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Original paper

On fuzzy approximation theorems for functions of two variables via statistical deferred Nörlund summability

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Abstract

This study introduces and investigates the concepts of deferred Nörlund statistical Riemann integrability and statistical deferred Nörlund Riemann summability for double sequences of fuzzy number-valued functions of two variables. An inclusion result is first established to clarify the relationship between these newly proposed notions in the bivariate setting. Building on this framework, new fuzzy Korovkin-type approximation theorems are developed using the four fundamental algebraic test functions 1, x, y and x^2+y^2 under the proposed means. To highlight the applicability of the results, an example is provided involving a fuzzy positive linear operator associated with bivariate Bernstein polynomials. Furthermore, the convergence behavior of these operators is illustrated graphically with the aid of MATLAB.

Keywords: Double sequence of fuzzy number-valued functions, double Riemann integrals, Deferred Nörlund mean, Bivariate Bernstein polynomials, Korovkin-type theorems.

1 Introduction

The study of convergence in double sequence spaces plays a central role in real and functional analysis and has gained considerable importance due to its applications in approximation theory and fuzzy analysis. A major advancement in this direction is the notion of statistical convergence, which generalizes classical convergence. Originally introduced for single sequences by Fast [7] and Steinhaus [29], statistical convergence was later extended to double sequences and has since become an active area of research. Its flexibility and applicability have led to numerous extensions and variants, particularly in summability theory and approximation theory.

Statistical convergence and its generalizations have been investigated in various functional settings. Alotaibi and Mursaleen [2] studied statistical convergence in random paranormed spaces, while Belen and Mohiuddine [3] introduced generalized weighted statistical convergence. The interaction between summability theory and fuzzy analysis was explored by Choudhary et al. [5] through Tauberian theorems for double sequences of fuzzy numbers via the De La Vallée Poussin mean. In approximation theory, Jena and Paikray [10] applied statistical probability convergence to Korovkin-type theorems. Further extensions involving deferred summability methods were proposed by Jena et al. [11], who studied statistical convergence of random variables via deferred Cesàro means, and by Jena et al. [13], who obtained Tauberian results for Cesàro summable double sequences of fuzzy numbers. The concept of generalized deferred Cesàro equi-statistical convergence and its application to approximation theorems was developed by Parida et al. [24].

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Parallel developments have taken place in fuzzy analysis, particularly for fuzzy-valued functions. Dutta et al. [6] introduced statistical deferred Cesàro summability, which serves as a useful generalization of classical summability methods. Statistical convergence in measure for double sequences of fuzzy-valued functions was studied by Hazarika et al. [8], while new notions of continuity in fuzzy normed spaces were introduced by Hazarika and Mohiuddine [9]. Mohiuddine et al. [18] investigated weighted statistical convergence of fuzzy sequences via difference operators and established fuzzy approximation theorems. Generalized statistically convergent fuzzy sequences were studied by Mursaleen et al. [19], and deferred statistical A-convergence of fuzzy sequences was examined by Nayak et al. [20]. Relatively equi-statistical fuzzy approximation theorems were obtained by Paikray et al. [22]. The foundational structure of fuzzy number spaces, which underlies these developments, was established by Wu and Ma [15, 16, 31].

The classical Korovkin approximation theorem [14] provides a powerful criterion for the convergence of sequences of positive linear operators on spaces of continuous functions. While originally formulated for single-variable real-valued functions, it was later extended to functions of two variables by enlarging the test function set. The incorporation of summability methods such as statistical convergence, Nörlund summability, and their deferred variants has significantly weakened the convergence requirements in Korovkin-type theorems. In this direction, Srivastava et al. [27, 28] developed several Korovkin-type approximation results based on deferred summability methods, including double sequences.

More recently, attention has focused on fuzzy approximation theorems for functions of single and double variables under advanced summability frameworks. Parida et al. [25] introduced statistical deferred weighted Riemann summability and established fuzzy approximation results. A deferred Nörlund summability approach to Korovkin-type theorems was proposed by Parida et al. [23]. Statistical gauge integrable functions and their applications to Korovkin-type approximation theorems were studied by Jena et al. [12]. Furthermore, Mahapatra et al. [17] investigated fuzzy approximation theorems for functions of two variables via statistical deferred Nörlund summability and Riemann integrability. These results highlight the deep interplay between fuzzy analysis, double sequences, and deferred summability methods, forming the foundation of the present study.

Motivated essentially by the above investigation and studies, we introduce the concepts of deferred Nörlund statistical Riemann integrability and statistical deferred Nörlund Riemann summability for double sequences of fuzzy number-valued functions of two variables. First, we establish an inclusion theorem to highlight the relationship between these novel and valuable concepts in the bivariate setting. Building upon these foundational ideas, we develop new versions of fuzzy Korovkin-type theorems utilizing the standard algebraic test functions 1, x, y and x^2+y^2, under our proposed means. Finally, to demonstrate the practical significance of our results, we provide an example involving a fuzzy positive linear operator associated with bivariate Bernstein polynomials. Furthermore, we employ MATLAB to illustrate the convergence behavior of these operators graphically.

2 Preliminaries

Let $\mathcal{E} = \{\gamma : \mathbb{R}^2 \to [0,1]\}$ denote the set of functions that satisfy the following conditions:

(i) Normality: There exists $(t_0, s_0) \in \mathbb{R}^2$ such that $\gamma(t_0, s_0) = 1$,

(ii) Fuzzy Convexity: γ is a fuzzy convex function on $\mathbb{R}^2$,

(iii) Upper Semi-Continuity: γ is upper semi-continuous on $\mathbb{R}^2$,

(iv) Compact Support: The set

$$[\gamma]^0 = \left\{\overline{(t,s) \in \mathbb{R}^2 \text{ such that } \gamma(t,s) > 0}\right\},$$

is compact.

A function $\gamma \in \mathcal{E}$ satisfying these properties is called a fuzzy number of two variables, and $\mathcal{E}$ represents the fuzzy number space in two dimensions.

Let $\gamma \in \mathcal{E}$, and define

$$[\gamma]^\Lambda = \{(t,s) \in \mathbb{R}^2 : \gamma(t,s) \geqq \Lambda\},$$

which represents a nonempty, closed, and bounded subset of $\mathbb{R}^2$ (i.e., a compact set) for $\Lambda \in [0,1]$.

Let us now review some fundamental properties of fuzzy numbers of two variables.

Let $\gamma_{\pm}^{\Lambda}, \beta_{\pm}^{\Lambda} \in \mathbb{R}^2$ denote the lower and upper bounds of the Λ-level sets of $\gamma, \beta \in \mathcal{E}$, where $\Lambda \in [0,1]$ and $k \in \mathbb{R}$. Then for $\gamma, \beta : \mathbb{R}^2 \to [0,1]$, we have:

(i) Addition:

$$(\gamma+\beta)(x,y) = \sup_{(x,y)=(t_1+s_1,\, t_2+s_2)} \min\{\gamma(t_1,t_2),\, \beta(s_1,s_2)\},$$

for $(x,y) \in \mathbb{R}^2$.

(ii) Scalar Multiplication:

$$k\gamma(x,y) = \gamma\left(\tfrac{x}{k}, \tfrac{y}{k}\right), \quad (k \neq 0).$$

(iii) Zero Scalar Multiplication:

$$0\gamma(x,y) = \bar{0}, \quad \text{where} \quad \bar{a}(x,y) = \begin{cases} 1 & (x,y) = (a_1,a_2), \\ 0 & (\text{otherwise}). \end{cases}$$

(iv) Λ-Cut of Sum:

$$[\gamma+\beta]^{\Lambda} = [\gamma]^{\Lambda} + [\beta]^{\Lambda} = \left[\gamma_{-}^{\Lambda} + \beta_{-}^{\Lambda},\, \gamma_{+}^{\Lambda} + \beta_{+}^{\Lambda}\right],$$

where the addition is interpreted in $\mathbb{R}^2$.

(v) Λ-Cut of Scalar Multiplication:

$$[k\gamma]^{\Lambda} = k[\gamma]^{\Lambda} = \left[k\gamma_{-}^{\Lambda},\, k\gamma_{+}^{\Lambda}\right], \quad (k \geqq 0).$$

(vi) Order Relation:

$$\gamma \preceq \beta \iff [\gamma]^{\Lambda} \leqq [\beta]^{\Lambda}, \quad \forall\, \Lambda \in [0,1].$$

The metric $\mathcal{D}$ is defined as

$$\mathcal{D} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^{+},$$

given by

$$\mathcal{D}(\gamma,\beta) = \sup_{0 \leqq \Lambda \leqq 1} \max\left\{\|\gamma_{-}^{\Lambda} - \beta_{-}^{\Lambda}\|,\, \|\gamma_{+}^{\Lambda} - \beta_{+}^{\Lambda}\|\right\}.$$

Note that the metric space $(\mathcal{E}, \mathcal{D})$ is complete (see [31]).

Let $\mathcal{D}^*(\tilde{f}, \tilde{g})$ denote the distance between the fuzzy number-valued functions $\tilde{f}, \tilde{g} : [a,b] \times [c,d] \to \mathcal{E}$ defined as follows:

$$\mathcal{D}^*(\tilde{f},\tilde{g}) = \sup_{(t,s)\in[a,b]\times[c,d]} \sup_{0 \leqq \Lambda \leqq 1} \max\left\{\|\tilde{f}_{-}^{\Lambda}(t,s) - \tilde{g}_{-}^{\Lambda}(t,s)\|,\, \|\tilde{f}_{+}^{\Lambda}(t,s) - \tilde{g}_{+}^{\Lambda}(t,s)\|\right\}.$$

Pringsheim [26] was the first to systematically study the convergence of double sequences in 1900. A double sequence $(u_{m,n})$ is said to converge (or P-convergent) to a limit ℓ if, for every $\epsilon > 0$, there exist integers $m_0, n_0 \in \mathbb{N}$ such that the inequality $|u_{m,n} - \ell| < \epsilon$ holds whenever $m \geqq m_0$ and $n \geqq n_0$. This property is denoted by $P\ \lim u_{m,n} = \ell$.

Likewise, $(u_{m,n})$ is bounded, if there exists a constant $K > 0$ such that $|u_{m,n}| \leqq K$ for all $m,n \in \mathbb{N}$. It is worth noting that, unlike the case of single sequences, P-convergence of a double sequence does not necessarily imply boundedness. Moreover, $(u_{m,n})$ is said to be non-increasing in the Pringsheim sense, if it satisfies the conditions $u_{m+1,n} \leqq u_{m,n}$

and $u_{m,n+1} \leqq u_{m,n}$ for all $m,n \in \mathbb{N}$.

We now recall the notion of statistical convergence of double sequences.

Let $\mathfrak{J}^* \subseteq \mathbb{N} \times \mathbb{N}$, and define

$$\mathfrak{J}^*_{m,n} = \{(\xi,\zeta) : \ \xi \leqq m, \ \zeta \leqq n, \ (\xi,\zeta) \in \mathfrak{J}^*\}.$$

The natural density $d(\mathfrak{J}^*)$ of $\mathfrak{J}^*$ is given by

$$d(\mathfrak{J}^*) = \lim_{m,n\to\infty} \frac{|\mathfrak{J}^*_{m,n}|}{mn} = \rho,$$

where ρ is a real and finite number, and $|\mathfrak{J}^*_{m,n}|$ represents the cardinality of $\mathfrak{J}^*_{m,n}$.

A double sequence $\{u_{m,n}\}$ is said to be statistically convergent to a fuzzy number ℓ if, for every $\epsilon > 0$, the set

$$\mathfrak{J}^*_\epsilon = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{D}(u_{m,n}, \ell) \geqq \epsilon\},$$

has a natural density of zero (see [21]). This implies that, for each $\epsilon > 0$,

$$d(\mathfrak{J}^*_\epsilon) = \lim_{m,n\to\infty} \frac{|\mathfrak{J}^*_\epsilon(m,n)|}{mn} = 0.$$

We express this as

$$\mathrm{stat}^{2D} \lim_{m,n\to\infty} \mathcal{D}(u_{m,n}, \ell) = 0.$$

Let $[a,b] \times [c,d] \subset \mathbb{R}^2$. For each $(m,n) \in \mathbb{N} \times \mathbb{N}$, there exists a fuzzy number-valued function

$$\tilde{g}_{m,n} : [a,b] \times [c,d] \ \to \ \mathcal{E}.$$

As a result, $\{\tilde{g}_{m,n}\}_{m,n\in\mathbb{N}}$ constitutes a double sequence of fuzzy number-valued functions defined on $[a,b] \times [c,d]$.

The Riemann sum for a double sequence $(\tilde{g}_{m,n})$ of fuzzy number-valued functions associated with a tagged partition $\dot{\mathcal{P}}$ of the rectangle $[a,b] \times [c,d]$ is defined as

$$\delta(\tilde{g}_{m,n}; \dot{\mathcal{P}}) := \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{g}_{i,j}(t_i, s_j)\, \Delta x_i \, \Delta y_j,$$

where (t_i, s_j) is the tags corresponding to the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}.$$

Next, we define the concept of Riemann integrability for a double sequence of fuzzy number-valued functions defined on the rectangle $[a,b] \times [c,d] \subset \mathbb{R}^2$.

A double sequence $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ of fuzzy number-valued functions is said to be Riemann integrable to a fuzzy number-valued function $\tilde{g}$ on the rectangle $[a,b] \times [c,d]$ if, for every $\epsilon > 0$, there exists $\sigma_\epsilon > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a,b] \times [c,d]$ with $|\dot{\mathcal{P}}| < \sigma_\epsilon$, the following inequality holds:

$$\mathcal{D}\big(\delta(\tilde{g}_{m,n}; \dot{\mathcal{P}}), \tilde{g}\big) < \epsilon.$$

We now present the definition of statistical convergence for Riemann integrable double sequences of fuzzy number-valued functions defined on the rectangle $[a,b] \times [c,d]$.

A double sequence $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ of fuzzy number-valued functions is said to be statistically Riemann integrable to a fuzzy number-valued function $\tilde{g}$ on the rectangle $[a,b] \times [c,d]$ if, for every $\epsilon > 0$ and for each $(t,s) \in [a,b] \times [c,d]$, there exists a threshold $\sigma_\epsilon > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a,b] \times [c,d]$ satisfying $|\dot{\mathcal{P}}| < \sigma_\epsilon$, the set

$$\mathfrak{J}^*{}_\epsilon = \{(m,n) \in \mathbb{N} \times \mathbb{N} \ : \ \mathcal{D}(\delta(\tilde{g}_{m,n}; \dot{\mathcal{P}}), \tilde{g}) \geqq \epsilon\},$$

has natural density zero. In other words, for every $\epsilon > 0$,

$$d(\mathfrak{J}^*{}_\epsilon) = \lim_{m,n\to\infty} \frac{|\mathfrak{J}^*{}_\epsilon(m,n)|}{mn} = 0.$$

This condition is expressed as

$$\text{stat}^{2D}_{\text{Rie}} \lim_{m,n\to\infty} \mathcal{D}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}),\tilde{g}) = 0.$$

The following example illustrates that every Riemann integrable double sequence of fuzzy number-valued functions is also statistically Riemann integrable. However, the reverse implication does not necessarily hold.

Example 2.1. *If $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ is Riemann integrable to a fuzzy number $\tilde{g}$ on $[0,1]\times[0,1]$, then it is statistically Riemann integrable to $\tilde{g}$; however, there exists a statistically Riemann integrable double sequence which is not Riemann integrable.*

By Riemann integrability, for every $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that for every tagged partition $\dot{\mathcal{P}}$ of $[0,1]\times[0,1]$ with $|\dot{\mathcal{P}}| < \sigma_\varepsilon$ and for all *(m,n),*

$$\mathcal{D}\big(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}),\tilde{g}\big) < \varepsilon.$$

Hence the exceptional index set is empty and thus has natural density 0. Therefore statistical Riemann integrability holds.

Let $\bar{a}$ denote the crisp fuzzy number at $a \in \mathbb{R}$, i.e.,

$$\bar{a}(x) = \begin{cases} 1, & x = a, \\ 0, & \textit{otherwise}. \end{cases}$$

Define a double sequence $\{\tilde{g}_{m,n}\}_{m,n\in\mathbb{N}}$ on $[0,1]\times[0,1]$ by

$$\tilde{g}_{m,n}(x,y) = \begin{cases} \bar{1}, & \textit{if } m \textit{ and } n \textit{ are perfect squares}, \\ \bar{0}, & \textit{otherwise}. \end{cases}$$

Let $S = \{(m,n) \in \mathbb{N}^2 :\ m = \alpha^2,\ n = \beta^2 \text{ for some } \alpha,\beta \in \mathbb{N}\}$. Since the set of squares has natural density 0 in $\mathbb{N}$, it follows that S has natural density 0 in $\mathbb{N}^2$.

For any tagged partition $\dot{\mathcal{P}}$ of $[0,1]\times[0,1]$, the double Riemann sum satisfies

$$\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}) = \begin{cases} \bar{1}\cdot\sum_{i,j} \Delta x_i\,\Delta y_j = \bar{1}, & \textit{if } (m,n) \in S, \\ \bar{0}, & \textit{if } (m,n) \notin S, \end{cases}$$

because on $[0,1]\times[0,1]$ we have $\sum_{i,j}\Delta x_i\,\Delta y_j = 1$.

Taking $\tilde{g} = \bar{0}$, we get

$$\mathcal{D}\big(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}),\tilde{g}\big) = \begin{cases} \mathcal{D}(\bar{1},\bar{0}) = 1, & (m,n) \in S, \\ 0, & (m,n) \notin S. \end{cases}$$

Hence, for any $0 < \varepsilon \leqq 1/2$, the exceptional set

$$\mathfrak{J}^*_\varepsilon = \{(m,n) \in \mathbb{N}^2 :\ \mathcal{D}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}),\bar{0}) \geqq \varepsilon\},$$

is exactly S, which has natural density 0. Therefore $\{\tilde{g}_{m,n}\}$ is statistically Riemann integrable to $\bar{0}$.

However, the sequence is not Riemann integrable to $\bar{0}$, because for every tagged partition $\dot{\mathcal{P}}$ of $[0,1]\times[0,1]$ we have

$$\mathcal{D}\big(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}),\bar{0}\big) = 1 \quad \textit{for infinitely many } (m,n) \in S,$$

so no σ_ε can make the inequality hold for all indices (m,n).

This proves that statistical Riemann integrability does not imply Riemann integrability in the two-variable double-sequence setting.

3 Statistical Riemann integrability with deferred Nörlund mean

Let $(\phi_{m,n})$ and $(\varphi_{m,n})$ be double sequences in $\mathbb{Z}^{0+}\times\mathbb{Z}^{0+}$ such that $\phi_{m,n}<\varphi_{m,n}$ and $\lim_{m,n\to\infty}\varphi_{m,n}=+\infty$. Moreover, let $(p_{m,n})$ be a double sequence of non-negative real numbers, where

$$P_{m,n}=\sum_{\mu=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\nu=\phi_{m,n}+1}^{\varphi_{m,n}}p_{\varphi_{m,n}-\mu,\,\varphi_{m,n}-\nu}.$$

The deferred Nörlund summability mean for the Riemann sum of a double sequence of fuzzy number-valued functions, $\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})$, with respect to a tagged partition $\dot{\mathcal{P}}$ of $[a,b]\times[c,d]$, is defined as

$$\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}))=\frac{1}{P_{m,n}}\sum_{\lambda=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\kappa=\phi_{m,n}+1}^{\varphi_{m,n}}p_{\varphi_{m,n}-\lambda,\,\varphi_{m,n}-\kappa}\;\delta(\tilde{g}_{\lambda,\kappa};\dot{\mathcal{P}}). \tag{1}$$

We introduce the notions of statistical Riemann integrability and statistical Riemann summability for a double sequence of fuzzy number-valued functions based on the deferred Nörlund mean.

Definition 3.1. *Let $(\phi_{m,n})$ and $(\varphi_{m,n})$ be double sequences in $\mathbb{Z}^{0+}\times\mathbb{Z}^{0+}$, and let $(p_{m,n})$ be a double sequence of non-negative real numbers. A double sequence $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ of fuzzy number-valued functions of two variables is said to be deferred Nörlund statistically Riemann integrable ($DNFR^{2D}_{\text{stat}}$) to a fuzzy number-valued function $\tilde{g}$ on $[a,b]\times[c,d]$ if, for every $\epsilon>0$, there exists $\sigma_\epsilon>0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a,b]\times[c,d]$ satisfying $|\dot{\mathcal{P}}|<\sigma_\epsilon$, the set*

$$\{(i,j): i,j\leqq P_{m,n}\quad \text{and}\quad p_{i,j}\,\mathcal{D}(\delta(\tilde{g}_{i,j};\dot{\mathcal{P}}),\tilde{g})\geqq\epsilon\},$$

has zero double natural density. That is, for each $\epsilon>0$,

$$\lim_{m,n\to\infty}\frac{|\{(i,j): i,j\leqq P_{m,n}\quad \text{and}\quad p_{i,j}\,\mathcal{D}(\delta(\tilde{g}_{i,j};\dot{\mathcal{P}}),\tilde{g})\geqq\epsilon\}|}{P_{m,n}}=0.$$

This condition is expressed as

$$DNFR^{2D}_{\text{stat}}\lim_{m,n\to\infty}\mathcal{D}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}),\tilde{g})=0.$$

Definition 3.2. *Let $(\phi_{m,n})$ and $(\varphi_{m,n})$ be double sequences in $\mathbb{Z}^{0+}\times\mathbb{Z}^{0+}$, and let $(p_{m,n})$ be a double sequence of non-negative real numbers. A double sequence $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ of fuzzy number-valued functions is said to be statistically deferred Nörlund Riemann summable ($stat^{2D}_{\text{DNFR}}$) to a fuzzy number-valued function $\tilde{g}$ on the rectangle $[a,b]\times[c,d]$ if, for every $\epsilon>0$, there exists $\sigma_\epsilon>0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a,b]\times[c,d]$ with mesh size $|\dot{\mathcal{P}}|<\sigma_\epsilon$, the set*

$$\{(m,n)\in\mathbb{N}^2: m,n\leqq k\quad \text{and}\quad \mathcal{D}(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\tilde{g})\geqq\epsilon\},$$

has natural density zero in $\mathbb{N}^2$.

That is, for every $\epsilon>0$,

$$\lim_{k\to\infty}\frac{\left|\{(m,n): m,n\leqq k\ \text{and}\ \mathcal{D}(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\tilde{g})\geqq\epsilon\}\right|}{k^2}=0.$$

We denote this as

$$stat^{2D}_{\text{DNFR}}\lim_{m,n\to\infty}\mathcal{D}(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\tilde{g})=0.$$

Every double sequence of fuzzy number-valued functions of two variables that is deferred Nörlund statistically Riemann integrable is also statistically deferred Nörlund Riemann summable. However, the reverse implication does not necessarily hold.

Theorem 3.3. *Let $(\phi_{m,n})$ and $(\varphi_{m,n})$ be double sequences in $\mathbb{Z}^{0+}\times\mathbb{Z}^{0+}$, and let $(p_{m,n})$ be a double sequence of non-negative real numbers. If a double sequence $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ of fuzzy number-valued functions of two variables is deferred Nörlund statistically Riemann integrable to a fuzzy number-valued function $\tilde{g}$ on $[a,b]\times[c,d]$, then it is also statistically deferred Nörlund Riemann summable to the same fuzzy number-valued function $\tilde{g}$ on $[a,b]\times[c,d]$. However, the reverse implication does not necessarily hold.*

Proof. Suppose the double sequence $(\tilde{g}_{m,n})_{m,n\in\mathbb{N}}$ is deferred Nörlund statistically Riemann integrable to a fuzzy number-valued function $\tilde{g}$ on $[a,b]\times[c,d]$. Then, according to Definition 3.1, we obtain the following result:

$$\lim_{m,n\to\infty}\frac{\left|\{(\xi,\eta):(\xi,\eta)\leqq P_{m,n}\quad\text{and}\quad p_{\xi,\eta}\mathcal{D}(\delta(\tilde{g}_{\xi,\eta};\dot{\mathcal{P}}),\tilde{g})\geqq\epsilon\}\right|}{P_{m,n}}=0.$$

Now, consider the following sets:

$$\mathcal{L}_{\epsilon}=\{(\xi,\eta):(\xi,\eta)\leqq P_{m,n}\quad\text{and}\quad p_{\xi,\eta}\mathcal{D}(\delta(\tilde{g}_{\xi,\eta};\dot{\mathcal{P}}),\tilde{g})\geqq\epsilon\},$$

and

$$\mathcal{L}_{\epsilon}^{c}=\{(\xi,\eta):(\xi,\eta)\leqq P_{m,n}\quad\text{and}\quad p_{\xi,\eta}\mathcal{D}(\delta(\tilde{g}_{\xi,\eta};\dot{\mathcal{P}}),\tilde{g})<\epsilon\}.$$

We then have

$$\begin{aligned}\mathcal{D}\Big(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\tilde{g}\Big)&=\frac{1}{P_{m,n}}\sum_{\lambda=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\mu=\phi_{m,n}+1}^{\varphi_{m,n}}p_{\varphi_{m,n}-(\lambda,\mu)}\,\mathcal{D}(\delta(\tilde{g}_{\lambda,\mu};\dot{\mathcal{P}}),\tilde{g})\\&\leqq\frac{1}{P_{m,n}}\sum_{\lambda=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\mu=\phi_{m,n}+1}^{\varphi_{m,n}}p_{\varphi_{m,n}-(\lambda,\mu)}\mathcal{D}\big(\delta(\tilde{g}_{\lambda,\mu};\dot{\mathcal{P}}),\tilde{g}\big)\\&\quad+\frac{1}{P_{m,n}}\sum_{\lambda=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\mu=\phi_{m,n}+1}^{\varphi_{m,n}}\mathcal{D}\big(p_{\varphi_{m,n}-(\lambda,\mu)}\tilde{g},\tilde{g}\big)\\&\leqq\frac{1}{P_{m,n}}\sum_{\substack{\lambda,\mu=\phi_{m,n}+1\\(\xi,\eta)\in\mathcal{L}_{\epsilon}}}^{\varphi_{m,n}}p_{\varphi_{m,n}-(\lambda,\mu)}\,\mathcal{D}\big(\delta(\tilde{g}_{\lambda,\mu};\dot{\mathcal{P}}),\tilde{g}\big)\\&\quad+\frac{1}{P_{m,n}}\sum_{\substack{\lambda,\mu=\phi_{m,n}+1\\(\xi,\eta)\in\mathcal{L}_{\epsilon}^{c}}}^{\varphi_{m,n}}p_{\varphi_{m,n}-(\lambda,\mu)}\,\mathcal{D}\big(\delta(\tilde{g}_{\lambda,\mu};\dot{\mathcal{P}}),\tilde{g}\big)\\&\quad+|\tilde{g}|\left(\frac{1}{P_{m,n}}\sum_{\lambda=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\mu=\phi_{m,n}+1}^{\varphi_{m,n}}p_{\varphi_{m,n}-(\lambda,\mu)}-1\right)\\&\leqq\frac{1}{P_{m,n}}|\mathcal{L}_{\epsilon}|+\frac{1}{P_{m,n}}|\mathcal{L}_{\epsilon}^{c}|=0.\end{aligned}$$

This implies that

$$\mathcal{D}(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\tilde{g})<\epsilon.$$

Therefore, the double sequence of fuzzy number-valued functions $(\tilde{g}_{m,n})$ is statistically deferred Nörlund Riemann summable to the fuzzy number-valued function $\tilde{g}$ on the rectangle $[a,b]\times[c,d]$. □

Since the converse statement does not always hold, the following example demonstrates that a double sequence of fuzzy number-valued functions that is statistically deferred Nörlund Riemann summable is not necessarily deferred Nörlund statistically Riemann integrable on $[a,b]\times[c,d]$.

Example 3.4. *Fix the rectangle $[a,b]\times[c,d]$ and let $\bar{0}$ and $\bar{1}$ denote the crisp fuzzy numbers at 0 and 1, respectively. Define a double sequence of fuzzy number-valued functions by*

$$\tilde{g}_{m,n}(x,y):=\begin{cases}\bar{1}, & \textit{if } m\in B \textit{ and } n\in B,\\ \bar{0}, & \textit{otherwise},\end{cases}\qquad (m,n\in\mathbb{N},\ (x,y)\in[a,b]\times[c,d]),$$

where $B\subset\mathbb{N}$ is chosen as follows. Partition $\mathbb{N}$ into contiguous blocks $I_r=\{N_r,\ldots,N_r+L_r-1\}$, $r\in\mathbb{N}$, with $N_1=1$, $N_{r+1}=N_r+L_r$, and lengths

$$L_{2r-1}=r,\qquad L_{2r}=r^4\quad(r\in\mathbb{N}).$$

Set $B := \bigcup_{r\geqq 1} I_{2r-1}$*, i.e.* B *is the union of the short (odd-indexed) blocks. Thus the complement* B^c *is the union of the long (even-indexed) blocks whose lengths dominate:*

$$\frac{|B\cap[1,K]|}{K}=0 \qquad (K\to\infty),$$

while each odd block I_{2r-1} *is fully occupied.*

Let $p_{i,j}=1$ *and, for each index* (m,n) *belonging to a block* $J\times J$ *(with* $J=I_s$ *for some* s*), define the deferred Nörlund window by*

$$\phi_{m,n}=\min J-1, \qquad \varphi_{m,n}=\max J,$$

so that

$$P_{m,n}=\sum_{\mu=\phi_{m,n}+1}^{\varphi_{m,n}}\sum_{\nu=\phi_{m,n}+1}^{\varphi_{m,n}}1=|J|^2=L_s^2,$$

and the mean

$$\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}))=\frac{1}{P_{m,n}}\sum_{\lambda,\kappa\in J}\delta(\tilde{g}_{\lambda,\kappa};\dot{\mathcal{P}}),$$

is the un-weighted average of the Riemann sums over the square index window $J\times J$*.*

(A) If $(m,n)\in I_{2r}\times I_{2r}$ *(a long block), then the averaging window is* $J=I_{2r}$ *and, by construction,* $\tilde{g}_{\lambda,\kappa}\equiv\bar{0}$ *for all* $(\lambda,\kappa)\in J\times J$*. Hence*

$$\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}}))=\bar{0}, \qquad \mathcal{D}\Big(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\bar{0}\Big)=0.$$

The set of indices (m,n) *that lie in such long blocks has natural density* 1 *in* $\mathbb{N}^2$ *because* $L_{2r}\gg L_{2r-1}$ *and*

$$\frac{\big|(I_{2r}\times I_{2r})\cap([1,K]\times[1,K])\big|}{K^2} \text{ dominates as } K\to\infty.$$

Therefore, for every $\epsilon>0$*, the set*

$$\Big\{(m,n):\ \mathcal{D}\big(\mathcal{N}(\delta(\tilde{g}_{m,n};\dot{\mathcal{P}})),\bar{0}\big)\geqq\epsilon\Big\},$$

has double natural density 0*. That is,* $(\tilde{g}_{m,n})$ *is* $\mathrm{stat}^{2D}_{\mathrm{DNFR}}$*-summable to* $\bar{0}$ *on* $[a,b]\times[c,d]$*.*

(B) Fix any tagged partition $\dot{\mathcal{P}}$ *of* $[a,b]\times[c,d]$*. For* $(\lambda,\kappa)\in I_{2r-1}\times I_{2r-1}$ *(a short block), we have* $\tilde{g}_{\lambda,\kappa}\equiv\bar{1}$*, hence*

$$\delta(\tilde{g}_{\lambda,\kappa};\dot{\mathcal{P}})=\bar{1} \quad \text{and} \quad \mathcal{D}\big(\delta(\tilde{g}_{\lambda,\kappa};\dot{\mathcal{P}}),\bar{0}\big)=1.$$

Now take any $(m,n)\in I_{2r-1}\times I_{2r-1}$ *and note that* $P_{m,n}=L_{2r-1}^2=r^2$*. Among the indices* (ξ,η) *with* $\xi,\eta\leqq P_{m,n}$*, there are at least* $\big|I_{2r-1}\times I_{2r-1}\big|=L_{2r-1}^2=r^2$ *pairs for which* $\mathcal{D}(\delta(\tilde{g}_{\xi,\eta};\dot{\mathcal{P}}),\bar{0})=1$*. Consequently,*

$$\frac{\Big|\{(\xi,\eta):\ \xi,\eta\leqq P_{m,n},\ \mathcal{D}(\delta(\tilde{g}_{\xi,\eta};\dot{\mathcal{P}}),\bar{0})\geqq\frac{1}{2}\}\Big|}{P_{m,n}} \ \geqq\ \frac{r^2}{r^2}\ =\ 1,$$

for all such (m,n)*. Hence the exceptional set in the definition of* $\mathrm{DNFR}^{2D}_{\mathrm{stat}}$ *has (limsup) density bounded away from* 0*, so* $\mathrm{DNFR}^{2D}_{\mathrm{stat}}$*-integrability to* $\bar{0}$ *fails.*

Combining (A) and (B), we have constructed a double sequence $(\tilde{g}_{m,n})$ *which is statistically deferred Nörlund Riemann summable to* $\bar{0}$ *but not deferred Nörlund statistically Riemann integrable on* $[a,b]\times[c,d]$*.*

We now present the geometrical and computational analysis of Example 3.4 given below. The Figure 1 shows the

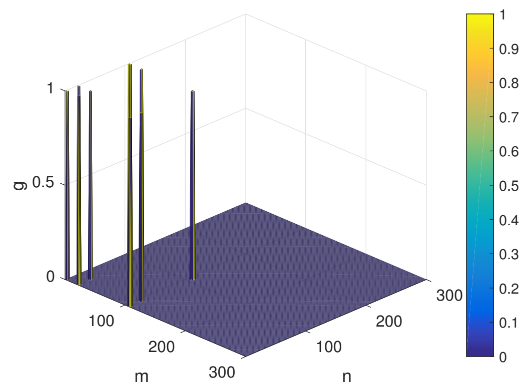

Figure 1: $\tilde{g}_{m,n} = \mathbf{1}_B(m)\mathbf{1}_B(n)$ (short blocks only)

raw values of the constructed double sequence

$$\tilde{g}_{m,n} = \begin{cases} 1, & m \in B,\, n \in B, \\ 0, & \textit{otherwise}, \end{cases} \qquad (m, n \in \mathbb{N})$$

from above mentioned Example 3.4.

The islands of height 1 *correspond exactly to regions where both indices lie in short (odd) blocks. These islands appear sparsely and shrink in relative size as* m, n *grow, since the long (even) blocks dominate. Thus, the Figure 1 illustrates the local presence of* $\bar{1}$ *values but also their asymptotic insignificance. The Figure 2 depicts the deferred Nörlund mean*

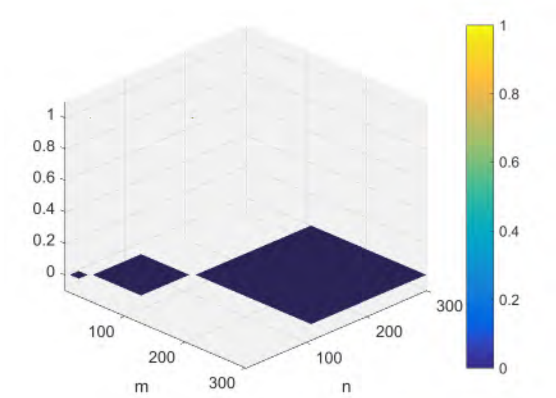

Figure 2: DNFR mean: 1 on short diagonal blocks, 0 on long diagonal blocks, not a number elsewhere

$$\mathcal{N}(\delta(\tilde{g}_{m,n}; \dot{\mathcal{P}})),$$

where the averaging window is the square $J \times J$ *determined by the block containing* (m, n).

If J *is a long (even) block, then* $\tilde{g}_{m,n} = \bar{0}$*, and the mean equals* $\bar{0}$*. If* J *is a short (odd) block, then* $\tilde{g}_{m,n} = \bar{1}$*, and the mean equals* $\bar{1}$*. Thus, the Figure 2 shows solid plateaus: height* 0 *on long diagonal blocks and height* 1 *on short diagonal blocks. This illustrates why* $(\tilde{g}_{m,n})$ *is DNFR-summable to* $\bar{0}$*: the long blocks dominate in density, even though the short blocks persist.*

The Figure 3 illustrates the normalized count

$$\frac{|(B \cap [1, m]) \times (B \cap [1, n])|}{mn},$$

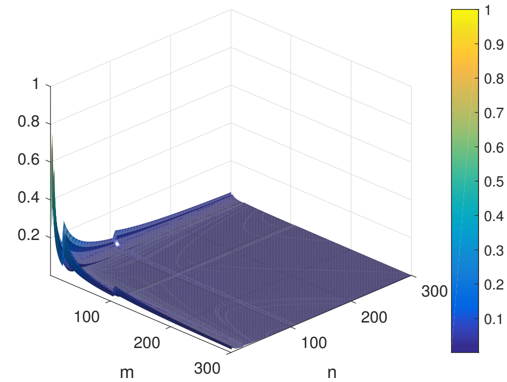

Figure 3: Cumulative double density of $B \times B$ (vanishes as $m, n \to \infty$)

the double natural density of the set $B \times B$. The surface decreases toward 0 as $m, n \to \infty$, confirming that the contribution of short blocks vanishes asymptotically. Hence, the exceptional set where $\tilde{g}_{m,n} = \bar{1}$ has double natural density zero.

4 Fuzzy Korovkin-type theorems

Let $\tilde{g} : [a, b] \times [c, d] \to \mathcal{E}$ be a fuzzy number-valued function. We define $\tilde{g}$ as continuous at a point $(t_0, s_0) \in [a, b] \times [c, d]$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $(t_i, s_j) \to (t_0, s_0)$ and

$$\mathcal{D}((t_i, s_j), (t_0, s_0)) < \delta,$$

it follows that

$$\mathcal{D}(\tilde{g}(t_i, s_j), \tilde{g}(t_0, s_0)) < \epsilon.$$

Furthermore, if $\tilde{g}$ satisfies this condition at every $(t, s) \in [a, b] \times [c, d]$, then it is said to be fuzzy continuous over the entire rectangle $[a, b] \times [c, d]$.

Let $\mathcal{C}_{\mathcal{L}}([a, b] \times [c, d])$ represent the set of all continuous fuzzy number-valued functions defined over the rectangle $[a, b] \times [c, d]$.

Now, suppose $\mathfrak{L} : \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d]) \to \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d])$ is a fuzzy linear operator. This means that for any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ and functions $\tilde{g}_1, \tilde{g}_2 \in \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d])$, the operator satisfies the following linearity condition:

$$\mathfrak{L}(\lambda_1 \odot \tilde{g}_1 \oplus \lambda_2 \odot \tilde{g}_2; x, y) = \lambda_1 \odot \mathfrak{L}(\tilde{g}_1; x, y) \oplus \lambda_2 \odot \mathfrak{L}(\tilde{g}_2; x, y),$$

where $\oplus$ denotes the addition of fuzzy numbers, while $\odot$ represents the scalar multiplication of a fuzzy number.

Furthermore, we say that $\mathfrak{L}$ is a positive fuzzy linear operator if it satisfies the condition

$$\tilde{g}_1(x, y) \preceq \tilde{g}_2(x, y) \implies \mathfrak{L}(\tilde{g}_1; x, y) \preceq \mathfrak{L}(\tilde{g}_2; x, y),$$

for all $\tilde{g}_1, \tilde{g}_2 \in \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d])$ and $(x, y) \in [a, b] \times [c, d]$, where $\preceq$ denotes the fuzzy ordering.

Theorem 4.1. *Let $(\phi_{i,j})$ and $(\varphi_{i,j}) \in \mathbb{Z}^{0+} \times \mathbb{Z}^{0+}$, and let $\mathfrak{L}_{i,j} : \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d]) \to \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d])$ $((i, j) \in \mathbb{N}^2)$ be a double sequence of fuzzy number-valued positive linear operators. Moreover, let $\{\mathfrak{L}^*_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ represent the corresponding double sequence of positive linear operators mapping from $\mathcal{C}([a, b] \times [c, d])$ into itself, with the relationship*

$$\{\mathfrak{L}_{i,j}(\tilde{g}; x, y)\}^{\Lambda}_{\pm} = \mathfrak{L}^*_{i,j}(\tilde{g}^{\Lambda}_{\pm}; x, y), \tag{2}$$

holding for all $(x, y) \in [a, b] \times [c, d]$, $\Lambda \in [0, 1]$, and $(i, j) \in \mathbb{N}^2$. Then, for $\tilde{g} \in \mathcal{C}_{\mathcal{L}}([a, b] \times [c, d])$, the following equivalence holds:

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}^* \left(\mathfrak{L}_{i,j}(\tilde{g}; x, y), \tilde{g}(x, y)\right) = 0, \tag{3}$$

if and only if

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\left(\mathfrak{L}^*_{i,j}(1;x,y),1\right)=0, \tag{4}$$

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\left(\mathfrak{L}^*_{i,j}(x;x,y),x\right)=0, \tag{5}$$

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\left(\mathfrak{L}^*_{i,j}(y;x,y),y\right)=0, \tag{6}$$

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\left(\mathfrak{L}^*_{i,j}(x^2+y^2;x,y),x^2+y^2\right)=0. \tag{7}$$

Proof. Let $\tilde{g}\in\mathcal{C}_{\mathcal{L}}([a,b]\times[c,d])$, $(x,y)\in[a,b]\times[c,d]$ and $\Lambda\in[0,1]$. Since $g^{\Lambda}_{\pm}(x,y)\in\mathcal{C}([a,b]\times[c,d])$, for each $\varepsilon>0$, there exists $\delta>0$ such that

$$\begin{aligned}&|\tilde{g}^{\Lambda}_{\pm}(u,v)-\tilde{g}^{\Lambda}_{\pm}(x,y)|<\varepsilon, \text{ whenever}\\ &\sqrt{(u-x)^2+(v-y)^2}<\delta \quad (\forall\ (x,y),(u,v)\in[a,b]\times[c,d]).\end{aligned} \tag{8}$$

Next, since $\tilde{g}$ is fuzzy bounded, we have

$$|\tilde{g}^{\Lambda}_{\pm}(x,y)|\leqq\mathcal{M}^{\Lambda}_{\pm} \quad ((x,y)\in[a,b]\times[c,d]).$$

Clearly, it follows that

$$|\tilde{g}^{\Lambda}_{\pm}(u,v)-\tilde{g}^{\Lambda}_{\pm}(x,y)|\leqq 2\mathcal{M}^{\Lambda}_{\pm} \quad ((x,y),(u,v)\in[a,b]\times[c,d]).$$

Let us choose

$$\theta\big((u,v),(x,y)\big)=(u-x)^2+(v-y)^2.$$

Then,

$$|\tilde{g}^{\Lambda}_{\pm}(u,v)-\tilde{g}^{\Lambda}_{\pm}(x,y)|<\varepsilon+\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\,\theta\big((u,v),(x,y)\big),$$

which yields

$$-\varepsilon-\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\theta\big((u,v),(x,y)\big)<\tilde{g}^{\Lambda}_{\pm}(u,v)-\tilde{g}^{\Lambda}_{\pm}(x,y)<\varepsilon+\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\theta\big((u,v),(x,y)\big). \tag{9}$$

Now the operator $\mathfrak{L}^*_{i,j}$ is linear and monotone. Thus, by applying the operator $\mathfrak{L}^*_{i,j}(1;x,y)$ in (9), we get

$$\begin{aligned}\mathfrak{L}^*_{i,j}(1;x,y)\left(-\varepsilon-\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\theta\big((u,v),(x,y)\big)\right)&<\mathfrak{L}^*_{i,j}\big(\tilde{g}^{\Lambda}_{\pm}(u,v)-\tilde{g}^{\Lambda}_{\pm}(x,y);\,x,y\big)\\&<\mathfrak{L}^*_{i,j}(1;x,y)\left(\varepsilon+\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\theta\big((u,v),(x,y)\big)\right).\end{aligned} \tag{10}$$

Since (x,y) is fixed, the term $\tilde{g}^{\Lambda}_{\pm}(x,y)$ is a constant number. We thus obtain

$$\begin{aligned}-\varepsilon\,\mathfrak{L}^*_{i,j}(1;x,y)-\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\mathfrak{L}^*_{i,j}(\theta;x,y)&<\mathfrak{L}^*_{i,j}(\tilde{g}^{\Lambda}_{\pm};x,y)-\tilde{g}^{\Lambda}_{\pm}(x,y)\,\mathfrak{L}^*_{i,j}(1;x,y)\\&<\varepsilon\,\mathfrak{L}^*_{i,j}(1;x,y)+\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\mathfrak{L}^*_{i,j}(\theta;x,y).\end{aligned} \tag{11}$$

Also, we know that

$$\mathfrak{L}^*_{i,j}(\tilde{g}^{\Lambda}_{\pm};x,y)-\tilde{g}^{\Lambda}_{\pm}(x,y)=\Big[\mathfrak{L}^*_{i,j}(\tilde{g}^{\Lambda}_{\pm};x,y)-\tilde{g}^{\Lambda}_{\pm}(x,y)\,\mathfrak{L}^*_{i,j}(1;x,y)\Big]+\tilde{g}^{\Lambda}_{\pm}(x,y)\Big[\mathfrak{L}^*_{i,j}(1;x,y)-1\Big]. \tag{12}$$

Using (11) and (12), we get

$$\mathfrak{L}^*_{i,j}(\tilde{g}^{\Lambda}_{\pm};x,y)-\tilde{g}^{\Lambda}_{\pm}(x,y)<\varepsilon\,\mathfrak{L}^*_{i,j}(1;x,y)+\frac{2\mathcal{M}^{\Lambda}_{\pm}}{\delta^2}\mathfrak{L}^*_{i,j}(\theta;x,y)+\tilde{g}^{\Lambda}_{\pm}(x,y)\,[\mathfrak{L}^*_{i,j}(1;x,y)-1]. \tag{13}$$

Now, we compute $\mathfrak{L}^*_{i,j}(\theta;x,y)$ as follows:

$$\begin{aligned}\mathfrak{L}^*_{i,j}(\theta;x,y) &= \mathfrak{L}^*_{i,j}\big((u-x)^2+(v-y)^2,\ x,y\big)\\ &= \mathfrak{L}^*_{i,j}(u^2,x,y) - 2x\,\mathfrak{L}^*_{i,j}(u,x,y) + x^2\mathfrak{L}^*_{i,j}(1,x,y)\\ &\quad + \mathfrak{L}^*_{i,j}(v^2,x,y) - 2y\,\mathfrak{L}^*_{i,j}(v,x,y) + y^2\mathfrak{L}^*_{i,j}(1,x,y)\\ &= [\mathfrak{L}^*_{i,j}(u^2,x,y)-x^2] - 2x[\mathfrak{L}^*_{i,j}(u,x,y)-x] + x^2[\mathfrak{L}^*_{i,j}(1,x,y)-1]\\ &\quad + [\mathfrak{L}^*_{i,j}(v^2,x,y)-y^2] - 2y[\mathfrak{L}^*_{i,j}(v,x,y)-y] + y^2[\mathfrak{L}^*_{i,j}(1,x,y)-1].\end{aligned}$$

Using (13), we then obtain

$$\begin{aligned}\mathfrak{L}^*_{i,j}(\tilde{g}^\Lambda_\pm;x,y) - \tilde{g}^\Lambda_\pm(x,y) &< \varepsilon\,\mathfrak{L}^*_{i,j}(1;x,y)\\ &\quad + \frac{2\mathcal{M}^\Lambda_\pm}{\delta^2}\Big\{[\mathfrak{L}^*_{i,j}(u^2+v^2,x,y)-x^2+y^2] - 2x[\mathfrak{L}^*_{i,j}(u,x,y)-x]\\ &\quad + x^2[\mathfrak{L}^*_{i,j}(1,x,y)-1] - 2y[\mathfrak{L}^*_{i,j}(v,x,y)-y] + y^2[\mathfrak{L}^*_{i,j}(1,x,y)-1]\Big\}\\ &\quad + \tilde{g}^\Lambda_\pm(x,y)[\mathfrak{L}^*_{i,j}(1;x,y)-1].\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we can write

$$\begin{aligned}\big|\mathfrak{L}^*_{i,j}(\tilde{g}^\Lambda_\pm;x,y) - \tilde{g}^\Lambda_\pm(x,y)\big| &\leqq \varepsilon + \Big(\varepsilon + \frac{2\mathcal{M}^\Lambda_\pm h^2}{\delta^2} + \mathcal{M}^\Lambda_\pm\Big)\big|\mathfrak{L}^*_{i,j}(1;(x,y))-1\big|\\ &\quad + \frac{4\mathcal{M}^\Lambda_\pm h}{\delta^2}\Big(\big|\mathfrak{L}^*_{i,j}(u;(x,y))-x\big| + \big|\mathfrak{L}^*_{i,j}(v;(x,y))-y\big|\Big)\\ &\quad + \frac{2\mathcal{M}^\Lambda_\pm}{\delta^2}\Big(\big|\mathfrak{L}^*_{i,j}(u^2+v^2;(x,y))-x^2+y^2\big|\Big),\end{aligned}$$

where $h=\max\{|a|,|b|,|c|\}$.

Consequently, setting

$$\mathcal{H}^\Lambda_\pm(\varepsilon) = \max\bigg(\varepsilon + \frac{2\mathcal{M}^\Lambda_\pm h^2}{\delta^2} + \mathcal{M}^\Lambda_\pm,\ \frac{4\mathcal{M}^\Lambda_\pm h}{\delta^2},\ \frac{2\mathcal{M}^\Lambda_\pm}{\delta^2}\bigg),$$

we obtain the compact bound

$$\begin{aligned}\big|\mathfrak{L}^*_{i,j}(\tilde{g}^\Lambda_\pm;x,y) - \tilde{g}^\Lambda_\pm(x,y)\big| &\leqq \varepsilon + \mathcal{H}^\Lambda_\pm(\varepsilon)\,\Big(\big|\mathfrak{L}^*_{i,j}(1;(x,y))-1\big|\\ &\quad + \big|\mathfrak{L}^*_{i,j}(u;(x,y))-x\big| + \big|\mathfrak{L}^*_{i,j}(v;(x,y))-y\big|\\ &\quad + \big|\mathfrak{L}^*_{i,j}(u^2+v^2;(x,y))-(x^2+y^2)\big|.\end{aligned} \tag{14}$$

Now it follows from the identity analogous to (2) that

$$\begin{aligned}\mathcal{D}^*(\mathfrak{L}_{i,j}(\tilde{g}),\tilde{g}) &= \sup_{(x,y)\in[a,b]\times[c,d]} \mathcal{D}\big(\mathfrak{L}^*_{i,j}(\tilde{g};(x,y)),\tilde{g}(x,y)\big)\\ &= \sup_{(x,y)\in[a,b]\times[c,d]}\ \sup_{\Lambda\in[0,1]} \max\Big\{\big|\mathfrak{L}^*_{i,j}(\tilde{g}^\Lambda_-;(x,y)) - \tilde{g}^\Lambda_-(x,y)\big|,\ \big|\mathfrak{L}^*_{i,j}(\tilde{g}^\Lambda_+;(x,y)) - \tilde{g}^\Lambda_+(x,y)\big|\Big\}.\end{aligned}$$

Combining this with (14) and taking suprema over (x,y) and Λ yields an estimate of $\mathcal{D}^*(\mathfrak{L}_{i,j}(\tilde{g}),\tilde{g})$ in terms of the sup-norm errors of the operators on the test functions $1,x,y,x^2+y^2$. Considering (14) with the last equality, one can easily write

$$\begin{aligned}\mathcal{D}^*(\mathfrak{L}_{i,j}(\tilde{g}),\tilde{g}) \leqq \sup_{(x,y)\in[a,b]\times[c,d]} \varepsilon + \mathcal{M}(\varepsilon)\Bigg(&\sup_{(x,y)\in[a,b]\times[c,d]} \big|\mathfrak{L}^*_{i,j}(1;(x,y))-1\big|\\ &+ \sup_{(x,y)\in[a,b]\times[c,d]} \big|\mathfrak{L}^*_{i,j}(u;(x,y))-x\big| + \sup_{(x,y)\in[a,b]\times[c,d]} \big|\mathfrak{L}^*_{i,j}(v;(x,y))-y\big|\\ &+ \sup_{(x,y)\in[a,b]\times[c,d]} \big|\mathfrak{L}^*_{i,j}(u^2+v^2;(x,y))-x^2+y^2\big|\Bigg),\end{aligned}$$

where

$$\mathcal{M}(\varepsilon) = \sup_{\Lambda \in [0,1]} \max\left\{\mathcal{M}_{-}^{\Lambda}(\varepsilon),\ \mathcal{M}_{+}^{\Lambda}(\varepsilon)\right\}.$$

Therefore,

$$\begin{aligned} p_{\varphi_{i,j}-\lambda}\mathcal{D}^*(\mathfrak{L}_{i,j}(\tilde{g}),\tilde{g}) \leqq p_{\varphi_{i,j}-\lambda} \sup_{(x,y)\in[a,b]\times[c,d]} \varepsilon + \mathcal{H}(\varepsilon)\Bigg(& p_{\varphi_{i,j}-\lambda} \sup_{(x,y)\in[a,b]\times[c,d]} \left|\mathfrak{L}_{i,j}^*(1;(x,y)) - 1\right| \\ & + p_{\varphi_{i,j}-\lambda} \sup_{(x,y)\in[a,b]\times[c,d]} \left|\mathfrak{L}_{i,j}^*(u;(x,y)) - x\right| \\ & + p_{\varphi_{i,j}-\lambda} \sup_{(x,y)\in[a,b]\times[c,d]} \left|\mathfrak{L}_{i,j}^*(v;(x,y)) - y\right| \\ & + p_{\varphi_{i,j}-\lambda} \sup_{(x,y)\in[a,b]\times[c,d]} \left|\mathfrak{L}_{i,j}^*(u^2+v^2;(x,y)) - x^2 + y^2\right| \Bigg). \end{aligned} \tag{15}$$

Next, for given $\kappa > 0$, choose $\varepsilon > 0$ such that

$$p_{\varphi_{i,j}-\lambda} \sup_{(x,y)\in[a,b]\times[c,d]} \varepsilon < \kappa.$$

Then, we can write

$$\Theta_{i,j}(x,y;\varepsilon) = \left|\left\{(i,j) : (i,j) \leqq (P_i,Q_j) \quad \text{and} \quad p_{\varphi_{i,j}-\lambda}\mathcal{D}^*\left(\mathfrak{L}_{i,j}(\tilde{g}),\tilde{g}\right) \geqq \varepsilon'\right\}\right|,$$

and

$$\Theta_{r,i,j}(x,y;\varepsilon) = \left|\left\{(i,j) : (i,j) \leqq (P_i,Q_j) \quad \text{and} \quad p_{\varphi_{i,j}-\lambda}\mathcal{D}\left(\mathfrak{L}_{i,j}^*(\tilde{g}_r(x,y)),\tilde{g}_r(x,y)\right) \geqq \frac{\varepsilon'-\varepsilon}{3\mathcal{H}_{\pm}^{\Lambda}}\right\}\right|,$$

where $r = 0,1,2,3$.

From (15), it follows that

$$\Theta_{i,j}(x,y;\varepsilon) \leqq \sum_{r=0}^{3} \Theta_{r,i,j}(x,y;\varepsilon).$$

Thus, we fairly have

$$\frac{\|\Theta_{i,j}(x,y;\varepsilon)\|}{P_iQ_j} \leqq \sum_{r=0}^{3} \frac{\|\Theta_{r,i,j}(x,y;\varepsilon)\|}{P_iQ_j}. \tag{16}$$

Hence, based on Definition 3.1 (adapted to the two-variable case) and the assumptions outlined for the implications in equations (4)-(7), the right-hand side of (16) approaches zero as $i,j \to \infty$. Therefore, we obtain the following result

$$\lim_{i,j\to\infty} \frac{\|\Theta_{i,j}(x,y;\varepsilon)\|}{P_iQ_j} = 0 \quad (\varepsilon > 0).$$

As a result, the implication in equation (3) holds true in the two-variable case. □

Theorem 4.2. *Let $(\phi_{i,j})$ and $(\varphi_{i,j}) \in \mathbb{Z}^{0+} \times \mathbb{Z}^{0+}$ be double sequences, and let*

$$\mathfrak{L}_{i,j} : \mathcal{C}_{\mathcal{L}}([a,b]\times[c,d]) \longrightarrow \mathcal{C}_{\mathcal{L}}([a,b]\times[c,d]) \qquad ((i,j)\in\mathbb{N}^2),$$

be a double sequence of fuzzy number-valued positive linear operators. Also, let $\{\mathfrak{L}_{i,j}^*\}_{(i,j)\in\mathbb{N}^2}$ be the corresponding double sequence of positive linear operators from $\mathcal{C}([a,b]\times[c,d])$ into itself satisfying the compatibility condition

$$\{\mathfrak{L}_{i,j}(\tilde{g};x,y)\}_{\pm}^{\Lambda} = \mathfrak{L}_{i,j}^*(\tilde{g}_{\pm}^{\Lambda};x,y), \tag{17}$$

for all $(x,y) \in [a,b]\times[c,d]$, $\Lambda \in [0,1]$, and $(i,j) \in \mathbb{N}^2$. Then for every $\tilde{g} \in \mathcal{C}_{\mathcal{L}}([a,b]\times[c,d])$ the following equivalence holds:

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}^*\big(\mathfrak{L}_{i,j}(\tilde{g};x,y), \tilde{g}(x,y)\big) = 0, \tag{18}$$

if and only if

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{i,j}(1;x,y), 1\big) = 0, \tag{19}$$

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{i,j}(x;x,y), x\big) = 0, \tag{20}$$

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{i,j}(y;x,y), y\big) = 0, \tag{21}$$

$$\mathrm{DNFR}^{2D}_{\mathrm{stat}} \lim_{i,j\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{i,j}(x^2+y^2;x,y), x^2+y^2\big) = 0. \tag{22}$$

Proof. In a similar manner to the proof of Theorem 4.1, Theorem 4.2 can be established. Therefore, we omit the detailed steps of the proof in the case of double sequences. □

Based on Theorem 4.2, we provide an example of a double sequence of positive linear operators (25) that does not align with the statistical version of the deferred Nörlund Riemann integrable double sequence of fuzzy number-valued functions (as outlined in Theorem 4.1), but instead works effectively with Theorem 4.2. From this, we conclude that Theorem 4.2 represents a significant extension of the statistical Nörlund Riemann integrable double sequence of fuzzy number-valued functions presented in Theorem 4.1.

We now introduce the bivariate operator defined by

$$\varpi_1(1+\varpi_1 D_1) + \varpi_2(1+\varpi_2 D_2) \qquad \left(D_1 = \frac{\partial}{\partial\varpi_1},\ D_2 = \frac{\partial}{\partial\varpi_2}\right), \tag{23}$$

which can be viewed as the natural extension of the univariate operator

$$\varpi(1+\varpi D) \qquad \left(D = \frac{d}{d\varpi}\right),$$

that was previously employed by Al-Salam [1] and more recently by Viskov and Srivastava [30]. In the two-variable case, this operator acts on functions $f(\varpi_1,\varpi_2)$ in such a way that it preserves the structural properties of the single-variable operator while introducing interactions between the two coordinate directions through the differential operators D_1 and D_2. This provides a more general framework for studying double sequences of positive linear operators in the fuzzy number-valued setting.

Example 4.3. *Let $\tilde{g} \in \mathcal{C}([0,1]^2)$ and $k,l \in \mathbb{N}$. The bivariate Bernstein polynomial [4] is defined as:*

$$\mathfrak{B}_{k,l}(\tilde{g};u,v) = \sum_{\varrho=0}^{k}\sum_{\sigma=0}^{l} \tilde{g}\Big(\frac{\varrho}{k},\frac{\sigma}{l}\Big)\binom{k}{\varrho}\binom{l}{\sigma} u^{\varrho}(1-u)^{k-\varrho} v^{\sigma}(1-v)^{l-\sigma}, \quad (u,v\in[0,1]). \tag{24}$$

Based on the double sequence $(\tilde{g}_{\varrho,\sigma})$ introduced in Example 3.4, we define the following positive linear operator:

$$\mathfrak{L}^*_{\varrho,\sigma}(\tilde{g};u,v) = [1+\tilde{g}_{\varrho,\sigma}]\, uv\,(1+uD_u)(1+vD_v)\,\mathfrak{B}_{\varrho,\sigma}(\tilde{g};u,v), \tag{25}$$

where $D_u = \frac{\partial}{\partial u}$ and $D_v = \frac{\partial}{\partial v}$.

1. *Constant function 1:*

$$\mathfrak{L}^*_{\varrho,\sigma}(1;u,v) = [1+\tilde{g}_{\varrho,\sigma}]\, uv. \tag{26}$$

2. *Linear functions u and v:*

$$\mathfrak{L}^*_{\varrho,\sigma}(u;u,v) = [1+\tilde{g}_{\varrho,\sigma}]\, uv\,(1+u), \tag{27}$$

$$\mathfrak{L}^*_{\varrho,\sigma}(v;u,v) = [1+\tilde{g}_{\varrho,\sigma}]\, uv\,(1+v). \tag{28}$$

3. *Quadratic function $u^2 + v^2$:*

$$\mathfrak{L}^*_{\varrho,\sigma}(u^2+v^2;u,v) = [1+\tilde{g}_{\varrho,\sigma}]\Big[u^2\Big(2-\frac{3u}{\varrho}\Big)v + u\,v^2\Big(2-\frac{3v}{\sigma}\Big)\Big]. \tag{29}$$

Consequently, we have the bivariate analogues of the statistical limits:

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}(\mathfrak{L}^*_{\varrho,\sigma}(1;u,v),1) = 0, \tag{30}$$

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}(\mathfrak{L}^*_{\varrho,\sigma}(u;u,v),u) = 0, \tag{31}$$

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}(\mathfrak{L}^*_{\varrho,\sigma}(v;u,v),v) = 0, \tag{32}$$

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}(\mathfrak{L}^*_{\varrho,\sigma}(u^2+v^2;u,v),u^2+v^2) = 0. \tag{33}$$

That is, the sequence $\mathfrak{L}^*_{\varrho,\sigma}(\tilde{g};u,v)$ satisfies the bivariate conditions analogous to (19)–(22). Therefore, by the bivariate extension of Theorem 4.2, we have

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}^*(\mathfrak{L}_{\varrho,\sigma}(\tilde{g};u,v),\tilde{g}(u,v)) = 0. \tag{34}$$

The double sequence $(\tilde{g}_{\varrho,\sigma})$ of fuzzy number-valued functions is statistically deferred Nörlund Riemann summable but not deferred Nörlund statistically Riemann integrable. Thus, the bivariate operators (25) defined above fulfill the conditions of Theorem 4.2 but do not satisfy the criteria for the statistical version of the deferred Nörlund Riemann integrable sequence of fuzzy number-valued functions as in Theorem 4.1.

We now describe the convergence role of each bivariate positive linear operator associated with the test functions $\{1,u,v,u^2+v^2\}$ in the fuzzy DNFR framework. Throughout, $\mathcal{D}$ (resp. $\mathcal{D}^*$) denotes the fuzzy metric used for point-wise (resp. operator-level) comparison, and stat-DNFR abbreviates statistically deferred Nörlund–Riemann summability.

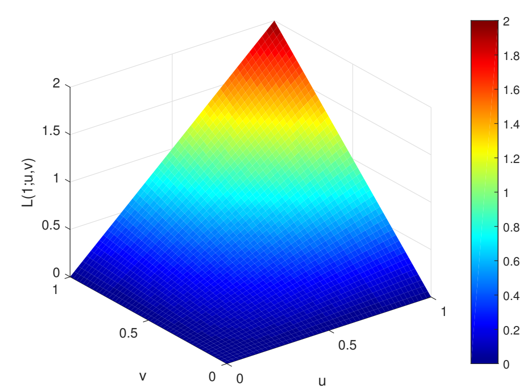

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Figure 4: Operator on Constant Function $\mathfrak{L}^*_{\varrho,\sigma}(1;u,v)$

The Figure 4 convergence on the constant function verifies the operator's normalization and mass preservation. In the Korovkin scheme, reproducing constants is indispensable since it anchors the approximation scale in the fuzzy metric. Under the assumed stat-DNFR behavior of $(\tilde{g}_{\varrho,\sigma})$, we have

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{\varrho,\sigma}(1;u,v),1\big) = 0,$$

which ensures that the operator family does not introduce systematic bias. Boundary effects such as $uv = 0$ on the axes are compensated statistically via the deferred Nörlund averaging.

The Figure 5 convergence on u guarantees first-order fidelity along the u-direction. Together with constants, this forms a necessary condition for barycentric consistency in the fuzzy sense. The stat-DNFR limit

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{\varrho,\sigma}(u;u,v),u\big) = 0,$$

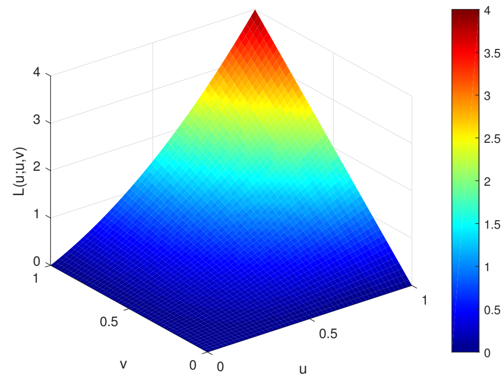

Figure 5: Convergence Behavior of the Operator $\mathfrak{L}^*_{\varrho,\sigma}(u;u,v)$

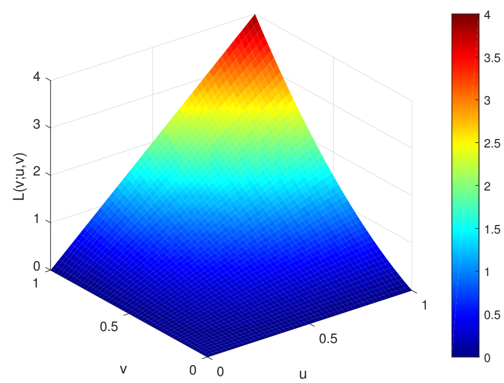

Figure 6: Convergence Behavior of the Operator $\mathfrak{L}^*_{\varrho,\sigma}(v;u,v)$

ensures that any perturbations due to $\tilde{g}_{\varrho,\sigma}$ vanish statistically, yielding asymptotic linear reproduction in u.

Analogous to the u-case, the Figure 6 convergence on v establishes first-order fidelity along the v-direction. This, with constants and u, forms the minimal set controlling linear behavior in both coordinates. The stat-DNFR convergence

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{\varrho,\sigma}(v;u,v), v\big) = 0,$$

ensures that no persistent directional bias remains under fuzzy DNFR summation.

The Figure 7 quadratic test governs second-order fidelity and measures the operator's responsiveness to curvature. The vanishing terms $\frac{u}{\varrho}$ and $\frac{v}{\sigma}$ represent the variance corrections typical of Bernstein-type operators. As $\varrho, \sigma \to \infty$, these terms disappear statistically, yielding

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{\varrho,\sigma}(u^2+v^2;u,v), u^2+v^2\big) = 0.$$

Thus, curvature fidelity is established in the fuzzy metric.

Since the operators are positive and linear, and the stat-DNFR limits hold for the bivariate Korovkin set $\{1, u, v, u^2 + v^2\}$, the bivariate Korovkin theorem in the fuzzy DNFR setting implies

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}^*(\mathfrak{L}_{\varrho,\sigma}(\tilde{g};u,v), \tilde{g}(u,v)) = 0, \quad \text{for all } \tilde{g} \in \mathcal{C}([0,1]^2).$$

Thus, constants ensure normalization, linears guarantee first-order fidelity in u and v, quadratic terms control second-order (curvature) fidelity. Together, these confirm the fuzzy stat-DNFR convergence of the operator sequence on $\mathcal{C}([0,1]^2)$.

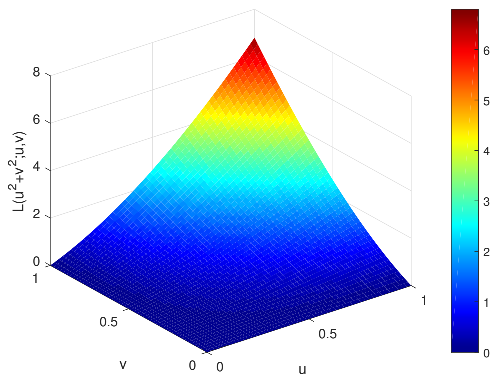

Figure 7: Convergence Behavior of the Operator $\mathfrak{L}^*_{\varrho,\sigma}(u^2+v^2;u,v)$

5 Conclusion

In this final section, we emphasize the practical significance and theoretical advantages of Theorem 4.2 in comparison with Theorem 4.1, as well as its improvements upon classical fuzzy Korovkin-type approximation results in the bivariate setting. While Theorem 4.1 establishes convergence criteria based on statistical deferred Nörlund Riemann integrability, Theorem 4.2 introduces statistical deferred Nörlund Riemann summability as a broader and more flexible framework. This generalization allows for a wider class of fuzzy-number-valued functions of two variables, represented through double sequences, to be approximated effectively, even in situations where classical summability methods fail to ensure convergence.

Consider the double sequence $(\tilde{g}_{\varrho,\sigma})_{\varrho,\sigma\in\mathbb{N}}$ of functions mentioned in Example 3.4. Moreover, assume that $(\tilde{g}_{\varrho,\sigma})$ is statistically deferred Nörlund Riemann summable, so that we have the following limit

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}\big(\delta(\tilde{g}_{\varrho,\sigma};\dot{\mathcal{P}}),\bar{0}\big) \quad \text{on} \quad [0,1]\times[0,1].$$

Then, we obtain

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}\big(\mathfrak{L}^*_{\varrho,\sigma}(\tilde{g}_\nu;u,v),\,\tilde{g}_\nu(u,v)\big)=0, \quad (\nu=0,1,2,3), \tag{35}$$

where the bivariate test functions are given by

$$\tilde{g}_0(u,v)=1, \quad \tilde{g}_1(u,v)=u, \quad \tilde{g}_2(u,v)=v, \quad \tilde{g}_3(u,v)=u^2+v^2.$$

Thus, by Theorem 4.2, we immediately deduce that

$$\mathrm{stat}^{2D}_{\mathrm{DNFR}} \lim_{\varrho,\sigma\to\infty} \mathcal{D}^*\big(\mathfrak{L}_{\varrho,\sigma}(\tilde{g};u,v),\,\tilde{g}(u,v)\big)=0. \tag{36}$$

The double sequence $(\tilde{g}_{\varrho,\sigma})$ of fuzzy number-valued functions is statistically deferred Nörlund Riemann summable, but it is neither deferred Nörlund statistically Riemann integrable nor classically Riemann integrable. Consequently, our fuzzy Korovkin-type approximation result in Theorem 4.2 holds for the bivariate operators defined in equation (25), while both the classical and statistical versions of the deferred Nörlund Riemann integrable double sequence of fuzzy number-valued functions do not apply to these operators.

From this, we conclude that Theorem 4.2 serves as a significant extension of both Theorem 4.1 and the classical Korovkin-type approximation theorem [14] to the setting of fuzzy bivariate approximation with double sequences.

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