recent results by Chen

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and Wang [2], where they presented an algorithm that can be applied to find the logical representation of all minimal solutions, and then they concluded that there may not be any polynomial-time algorithm that can find all minimal solutions to FRE (with a max-min composition). In addition, Markovskii pointed out that solving the max-product FRE is strongly relevant to the covering problem, which is an NP-hard problem [35]. Interestingly, the same result remains true even if a t-norm is used rather than a minimum or a product operator [3, 31, 32]. There have been many researchers who have investigated the solvability of FRE, which is defined by using different compositions of max-t over the past decade [13, 14, 16, 19, 21, 39, 42, 45, 46, 49, 51]. It is also worth mentioning that some other researchers have contributed to the field of fuzzy relational inequalities (FRI) [11, 12, 18, 22, 23, 30, 53]. Li and Yang [30] studied FRI with addition-min composition and presented an algorithm to search for minimal solutions. They applied FRI to the data transmission mechanism in a BitTorrent-like peer-to-peer file sharing system. Guo et al. [22] investigated a type of FRI problem and the relationship between minimal solutions and FRI paths. They also introduced some rules for reducing problems and presented an algorithm for solving optimization problems using FRI constraints.

The optimization problem under FRE and FRI is currently the subject of a lot of research, and it is the most interesting problem compared to other similar problems [1, 7, 11–13, 15, 17–19, 21–25, 28, 33, 38, 43, 47, 53]. It has been found that a variety of approaches can be applied to the problem by mapping it into an integer linear programming problem that can be solved using well-developed techniques. Other algorithms, however, benefit from the resolution of the feasible region, as well as from the formulation of certain necessary and sufficient conditions for the simplification and optimization of the process. As far as these methods are concerned, most of them are based on the analytical results provided by Sanchez [41] and Pedrycz [26]. A linear optimization problem involving FRE constraints combined with max-min operations was mapped by Fang and Li to an integer programming problem by applying a branch-and-bound method using jump-tracking techniques to address the problem [8]. According to Wu et al., Fang and Li's method could be improved by limiting the search domain and simplifying the process by three rules, resulting in a necessary condition [48]. New theoretical results were presented by Chang and Shieh regarding the linear optimization problem, which was constrained by fuzzy max-min relation equations [1]. Their work improved an upper bound on the optimal objective value, simplified the problem in several ways, and proposed a reduction rule for the solution tree. Numerous scholars have also explored the linear optimization problem in terms of max-product operations [24, 34]. In Loetamonphong and Fang's approach, the objective function was broken down into two sub-problems, the negative coefficients were separated from the positive coefficients, and then the optimal solutions of both sub-problems were combined to obtain the optimal solution [34]. Furthermore, extensions of linear optimization about to the FRE have been investigated; this involves replacing the max-min and max-product compositions with fuzzy compositions, such as max-average compositions [47] or max-t-norm compositions [13, 19, 21, 25, 28, 43].

Recently, a number of fascinating forms of extension of linear programming have been developed and applied to the system of fuzzy relations [4, 6, 9, 29, 33, 50]. Accordingly, Wu et al. provided a method that was shown to be efficient for optimizing a linear fractional programming problem subject to FRE with max-Archimedean t-norm composition [50]. As a result of considering fuzzy coefficients, Dempe and Ruziyeva were able to extend the fuzzy linear optimization problem [4]. A further study was conducted by Dubey et al. in which fuzzy sets were used to model linear programming problems containing interval uncertainty [6]. The linear optimization of bipolar FRE was also the focus of a study carried out by some researchers where FRE was defined with max-min composition [9] and max-Lukasiewicz composition [29, 33]. For example, in [29], the authors introduced a linear optimization problem subjected to a system of bipolar FRE defined as $X\left(A^{+}, A^{-}, b\right)=\{x \in[0,1]^{m} : x \circ A^{+} \vee \tilde{x} \circ A^{-}=b\}$ where $\tilde{x}_i=1-x_i$ for each component of $\tilde{x}=\left(\tilde{x}_i\right)_{1 \times m}$ and the notations "$\vee$" and "$\circ$" denote max operation and the max-Lukasiewicz composition, respectively. They translated the original problem into a $0-1$ integer linear problem, which is then solved using well-developed techniques. On the other hand, the bipolar linear optimization problem mentioned above has been solved analytically using some structural properties and resolution of the feasible region [33].

The optimization problem subjected to various versions of FRI is widely available in the literature as well [11, 12, 17, 18, 22, 23, 52, 53]. In [18], the authors studied the linear optimization with constraints formed by $X\left(A, D, b^{1}, b^{2}\right)=\left\{x \in[0,1]^{n} : A \varphi x \leq b^{1}, D \varphi x \geq b^{2}\right\}$ where φ represents an operator with convex solutions (e.g., non-decreasing or non-increasing operator). They showed that the feasible region can be expressed as the union of a finite number of convex sets. Yang studied the optimal solution of minimizing a linear objective function subject to an FRI where the constraints defined as $a_{i1} \wedge x_1+a_{i2} \wedge x_2+\cdots+a_{in} \wedge x_n \geq b_i$ for $i=1, \ldots, m$ and $a \wedge b=\min\{a, b\}$ [52]. He presented an algorithm based on some properties of the minimal solutions of the FRI. Also, in [53], the authors introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities with application in the optimization management model for wireless communication emission base stations. They also presented an algorithm based on the resolution of the feasible region.

The FRI_FC problem was introduced in [17] as the following mathematical model of where φ is the minimum t-norm:

$$\begin{aligned} \min \quad & c^T x \\ & A\varphi x \circ b \\ & x \in [0,1]^n \end{aligned} \tag{1}$$

where $A = (a_{ij})_{m\times n}$ is a fuzzy matrix and $b = (b_i)_{m\times 1}$ is a fuzzy vector such that $0 \le a_{ij} \le 1$ and $0 \le b_i \le 1$ for each $i \in I = \{1, 2, \ldots, m\}$ and each $j \in J = \{1, 2, \ldots, n\}$, the constant vector $c = (c_j)_{n\times 1}$ and the unknown vector $x = (x_j)_{n\times 1}$ are in $\mathbb{R}^n$, $A\varphi x \circ b$ denotes a fuzzy max-φ composition and "$\circ$" denotes a relaxed or fuzzy version of the ordinary inequality "$\le$". So, the problem (1) can also be viewed as a generalization of the following problem with the ordinary inequality [18]:

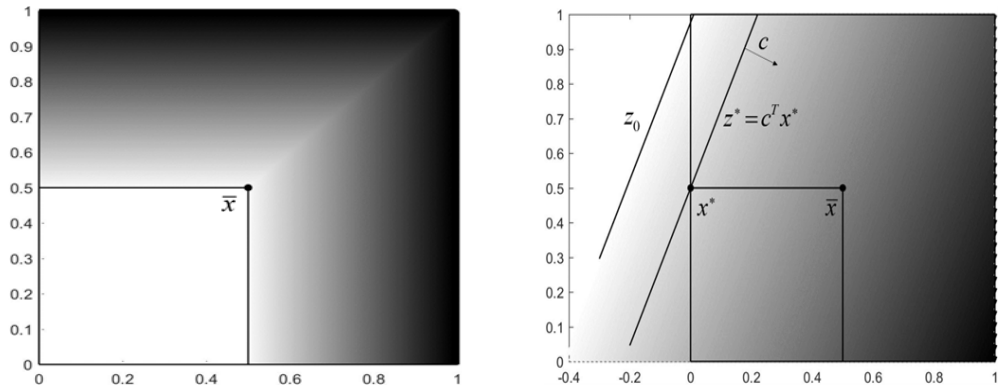
$$\begin{aligned} \min \quad & c^T x \\ & A\varphi x \le b \\ & x \in [0,1]^n \end{aligned} \tag{2}$$

In this paper, we consider problem (2), in which φ represents the product t-norm. Once the main problem has been resolved, we are left with equivalent forms of the main problem that are linear in both terms of the problem and the constraint definition. Once the problem has been linearized, we are able to find the solution by applying a common algorithm like Simplex. Based on the fact that solving FRI-FC problems falls under the NPH category of complexity, this paper presents an effective method of treating NPH problems.

In [17], the authors propose an algorithm to find a super-optimum for problem 1 (1). The authors in [20] presented a modified PSO algorithm for solving the most general version of problem (1) where φ is an arbitrary continuous t-norm. According to their results, the modified PSO can provide high-quality solutions with a low average error (good accuracy) and a low standard deviation (good stability). Additionally, it was shown that the modified PSO generates better solutions than the max-min algorithm [17] and the original PSO [27] when φ is considered as the min t-norm and an arbitrary t-norm, respectively.

In contrast to most optimization algorithms, which use only feasible solutions as the search domain, the FRI-FC problem takes advantage of both feasible and infeasible points. Therefore, relaxed or perturbed inequalities (denoted by "$\circ$") replace ordinary ones (denoted by "$\le$") in the constraints of problem (1). Clearly, it is not reasonable to consider all the infeasible points with the same evaluation. For example, an infeasible point (however, with a good objective value) may be so far from a feasible region that it cannot reasonably be considered a solution to the problem. Thus, to evaluate the feasibility (or infeasibility) magnitude of the points, the feasibility property is defined as a fuzzy set so that closer points to the feasible set attain larger membership values. Figure 1(a) illustrates the notion of the fuzzy set "feasibility" by using grayscale colors varying from white to black, in which the brighter parts include the points with larger membership values for "feasibility" (the white square - including all points between zero vector $\mathbf{0}$ and maximum solution $\bar{x}$ - depicts the feasible region of problem (2)). The consideration of such infeasible points as admissible selections for decision makers can also be viewed as perturbing some constraint(s) such that the feasible solutions set becomes greater and includes new points with likely better objective function values. In practice, the applied aspects of the problems sometimes allow some restrictions or limitations to be perturbed while preserving the purposes of decision makers.

Likewise, the quality level or satisfaction amount of the objective value is defined in a fuzzy set. To define the fuzzy set of "satisfaction", the optimal objective value of problem (2), say $z^* = c^T x^*$, is assumed to be the initial or current optimal value. Moreover, we consider a predetermined value $z_0\,(z_0 < z^*)$ as the best case and set its degree of membership (in other words, its satisfaction amount) equal to one. However, the best case z_0 is an approximate value we wish to achieve at some near-feasible point (since for each feasible solution x in problem (2), we have $z_0 < z^* \le z = c^T x$). The value z_0 may be determined from previous experiments or by human experts. Anyway, the satisfaction amount for each objective value $z = c^T x$ (especially for z^*) is realized by considering the difference between z_0 and z (see Figure 1(b)). Ultimately, the total values are obtained by evaluating the feasibility amount of the points as well as the quality level of their objective values, and a point with the highest total value is introduced as an optimal solution. Briefly, the target of the FRI-FC problem is to find an infeasible point with acceptable infeasibility and a better objective value than z^* by taking advantage of the flexibility of the constraint(s). Similar to [17, 20], we refer to such solutions as Super-Optima in this paper. If there is no super-optimum, then the algorithm gives the same optimal solution obtained by other algorithms using only the feasible region as the search domain. In this case, to find a super-optimum, a decision maker has to consider more freedom for constraints to be perturbed (if it is possible based on the problem structure and the views of human experts).



(a) (b)

Figure 1: **(a)** Fuzzy evaluation of feasibility. **(b)** Fuzzy evaluation of objective values.

In this paper, we present a linearization approach for solving problem (1) in which φ is the product t-norm. In the proposed approach, the primal problem is initially converted into an equivalent linear optimization. Based on the linearity of the equivalent model, to solve the problem, we can use many efficient methods, such as Dantzig's simplex algorithm, Karmarkar's algorithm, and interior-point methods. It is shown that the proposed linearization approach is an efficient and fast method that can find better solutions than those obtained by other related algorithms. In contrast to the max-min algorithm [17], which stops once the first super-optimum is found, the proposed algorithm can find the best super-optimum. Moreover, rather than the modified PSO [20], the current approach provides exact solutions to problem 1.

The rest of the paper is organized as follows. Section 2 takes a look at some basic results from the feasible solutions set for problem (2). In Section 3, the fuzzy constraints of problem (1) are precisely defined by employing some membership functions. Then, problem (1) is transformed into an equivalent problem with linear constraints. Two simplification rules are presented in Section 4, and experimental results are demonstrated in Section 5.

2 Some previous theoretical results

Consider problem (2) in which φ is the product t-norm and let x^* be its optimal solution with objective value $z^* = c^T x^*$. Additionally, suppose there exists at least a flexible constraint in the problem, i.e., there are some points that violate the constraints and are still permissible to be solutions for the problem based on the decision maker's view. Based on these assumptions, we can define problem (1) and find a better solution than x^* for problem (2) by taking advantage of some constraints' flexibility. However, if the flexibility of the constraints is not sufficient, x^* is given again as an optimal solution. In this case, to find a super-optimum, a decision maker has to consider more freedom to perturb constraints. At first, we mention some previously obtained results used throughout the paper. For simplicity, we let $S(A, b)$ denote the feasible solutions set of problem (2), that is, $S(A, b) = \{x \in [0, 1]^n : A\varphi x \leq b\}$. So, similar to problem (1), we can rewrite $S(A, b)$ in terms of rows $a_i (i \in I)$ of matrix A as $S(A, b) = \{x \in [0, 1]^n : a_i \varphi x \leq b_i,\ i \in I\}$ where the constraints mean $a_i \varphi x = \max j = 1^n \{\varphi(a_{ij}, x_j)\} = \max j = 1^n \{a_{ij} x_j\} \leq b_i$, $\forall i \in I$.

Definition 2.1. (a) *For each $j \in J$, let $I(j) = \{i \in I : a_{ij} > b_i\}$.*
(b) *Let $\bar{x} = (\bar{x}_j)_{1\times n}$ be an n-dimensional vector in $[0, 1]^n$ whose components are defined as $\bar{x}_j = \min_{i\in I(j)} \{b_i/a_{ij}\}$, $\forall j \in J$.*

Theorem 2.2. (a) *$S(A, b) = [\mathbf{0}, \bar{x}]$. In other words, $S(A, b)$ is a cube including all points $x \in [0, 1]^n$ such that $0 \leq x_j \leq \bar{x}_j$, $\forall j \in J$.*
(b) *Solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ as defined below is the optimal problem solution (2).*

$$x_j^* = \begin{cases} \bar{x}_j & , c_j < 0 \\ 0 & , c_j \geq 0 \end{cases} \tag{3}$$

Proof. See Theorems 3 and 4 in [20]. □

Theorem 2.3. *(First Simplification) Consider problem (2) and let $A' = \left(a'_{ij}\right)_{m\times n}$ be a matrix resulted from matrix*

$A = (a_{ij})_{m\times n}$ *as follows. For* $i \in I$ *and* $j_0 \in J$*, if there exist* $k \in I$ *such that* $a_{ij_0} \geq b_i$*,* $a_{kj_0} > b_k$ *and* $b_i/a_{ij_0} > b_k/a_{kj_0}$*, we set* $a'_{ij_0} = 0$*; otherwise,* $a'_{ij_0} = a_{ij_0}$*. Then,* $S(A', b) = S(A, b)$*.*

Proof. Suppose that $a_{ij_0} \geq b_i, a_{kj_0} > b_k$ and $b_i/a_{ij_0} > b_k/a_{kj_0}$. We show that "resetting a_{ij_0} to zero" has no effect on $S(A, b)$; that is, $a_i\varphi x = \max_{j=1}^{n}\{a_{ij}x_j\} \leq b_i$ is equivalent to $\max_{j=1,\ j\neq j_0}^{n}\{a_{ij}x_j\} \leq b_i$, $\forall x \in S(A, b)$. To this end, it is sufficient to prove $a_{ij_0}x_{j_0} < b_i$, $\forall x \in S(A, b)$. From $\max_{j=1}^{n}\{a_{ij}x_j\} \leq b_i$, it is obvious that $a_{ij_0}x_{j_0} > b_i$ never holds. So, assume that $a_{ij_0}x_{j_0} = b_i$. So, $x_{j_0} = b_i/a_{ij_0} > b_k/a_{kj_0}$. Therefore, $a_{kj_0}x_{j_0} > b_k$, which implies $\max_{j=1}^{n}\{a_{kj}x_j\} > b_k$. It contradicts $x \in S(A, b)$ and the proof is complete. □

3 A mathematical model for fuzzy constraints and equivalent problems

In this section, we consider the constraints and objective functions of problem (1) as fuzzy sets by defining their associated membership functions. Via this approach, all points (whether feasible or not) in problem (2) are treated as feasible ones in 1 with different degrees of membership in the interval $[0, 1]$. In fact, a point that belongs to the feasible set of 2 also belongs to 1 with the degree of membership equal to one, and a further point from the feasible set of 2 gets a lower degree of membership. To treat the ith fuzzy inequality of problem (1), we employ the same membership functions used in [17] as follows:

$$\mu(a_i\varphi x) = \begin{cases} 1 & ,\ a_i\varphi x \leq b_i \\ 1 - \frac{a_i\varphi x - b_i}{d_i} & ,\ b_i \leq a_i\varphi x \leq b_i + d_i,\ i = 1, 2, \ldots, m \\ 0 & ,\ a_i\varphi x \geq b_i + d_i \end{cases} \tag{4}$$

where each $d_i (i = 1, 2, \ldots, m)$ is initially determined as the limit of the admissible violation of the i'th inequality. From relation 4, the i'th membership function is equal to 1 if the ith constraint of problem (2) is well satisfied, 0 if the constraint is violated beyond its admissible limit d_i, and decreases linearly from 1 to 0.

Definition 3.1. *For each* $x \in [0, 1]^n$*, the crisp constraints violation vector at point* x *is an* $m \times 1$ *vector* $CCV(x) = (CCV(x)_i)_{m\times 1}$ *whose* i*'th component is defined as follows:*

$$CCV(x)_i = \max\{0, a_i\varphi x - b_i\}. \tag{5}$$

Also, we define the fuzzy constraints violation vector at point x *as an* $m \times 1$ *vector* $FCV(x) = (FCV(x)_i)_{m\times 1}$ *whose* i*'th component is*

$$FCV(x)_i = \max\{0, a_i\varphi x - (b_i + d_i)\}. \tag{6}$$

From the above-mentioned definition, relations 5 and 6 show how much point x violates ith constraint of problem (2) (i.e., $a_i\varphi x \leq b_i$) and that of problem (1) (i.e., $a_i\varphi x \circ b_i$), respectively. Similar to 4, for the objective function of the problem (1), we define

$$\mu\left(c^T x\right) = \begin{cases} 1 & ,\ c^T x \leq z_0 \\ 1 - \frac{c^T x - z_0}{d_0} & ,\ z_0 \leq c^T x \leq z_0 + d_0 \\ 0 & ,\ c^T x \geq z_0 + d_0 \end{cases} \tag{7}$$

where $z_0 = c^T x^* - vd_0$ for parameters $v \in (0, 1)$ and $d_0 \geq 0$. From relation 7, we have $\mu(z_0) = 1$, $\mu\left(c^T x^*\right) = 1 - v$ and $\mu(z_0 + d_0) = 0$. The parameters v and d_0 determine the symmetry and length of the interval $[z_0, z_0 + d_0] = \left[c^T x^* - vd_0,\ c^T x^* + (1 - v)d_0\right]$, respectively. If $v \approx 0$ ($v \approx 1$) then $\mu\left(c^T x^*\right) \approx 1$($\mu\left(c^T x^*\right) \approx 0$), that means the optimal solution of problem (2), x^*, has a high (low) degree of satisfaction. If $v = 0.5$, then this interval is converted into the closure of the symmetric ε-neighborhood around $c^T x^*$ with radius $\varepsilon = 0.5$. In this case, point x^* is interpreted as a solution where the satisfaction amount of its objective value is mediocre. Also, parameter d_0 determines the length of the interval in which we want to survey the attainability or non-attainability of a better objective function value than $c^T x^*$; that is, we want to find a solution with a better objective function value than $c^T x^*$ in this interval. Obviously, if d_0 is selected very small ($d_0 \approx 0$), then x^* can be considered as the best solution. In other words, this interval is an approximate domain in which we expect to be able to find a better objective function value than $c^T x^*$ by partly perturbing some constraints. In [20], a discussion was provided on the initial setup of the parameters v and d_0, and their influences on the algorithm's convergence. Also, we refer the reader to [17] for a more detailed analysis of the influences of these parameters and some theoretical and experimental aspects that should be considered in the problems.

By considering relations 4 and 7, we can express both the feasibility and optimality amounts of a point by only one variable, as defined in the following definition.

Definition 3.2. *For each $x \in [0,1]^n$, let $\mu_o(x) = \mu\left(c^T x\right)$ (i.e., the optimality value of x) and $\mu_F(x) = \min_{i=1}^{m}\{\mu\left(a_i \varphi x\right)\}$ (i.e., the feasibility value of x). Then, the total value of x is defined by:*

$$\mu_T(x) = \min\{\mu_o(x), \mu_F(x)\} = \min\left\{\mu\left(c^T x\right), \min_{i=1}^{m}\{\mu\left(a_i \varphi x\right)\}\right\}.$$

Definition 3.2 defines the whole space $S = [0,1]^n$ as a fuzzy set by associating the membership value $\mu_T(x)$ to each $x \in S$. Based on this fact, our purpose becomes equivalent to finding a point with the greatest total value among all $x \in [0,1]^n$. In other words, we can express problem (1) as the following problem

$$\max_{x \in S} \mu_T(x), \tag{8}$$

or equivalently

$$\max_{x \in [0,1]^n}\left\{\min\left\{\mu\left(c^T x\right), \min_{i=1}^{m}\{\mu\left(a_i \varphi x\right)\}\right\}\right\}. \tag{9}$$

Therefore, by considering relations 4 and 7, problem (9) is rewritten as

$$\max_{x \in [0,1]^n}\left\{\min\left\{B_0 - D_0\left(c^T x\right), \min_{i=1}^{m}\{B_i - D_i\left(a_i \varphi x\right)\}\right\}\right\}, \tag{10}$$

where $D_i = \frac{1}{d_i}$ for $i \in I \bigcup \{0\}, B_0 = 1 + \frac{z_0}{d_0}$ and $B_i = 1 + \frac{b_i}{d_i}$ for $i \in I$. Now, by introducing the auxiliary variable λ, problem (10) can be transformed into the following equivalent programming problem:

$$\begin{array}{ll} \max & \lambda \\ & D_i\left(a_i \varphi x\right) + \lambda \leq B_i,\ i \in I \\ & D_0\left(c^T x\right) + \lambda \leq B_0 \\ & x \in [0,1]^n, \end{array} \tag{11}$$

or equivalently,

$$\begin{array}{ll} \max & \lambda \\ & D_i\left(\max_{j=1}^{n}\{a_{ij} x_j\}\right) + \lambda \leq B_i,\ i \in I \\ & D_0\left(c^T x\right) + \lambda \leq B_0 \\ & x \in [0,1]^n. \end{array} \tag{12}$$

In problem (12), each constraint $D_i\left(\max_{j=1}^{n}\{a_{ij} x_j\}\right) + \lambda \leq B_i$ is equivalent to n constraints $D_i a_{ij} x_j + \lambda \leq B_i$ $(j \in J)$. Due to this fact, problem (12) can be converted into the following optimization problem in which all the constraints are linear.

$$\begin{array}{ll} \max & \lambda \\ & D_i a_{ij} x_j + \lambda \leq B_i, i \in I,\ j \in J \\ & D_0\left(c^T x\right) + \lambda \leq B_0 \\ & x \in [0,1]^n. \end{array} \tag{13}$$

Heretofore, we show that problem (1) is equivalent to problem (13), which is a linear programming problem. So, there are many efficient approaches, such as Dantzig's simplex algorithm, Karmarkar's algorithm, and interior-point methods that can be used for solving problem (13).

As mentioned before, the modified PSO algorithm [20] was presented for solving problem (1) defined by an arbitrary continuous t-norm. Furthermore, by converting problem (1) into an equivalent problem (8), we can use many heuristic algorithms proposed for solving unconstrained optimization problems or those problems with some lower and upper bounds on the variables as their constraints. In Section 5, we apply the Simplex algorithm to problem (13) and compare the generated results to those obtained by the modified PSO [20] and some well-known meta-heuristic methods that have been applied to many practical optimization problems.

4 Simplification rules

As mentioned above, problems 12 and 13 are mathematically equivalent. So, by simplifying problem (12), we will have a more simplified linear optimization model (problem (13)) for solving. In this section, two simplification rules are presented to convert problem (12) into a more simplified equivalent form.

Definition 4.1. *Let* $\lambda_0(x) = B_0 - D_0\left(c^T x\right)$, $\lambda_i(x) = B_i - D_i\left(\max_{j=1}^{n}\{a_{ij}x_j\}\right)$, $\forall i \in I$, $\lambda_{ij}\left(x_j\right) = B_i - D_i a_{ij} x_j$, $\forall i \in I$ *and* $\forall j \in J$, *and* $\Lambda(x) = \min_{i=0}^{m}\{\lambda_i(x)\}$.

Corollary 4.2. *Functions* $\lambda_i(x)$, $\forall i \in I$, *are non-increasing continuous functions. Functions* $\lambda_{ij}\left(x_j\right)$, $\forall i \in I$, *and* $\forall j \in J$, *are strictly decreasing and function* $\lambda_0(x)$ *is strictly decreasing with respect to component* x_j *if* $c_j > 0$ *and strictly increasing if* $c_j < 0$.

By Definition 4, problem (12) can be expressed as follows:

$$\max_{x\in[0,1]^n}\left\{\min_{i=0}^{m}\{\lambda_i(x)\}\right\}, \tag{14}$$

or equivalently,

$$\max_{x\in[0,1]^n}\{\Lambda(x)\}. \tag{15}$$

Theorem 4.3 provides a simplification rule for solving problem (14) by finding some components of the optimal solution.

Theorem 4.3. *Suppose that* x^* *is an optimal solution of problem (14). Then,* $x_j^* = 0$ *for each* $j \in J$ *such that* $c_j > 0$.

Proof. Since 12 and 15 are equivalent problems, x^* is also an optimal solution for 15. Thus, $\Lambda\left(x^*\right) \geq \Lambda(x)$ for each $x \in [0,1]^n$. By contradiction, suppose that $x_{j_0}^* > 0$ for some $j_0 \in J$ such that $c_{j_0} > 0$. Let $x' \in [0,1]^n$ such that $x'_j = x_j^*$ if $c_j < 0$ and $x'_j = 0$ if $c_j > 0$. Then, from Corollary 4.2 , we have $\lambda_i\left(x'\right) \geq \lambda_i\left(x^*\right)$, $\forall i \in I \bigcup \{0\}$, and therefore $\Lambda\left(x'\right) \geq \Lambda\left(x^*\right)$ which contradicts the optimality of x^*. □

Corollary 4.4. *(Second Simplification) For each* $j \in J$ *such that* $c_j > 0$, *we can assign* $x_j^* = 0$ *and remove the* j*'th column of matrix A. Therefore, to solve problem (12) (or problem (13)), it is sufficient to consider only the columns of matrix A that belong to* $J' = \{j \in J : c_j < 0\}$

Lemma 4.5. *Let* $i \in I$. *Then,* $\lambda_i(x) = \min_{j\in J}\{\lambda_{ij}\left(x_j\right)\}$, $\forall x \in [0,1]^n$.

Proof. Let $x \in [0,1]^n$ and $\lambda_{ij_0}\left(x_{j_0}\right) = \min_{j\in J}\{\lambda_{ij}\left(x_j\right)\}$. So, from Definition 4 we have $B_i - D_i a_{ij_0} x_{j_0} \leq B_i - D_i a_{ij} x_j$, $\forall j \in J$. Therefore, $a_{ij}x_j \leq a_{ij_0}x_{j_0}, \forall j \in J$, which together with Definition 4 imply that $\lambda_i(x) = B_i - D_i\left(\max_{j=1}^{n}\{a_{ij}x_j\}\right) = B_i - D_i a_{ij_0} x_{j_0} = \lambda_{ij_0}\left(x_{j_0}\right)$. □

From Corollary 4.4 and Lemma 4.5, problem (14) is converted into the following simplified problem:

$$\max_{x\in[0,1]^n}\left\{\min\left\{\lambda_0(x), \min_{i\in I, j\in J'}\{\lambda_{ij}\left(x_j\right)\}\right\}\right\}. \tag{16}$$

Lemma 4.6. *Let* $i \in I$ *and* $j \in J'$. *If* $a_{ij} \leq b_i$, *then* $\lambda_{ij}\left(x_j\right) \geq 1$, $\forall x_j \in [0,1]$.

Proof. Suppose that $a_{ij} \leq b_i$. Thus, $a_{ij}x_j \leq b_i$ and therefore $1 - \frac{a_{ij}x_j - b_i}{d_i} \geq 1$, $\forall x_j \in [0,1]$. Now, by substituting $D_i = 1/d_i$ and $B_i = 1 + b_i/d_i$ in Definition 4, we have $\lambda_{ij}\left(x_j\right) = 1 - \frac{a_{ij}x_j - b_i}{d_i} \geq 1$. □

Theorem 4.7. *Suppose the second simplification (Corollary 4.4) is done and* $i \in I$. *Then,* $\lambda_i(x) = \min_{j\in J_i'}\{\lambda_{ij}\left(x_j\right)\}$ *where* $J_i' = \{j \in J' : a_{ij} > b_i\}$.

Proof. From Corollary 4.4 and Lemma 4.5, $\lambda_i(x) = \min_{j\in J'}\{\lambda_{ij}\left(x_j\right)\}$, $\forall i \in I$. Hence, $\lambda_i(x) = \min\{\min_{j\in J_i'}\{\lambda_{ij}(x_j)\}, \min_{j\notin J_i'}\{\lambda_{ij}(x_j)\}\}$, $\forall i \in I$. Since $j \notin J_i'$ implies $a_{ij} \leq b_i$, from Lemma 4.6 we obtain $\lambda_{ij}\left(x_j\right) \geq 1$, $\forall j \in J' - J_i'$ and $\forall x_j \in [0,1]$. Therefore, $\min_{j\notin J_i'}\{\lambda_{ij}\left(x_j\right)\} \geq 1$ and then $\lambda_i(x) = \min_{j\in J_i'}\{\lambda_{ij}\left(x_j\right)\}$. □

From Theorem 4.7, problem (16) is converted into the more simplified form 17 as follows:

$$\max_{x\in[0,1]^n}\left\{\min\left\{\lambda_0(x), \min_{i\in I, j\in J_i'}\{\lambda_{ij}\left(x_j\right)\}\right\}\right\}. \tag{17}$$

Corollary 4.8. *(Third Simplification)* ***(a)*** *Suppose that $i_0 \in I$ and $j_0 \notin J'_i$. Then, "resetting $a_{i_0 j_0}$ to zero" has no effect on the optimal solution of problem (14) (or that of problem (12)). As a result, if $j_0 \notin J'_i$, $\forall i \in I$, then we can remove the j_0'th column of matrix A.* ***(b)*** *Let x^* be an optimal solution of problem (14). If $j_0 \notin J'_i$, $\forall i \in I$, then $x^*_{j_0} = 1$.*

Proof. (a) Since $j_0 \notin J'_{i_0}$, then $\lambda_{i_0 j_0}(x_{j_0})$ does not appear in 17 and therefore we can take $\lambda_{i_0 j_0}(x_{j_0})$ out of consideration. So, the value of $a_{i_0 j_0}$ plays no role in 17, and can be assigned to zero.
(b) By contradiction, suppose $x^*_{j_0} < 1$. Let $x' \in [0,1]^n$ such that $x'_{j_0} = 1$ and $x'_j = x^*_j$, $\forall j \in J'_i$. Then,

$$\min_{i \in I, j \in J'_i} \left\{ \lambda_{ij}\left(x'_j\right) \right\} = \min_{i \in I, j \in J'_i} \left\{ \lambda_{ij}\left(x^*_j\right) \right\},$$

and we also have $\lambda_0(x') > \lambda_0(x^*)$ by Corollary 4.2. Hence, $\Lambda(x') \geq \Lambda(x^*)$ which contradicts the optimality of x^*. □

5 Numeral examples and applications

At first, this section presents an application that utilizes FRI-FC problems to enhance quality while maintaining a minimum cost [17], and continues with experimental results illustrating our algorithm's performance. In Section 5.2, the proposed algorithm is applied to solving the transformed equivalent problem (problem (13)) for the test problems described in Appendix A. The test problems have been randomly generated in different sizes by using the product t-norm. For each test problem, we have a pair of problems associated with each other; one FRI-FC problem (problem (1)) and one FRI problem with ordinary (crisp) inequalities (problem (2)). From Theorem 2.2(a), we know that problem (2) is feasible and $S(A,b) = [\mathbf{0}, \bar{x}]$ where the maximum solution $\bar{x}$ is obtained from Definition 1. Using this result, the feasible optimal solution x^* of problem (2) is directly given by Theorem 2.2(b). To perform a fair comparison, we use the same parameters for each experiment, that is, we set $v = 0.5$ and $d_i = 0.1$ for $i = 0, 1, .., m$. Therefore, from 4 and 7 we have $\mu_T(z_0) = 1$ and $\mu_T\left(c^T x^*\right) = 1 - v = 0.5$. The equality $\mu_T\left(c^T x^*\right) = 0.5$ means that x^* (the optimal solution of problem (2)) has a mediocre objective value for us. Actually, the methods aim to find a solution x^{**} such that $c^T x^{**} \in [z_0, z_0 + d_0]$ and $c^T x^{**}$ is as close as possible to z_0 (the best case) by perturbing the constraints within the range determined by $d_i = 0.1, i = 0, 1, .., m$. Finally, a comparison is made between the current method, the Modified PSO (MPSO) [20], Original PSO (OPSO) [27], Continuous Ant Colony Optimization (CACO) [44], Differential Evolution (DE) [37] and Harmony Search (HS) [10] algorithms. For this purpose, in Section 5.3, we apply these methods to the ten test problems described in Appendix A.

5.1 Application example: Quality enhancement of a school with minimum cost

Consider a city divided into several educational zones, as in Figure 2. A particular school, say *School A*, intends to increase its enrollment by improving its overall quality.

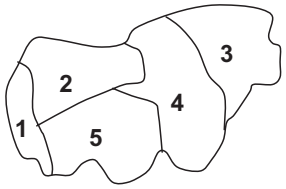
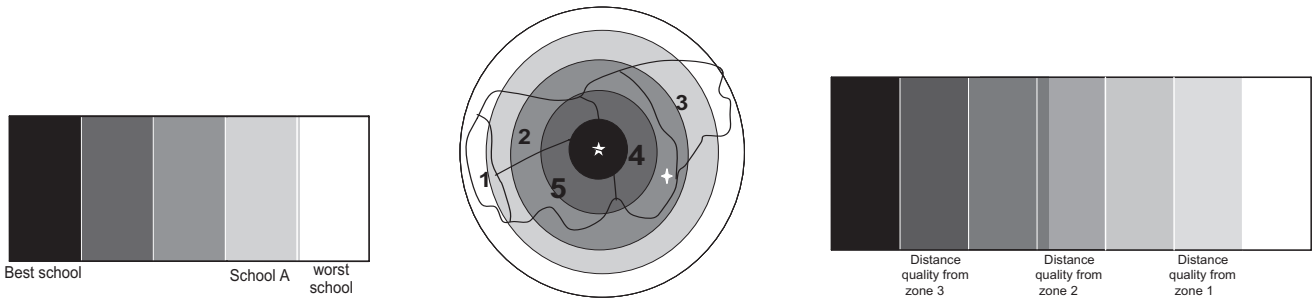


Figure 2: Educational zones.

Parents in each zone evaluate the school according to multiple criteria, as shown in Figures 3(a-c):

1. distance to residence (it is preferred to have a shorter distance to residence),
2. proximity to the city center (locations which are closer to the city center are deemed to be more prestigious),
3. quality of athletics and recreational facilities (In the case of modern facilities, they are more appealing).

The school can invest resources in these criteria-for instance, by expanding bus routes (criterion 1), hiring educational consultants (criterion 2), or improving athletic programs (criterion 3). Each enhancement entails a cost, and the schoolmaster aims to allocate resources to enhance quality while keeping the total expenditure within a limited budget.



(a) (b) (c)

Figure 3: **(a)** Evaluation of school A by comparing with all schools in zone 1 from the viewpoint of the athletic-recreational facilities. **(b)** Evaluation of school A from the viewpoint of its place in the city. (school A: ✧ and city center: ☆) **(c)** evaluation of school A based its distance from zones 1, 2 and 3.

We introduce decision variables $x_j \in [0,1]$ for $j = 1,2,3$, where:

x_1: represents improvement in accessibility (transportation),

x_2: represents improvement in prestige and educational support,

x_3: represents improvement in athletic facilities.

Here, $x_j = 0$ indicates no improvement, while $x_j = 1$ corresponds to the maximum feasible improvement.

For each zone i, parents require that the school reach at least a satisfaction level $b_i \in [0,1]$. If the obtained satisfaction is below b_i, parents perceive School A as less competitive compared to alternatives; if the requirement is met or exceeded, the zone's expectations are considered satisfied.

The satisfaction of zone i with respect to criterion j depends on the base quality a_{ij} and the level of improvement x_j. This is expressed by fuzzy relational inequalities of the form:

$$\max_j \min(a_{ij},\, x_j) \;\geq\; b_i,\;\; i = 1,2,3.$$

In this expression:

- a_{ij} denotes the initial quality of criterion j in zone i,
- x_j denotes the chosen improvement level for criterion j,
- $\min(a_{ij}, x_j)$ reflects the satisfaction degree, limited by the weaker of the base quality and improvement,
- the outer $\max_j$ represents the fact that for each zone, it is sufficient if *at least one criterion* reaches the desired level b_i.

The base qualities are represented in a fuzzy relation matrix:

$$A = (a_{ij}),\;\; a_{ij} \in [0,1],$$

where each row corresponds to a zone and each column to a criterion. With the threshold vector $b = (b_1, b_2, b_3)^T$, the system of fuzzy relational inequalities is written compactly as:

$$A \circ x^T \;\geq\; b^T,$$

where "$\circ$" denotes the max–min composition.

Each improvement x_j is associated with a cost $c_j > 0$. The schoolmaster's objective is to minimize the total cost of improvements subject to fuzzy relational inequality constraints. Formally, the model can be expressed as follows:

$$\begin{aligned} \min \quad & \textstyle\sum_{j=1}^{3} c_j x_j \\ & A \circ x^T \;\geq\; b^T \\ & x_j \in [0,1], \quad j = 1,2,3. \end{aligned}$$

This formulation provides a fuzzy optimization framework for quality enhancement of the school: it captures the trade-off between resource allocation and satisfaction of parental requirements across all zones, using max-min fuzzy relational inequalities.

5.2 Results of the linearization approach

In this section, the proposed algorithm is applied to the test problems described in Appendix A. Table 1 includes the feasibility values $\mu_F(x^{**})$, optimality values $\mu_O(x^{**})$ and total values $\mu_T(x^{**})$ for the best solutions x^* found by the linearization algorithm. In each case, to simplify the comparison between the current optimum x (the optimal solution for problem (2)) and the best super-optimum x^* found by the proposed algorithm, values $c^T x$ and $c^T x^{**}$ have been reported. Also, we reported admissible ranges $[z_0, z_0 + \varepsilon_0]$ for each test problem to determine if the proposed method can find a super-optimum $x^{**} \in [z_0, z_0 + d_0]$ such that $c^T x^* < c^T x$. Additionally, for each test problem, Table 2 presents the optimal solutions x^* of problem (2), solutions x^{**} found by the current algorithm, and the crisp constraints violation vectors at points x^{**}.

Table 1: A comparison between the quality of the best super-optima (i.e., solutions x^{**}) found by the proposed linearization approach and that of the optimal solutions of problem (2) (i.e., solutions x^*) for the test problems A. 1 - A.10.

Test problems	$\mu_F(x^{**})$	$\mu_0(x^{**})$	$\mu_T(x^{**})$	$c^T x^{**}$	$c^T x^*$	$[z_0, z_0 + d_0]$
A. 1	0.9910	0.9910	0.9910	- 0.9232	-0.8741	[- 0.9241 , - 0.8241]
A. 2	0.9933	0.9933	0.9933	- 11.3722	- 11.3228	[- 11.3728, - 11.2728]
A. 3	0.9765	0.9765	0.9765	- 1.3501	- 1.3024	[- 1.3524 , - 1.2524]
A. 4	0.9916	0.9916	0.9916	- 9.7886	- 9.7395	[- 9.7895 , - 9.6895]
A. 5	0.9793	0.9793	0.9793	- 1.4395	- 1.3916	[- 1.4416 , - 1.3416]
A. 6	0.9809	0.9809	0.9809	- 0.1638	- 0.1157	[- 0.1657 , - 0.0657]
A. 7	0.9647	0.9647	0.9647	- 0.0820	-0.0356	[- 0.0856, 0.0144]
A. 8	0.9373	0.9373	0.9373	- 0.1337	-0.0899	[- 0.14, - 0.04]
A. 9	0.9907	0.9907	0.9907	- 1.0552	- 1.0061	[- 1.0561 , - 0.9561]
A. 10	0.9914	0.9914	0.9914	- 1.7222	-1.6731	[- 1.7231 , - 1.6231]

As shown in Table 1, we have $\mu_T(x^*) > \mu_T(c^T x) = 0.5$ in all the cases, which means the proposed algorithm can find solutions with higher quality than x^*. According to Table 1, objective values $c^T x^*$ belong to the admissible intervals $[z_0, z_0 + d_0]$ (and are very close to the best cases z_0) and are strictly less than $c^T x$ for all the test problems. Therefore, the linearization method produces better solutions (i.e., solutions with a lower objective values) than the optimal solutions of problem (2). Table 2 includes solutions x^*, x^{**} and the crisp constraints violation vectors $CCV(x^{**})$ for each test problem. The results, in Table 2, show that the proposed method produces optimal solutions with admissible infeasibilities; more precisely, for each $i \in I$, we have $CCV(x^{**})_i \leq d_i = 0.1$. As a key result, we see that the best super-optimum x^{**} satisfies $\mu_F(x^{**}) = \mu_O(x^{**})$ for each test problem. As will be shown in the next section, this equality does not hold for each super-optimum.

5.3 Linearization approach versus the other related methods

In this section, a comparison is made between the current linearization method and the modified PSO algorithm [20] proposed for solving the FRI-FC problems. As mentioned before, since problem (1) is equivalent to problem (9), many heuristic algorithms may be used for solving the problem. So, the generated solutions for the linearization method and modified PSO are also compared with some well-known meta-heuristics such as Original PSO (OPSO) [27], Continuous Ant Colony Optimization (CACO) [44], Differential Evolution (DE) [37] and Harmony Search (HS) [10] algorithms. For the heuristic algorithms, 30 experiments are performed for each test problem. The maximum number of iterations is 100. The PSO algorithms used in each case have the following parameters. Swarm size is set to 10, $c_1 = c_2 = 2$, and the inertia factor w decreases linearly from 1 to 0 by a damping factor of 0.99 [20, 27]. For CACO, $m = 10$ (number of ants used in an iteration), $\zeta = 0.85$ (the speed of convergence), $q = 10^{-4}$ (locality of the search process) and $k = 50$ (archive size) [44], and for DE, $Cr = 0.95$ (crossover rate) and $F = 0.5$ (mutation scaling factor) [37], and for HS, $HMS = 10$ (size of harmony memory), $HMCR = 0.825$ (harmony memory consideration rate) and $PAR = 0.35$ (Pitch Adjustment Rate) [10].

Table 2: x^{**}, The best super-optima, and their crisp constraints violation vectors $CCV(x^{**})$.

A. 1	$x^* = [0.1859, 0.115, 0.0165, 0, 0, 0]$	
	$x^{**} = [0.1964, 0.1215, 0.0174, 0, 0, 0]$	, CCV $(x^{**}) = [0, 0.0009, 0, 0]$
A. 2	$x^* = [0, 0.8702, 0, 0, 0.6468, 0.5835]$	
	$x^{**} = [0, 0.8731, 0, 0, 0.6506, 0.5854]$	, CCV $(x^{**}) = [0, 0.0007, 0.0007, 0, 0]$
A. 3	$x^* = [0, 0.0979, 0, 0.1334, 0, 0.1532]$	
	$x^{**} = [0, 0.1015, 0, 0.1383, 0, 0.1588]$	, CCV $(x^{**}) = [0, 0, 0, 0, 0.0023, 0]$
A. 4	$x^* = [0.2069, 0, 0, 0.0339, 0, 0, 0.8705]$	
	$x^{**} = [0.2123, 0, 0, 0.0348, 0, 0, 0.8717]$	, $CCV\ (x^{**}) = [0.0008, 0, 0, 0.0008, 0]$
A. 5	$x^* = [0.3282, 0, 0, 0.1228, 0, 0, 0.1221, 0]$	
	$x^{**} = [0.3319, 0, 0, 0.1302, 0, 0, 0.1294, 0]$	, CCV $(x^{**}) = [0, 0.0021, 0.0021, 0, 0, 0]$
A. 6	$x^* = [0, 0, 0, 0, 0.0058, 0.0067, 0.0056]$	
	$x^{**} = [0, 0, 0, 0, 0.0082, 0.0094, 0.0079]$	, CCV $(x^{**}) = [0, 0, 0, 0.0019, 0, 0, 0]$
A. 7	$x^* = [0.0057, 0, 0, 0, 0, 0, 0, 0, 0, 0.0043]$	
	$x^{**} = [0.0131, 0, 0, 0, 0, 0, 0, 0, 0, 0.0099]$	, CCV $(x^{**}) = [0, 0, 0, 0, 0, 0, 0.0035]$
A. 8	$x^* = [0.0613, 0, 0, 0, 0, 0.0163, 0, 0, 0, 0]$	
	$x^{**} = [0.0910, 0, 0, 0, 0, 0.0242, 0, 0, 0, 0]$	, $CCV\ (x^{**}) = [0, 0, 0, 0, 0.0063, 0, 0, 0]$
A. 9	$x^* = [0, 0.0234, 0.1036, 0, 0.0235, 0.0228, 0, 0, 0]$	
	$x^{**} = [0, 0.0245, 0.1087, 0, 0.0246, 0.0240, 0, 0, 0]$	, $CCV\ (x^{**}) = [0, 0.0009, 0, 0, 0, 0, 0, 0, 0]$
A. 10	$x^* = [0, 0.0295, 0, 0, 0.0335, 0.0633, 0, 0, 0, 0.1846]$	
	$x^{**} = [0, 0.0304, 0, 0, 0.0345, 0.0652, 0, 0, 0, 0.1900]$	, $CCV\ (x^{**}) = [0.0009, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

Table 3 presents the total values of the best solutions obtained by the algorithms. As shown in this table, CACO, DE, and HS have the worst results (i.e., solutions with the least total values $\mu_T(x) = 0$) in most cases (70% of the test problems). Moreover, for the remaining 30% of the cases, they could not necessarily find a super-optimum (i.e., solutions x such that $\mu_T(x) > \mu_T(x^*)$); A.3 for CACO, and A.4 for DE and HS. Also, the OPSO produced the worst solutions in half of the cases and could find super-optima only for 50% of the test problems. On the other hand, the modified PSO could find a super-optimum for each test problem that is very close to the best super-optimum x^{**} obtained by the linearization approach. Particularly, for test problem A.8, we have $\mu_T(x^*_{MPSO}) = \mu_T(x^{**})$. However, for test problem A.4, although $\mu_T(x^*_{MPSO}) > \mu_T(x^*)$, the distance between $\mu_T(x^*_{MPSO})$ and $\mu_T(x^{**})$ is very large in average.

Table 3: A comparison between the total values of the best solutions x^{**}, x^*_{MPSO}, x^*_{QPSO}, x^*_{CACO}, x^*_{DE} and x^*_{HE} found by the linearization approach, MPSO, OPSO, CACO, DE and HS algorithms, respectively, for the test problems of Appendix A.

Test problems	$\mu_T\,(x^{**})$	$\mu_T\,(x^*_{\text{MPSO}})$	$\mu_T\,(x^*_{\text{OPSO}})$	$\mu_T\,(x^*_{\text{CACO}})$	$\mu_T\,(x^*_{DE})$	$\mu_T\,(x^*_{HS})$
A. 1	0.9910	0.9904	0.9608	0.9632	0.5638	0.7519
A. 2	0.9933	0.9924	0.9929	0.9895	0.948	0.948
A. 3	0.9765	0.9717	0.6582	0.0834	0	0
A. 4	0.9916	0.9901	0.9494	0	0.2619	0.1217
A. 5	0.9793	0.9749	0	0	0	0
A. 6	0.9809	0.9788	0.939	0	0	0
A. 7	0.96472	0.96471	0	0	0	0
A. 8	0.9373	0.9373	0	0	0	0
A. 9	0.9907	0.987	0	0	0	0
A. 10	0.9914	0.9875	0	0	0	0

In Table 4, the results have been averaged over 30 runs, and the average best-so-far (Avg), median best-so-far (Mdn) in the last iteration, and the standard deviations (Sd) are reported for the heuristic algorithms. The results, in Table 4, show that the modified PSO produces better solutions with a higher convergence rate when compared with the other heuristic algorithms. As this table illustrates, 70% of the average best-so-far values found by CACO, DE, and HS

are zero. As mentioned earlier, this is due to unfavorable points having zero total values. Due to this fact, it can be concluded that these algorithms could not escape from these poor solutions for 70% of the problems. Also, the OPSO could avoid trapping in such unfavorable points in only half of the cases. Figures 4-10 depict the performances of the algorithms averaged over 30 runs for some test problems. Dashed lines $y = \mu_T(x^*)$ and $y = \mu_T(x^{**})$ determine the least and the most accuracy required, respectively.

Table 4: A comparison between the results found by the MPSO, OPSO, CACO, DE and HS algorithms for the test problems of Appendix A. The results have been averaged over 30 runs. Maximum number of iterations= 100.

Test problems		MPSO	OPSO	CACO	DE	HS
A. 1	Avg	0.9759	0.091549	0.059136	0.018792	0.045539
	Mdn	0.9787	0	0	0	0
	Sd	0.0035	2.7014	1.3873	1.3413	1.5425
A. 2	Avg	0.93161	0.34901	0.6167	0.49248	0.49248
	Mdn	0.97558	0	0.84622	0.57185	0.57185
	Sd	0.58775	2.8787	2.5854	0.89195	0.89195
A. 3	Avg	0.83368	0.038722	0.0027787	0	0
	Mdn	0.87382	0	0	0	0
	Sd	0.62053	2.7763	1.885	1.9039	1.7758
A. 4	Avg	0.5177	0.031648	0	0.0087283	0.0040561
	Mdn	0.50478	0	0	0	0
	Sd	0.40083	2.1511	1.8494	1.8822	2.1892
A. 5	Avg	0.92709	0	0	0	0
	Mdn	0.96284	0	0	0	0
	Sd	0.092625	4.3403	2.0897	1.9291	2.5145
A. 6	Avg	0.93838	0.031301	0	0	0
	Mdn	0.95304	0	0	0	0
	Sd	0.0024304	1.6968	1.4244	1.505	1.68
A. 7	Avg	0.96435	0	0	0	0
	Mdn	0.96422	0	0	0	0
	Sd	1.2739e − 05	4.0588	2.4427	2.0715	2.7796
A. 8	Avg	0.93015	0	0	0	0
	Mdn	0.93718	0	0	0	0
	Sd	0.0020195	5.0862	3.454	3.3177	2.7028
A. 9	Avg	0.94184	0	0	0	0
	Mdn	0.92721	0	0	0	0
	Sd	0.0027109	2.4319	1.4109	1.1276	0.90631
A. 10	Avg	0.91627	0	0	0	0
	Mdn	0.95904	0	0	0	0
	Sd	0.0082584	2.3339	1.7099	1.796	1.788

Table 5 contains the error of the solutions found by the algorithms (for heuristic algorithms, the results are the average error of 30 runs over the last iterations). And the error is given by $1/2\{(1/m)\sum_{i=1}^{m} CCV(x)_i + |z_0 - c^T x|\}$, where $(1/m)\sum_{i=1}^{m} CCV(x)_i$ is the average of all the crisp constraint violations, and $|z_0 - c^T x|$ is the difference between z_0 (the best desirable value) and optimal values obtained by applying the algorithms to the test problems. From this table, we can note that the linearization approach produces the best super-optima with the least errors and therefore with the least constraint violations (also see Table 2). Additionally, the time (in seconds) involved in each method's execution is reported in Table 6. Based on the results in Table 6, the current approach is the fastest algorithm compared to the other algorithms in each test problem. The linear approach takes less than 0.996 seconds to finish an experiment.

Table 7 shows the results of a paired t-test to illustrate significant differences between the algorithms. In the paired t-test, the result is a value called the p-value, and if it is close to zero, it indicates a significant difference between the two methods. Also, a confidential interval implies an interval of difference in which 95% of values lie. In Table 7, Sd, SEM, and CI stand for standard deviation of differences, standard error mean of differences, and confidence interval, respectively. As illustrated in Table 7, the p-values are mostly lower than 10^{-7}, which indicates a significant difference between the algorithms. These results show that the differences between the optimal solutions found by the

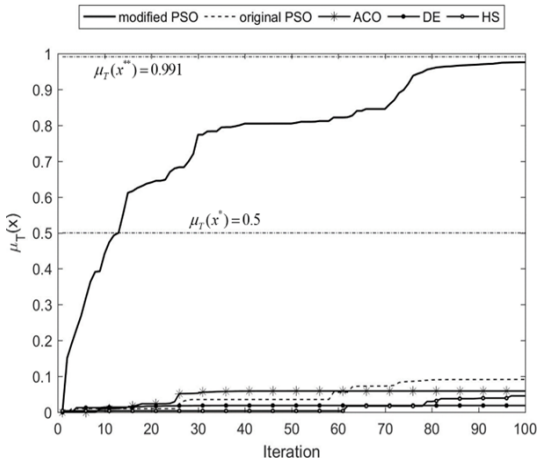


Figure 4: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.1 and their differences from the best super-optimum found by the linearization approach.

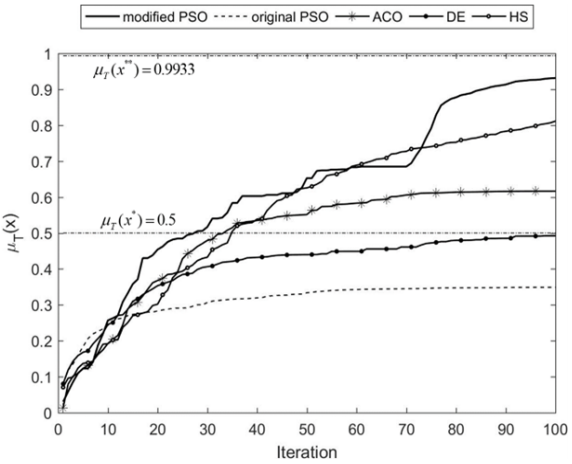


Figure 5: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.2 and their differences from the best super-optimum found by the linearization approach.

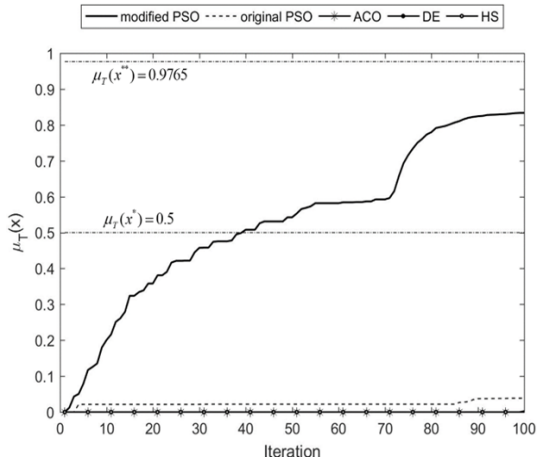


Figure 6: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.3 and their differences from the best super-optimum found by the linearization approach.

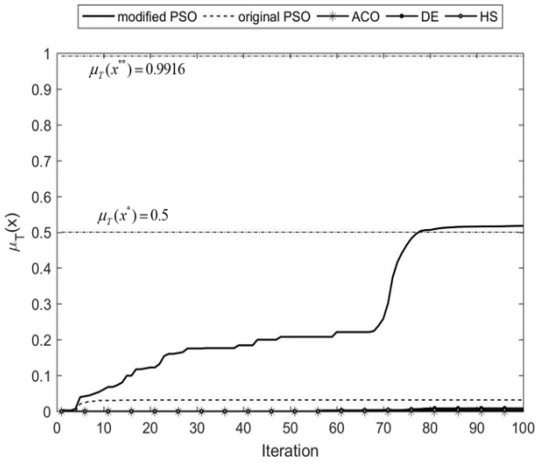


Figure 7: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.4 and their differences from the best super-optimum found by the linearization approach.

Table 5: Errors averaged over the last iterations of 30 runs.

Test problems	Proposed Method	MPSO	OPSO	CACO	DE	HS
A. 1	0.00056521	0.0022895	2.9873	2.1724	2.6526	2.3742
A. 2	0.00047085	0.12277	2.8629	1.4587	0.52682	0.52682
A. 3	0.0013702	0.12296	3.6869	3.3445	3.5706	3.9572
A. 4	0.00058783	0.31098	3.3696	3.0948	3.1164	3.6959
A. 5	0.0013822	0.019544	7.9953	7.3276	7.544	7.7492
A. 6	0.0010926	0.0035921	2.3294	1.6659	2.3062	2.6638
A. 7	0.0020163	0.0020349	8.5333	8.5898	8.7956	9.5084
A. 8	0.0035273	0.0038748	10.7297	13.4616	12.9967	12.9733
A. 9	0.00051485	0.003506	2.9832	2.0228	1.9556	1.7978
A. 10	0.00047489	0.0061162	3.6554	2.6794	2.7964	2.9454
MSE	2.2866e − 06	0.012737	32.1624	35.035	34.9926	37.3003

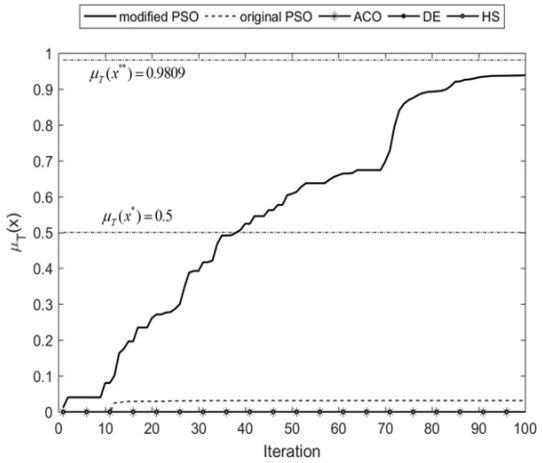


Figure 8: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.6 and their differences from the best super-optimum found by the linearization approach.

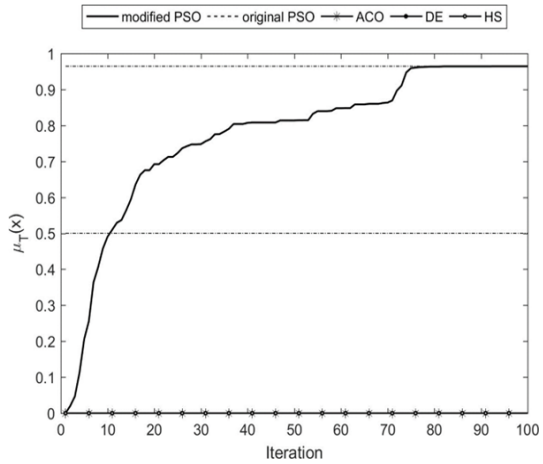


Figure 9: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.7 and their differences from the best super-optimum found by the linearization approach.

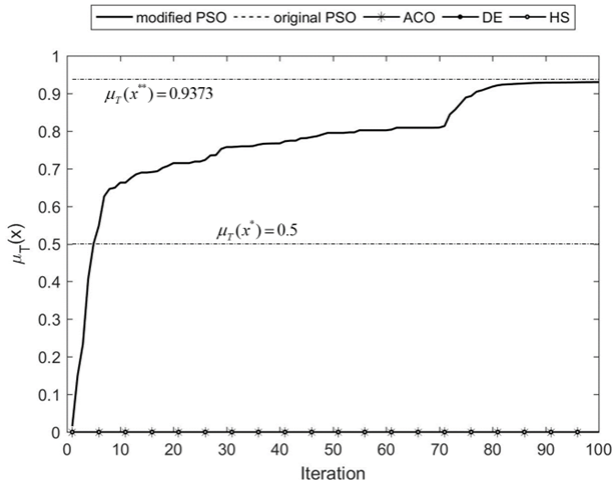


Figure 10: The results (averaged over 30 runs) of the MPSO, OPSO, CACO, DE and HS algorithms on test problem A.8 and their differences from the best super-optimum found by the linearization approach.

Table 6: Time involved in the execution of the methods.

Test problems	Proposed Method	MPSO	OPSO	CACO	DE	HS
A. 1	0.036123	0.28818	0.15882	0.19304	0.14393	0.20819
A. 2	0.099543	0.31729	0.17306	0.2198	0.1677	0.1677
A. 3	0.071229	0.33334	0.18697	0.23123	0.17436	0.2457
A. 4	0.063235	0.35112	0.19826	0.23182	0.17636	0.25039
A. 5	0.092431	0.3857	0.21215	0.2583	0.19873	0.28562
A. 6	0.088895	0.39704	0.21742	0.26146	0.20715	0.28364
A. 7	0.041282	0.55216	0.25899	0.31076	0.24577	0.34284
A. 8	0.026400	0.516	0.28584	0.33663	0.27376	0.3705
A. 9	0.042752	0.5221	0.28127	0.33833	0.27599	0.36654
A. 10	0.035890	0.59621	0.32267	0.38421	0.32517	0.41314

OPSO, CACO, DE, and HS algorithms and the best super-optima found by the linearization approach are statistically significant. Moreover, for the modified PSO, the results indicate the closeness between these solutions and the best super-optima.

Table 7: Paired t-test results based on output value and with degree freedom of 7.

Test problems	Difference between proposed method and MPSO	Difference between proposed method and OPSO	Difference between proposed method and CACO	Difference between proposed method and DE	Difference between proposed method and HS
A. 1	0.0151	0.89941	0.93182	0.97217	0.94542
A. 2	0.06166	0.64426	0.37657	0.50079	0.50079
A. 3	0.14283	0.93779	0.97373	0.97651	0.97651
A. 4	0.4739	0.95995	0.9916	0.98287	0.98754
A. 5	0.05218	0.97927	0.97927	0.97927	0.97927
A. 6	0.0425	0.94958	0.98088	0.98088	0.98088
A. 7	0.00037	0.96472	0.96472	0.96472	0.96472
A. 8	0.00714	0.93729	0.93729	0.93729	0.93729
A. 9	0.04889	0.99073	0.99073	0.99073	0.99073
A. 10	0.0751	0.99137	0.99137	0.99137	0.99137
Average	0.091967	0.92544	0.9118	0.92766	0.92545
Sd	0.14025	0.10266	0.18927	0.15079	0.15035
SEM	0.044352	0.032465	0.059853	0.047683	0.047546
p-value	0.067972	$3.9164e-10$	$9.861e-08$	$1.1585e-08$	$1.1534e-08$
95% Cl	(-0.00836,0.1923)	(0.852,0.99888)	(0.7764,1.0472)	(0.81979,1.0355)	(0.8179,1.033)

6 Discussion and future prospects

This study has introduced an exact linearization framework for solving fuzzy relational inequalities with fuzzy constraints (FRI-FC) under the product t-norm. The proposed approach transforms the original nonlinear formulation into an equivalent linear optimization problem, enabling the use of established linear programming techniques such as the Simplex algorithm. Numerical experiments confirmed that the method consistently identifies exact super-optima within very short computation times. Compared with heuristic and metaheuristic algorithms, which often suffer from premature convergence or solution instability, the proposed method demonstrated superior accuracy, robustness, and efficiency. A notable aspect of the framework is its treatment of feasibility. Unlike traditional optimization approaches that restrict the search to strictly feasible solutions, the fuzzy constraint formulation permits controlled perturbations of constraints. This relaxation, evaluated through appropriately defined membership functions, allows the algorithm to identify super-optima that balance minor infeasibility with improved objective values. Such a feature reflects practical decision-making scenarios, where strict adherence to all constraints may be less critical than achieving satisfactory overall performance.

Despite these promising results, several avenues remain open for further investigation. The present work is limited to the product t-norm, and extending the methodology to other t-norms, such as the minimum or Lukasiewicz operators, would broaden its theoretical and practical applicability. In addition, the current formulation is restricted to linear objective functions. Since many real-world optimization problems involve nonlinear objectives, adapting or generalizing the linearization framework to such cases represents an important direction for future research. Another potential extension lies in the study of alternative inequality types. In particular, incorporating "$\geq$" inequalities and developing bipolar generalizations of FRI-FC would enrich the theoretical structure of the problem class and provide additional modeling flexibility. Furthermore, systematic investigation of larger-scale problems and more diverse test instances would yield deeper insights into the scalability and general performance of the method.

In conclusion, while the proposed algorithm achieves exact and efficient solutions for FRI-FC problems under the product t-norm, it also provides a foundation for further methodological advances. Extensions to alternative t-norms, nonlinear objectives, and generalized inequality formulations are natural directions that can significantly enhance the scope and impact of fuzzy optimization research.

Conclusion

This paper presents a new algorithm for solving fuzzy relational inequalities with fuzzy constraints (FRI-FC) defined by the product t-norm. The linear optimization problem for the regions formed by FRI-FC was studied, and a set of equivalent optimization problems was introduced by the theoretical aspects of fuzzy relational inequalities. It is shown that the main problem can be converted into an equivalent linear model, allowing efficient algorithms such as Simplex to solve it. The problem was further simplified using three simplification operations. A comparison was made between the proposed method and the modified PSO, which solves linear optimization problems subjected to FRI-FC regions. Furthermore, since the main problem could be transformed into an unconstrained optimization problem, the solutions of the linearization method and the modified PSO were compared to the results obtained by some well-known heuristic algorithms such as the original PSO (OPSO), continuous Ant Colony Optimization (CACO), Differential Evolution (DE), and Harmony Search (HS). In contrast to the OPSO, CACO, DE, and HS algorithms, the solutions found by the modified PSO were mostly very close to the best super-optima generated by the linearization method. From the results of errors and the time of execution of the proposed method, we can observe that our algorithm finds the best super-optima with the least constraints violation and finishes an experiment in less than 0.996 seconds. As a result, some of the heuristic algorithms tested in this study could not escape from poor solutions, whereas our linearization approach was able to do so, and also, it was able to find higher quality solutions, as well. Future research can proceed along several directions. One avenue is to evaluate the proposed algorithm on other classes of FRI-FC problems defined by alternative t-norms, as well as on problems involving nonlinear objective functions. Moreover, it would be valuable to extend the current formulation to incorporate "$\geq$" inequalities and to further advance the framework of FRI-FC through generalized models, such as bipolar or latticized formulations.

Appendix A

Test Problem A.1

$$c^T = \begin{bmatrix} -1.7005 & -4.3370 & -3.5848 & 7.7951 & 0.4787 & 9.3360 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.1359 & 0.8372 & 0.1439 & 0.8102 & 0.8317 & 0.0801 \\ 0.0866 & 0.1400 & 0.9757 & 0.1262 & 0.7061 & 0.8810 \\ 0.7325 & 0.0336 & 0.3292 & 0.6075 & 0.6609 & 0.6825 \\ 0.8851 & 0.3458 & 0.6788 & 0.5612 & 0.8223 & 0.3284 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.2014 & 0.0161 & 0.6792 & 0.8360 \end{bmatrix}.$$

Test Problem A.2

$$c^T = \begin{bmatrix} 6.1322 & -1.5004 & 4.2041 & 0.1303 & -7.0502 & -9.3530 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.5847 & 0.1338 & 0.4806 & 0.2675 & 0.3970 & 0.3215 \\ 0.1819 & 0.1038 & 0.0074 & 0.0476 & 0.1761 & 0.0927 \\ 0.2044 & 0.2334 & 0.1171 & 0.3456 & 0.2832 & 0.3481 \\ 0.0659 & 0.4362 & 0.5850 & 0.3090 & 0.7723 & 0.4813 \\ 0.8992 & 0.0619 & 0.7021 & 0.0030 & 0.8549 & 0.8145 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.9366 & 0.1139 & 0.2031 & 0.8282 & 0.8752 \end{bmatrix}.$$

Test Problem A.3

$$c^T = \begin{bmatrix} 6.8522 & -1.8291 & 8.674 & -6.4465 & 5.9379 & -1.7184 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.3248 & 0.1426 & 0.1232 & 0.8051 & 0.1083 & 0.2499 \\ 0.7076 & 0.2252 & 0.7964 & 0.0571 & 0.6141 & 0.0879 \\ 0.1289 & 0.7975 & 0.0480 & 0.0046 & 0.0144 & 0.1980 \\ 0.5316 & 0.8095 & 0.8496 & 0.1967 & 0.5240 & 0.0389 \\ 0.6753 & 0.6560 & 0.3397 & 0.4811 & 0.3146 & 0.4192 \\ 0.0151 & 0.7319 & 0.4089 & 0.6728 & 0.6220 & 0.7515 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.6034 & 0.4401 & 0.5971 & 0.1162 & 0.0642 & 0.8811 \end{bmatrix}.$$

Test Problem A.4

$$c^T = \begin{bmatrix} -5.9968 & 1.6453 & 72174 & -6.3971 & 2.9091 & 1.7317 & -9.5134 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.1556 & 0.7437 & 0.3357 & 0.9491 & 0.8195 & 0.7906 & 0.0144 \\ 0.8460 & 0.1222 & 0.4980 & 0.7749 & 0.7375 & 0.4606 & 0.3457 \\ 0.6406 & 0.5979 & 0.9258 & 0.2752 & 0.2663 & 0.4620 & 0.6144 \\ 0.5399 & 0.3409 & 0.4983 & 0.9553 & 0.7804 & 0.4636 & 0.7174 \\ 0.1448 & 0.1067 & 0.8405 & 0.8741 & 0.6756 & 0.8831 & 0.6736 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.0322 & 0.4687 & 0.8801 & 0.6245 & 0.8080 \end{bmatrix}.$$

Test Problem A.5

$$c^T = \begin{bmatrix} -2.2349 & 9.8684 & 6.0145 & -0.1409 & 4.4755 & 5.1849 & -5.2513 & 7.4044 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.6685 & 0.8277 & 0.7281 & 0.1500 & 0.0963 & 0.9766 & 0.0665 & 0.0789 \\ 0.1002 & 0.7566 & 0.3847 & 0.2817 & 0.6514 & 0.7538 & 0.2835 & 0.4277 \\ 0.5461 & 0.8736 & 0.7027 & 0.3276 & 0.9032 & 0.3744 & 0.5540 & 0.5274 \\ 0.5924 & 0.2766 & 0.5094 & 0.3399 & 0.3381 & 0.9006 & 0.5930 & 0.4494 \\ 0.9426 & 0.0540 & 0.7771 & 0.8831 & 0.1892 & 0.4538 & 0.4534 & 0.3912 \\ 0.3466 & 0.2082 & 0.2217 & 0.2299 & 0.7907 & 0.0611 & 0.3830 & 0.9836 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.9642 & 0.0346 & 0.1792 & 0.7077 & 0.9931 & 0.3542 \end{bmatrix}.$$

Test Problem A.6

$$c^T = \begin{bmatrix} 3.9815 & 7.8181 & 9.1858 & 0.9443 & -7.2275 & -7.0141 & -4.8498 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.8407 & 0.2511 & 0.9172 & 0.0540 & 0.0119 & 0.6020 & 0.2290 \\ 0.2543 & 0.6160 & 0.2858 & 0.5308 & 0.3371 & 0.2630 & 0.9133 \\ 0.8143 & 0.4733 & 0.7572 & 0.7792 & 0.1622 & 0.6541 & 0.1524 \\ 0.2435 & 0.3517 & 0.7537 & 0.9340 & 0.7943 & 0.6892 & 0.8258 \\ 0.9293 & 0.8308 & 0.3804 & 0.1299 & 0.3112 & 0.7482 & 0.5383 \\ 0.3500 & 0.5853 & 0.5678 & 0.5688 & 0.5285 & 0.4505 & 0.9961 \\ 0.1966 & 0.5497 & 0.0759 & 0.4694 & 0.1656 & 0.0838 & 0.0782 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.4427 & 0.1067 & 0.9619 & 0.0046 & 0.7749 & 0.8173 & 0.8687 \end{bmatrix}.$$

Test Problem A.7

$$c^T = \begin{bmatrix} -6.225 & 7.3391 & 7.0215 & 6.5683 & 1.2437 & 5.2404 & 4.1114 & 1.4945 & 7.7537 & -0.0706 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.8376 & 0.5607 & 0.4054 & 0.9161 & 0.7397 & 0.0716 & 0.3919 & 0.2039 & 0.2089 & 0.5333 \\ 0.0516 & 0.9841 & 0.0849 & 0.2607 & 0.1426 & 0.3386 & 0.1199 & 0.4257 & 0.5544 & 0.3795 \\ 0.4259 & 0.7705 & 0.8464 & 0.3001 & 0.4591 & 0.6629 & 0.3465 & 0.5807 & 0.0754 & 0.7783 \\ 0.2721 & 0.0729 & 0.5261 & 0.4080 & 0.6138 & 0.5058 & 0.5298 & 0.0238 & 0.5426 & 0.8281 \\ 0.7275 & 0.3988 & 0.5304 & 0.8283 & 0.3916 & 0.9934 & 0.1053 & 0.7491 & 0.2099 & 0.8877 \\ 0.8102 & 0.0718 & 0.8724 & 0.4534 & 0.4194 & 0.4759 & 0.3741 & 0.0360 & 0.7298 & 0.2837 \\ 0.4767 & 0.1241 & 0.7166 & 0.9912 & 0.3778 & 0.8198 & 0.6201 & 0.2147 & 0.1651 & 0.6314 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.1699 & 0.3880 & 0.1850 & 0.2983 & 0.0652 & 0.7516 & 0.0027 \end{bmatrix}.$$

Test Problem A.8

$$c^T = \begin{bmatrix} -0.6133 & 9.5958 & 8.9277 & 2.5328 & 4.5288 & -3.2146 & 8.4352 & 9.0807 & 9.8011 & 2.4952 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.9576 & 0.1524 & 0.4591 & 0.1542 & 0.6114 & 0.3348 & 0.5124 & 0.5708 & 0.5980 & 0.4008 \\ 0.0317 & 0.1542 & 0.0775 & 0.4483 & 0.4208 & 0.2367 & 0.2025 & 0.2892 & 0.7608 & 0.7968 \\ 0.4176 & 0.9340 & 0.5365 & 0.5206 & 0.3109 & 0.6503 & 0.9856 & 0.7474 & 0.0019 & 0.3848 \\ 0.5935 & 0.2486 & 0.9793 & 0.3124 & 0.2278 & 0.9957 & 0.1159 & 0.8994 & 0.6739 & 0.7614 \\ 0.2106 & 0.3407 & 0.2622 & 0.4472 & 0.3391 & 0.7915 & 0.3352 & 0.6066 & 0.1036 & 0.2750 \\ 0.2554 & 0.2953 & 0.2070 & 0.3073 & 0.8167 & 0.2889 & 0.1124 & 0.7823 & 0.5541 & 0.2769 \\ 0.6388 & 0.2688 & 0.6292 & 0.7254 & 0.9561 & 0.5072 & 0.0170 & 0.4677 & 0.3709 & 0.7724 \\ 0.5010 & 0.2386 & 0.5260 & 0.3181 & 0.2825 & 0.2465 & 0.6896 & 0.1970 & 0.4726 & 0.8429 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.9750 & 0.0237 & 0.5693 & 0.5947 & 0.0129 & 0.7476 & 0.8917 & 0.9195 \end{bmatrix}.$$

Test Problem A.9

$$c^T = \begin{bmatrix} 2.7276 & -9.1572 & -7.1452 & 0.9314 & -1.9806 & -0.2452 & 8.1128 & 2.4546 & 2.8567 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.6637 & 0.7432 & 0.8188 & 0.5660 & 0.6163 & 0.6756 & 0.8136 & 0.9008 & 0.1373 \\ 0.4567 & 0.8137 & 0.1834 & 0.5090 & 0.8100 & 0.8336 & 0.0360 & 0.6520 & 0.7940 \\ 0.9142 & 0.6858 & 0.3362 & 0.4732 & 0.9350 & 0.3233 & 0.1749 & 0.3490 & 0.6930 \\ 0.3936 & 0.8700 & 0.9014 & 0.4516 & 0.8772 & 0.2706 & 0.2962 & 0.7862 & 0.8683 \\ 0.5089 & 0.8362 & 0.9459 & 0.6415 & 0.7242 & 0.3912 & 0.0342 & 0.8280 & 0.1411 \\ 0.1390 & 0.9381 & 0.4557 & 0.4086 & 0.4804 & 0.7904 & 0.2785 & 0.5207 & 0.4494 \\ 0.1239 & 0.4953 & 0.3745 & 0.3644 & 0.8924 & 0.5274 & 0.5470 & 0.6684 & 0.5228 \\ 0.6215 & 0.6577 & 0.2243 & 0.2977 & 0.8589 & 0.8018 & 0.0108 & 0.4651 & 0.9415 \\ 0.6028 & 0.2612 & 0.0686 & 0.7647 & 0.0017 & 0.2003 & 0.2626 & 0.1225 & 0.5671 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.6412 & 0.0190 & 0.1702 & 0.6046 & 0.2470 & 0.8351 & 0.7981 & 0.4645 & 0.6098 \end{bmatrix}.$$

Test Problem A.10

$$c^T = \begin{bmatrix} 0.1117 & -5.4486 & 4.4966 & 2.1297 & -0.4456 & -6.9081 & 6.4627 & 4.4522 & 5.6057 & -5.7421 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.7906 & 0.9957 & 0.0597 & 0.5569 & 0.8766 & 0.4642 & 0.7950 & 0.4590 & 0.5371 & 0.1593 \\ 0.4117 & 0.3598 & 0.0890 & 0.7960 & 0.6433 & 0.7929 & 0.9148 & 0.2328 & 0.5097 & 0.8664 \\ 0.8080 & 0.8941 & 0.6725 & 0.9067 & 0.6509 & 0.5584 & 0.0008 & 0.0250 & 0.5247 & 0.0786 \\ 0.0878 & 0.9908 & 0.1796 & 0.9319 & 0.1598 & 0.0141 & 0.8120 & 0.0548 & 0.7654 & 0.8247 \\ 0.0338 & 0.7684 & 0.8985 & 0.1536 & 0.8718 & 0.2336 & 0.8460 & 0.5478 & 0.6549 & 0.7402 \\ 0.8519 & 0.1464 & 0.0081 & 0.5379 & 0.8072 & 0.3497 & 0.9215 & 0.9925 & 0.1810 & 0.1014 \\ 0.2547 & 0.9694 & 0.0436 & 0.7910 & 0.0878 & 0.7878 & 0.4435 & 0.6274 & 0.8116 & 0.3670 \\ 0.1954 & 0.8315 & 0.7230 & 0.8740 & 0.4403 & 0.2030 & 0.0607 & 0.2924 & 0.2905 & 0.8977 \\ 0.7841 & 0.2572 & 0.5130 & 0.9844 & 0.5482 & 0.5547 & 0.2540 & 0.7913 & 0.8254 & 0.0586 \\ 0.1628 & 0.6455 & 0.8100 & 0.2105 & 0.3994 & 0.9538 & 0.9343 & 0.3079 & 0.1052 & 0.5025 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0.0294 & 0.9041 & 0.9020 & 0.9824 & 0.8485 & 0.1962 & 0.4210 & 0.2162 & 0.8787 & 0.3932 \end{bmatrix}.$$

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