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$\mathbb{S}$-spectral topology in MV-algebras

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Abstract

In this paper, we introduce new classes of ideals, called S-prime ideals and S-maximal ideals, based on an $\wedge$-closed system S. The connection between these ideals and classical prime ideals is examined, and it is shown that every S-prime ideal constitutes a specific type of prime ideal. Moreover, it is proven that any proper ideal disjoint from S is contained in an S-prime ideal.
The behavior of S-prime ideals is further analyzed in the setting of quotient MV-algebras, and their properties under isomorphisms are investigated. In the final part of the study, the complete $\wedge$-closed system $\mathbb{S}$ is introduced, and a new topology, called the $\mathbb{S}$-spectral topology, is defined using $\mathbb{S}$ and the family of $\mathbb{S}$-ideals. Several topological properties of this space, including the Hausdorff, T_0, and T_1 separation axioms, are discussed.

Keywords: MV-algebra, $\wedge$-closed system, S-prime ideal, $\mathbb{S}$-spectral topology.

1 Introduction

MV-algebras were introduced by C. C. Chang in 1958 [4, 5] as the algebraic counterpart of the Łukasiewicz infinite-valued propositional logic. To keep the brevity of the paper, we direct the reader to [4, 5, 6, 21] for further results on MV-algebras. In particular, significant emphasis has been placed on the ideal theory of MV-algebras [6, 7, 12, 13]. Chang introduced the concept of a prime ideal in MV-algebras.
Prime ideals play a fundamental role in MV-algebras. It has been shown that every proper ideal can be expressed as an intersection of prime ideals [21].
Bedrood et al. considered a generalization of prime ideals in MV-algebras. Specifically, they studied the property of being prime with respect to another ideal and explored various conditions for these ideals [2]. In another paper, Bedrood et al. introduced ideals based on minimal prime ideals and called them Z°-ideals. They examined these ideals in various types of MV-algebras and provided a new classification of MV-algebras.[1]
A. H. Movahed et al. introduced 2-absorbing ideals, a type of prime ideal in MV-algebras. They explored the relationships between 2-absorbing ideals and other types of ideals, such as prime ideals, obstinate ideals, primary ideals, maximal ideals, and others [17]. Also, A. H. Movahed et al. introduced new types of ideals based on maximal ideals, and through their investigation, they presented properties concerning both maximal and prime ideals [16, 18].
As mentioned, over the years, a lot of research has been conducted on specific types of prime ideals, which emphasizes their importance. In fact, they established a connection between algebra and topology using the Zariski topology.
Belluce et al. introduced the prime spectrum of an MV-algebra and studied the topological space on $Spec(A)$ in [3]. They defined the Zariski topological space for MV-algebras, and the space of prime ideals with Zariski topology gives a lot of information on the MV-algebra.
F. Forouzesh et al. introduced the spectral topology and quasi-spectral topology of proper prime A-ideals in MV-modules and proved several properties of them [9]. Additionally, F. Forouzesh et al. introduced the inverse topology

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on the set of all minimal prime ideals in MV-algebras[10].
One interesting property of prime ideals in MV-algebras is that the complement of every prime ideal is a $\wedge$-closed system [21]. This feature establishes an important connection between prime ideals and $\wedge$-closed systems in MV-algebras.
Based on this relationship between prime ideals and $\wedge$-closed systems, we have decided to introduce new ideals that are defined based on these properties. In this paper, we define S-prime ideals, which have a close relationship with $\wedge$-closed systems. These ideals possess unique properties that distinguish them from other ideals, and their study can significantly contribute to a better understanding of the structure of MV-algebras. From now on, S denotes an $\wedge$-closed system and $\mathbb{S}$ denotes a complete $\wedge$-closed system of A. (A complete $\wedge$-closed system is an $\wedge$-closed system in which the infimum of all elements exists in it.) A proper ideal I that does not intersect with S is called an S-prime ideal, if a and b are two elements of the MV-algebra A and $a \wedge b \in I$, then there exists an element $s \in S$ such that either $a \wedge s \in I$ or $b \wedge s \in I$. The set of all S-prime ideals is denoted by $Spec_S(A)$. This definition shows that S-prime ideals have distinct properties based on their relationship with $\wedge$-closed systems and prime ideals.
Any prime ideal that does not intersect with the $\wedge$-closed system S must be an S-prime ideal. However, we provide an example of a non-prime ideal in MV-algebras for which there exists an $\wedge$-closed system S that turns it into an S-prime ideal. In addition, we define S-maximal ideals in MV-algebras and provide a necessary and sufficient condition for an ideal to be an S-maximal ideal of A. In the following, it is proven that if the proper ideal I does not intersect with the complete $\wedge$-closed system $\mathbb{S}$, then I is an $\mathbb{S}$-prime ideal if and only if there exists $s \in \mathbb{S}$ such that $(I : s)$ is a prime ideal. It is demonstrated that every proper ideal I is equivalent to the intersection of all S-prime ideals that contain $(s] \wedge I$ for every $s \in S$.
We show that if an element a of an MV-algebra A does not belong to the ideal I, then there exists an S-prime ideal P containing I, and for some $s \in S$, the element $s \wedge a$ does not belong to P. Based on the open and closed sets of the spectral topology, we define the following sets for each subset X and each element a:

$$V_S(X) = \{P \in Spec_S(A) : (s] \cap X \subseteq P, \text{ for some } s \in S\} \text{ and } U_S(X) = Spec_S(A) \setminus V_S(X).$$

$$V_S(a) = \{P \in Spec_S(A) : s \wedge a \in P, \text{ for some } s \in S\} \text{ and } U_S(a) = Spec_S(A) \setminus V_S(a).$$

It is shown that the set $\tau = \{V_{\mathbb{S}}(I) : I \in Id(A)\}$ satisfies all the conditions of closed sets in the topology τ on $Spec_{\mathbb{S}}(A)$, and this topology is called the $\mathbb{S}$-spectral topology. Moreover, we prove that the set $\Gamma = \{U_{\mathbb{S}}(a) : a \in A\}$ is a basis for the $\mathbb{S}$-spectral topology on $Spec_{\mathbb{S}}(A)$. Finally, the conditions under which this topological space is compact, connected, Hausdorff, T_0, and T_1 have been studied.

2 Preliminaries

In this section, we recollect some definitions and results, which will be used in what follows.

Definition 2.1. [4] *An MV-algebra is a structure $(A, \oplus, ^*, 0)$, where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $x, y \in A$:*
$(MV1)$ $(A, \oplus, 0)$ *is an abelian monoid;*
$(MV2)$ $(x^*)^* = x$;
$(MV3)$ $0^* \oplus x = 0^*$;
$(MV4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Note that the abelian property of $\oplus$ an MV-algebra follows from the remaining properties (Kolařík); see [15].
From now on, A is an MV-algebra.
We denote $1 = 0^*$, and the operation $\odot$ is defined as follows:

$$x \odot y = (x^* \oplus y^*)^*.$$

Moreover, we can define an order $\leq$ in such a way, for any two elements $x, y \in A$, $x \leq y$ if and only if $x^* \oplus y = 1$ if and only if $x \odot y^* = 0$. This order is called *natural order* and makes A into a bounded distributive lattice such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

We will say that A is an MV-chain if it is linearly ordered.
An element a of A is called a Boolean element, if $a \oplus a = a$ and the collection of all Boolean elements is displayed with $B(A)$ and it is a Boolean algebra.

Definition 2.2. *Let* A *and* B *be two MV-algebras. A function $h : A \to B$ is called an MV-algebra homomorphism if it satisfies the following conditions:*

(h1) *For every $x, y \in A$, $h(x \oplus y) = h(x) \oplus h(y)$;*

(h2) *For every $x \in A$, $h(x^*) = (h(x))^*$.*

Lemma 2.3. [6] *The following conditions hold for all $x, y, z \in A$:*
(1) *If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z, x \wedge z \leqslant y \wedge z$.*
(2) *$x, y \leq x \oplus y$, $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$, and $x^n = x \odot x \odot \cdots \odot x \leq x$.*
(3) *If $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$.*
(4) *$x \wedge (y \oplus z) \leqslant (x \wedge y) \oplus (x \wedge z), x \wedge (x_1 \oplus \cdots \oplus x_n) \leqslant (x \wedge x_1) \oplus \cdots \oplus (x \wedge x_n)$, for all $x_1, \ldots, x_n \in A$; in particular $(mx) \wedge (ny) \leqslant mn(x \wedge y)$, for every $m, n \geq 0$.*
(5) *$(x \odot y^*) \wedge (y \odot x^*) = 0$.*

Theorem 2.4. [21] *For every element e in A; the following conditions are equivalent:*
(i) *$e \in B(A)$;*
(ii) *$e \vee e^* = 1$;*
(iii) *$e \wedge e^* = 0$;*
(iv) *$e \odot e = e$.*

Theorem 2.5. [21] *If $e \in B(A)$ and $x \in A$, then $e \oplus x = e \vee x$ and $e \odot x = e \wedge x$.*

Definition 2.6. [4] *An ideal of A is a nonempty subset I of A satisfying the following conditions:*
(I1) If $x \in I$, $y \in A$, and $y \leq x$, then $y \in I$,
(I2) If $x, y \in I$, then $x \oplus y \in I$.
If $I \neq A$, then I is a proper ideal of A.
We denote by $Id(A)$ the set of all ideals of A.

Remark 2.7. *Let $X \subseteq A$. We denote by $(X]$ the ideal generated by X, that is, the intersection of all ideals I such that $X \subseteq I$.*

Lemma 2.8. [6] (1) *If $X \subseteq A$ is a nonempty set, then*
$(X] = \{a \in A \mid a \leqslant x_1 \oplus x_2 \oplus \cdots \oplus x_n,$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X\}$. In particular, $(a] = \{x \in A \mid x \leqslant na,$ for some $n \in \mathbb{N}\}$, for every $a \in A$.
(2) *If I and J are two ideals of A, then*

$$I \wedge J = I \cap J,\ I \vee J = (I \cup J] = \{a \in A \mid a \leq x_1 \oplus x_2 \text{ for some } x_1 \in I \text{ and } x_2 \in J\}.$$

(3) *If $a, b \in A$, then $(a] \cap (b] = (a \wedge b]$ and $(a] \vee (b] = (a \oplus b]$.*
(4) *If J is an ideal and $\{I_\alpha\}_{\alpha \in \Gamma}$ is a family of ideals, then $J \wedge (\bigvee_{\alpha \in \Gamma} I_\alpha) = \bigvee_{\alpha \in \Gamma} (J \wedge I_\alpha)$.*

Definition 2.9. [19] *A proper ideal P of A is called a prime ideal of A if for all $x, y \in A$, either $x \odot y^* \in P$ or $y \odot x^* \in P$.*
We denote by $Spec(A)$ the set of all prime ideals of an MV-algebra A.

Definition 2.10. [4] *A proper ideal M of A is called a maximal ideal if there exists no other proper ideal J of A so that $M \subset J$.*
We denote by $Max(A)$ the set of all maximal ideals of A.

Obviously, every maximal ideal is a prime ideal.

Proposition 2.11. [19] *Let P be a proper ideal of A. Then the following conditions are equivalent:*
(1) *$P \in Spec(A)$.*
(2) *A/P is totally ordered.*
(3) *If $I \cap J \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$, for all $I, J \in Id(A)$.*
(4) *If $x \wedge y \in P$, then either $x \in P$ or $y \in P$.*

A nonempty subset S of an MV-algebra is called an $\wedge$-*closed system* in A if $1 \in S$ and it is closed under the operation $\wedge$.

Theorem 2.12. [21] *Let I be an ideal of A such that $I \cap S = \emptyset$. Then there exists a prime ideal P of A such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Definition 2.13. [1] *Let X be a nonempty subset of A, and I be an ideal of A. Put*

$$(I : X) = \{a \in A : \quad a \wedge x \in I, \ for \ all \ x \in X\}.$$

It is clear that $(I : X)$ is an ideal and $I \subseteq (I : X)$.

Definition 2.14. [11] *Let X be a nonempty subset of A. The set of all zero divisors of X is denoted by $Z(X)$ and is defined as follows:*

$$Z(X) = \{a \in A : x \wedge a = 0, \ for \ some \ x \in X \setminus \{0\}\}.$$

Obviously, $Z(X) \subseteq Z(A)$.

Definition 2.15. [4] *If I is an ideal of MV-algebra A and $x, y \in A$, then we let*

$$x \sim_I y \Longleftrightarrow (x \odot y^*) \oplus (y \odot x^*) \in I.$$

The congruence class of x with respect to $\sim_I$ will be denoted by x/I, i.e. $x/I = \{y \in A : x \sim_I y\}$; one can see that $x \in I$ if and only if $x/I = 0/I$.

We shall denote the quotient set $A/\sim_I$ by A/I. Since $\sim_I$ is a congruence on A, the MV-algebra operations on A/I give by $x/I \oplus y/I := (x \oplus y)/I$ and $(x/I)^* := x^*/I$ are well defined. Hence the system $(A/I, \oplus, *, 0/I)$ becomes an MV-algebra, called the quotient algebra of A by the ideal I. The assignment $x \to x/I$ defines a homomorphism h_I from A onto A/I; we remark that $Ker(h_I) = I$. Clearly, if $x, y \in A$, then $x/I \leq y/I$ if and only if $(x^* \oplus y)/I = 1/I$ if and only if $(x^* \oplus y)^* \in I$ if and only if $x \odot y^* \in I$.

Theorem 2.16. [1] *Suppose that I is an ideal and $P_1, P_2, ..., P_n$ are prime ideals of A, such that $I \subseteq \bigcup_{i=1}^{n} P_i$ and $n \geq 3$. Then there exists P_i such that $I \subseteq P_i$ and $1 \leq i \leq n$.*

Theorem 2.17. [21] *Let $I \in Id(A)$. Then $I = \cap\{P \in Spec(A) : I \subseteq P\}$.*

Definition 2.18. [20] 1) *A topological space X is called a T_0-space if for each distinct x_1 and x_2 of X there exists an open set U such that $x_1 \in U$ and $x_2 \notin U$.*
2) *A topological space X is called a T_1-space if for each distinct x_1 and x_2 of X there exist the open sets U_1 and U_2 such that $x_1 \in U_1, x_2 \notin U_1, x_1 \notin U_2$ and $x_2 \in U_2$.*

Note: For every subset X of the topological space, by $\overline{X}$ we mean the closure of X.

Theorem 2.19. [20] (1) *A topological space X is a T_0-space if and only if $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$, for all $x, y \in X$.*
(2) *A topological space X is a T_1-space if and only if $\overline{\{x\}} = \{x\}$ for each $x \in X$.*

3 S-prime ideals in MV-algebras

In this section, the notions of S-prime ideals are introduced, and several equivalent definitions are given. Their relationship with prime ideals is examined, and these ideals are explored in some types of MV-algebras.

Definition 3.1. *Let I be a proper ideal of A and S be an $\wedge$-closed system of A such that $I \cap S = \emptyset$. The ideal I is called an S-prime ideal if for any $a, b \in A$ with $a \wedge b \in I$, there exists $s \in S$ such that $s \wedge a \in I$ or $s \wedge b \in I$.*
The set of all S-prime ideals is denoted by $Spec_S(A)$.

Definition 3.2. *The S- dimension of A is the supremum of the length of all chains of S-prime ideals, and it is denoted by $Dim_S(A)$.*

Example 3.3. (1) *Let $l_3 = \{0, 1/2, 1\}$ and $l_2 = \{0, 1\}$. Consider the MV-algebra $A = l_3 \times l_2$ with operations $(a, b) \oplus (c, d) = (1, 1) \wedge (a + c, b + d) = (1 \wedge (a + c), 1 \wedge (b + d))$ and $(a, b)^* = (1 - a, 1 - b)$ as in* [14]. *Then $Id(A) = \{I = \{(0, 0)\},\ J = \{(0, 0), (1/2, 0), (1, 0)\},\ K = \{(0, 0), (0, 1)\},\ A\}$. Obviously, $S_1 = \{(1/2, 1), (1, 1)\}$ and $S_2 = \{(0, 1), (1, 1)\}$ are two $\wedge$-closed systems of A. The ideal $I = \{(0, 0)\}$ is not a prime ideal, nor is it an S_1-prime ideal, but it is an S_2-prime ideal.*
(2) *Let $C = \{0, c, 2c, 3c, \ldots, 1 - 2c, 1 - c, 1\}$ be the MV-algebra by the following operations:*

If $x = nc$ and $y = mc$, then $x \oplus y := (m+n)c$,
If $x = 1 - nc$ and $y = 1 - mc$, then $x \oplus y := 1$,
If $x = nc$ and $y = 1 - mc$ and $m \leq n$, then $x \oplus y := 1$,
If $x = nc$ and $y = 1 - mc$ and $n < m$, then $x \oplus y := 1 - (m-n)c$,
If $x = 1 - mc$ and $y = nc$ and $m \leq n$, then $x \oplus y := 1$,
If $x = 1 - mc$ and $y = nc$ and $n < m$, then $x \oplus y := 1 - (m-n)c$,
If $x = nc$, then $x^* := 1 - nc$,
If $x = 1 - nc$, then $x^* := nc$.
The structure $(C, \oplus, *, 0)$ is called the Chang' MV-algebra [21].
Then it has three ideals: $I_0 = \{0\}, I_1 = \{0, c, 2c, 3c, \ldots\}$ and $I_2 = C$.
Put $S = \{1, 1-c\}$. Obviously, I_0 and I_1 are two S-prime ideals and $I_0 \subseteq I_1$. Then $Dim_S(A) = 1$.

Proposition 3.4. (1) *Every prime ideal that does not intersect S is an S-prime ideal.*
(2) *If $Dim_S(A) = 0$, then every S-prime ideal is a prime ideal.*

Proof. (1) It is clear.
(2)Let P be an S-prime ideal. Then $P \cap S = \emptyset$. It follows from Theorem 2.12 that there exists $Q \in Spec(A)$ such that $Q \cap S = \emptyset$ and $P \subseteq Q$. By part (1), we get Q is an S-prime ideal. It follows from the hypotheses that $P = Q$. So, P is a prime ideal. □

In Example 3.3, it can be seen that the converse of Proposition 3.4 is not necessarily true.

Example 3.5. *Let $L_3 = \{0, \frac{1}{2}, 1\}$. Consider MV-algebra $A = L_3 \times L_3$ with this operations*

$$(x,y) \oplus (z,t) = (min\{1, x+z\}, min\{1, y+t\}) \quad \text{and} \quad (x,y)^* = (1-x, 1-y).$$

The MV-algebra A has only four ideals:

$$I_1 = \{(0,0)\}, I_2 = \{(0,0), (0, \frac{1}{2}), (0,1)\}, I_3 = \{(0,0), (\frac{1}{2}, 0), (1,0)\} \text{ and } I_4 = A.$$

Put $S = \{(1,1), (\frac{1}{2}, \frac{1}{2})\}$. It is clear that I_2 and I_3 are S-prime ideals, but I_0 is not an S-prime ideal.
So, from this example, it can be concluded that the intersection of two S-prime ideals is not necessarily an S-prime ideal, and also, an ideal contained in an S-prime ideal is not necessarily an S-prime ideal.

Proposition 3.6. *Let I and J be two proper ideals of A.*
(1) *If I is an S-prime ideal and $J \cap S \neq \emptyset$, then $I \cap J$ is an S-prime ideal of A.*
(2) *If I is a prime ideal such that $I \cap S = \emptyset$, then $(s] \cap I$ is an S-prime ideal, for every $s \in S$.*
(3) *Let $I \cap S = \emptyset$ and suppose there exists $s \in S$ such that $(s] \wedge I$ is a prime ideal. Then I is an S-prime ideal.*

Proof. (1) Obviously, $(I \cap J) \cap S = \emptyset$. Let $x \wedge y \in I \cap J$ where $x, y \in A$. Hence $x \wedge y \in I$ we get there exists $s \in S$ such that $x \wedge s \in I$ or $y \wedge s \in I$. Now, we consider two cases:
Case 1: if $s \in J$, then $x \wedge s \in J$ or $y \wedge s \in J$. Hence $x \wedge s \in I \cap J$ or $y \wedge s \in I \cap J$. We get $I \cap J$ is an S-prime ideal of A.
Case 2: if $s \notin J$. Since $J \cap S \neq \emptyset$ then there exists $t \in J \cap S$. Put $s' := t \wedge s$, hence $s' \in S$. It follows that $s' \wedge x \in I \cap J$ or $s' \wedge y \in I \cap J$.
(2) It follows from Proposition 3.4 (1) that I is an S-prime ideal. Put $J_s := (s]$, for every $s \in S$. Obviously, $J_s \cap S \neq \emptyset$, for every $s \in S$. By part 1 of this proposition, we get $J_s \cap I$ is a S-prime ideal, for every $s \in S$. Therefore $(s] \cap I$ is an S-prime ideal, for every $s \in S$.
(3) By contrary, let $a \wedge b \in I$, where $a, b \in A$ but $a \wedge s \notin I$ and $b \wedge s \notin I$, for every $s \in S$. Obviously, $(a \wedge b) \wedge s = a \wedge (b \wedge s) \in I \cap (s]$. By hypotheses, $a \in I \cap (s]$ or $b \wedge s \in I \cap (s]$. We consider two cases:
Case 1: if $a \in I \cap (s]$, then $a \in I$. Hence $a \wedge s \in I$, which is a contradiction.
Case 2: if $b \wedge s \in I \cap (s]$, then $b \wedge s \in I$, which is a contradiction.
We conclude there exists $s \in S$ such that $a \wedge s \in I$ and $b \wedge s \in I$. Therefore I is an S-prime ideal. □

Note: Obviously, if $f : A \to B$ be a homomorphism of MV-algebras and let S be a $\wedge$-closed system of A, then $f(S)$ is a $\wedge$-close system of B.

Theorem 3.7. *Let $f : A \to B$ be a homomorphism of MV-algebras and let I be a $f(S)$-prime ideal of B. Then $f^{-1}(I)$ is an S-prime ideal of A.*

Proof. Let $x \in f^{-1}(I) \cap S$. Then $f(x) \in I \cap f(S)$, which is a contradiction. Hence $f^{-1}(I) \cap S = \emptyset$. Now, let $x \wedge y \in f^{-1}(I)$, where $x, y \in A$. Then $f(x \wedge y) = f(x) \wedge f(y) \in I$. Hence, there exists $\alpha \in f(S)$ such that $\alpha \wedge f(x) \in I$ or $\alpha \wedge f(y) \in I$. On the other hand, there exists $s \in S$ such that $\alpha = f(s)$. We have $f(s) \wedge f(x) = f(s \wedge x) \in I$ or $f(s) \wedge f(y) = f(s \wedge y) \in I$. It follows that $s \wedge x \in f^{-1}(I)$ or $s \wedge y \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is an S-prime ideal of A. □

Proposition 3.8. *Let I and J be two ideals of A such that $I \subseteq J$. Then J is an S-prime ideal if and only if J/I is an S/I-prime ideal of A/I, where $S/I := \{s/I : s \in S\}$.*

Proof. Obviously, $J \cap S = \emptyset$ if and only if $J/I \cap S/I = \emptyset$.
Let $(a/I) \wedge (b/I) = (a \wedge b)/I \in J/I$. Then $a \wedge b \in J$. Hence, there exists $s \in S$ such that $s \wedge a \in J$ or $s \wedge b \in J$. We have $s/I \wedge a/I = (s \wedge a)I \in J/I$ or $s/I \wedge b/I = (s \wedge b)/I \in J/I$.
The converse is easy. □

Proposition 3.9. *Let I be a proper ideal of A.*
(1) If $I \cap S = \emptyset$, then there exists $P \in Spec_S(A)$ such that $I \subseteq P$.
(2) If $a \in A \setminus I$, then there exists $P \in Spec_S(A)$ such that $I \subseteq P$ and for some $s \in S, s \wedge a \notin P$.
(3) $I = \cap\{P \in Spec_S(A) : (s] \wedge I \subseteq P, \ \forall s \in S\}$.

Proof. (1) It follows from Theorem 2.12 and Proposition 3.4.
(2) Put $S = \{1, a\}$. It follows from part (1). Also, $s := 1$ hence $1 \wedge a = a \notin P$.
(3) Put $\Omega = \cap\{P \in Spec_S(A) : (s] \wedge I \subseteq P, \ \forall s \in S\}$. Since $1 \in S$, it is clear that $I \subseteq \Omega$. Now, let $x \in \Omega$ but $x \notin I$. Then by part (2), there exists $Q \in Spec_S(A)$ such that $I \subseteq Q$ and $s \wedge x \notin Q$, for some $s \in S$. It follows from $I \subseteq Q$ that $(s] \wedge I \subseteq I \subseteq Q$, for every $s \in S$. Hence $\Omega \subseteq Q$ thus, $x \in Q$. We obtain $x \wedge s \in Q$, for every $s \in S$, which is a contradiction. So $\Omega \subseteq I$. Therefore, $I = \cap\{P \in Spec_S(A) : (s] \wedge I \subseteq P, \ \forall s \in S\}$. □

Definition 3.10. *A proper ideal I that does not intersect with S is called an S-maximal ideal if, for any ideal J such that $I \subseteq J$, it holds that either $J \cap S \neq \emptyset$ or there exists an element $s \in S$ such that $(s] \wedge J \subseteq I$.*

Example 3.11. *Let $l_4 = \{0, 1/3, 2/3, 1\}$ and $l_2 = \{0, 1\}$. Consider the MV-algebra $A = l_4 \times l_2$ with operations $(a, b) \oplus (c, d) = (1, 1) \wedge (a + c, b + d) = (1 \wedge (a + c), 1 \wedge (b + d))$ and $(a, b)^* = (1 - a, 1 - b)$ as in* [14]. *Then $Id(A) = \{I_0 = \{(0, 0)\}, \ I_1 = \{(0, 0), (0, 1)\}, \ I_2 = \{(0, 0), (1/3, 0), (2/3, 0), (1, 0)\}, \ I_3 = A\}$. Now, consider the subset $S = \{(1/3, 0), (1, 1)\}$. Since $I_1 \cap S = \emptyset$, it is clear that I_1 is an S-maximal ideal of A.*

Theorem 3.12. *The ideal M is an $S-$maximal if and only if for each $a \notin M$ there exists $s \in S$ such that $s \wedge a \in M$ or $(M \vee (a]) \cap S \neq \emptyset$.*

Proof. Let M be an $S-$maximal and $a \notin M$. Obviously, $M \subsetneq M \vee (a]$. Then $(M \vee (a]) \cap S \neq \emptyset$ or there exists $s \in S$ such that $(s] \wedge (M \vee (a]) \subseteq M$.
If $(M \vee (a]) \cap S \neq \emptyset$, then the proof is complete.
Now, let there exist $s \in S$ such that $(s] \wedge (M \vee (a]) \subseteq M$. Then it follows Lemma 2.8

$$\begin{aligned}(a] \subseteq M \vee (a] &\Rightarrow (s] \wedge (a] \subseteq (s] \wedge (M \vee (a]) \\ &\Rightarrow (s] \wedge (a] \subseteq M \\ &\Rightarrow (s \wedge a] \subseteq M.\end{aligned}$$

Hence $s \wedge a \in M$.
Conversely, on contrary. Let M is not an S-maximal ideal. Then there exists $Q \in Id(A)$ such that $M \subsetneq Q, Q \cap S = \emptyset$ and $(s] \cap Q \nsubseteq M$, for all $s \in S$. Then there exists x of A such that $x \in (s] \cap Q$ and $x \notin M$. Hence there exists $n \in \mathbb{N}$ such that $x \leq ns$ and $x \in Q$. By hypothesis, there exists $s \in S$ such that $s \wedge x \in M$ or $(M \vee (x]) \cap S \neq \emptyset$.
Let $(M \vee (x]) \cap S \neq \emptyset$. On the other hand $(M \vee (x]) \cap S \subseteq (Q \vee (x]) \cap S$, then $(Q \vee (x]) \cap S \neq \emptyset$. Hence there exists $t \in (Q \vee (x]) \cap S$. We get $t \in S$ and there exists $m \in \mathbb{N}$ and $q \in Q$ such that $t \leq q \oplus mx$. It follows that $t \in Q \cap S$, which is a contradiction.
If $x \wedge s \in M$. Since $x = x \wedge ns \leq n(x \wedge s)$, we obtain $x \in M$, which is a contradiction.
Therefore M is an $S-$maximal ideal. □

Proposition 3.13. *Every S-maximal ideal is an S-prime ideal.*

Proof. Suppose that P is an S-maximal ideal and $a \wedge b \in P$ where $a, b \in A$. On contrary, let $s \wedge a \notin P$ and $s \wedge b \notin P$, for every $s \in S$. Then $P \subseteq P \vee (a]$ and $P \subseteq P \vee (b]$. By hypotheses, either $(P \vee (a]) \cap S \neq \emptyset$ or $(s] \wedge (P \vee (a]) \subseteq P$, for some $s \in S$. Also, we have either $(P \vee (b]) \cap S \neq \emptyset$ or $(s] \wedge (P \vee (b]) \subseteq P$, for some $s \in S$. Now, let $(s] \wedge (P \vee (a]) \subseteq P$ and let $x \in (s] \wedge (a]$. Then $x \in (s]$ and $x \in (a] \vee P$, so $x \in P$. Hence, $(s] \wedge (a] = (s \wedge a] \subseteq P$. We get $s \wedge a \in P$, which is a contradiction. Hence, $(P \vee (a]) \cap S \neq \emptyset$. Similarly, it can be proven that $(P \vee (b]) \cap S \neq \emptyset$. Then there exist $s_1, s_2 \in S$ such that $s_1 \leq p_1 \oplus n_1 a$ and $s_2 \leq p_2 \oplus n_2 a$, for some $p_1, p_2 \in P$ and $n_1, n_2 \in \mathbb{N}$. It follows that

$$\begin{aligned} s_1 \wedge s_2 &\leq (p_1 \oplus n_1 a) \wedge (p_2 \oplus n_2 a) \\ &\leq ((p_1 \oplus n_1 a) \wedge p_2) \oplus ((p_1 \oplus n_1 a) \wedge n_2 b) \\ &\leq (p_1 \wedge p_2) \oplus (p_2 \wedge n_1 a) \oplus (p_1 \wedge n_2 a) \oplus n_1 n_2 (a \wedge b). \end{aligned}$$

Hence, $s_1 \wedge s_2 \in P$, which is a contradiction. Therefore, there exists $s \in S$ such that $s \wedge a \in P$ or $s \wedge b \in P$. □

Example 3.14. *Let ${}^*\mathbb{R}$ be a nonprincipal ultrapower of the field of the real numbers with natural order and ε be a positive infinitesimal element of ${}^*\mathbb{R}$. Let $\varepsilon^2 = \varepsilon \cdot \varepsilon, ..., \varepsilon^n = \varepsilon \cdot \varepsilon \cdot ... \cdot \varepsilon(n - times)$, where $\cdot$ is the usual product in the field ${}^*\mathbb{R}$; then $\varepsilon^i > 0$ for any $i \in \mathbb{N}$ and $\varepsilon^i \ll \varepsilon^j$, for $i > j$.*
The unit interval ${}^*[0,1] \subseteq^* \mathbb{R}$ is an MV-algebra with the operations: $x \oplus y = min\{1, x + y\}$, $x^* = 1 - x$. Let $\mathbb{N}$ be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of ${}^*[0,1]$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n\}$ and E be the subalgebra $\bigcup_{n\in\mathbb{N}} E_n$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n, \ldots\}$ [8]. The ideals of E are $\{0\}, (\varepsilon], ..., (\varepsilon^i]...$, where $i \in \mathbb{N}$ and $(\varepsilon^i] \subseteq (\varepsilon^j]$, for any $i > j$. Obviously, $S = \{1, \varepsilon\}$ is a $\wedge$-closed system and $(\varepsilon^i]$, for every $i \geq 2$ is an S-prime ideal. It is claimed that $(\varepsilon^3]$ is not an S-maximal ideal. To demonstrate this, we need to consider the two ideals $(\varepsilon]$ and $(\varepsilon^2]$, since $(\varepsilon^3] \subseteq (\varepsilon]$ and $(\varepsilon^3] \subseteq (\varepsilon^2]$:
(1) *For $(\varepsilon]$, we have $(\varepsilon] \cap S \neq \emptyset$. Also $(\varepsilon] \cap (\varepsilon^3] = (\varepsilon^3]$.*
(2) *For $(\varepsilon^2]$, we have $(\varepsilon^2] \cap S = \emptyset$. Also, $(1] \cap (\varepsilon^2] = (\varepsilon^2] \nsubseteq (\varepsilon^3]$ and $(\varepsilon] \cap (\varepsilon^2] = (\varepsilon^2] \nsubseteq (\varepsilon^3]$.*
Therefore, $(\varepsilon^3]$ is an S-prime ideal but is not an S-maximal ideal.
In this example, $Dim_S(A) = \infty$.

Remark 3.15. • *Obviously, every maximal ideal that does not intersect S is an S-maximal ideal.*
• *If M is a maximal ideal and S-prime ideal, then M is an S-maximal ideal.*

We want to demonstrate in the following example that the infimum of all elements of a $\wedge$-closed system may either exist or not exist within it. An $\wedge$-closed system in which the infimum of all its elements exists within it is called a complete $\wedge$-closed system and is denoted by the symbol $\mathbb{S}$.

Example 3.16. (1) *Let $A = ([0,1], 0, \oplus, *)$ be the standard MV-algebra. It is clear that $S = (0,1]$ is an $\wedge$-closed system of A and $\bigwedge_{s\in S} s = 0$ but $0 \notin S$.*
(2) *Let A be the power set of $\mathbb{N}$, denoted by $P(\mathbb{N})$. Define $\oplus$ and $*$ as follows:*

$$X \oplus Y := X \cup Y \quad \text{and} \quad X^* := \mathbb{N} \setminus X, \ \forall X, Y \in P(\mathbb{N}).$$

Obviously, $A = (P(\mathbb{N}), \cup, *, \emptyset, \mathbb{N})$ is an MV-algebra and it is even a Boolean algebra. Put $X_n = \{1, n\}$ for every $n \in \mathbb{N}$. It is clear that $\mathbb{S} = \{X_1, ..., X_n, ...\}_{n\in\mathbb{N}}$ is a complete $\wedge$-closed system and $\bigwedge_{i\in\mathbb{N}} X_i = \{1\}$.

Proposition 3.17. *Let P be an $\mathbb{S}$-prime ideal. Then there exists $s_P \in \mathbb{S}$ such that $(P : s) \subseteq (P : s_P)$, for each $s \in \mathbb{S}$.*

Proof. Put $s_P = \bigwedge_{s\in\mathbb{S}} s$. Obviously, $s_P \in \mathbb{S}$. Now, let $x \in (P : s)$, then $x \wedge s \in P$. We get that there exists $\alpha \in \mathbb{S}$ such that $x \wedge \alpha \in P$ or $s \wedge \alpha \in P$. As $P \cap \mathbb{S} = \emptyset$, we obtain $x \wedge \alpha \in P$. It follows that $x \wedge s_P \in P$, hence $x \in (P : s_P)$. □

Note: $(P : s_P)$ is called the maximal element of $\{(P : s)\}_{s\in\mathbb{S}}$.

Proposition 3.18. *Assume that I is an ideal that does not intersect with $\mathbb{S}$. Then the following statements are equivalent:*
(1) *I is an $\mathbb{S}$-prime ideal of A.*
(2) *$(I : s)$ is a prime ideal of A, for some $s \in \mathbb{S}$.*

Proof. $1 \Rightarrow 2$) By part 5 of Lemma 2.3, we get $(x \odot y^*) \wedge (y \odot x^*) = 0$, for every $x, y \in A$. On the other hand, $0 \in I$ hence, $(x \odot y^*) \wedge (y \odot x^*) \in I$, for every $x, y \in A$. It follows from the hypothesis that there exists $s_{xy} \in \mathbb{S}$ such that $(x \odot y^*) \wedge s_{xy} \in I$ or $(y \odot x^*) \wedge s_{xy} \in I$, for every $x, y \in A$. Put $s_I = \bigwedge_{s \in \mathbb{S}} s$. Obviously $(x \odot y^*) \wedge s_I \in I$ or $(y \odot x^*) \wedge s_I \in I$. Hence $x \odot y^* \in (I : s_I)$ or $y \odot x^* \in (I : s_I)$ for every $x, y \in A$.
Now, by Definition 2.10, it suffices for us to prove that $(I : s_I)$ is a proper ideal. On contrary, let $(I : s_I) = A$. Then $1 \in (I : s_I)$, hence $1 \wedge s_I \in I$. It follows that $s_I \in I$ implies that $s_I \in I \cap \mathbb{S}$ hence $I \cap \mathbb{S} \neq \emptyset$, which is a contradiction. Thus $(I : s_I)$ is a proper ideal.
$2 \Rightarrow 1$) Let $a, b \in A$ such that $a \wedge b \in I$. By hypothesis, there exists $s \in \mathbb{S}$ such that $(I : s)$ is a prime ideal. Obviously, $s \wedge (a \wedge b) \in I$ then $a \wedge b \in (I : s)$. we get $a \in (I : s)$ or $b \in (I : s)$ hence $a \wedge s \in I$ or $b \wedge s \in I$. By hypothesis, $I \cap \mathbb{S} = \emptyset$ hence I is an $\mathbb{S}$-prime ideal of A.

□

Theorem 3.19. *Let P be an ideal of A such that P does not intersect $\mathbb{S}$ and define $\frac{\mathbb{S}}{P} := \{\frac{s}{P} : s \in \mathbb{S}\}$. If $Z(\frac{A}{P}) \cap \frac{\mathbb{S}}{P} = \emptyset$, then P is an $\mathbb{S}$-prime ideal if and only if P is a prime ideal.*

Proof. Let P is an $\mathbb{S}$-prime ideal. We claim $(P : s) = P$, for all $s \in \mathbb{S}$. Obviously, $P \subseteq (P : s)$, for all $s \in \mathbb{S}$. Now, let $s \in \mathbb{S}$ and $x \in (P : s)$. Then $x \wedge s \in P$. Hence $\frac{s}{P} \wedge \frac{x}{P} = \frac{0}{P}$. On the other hand $P \cap \mathbb{S} = \emptyset$ and $Z(\frac{A}{P}) \cap \frac{\mathbb{S}}{P} = \emptyset$, we get $\frac{s}{P} \neq \frac{0}{P}$ and $\frac{x}{P} = \frac{0}{P}$. Thus $x \in P$, hence $(P : s) \subseteq P$. We have $P = (P : s)$, for every $s \in \mathbb{S}$. It follows from Proposition 3.18 that P is a prime ideal.
The converse is clear. □

Theorem 3.20. *Let I be an ideal of A such that $I \cap \mathbb{S} = \emptyset$. Then the following statements are equivalent:*
(1) *I is an $\mathbb{S}$-prime ideal.*
(2) *If J and K are two ideals of A such that $J \cap K \subseteq I$, then $(s] \cap J \subseteq I$ or $(s] \cap K \subseteq I$ for some $s \in \mathbb{S}$.*

Proof. $1 \Rightarrow 2$) Let I be an $\mathbb{S}$-prime ideal and let J and K be two ideals of A such that $J \cap K \subseteq I$ but for every $t \in \mathbb{S}$ we have $(t] \cap J \nsubseteq I$ and $(t] \cap K \nsubseteq I$. It follows from Proposition 3.18 that there exists $s \in \mathbb{S}$ such that $(I : s)$ is a prime ideal of A. Also, by hypothesis, we obtain $(s] \cap J \nsubseteq I$ and $(s] \cap K \nsubseteq I$. Hence there exist $x \in (s] \cap J$ and $y \in (s] \cap K$ such that $x \notin I$ and $y \notin I$. Then there exist $m, n \in \mathbb{N}$ such that $x \leq ns$ and $y \leq ms$. We get $x \wedge y \in I \cap J$ implies that $x \wedge y \in I$. Hence $x \wedge y \wedge s \in J$ thus $x \wedge y \in (I : s)$. It follows that $x \in (I : s)$ or $y \in (I : s)$ we obtain $x \wedge s \in I$ or $y \wedge s \in I$. Without loss of generality, let $x \wedge s \in I$. Then $n(x \wedge s) \in I$. On the other hand $x = x \wedge ns \leq n(x \wedge s)$ implies that $x \in I$, which is a contradiction.
$2 \Rightarrow 1$) Let $a \wedge b \in I$. Then $(a \wedge b] \subseteq I$. It follows from Lemma 2.8(3) that $(a] \cap (b] \subseteq I$. By hypothesis, there exists $s \in \mathbb{S}$ such that $(s] \cap (a] \subseteq I$ or $(s] \cap (b] \subseteq I$. We have $s \wedge a \in I$ or $s \wedge b \in I$. □

Theorem 3.21. *Let $Spec_{\mathbb{S}}(A)$ be a finite set with n elements, where $n \geq 1$ and let I be a proper ideal of A. If $I \subseteq \bigcup_{i=1}^{n} P_i$, then there exist $s \in \mathbb{S}$ and $P_i \in Spec_{\mathbb{S}}(A)$ such that $(s] \wedge I \subseteq P_i$.*

Proof. It follows from Proposition 3.18 that there exists $s_i \in \mathbb{S}$ such that $(P_i : s_i)$ is a prime ideal, for every $P_i \in Spec_{\mathbb{S}}(A)$. On the other hand $I \subseteq \bigcup_{i=1}^{n} P_i \subseteq \bigcup_{i=1}^{n} (P_i : s_i)$. Now, by Theorem 2.16, we get that there exists j such that $1 \leq j \leq n$ and $I \subseteq (P_j : s_j)$. Hence $s_j \wedge a \in P_j$, for every $a \in I$. It is a claim that $(s_j] \wedge I \subseteq P_j$.
Let $x \in (s_j] \wedge I$. Then $x \in I$ and there exists $m \in \mathbb{N}$ such that $x \leq ms_j$. Hence $x \wedge ms_j = x$. On the other hand, $x = x \wedge ms_j \leq m(x \wedge s_j)$ thus $x \in P_j$. Hence $(s_j] \wedge I \subseteq P_j$. □

4 Topology by complete $\wedge$-closed system on MV-algebras

In this section, we will consider a complete $\wedge$-closed system $\mathbb{S}$ of A and the $\mathbb{S}$-spectral topology based on $\mathbb{S}$-prime ideals. In the following, we use $\mathbb{S}$ to represent a complete $\wedge$-closed system and S to represent an $\wedge$-closed system. The open and closed sets of this topology will be determined, and their properties will be examined. A basis for the topology will also be identified. Furthermore, key topological properties such as compactness, connectedness, and others will be studied in this topology.

Definition 4.1. *Let X be a subset of A and $a \in A$. Define*

$$V_S(X) = \{P \in Spec_S(A) : (s] \cap X \subseteq P, \quad \text{for some } s \in S\} \quad \text{and} \quad U_S(X) = Spec_S(A) \setminus V_S(X).$$

$$V_S(a) = \{P \in Spec_S(A) : s \wedge a \in P, \quad \text{for some } s \in S\} \quad \text{and} \quad U_S(a) = Spec_S(A) \setminus V_S(a).$$

Obviously, if $S = \{1\}$ and $I, J \in Id(A)$, then $V_S(I) = V(I)$ and $U_S(a) = U(a)$ where $V(I)$ and $U(a)$ respectively, are a closed subset and a basic element in the topology of the spectral topology on $Spec(A)$.

Lemma 4.2. *Let I and J be two subsets of A and $a \in A$. Then*
(1) *If $I \subseteq J$, Then $V_S(J) \subseteq V_S(I)$.*
(2) *$V_S(I) \subseteq V_S(I \cap J)$.*
(3) *If I and J are two ideals, then $V_S(I \cap J) = V_S(I) \cup V_S(J)$.*
(4) *$V_S(A) = \emptyset$ and $V_S(\{0\}) = Spec_S(A)$.*
(5) *$V_S(a) = V_S((a])$.*
(6) $\bigcap\limits_{P \in V_S(a)} P \subseteq (a]$.
(7) *Assume that I is an ideal. Then $V_S(I) = \emptyset$ if and only if $I \cap S \neq \emptyset$.*
(8) *Assume that Γ is a family of ideals of A. Then $V_S(\bigvee\limits_{\alpha \in \Gamma} I_\alpha) \subseteq \bigcap\limits_{\alpha \in \Gamma} V_S(I_\alpha)$.*

Proof. (1) Let $P \in V_S(J)$. Then $P \in Spec_S(A)$ and $(s] \wedge I \subseteq P$ for some $s \in S$. On the other hand, $I \subseteq J$ hence $(s] \cap I \subseteq (s] \cap J$. We get $P \in V_S(I)$.
(2) It is clear from part (1).
(3) Let $P \in V_S(I \cap J)$. Then $P \in Spec_S(A)$ and $(s] \wedge I \wedge J \subseteq P$ for some $s \in S$. It follows from Theorem 3.20 that there exists $s^{'} \in S$ such that $(s^{'}] \wedge J \subseteq P$ or $(s^{'}] \wedge I \subseteq P$ or $(s^{'}] \wedge (s] \subseteq P$. Then we have three cases:
Case 1: If $(s^{'}] \wedge (s] \subseteq P$, then $(s \wedge s^{'}] \subseteq P$. Hence, $s \wedge s^{'} \in P$, which is a contradiction.
Case 2: if $(s^{'}] \wedge J \subseteq P$, then $P \in V_S(J)$. Hence, $P \in V_S(I) \cup V_S(J)$.
Case 3: if $(s^{'}] \wedge I \subseteq P$, then $P \in V_S(I)$. Hence, $P \in V_S(I) \cup V_S(J)$.
Thus $V_S(I \cap J) \subseteq V_S(I) \cup V_S(J)$. Also, by part (2), we get $V_S(I) \cup V_S(J) \subseteq V_S(I \cap J)$. Therefore $V_S(I) \cup V_S(J) = V_S(I \cap J)$.
(4) It is clear.
(5) It is clear from Lemma 2.8.
(6) Obviously, $\{P \in Spec_S(A) : s \wedge a \in P, \ \forall s \in S\} \subseteq \{P \in Spec_S(A) : s \wedge a \in P, \ \exists s \in S\}$.
Hence $\cap\{P \in Spec_S(A) : s \wedge a \in P, \ \exists s \in S\} \subseteq \cap\{P \in Spec_S(A) : s \wedge a \in P, \ \forall s \in S\}$.
Then $\bigcap\limits_{P \in V_S(a)} P \subseteq \cap\{P \in Spec_S(A) : s \wedge a \in P, \ \forall s \in S\}$. On the other hand

$$\{P \in Spec_S(A) : s \wedge a \in P, \ \forall s \in S\} = \{P \in Spec_S(A) : (s] \wedge (a] = (s \wedge a] \subseteq P, \ \forall s \in S\}.$$

It follows from Proposition 3.9 (3) that $\bigcap\limits_{P \in V_S(a)} P \subseteq (a]$.
(7) Let $V_S(I) = \emptyset$ and $I \cap S = \emptyset$. It follows from Theorem 2.12 that there exists $P \in Spec(A)$ such that $I \subseteq P$ and $P \cap S = \emptyset$. We obtain $P \in Spec_S(A)$ and $(s] \wedge I \subseteq I \subseteq P$, for every $s \in S$. Thus $P \in V_S(I)$, which is a contradiction.
Conversely, let $I \cap S \neq \emptyset$ and there exists the ideal P such that $P \in V_S(I)$. It follows that there exists $s \in S$ such that $(s] \wedge I \subseteq P$. Also, we have that there exists $s^{'} \in I \cap S$. Obviously, $s \wedge s^{'} \in P$ hence $s \wedge s^{'} \in S \cap P$, which is a contradiction.
(8) Let $P \in V_S(\bigvee\limits_{\alpha \in \Gamma} I_\alpha)$. Then $P \in Spec_S(A)$ and $(s] \wedge \bigvee\limits_{\alpha \in \Gamma} I_\alpha \subseteq P$, for some $s \in S$. On the other hand $(s] \wedge I_\alpha \subseteq (s] \wedge \bigvee\limits_{\alpha \in \Gamma} I_\alpha$, hence $(s] \wedge I_\alpha \subseteq P$, for every $\alpha \in \Gamma$. Then $P \in V_S(I_\alpha)$, for every $\alpha \in \Gamma$. Thus $P \in \bigcap\limits_{\alpha \in \Gamma} V_S(I_\alpha)$. We get $V_S(\bigvee\limits_{\alpha \in \Gamma} I_\alpha) \subseteq \bigcap\limits_{\alpha \in \Gamma} V_S(I_\alpha)$. □

Lemma 4.3. *The following holds for any $a, b \in A$ and for any subsets I and J of A.*
(1) *If $U_S(a) \subseteq U_S(b)$, then* $\bigcap\limits_{P \in V_S(a)} P \subseteq (b] \cap (a]$.
(2) *$U_S(A) = Spec_S(A)$ and $U_S(\{0\}) = \emptyset$.*
(3) *If $a \leq b$, then $U_S(a) \subseteq U_S(b)$.*
(4) *$U_S(a) \cap U_S(b) = U_S(a \wedge b)$.*

(5) $U_S(a) \cup U_S(b) = U_S(a \vee b)$.
(6) $U_S(a \oplus b) = U_S(a) \cup U_S(b)$.
(7) $U_S((a]) = U_S(a)$.
(8) *If* $I \subseteq J$, *then* $U_S(I) \subseteq U_S(J)$.
(9) *If* I *and* J *are two ideals of* A, *then* $U_S(I \wedge J) = U_S(I) \cap U_S(J)$.
(10) *If* I *and* J *are two ideals of* A, *then* $U_S(I \vee J) = U_S(I) \cup U_S(J)$.

Proof. (1) Let $U_S(a) \subseteq U_S(b)$. Then $V_S(b) \subseteq V_S(a)$. Hence $\bigcap_{P \in V_S(a)} P \subseteq \bigcap_{P \in V_S(b)} P$. By Lemma 4.2(6), it follows that $\bigcap_{P \in V_S(a)} P \subseteq (b]$ and subsequently, by applying Lemma 4.2(6) once again, it follows that $\bigcap_{P \in V_S(a)} P \subseteq (b] \cap (a]$.
(2) It is clear.
(3) Let $P \in U_S(a)$. Then $P \in Spec_S(A)$ and $s \wedge a \notin P$, for all $s \in S$. On the other hand, $s \wedge a \leq s \wedge b$, thus $s \wedge b \notin P$. Hence $P \in U_S(b)$. Therefore $U_S(a) \subseteq U_S(b)$.
(4) Let $P \in U_S(a) \cap U_S(b)$ and let $P \notin U_S(a \wedge b)$. Then $P \in Spec_S(A)$ and there exists $x \in S$ such that $(a \wedge b) \wedge x \in P$. Then $a \wedge (b \wedge x) \in P$. It follows that there exists $y \in S$ such that $a \wedge y \in P$ or $b \wedge (x \wedge y) \in P$, which, based on the hypothesis, is a contradiction.
Now, let $P \in U_S(a \wedge b)$ but $P \notin U_S(a) \cap U_S(b)$. Without loss of generality, we can assume that there exists $x \in S$ such that $a \wedge x \in P$. Then $b \wedge a \wedge x \in P$, which is a contradiction.
Therefore, $U_S(a) \cap U_S(b) = U_S(a \wedge b)$.
(5) Let $P \in U_S(a) \cup U_S(b)$ and $P \notin U_S(a \vee b)$. Then there exists $s \in S$ such that $s \wedge (a \vee b) \in P$. On the other hand $s \wedge (a \vee b) = (s \wedge a) \vee (s \wedge b)$. Hence, $s \wedge a \in P$ and $s \wedge b \in P$, which is a contradiction. It follows that $U_S(a) \cup U_S(b) \subseteq U_S(a \vee b)$. Similarly, it can be proven that $U_S(a \vee b) \subseteq U_S(a) \cup U_S(b)$.
(6) Let $P \in U_S(a \oplus b)$. Then $P \in Spec_S(A)$ and $s \wedge (a \oplus b) \notin P$, for every $s \in S$. On the other hand $s \wedge (a \oplus b) \leq (s \wedge a) \oplus (s \wedge b)$, we obtain $(s \wedge a) \oplus (s \wedge b) \notin P$. Now, we consider four cases:
Case 1: if $(s \wedge a) \in P$ and $(s \wedge b) \in P$, then $(s \wedge a) \oplus (s \wedge b) \in P$, which is a contradiction.
Case 2: if $(s \wedge a) \notin P$ and $(s \wedge b) \in P$, then $P \in U_S(a) \cup U_S(b)$.
Case 3: if $(s \wedge a) \in P$ and $(s \wedge b) \notin P$, then $P \in U_S(a) \cup U_S(b)$.
Case 4: if $(s \wedge a) \notin P$ and $(s \wedge b) \notin P$, then $P \in U_S(a) \cup U_S(b)$.
Hence $U_S(a \oplus b) \subseteq U_S(a) \cup U_S(b)$.
It follows from $a \vee b \leq a \oplus b$, parts 3 and 5 that $U_S(a) \cup U_S(b) \subseteq U_S(a \oplus b)$.
Therefore $U_S(a) \cup U_S(b) = U_S(a \oplus b)$.
(7) It is clear from Lemma 2.8.
(8) It is clear.
(9) By part 8, we get $U_S(I \wedge J) \subseteq U_S(I) \cap U_S(J)$. Let $P \in U_S(I) \cap U_S(J)$ but $P \notin U_S(I \wedge J)$. Then there exists $s \in S$ such that $(s] \wedge I \wedge J \subseteq P$. It follows from Theorem 3.20 that there exists $s^{'} \in S$ such that $(s] \wedge (s^{'}] \wedge I \subseteq P$ or $(s^{'}] \wedge J \subseteq P$. Hence $(s \wedge s^{'}] \wedge I \subseteq P$ or $(s^{'}] \wedge J \subseteq P$, which is a contradiction. Thus $U_S(I) \cap U_S(J) \subseteq U_S(I \wedge J)$. Therefore $U_S(I) \cap U_S(J) = U_S(I \wedge J)$.
(10) By part 8, we get $U_S(I) \cup U_S(J) \subseteq U_S(I \vee J)$. Let $P \in U_S(I \vee J)$ but $P \notin U_S(I) \cup U_S(J)$. Then there exist $s, s^{'} \in S$ such that $(s] \wedge I \subseteq P$ and $(s^{'}] \wedge J \subseteq P$. It follows from Lemma 2.8 part (3) that $(s \wedge s^{'}] \wedge I \subseteq P$ and $(s \wedge s^{'}] \wedge J \subseteq P$, where $s \wedge s^{'} \in S$. We get $(s \wedge s^{'}] \wedge (I \vee J) \subseteq P$, which is a contradiction. Hence $U_S(I \vee J) \subseteq U_S(I) \cup U_S(J)$. Therefore $U_S(I \vee J) = U_S(I) \cup U_S(J)$. □

Note: In the following sections of the paper, the results are examined in the context of a complete $\wedge$-closed system.

Lemma 4.4. *Assume that* Γ *is a family of ideals of* A. *Then* $\bigcap_{\alpha \in \Gamma} V_{\mathbb{S}}(I_\alpha) \subseteq V_{\mathbb{S}}(\bigvee_{\alpha \in \Gamma} I_\alpha)$.

Proof. Let $P \in \bigcap_{\alpha \in \Gamma} V_{\mathbb{S}}(I_\alpha)$. Then $P \in V_{\mathbb{S}}(I_\alpha)$, for every $\alpha \in \Gamma$. Hence $P \in Spec_{\mathbb{S}}(A)$ and there exists $s_\alpha \in \mathbb{S}$ such that $(s_\alpha] \wedge I_\alpha \subseteq P$, for every $\alpha \in \Gamma$. Obviously, $s_\alpha \wedge x \in P$, for every $x \in I_\alpha$. Thus $I_\alpha \subseteq (P : s_\alpha)$, for every $\alpha \in \Gamma$. It follows from Proposition 3.17 that there exists $s \in \mathbb{S}$ such that $I_\alpha \subseteq (P : s_\alpha) \subseteq (P : s)$ for every $\alpha \in \Gamma$. Hence $I_\alpha \subseteq (P : s)$ for every $\alpha \in \Gamma$ implies that $\bigvee_{\alpha \in \Gamma} I_\alpha \subseteq (P : s)$ and $(s] \wedge I_\alpha \subseteq P$ for every $\alpha \in \Gamma$. Thus $\bigvee_{\alpha \in \Gamma} ((s] \wedge I_\alpha) \subseteq P$. It follows from Lemma 2.8(4) that $(s] \wedge (\bigvee_{\alpha \in \Gamma} (I_\alpha)) \subseteq P$.

$\bigvee_{\alpha \in \Gamma} I_\alpha \subseteq (P : s)$. Now, we want to show that $(s] \wedge (\bigvee_{\alpha \in \Gamma} I_\alpha) \subseteq P$. Let $y \in (s] \wedge (\bigvee_{\alpha \in \Gamma} I_\alpha)$. Then there exists $n \in \mathbb{N}$ such that $y \leq ns$. Hence $y = y \wedge ns \leq n(y \wedge s)$. On the other hand $y \in \bigvee_{\alpha\in\Gamma} I_\alpha$, we obtain $y \wedge s \in P$. Thus $n(y \wedge s) \in P$. Then $y \in P$. It follows that $(s] \wedge (\bigvee_{\alpha\in\Gamma} I_\alpha) \subseteq P$. Hence $P \in V_{\mathbb{S}}(\bigvee_{\alpha\in\Gamma} I_\alpha)$. We have $\bigcap_{\alpha\in\Gamma} V_{\mathbb{S}}(I_\alpha) \subseteq V_{\mathbb{S}}(\bigvee_{\alpha\in\Gamma} I_\alpha)$. □

Corollary 4.5. *Assume that* Γ *is a family of ideals of* A*. Then* $\bigcap_{\alpha\in\Gamma} V_{\mathbb{S}}(I_\alpha) = V_{\mathbb{S}}(\bigvee_{\alpha\in\Gamma} I_\alpha)$*.*

Proof. It is clear from part 8 of Lemma 4.2 and Lemma 4.4. □

Remark 4.6. *By Lemma 4.2 and Corollary 4.5, it is clear that the set* $\tau = \{V_{\mathbb{S}}(I) : I \in Id(A)\}$ *satisfies all conditions of closed sets for the topology* τ *on* $Spec_{\mathbb{S}}(A)$*. This topology is said to be an* $\mathbb{S}$*-spectral topology. Thus any open set of* $Spec_{\mathbb{S}}(A)$ *has the form* $U_{\mathbb{S}}(I) = Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}(I)$*, for any ideal* I *of* A*.*

Theorem 4.7. *The set* $\Gamma = \{U_{\mathbb{S}}(a) : a \in A\}$ *is a basis for* $\mathbb{S}$*-spectral topology on* $Spec_{\mathbb{S}}(A)$*.*

Proof. Let $P \in U_{\mathbb{S}}(I)$. Then $P \in Spec_{\mathbb{S}}(A)$ and $I \wedge (s] \nsubseteq P$, for every $s \in \mathbb{S}$. It follows from $P \in Spec_{\mathbb{S}}(A)$ and Proposition 3.17 that there exists $s_P \in \mathbb{S}$ such that $(P : s) \subseteq (P : s_P)$, for every $s \in \mathbb{S}$.
On the other hand, $I \wedge (s_P] \nsubseteq P$, thus there exists the subset $\mathbb{A} = \{x_j : x_j \in A\}$ of A such that $x_j \in I \wedge (s_P]$ and $x_j \notin P$. We want to show that $U_{\mathbb{S}}(I) = \bigcup_{x_j\in\mathbb{A}} U_{\mathbb{S}}(x_j)$.
Since $x_j \in I \wedge (s']$, hence $x_j \in I$ and there exists $n_j \in \mathbb{N}$ such that $x_j \leq n_j s_P$. It follows that $x_j = x_j \wedge (n_j s_P) \leq n_j(x_j \wedge s_P)$. We claim that $x_j \notin (P : s_P)$. Let $x_j \in (P : s_P)$. Then $x_j \wedge s_P \in P$, we get $n_j(x_j \wedge s_P) \in P$. Hence $x_j \in P$, which is a contradiction. Then $x_j \notin (P : s_P)$. We obtain $x_j \notin (P : s)$, for every $s \in \mathbb{S}$. Thus $x_j \wedge s \notin P$, for every $s \in \mathbb{S}$. It follows that $P \in U_{\mathbb{S}}(x_j)$. Then $U_{\mathbb{S}}(I) \subseteq \bigcup_{x_j\in\mathbb{A}} U_{\mathbb{S}}(x_j)$.
Let $Q \in \bigcup_{x_j\in\mathbb{A}} U_{\mathbb{S}}(x_j)$. Then there exists $j \in J$ such that $Q \in U_{\mathbb{S}}(x_j)$. Hence $x_j \wedge s \notin Q$, for all $s \in \mathbb{S}$. It follows from Lemma 2.8 that $(x_j] \wedge (s] = (x_j \wedge s] \nsubseteq Q$. On the other hand $x_j \in I$, then $I \wedge (s] \nsubseteq Q$. We obtain $Q \in U_{\mathbb{S}}(I)$. Thus $\bigcup_{x_j\in\mathbb{A}} U_{\mathbb{S}}(x_j) \subseteq U_{\mathbb{S}}(I)$.
Therefore $\bigcup_{x_j\in\mathbb{A}} U_{\mathbb{S}}(x_j) = U_{\mathbb{S}}(I)$. □

Note: Let f be an injective function from A to B. It is clear that for every X and Y in A, the following holds $f(X \cap Y) = f(X) \cap f(Y)$.

Theorem 4.8. *Let* A_1 *and* A_2 *be two complete MV-algebras and let* $f : A_1 \to A_2$ *be an injective map. The map* $\psi : Spec_{f(\mathbb{S})}(A_2) \to Spec_{\mathbb{S}}(A_1)$ *is defined by* $\psi(Q) = f^{-1}(Q)$*, for every* $Q \in Spec_{f(\mathbb{S})}(A_2)$*. This map is continuous.*

Proof. The map ψ is clearly well-defined. Now, let I be an ideal of A_1. Then

$$\begin{aligned}\psi^{-1}(V_{\mathbb{S}}(I)) &= \{Q \in Spec_{f(\mathbb{S})}(A_2) : f^{-1}(Q) \in V_{\mathbb{S}}(I)\}\\ &= \{Q \in Spec_{f(\mathbb{S})}(A_2) : (s] \wedge I \subseteq f^{-1}(Q) \text{ for some } s \in \mathbb{S}\}\\ &= \{Q \in Spec_{f(\mathbb{S})}(A_2) : f((s] \wedge I) \subseteq Q \text{ for some } s \in \mathbb{S}\}\\ &= \{Q \in Spec_{f(\mathbb{S})}(A_2) : f((s]) \wedge f(I) \subseteq Q \text{ for some } s \in \mathbb{S}\}\\ &= V_{f(\mathbb{S})}(f(I)).\end{aligned}$$

Therefore, ψ is continuous. □

Next, we present some simple properties of ideals that are needed to prove the topological properties.

Lemma 4.9. *Let* I *and* J *be two ideals of* A *and* $x, y \in A$*.*
(1) *If* $x \wedge I \subseteq J$*, then* $(x] \wedge I \subseteq J$*.*
(2) $(I : (x]) \subseteq (I : x)$*.*
(3) $(x] \wedge (y] \wedge (I : y) \subseteq (x] \wedge I$*.*
(4) $(x \wedge y] \wedge (I : y) \subseteq (x] \wedge I$*.*
(5) $(I : x) = ((I : x) : x)$*.*

Proof. (1) Let $\alpha \in (x] \wedge I$. Then $\alpha \in I$ and there exists $n \in \mathbb{N}$ such that $\alpha \leq nx$. On the other hand, $\alpha = \alpha \wedge nx \leq n(\alpha \wedge x)$. By hypothesis, we get $\alpha \in J$.
(2) Let $\alpha \in (I : (x])$. Then $\alpha \wedge (x] \subseteq I$. Hence $\alpha \wedge x \in I$. We get $\alpha \in (I : x)$.
(3) Let $\alpha \in (x] \wedge (y] \wedge (I : y)$. Then $\alpha \in (x], \alpha \wedge y \in I$ and, there exists $n \in \mathbb{N}$ such that $\alpha \leq ny$. Hence $\alpha = \alpha \wedge ny \leq n(\alpha \wedge y)$. We obtain $\alpha \in I$. Thus $\alpha \in (x] \wedge I$.
(4) It is clear by (3) and Lemma 2.8 (3).
(5) Let $a \in (I : x)$. Then $a \wedge x \in I$. On the other hand $I \subseteq (I : x)$, hence $a \wedge x \in (I : x)$. Thus $a \in ((I : x) : x)$. We have $(I : x) \subseteq ((I : x) : x)$. Now, let $a \in ((I : x) : x)$. Then $a \wedge x \in (I : x)$, hence $a \wedge x \wedge x \in I$. We obtain $a \wedge x \in I$ implies that $a \in (I : x)$. Thus $((I : x) : x) \subseteq (I : x)$. Therefore $(I : x) = ((I : x) : x)$. □

Proposition 4.10. *Let $Y \subseteq Spec_{\mathbb{S}}(A)$. Then $\overline{Y} = V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_p))$.*

Proof. By Proposition 3.17, for every $P \in Y$ there exists $s_P \in S$ such that $(P : s_P)$ is the maximal element of $\{(P : s)\}_{s \in \mathbb{S}}$. Now, if $P^{'} \in Y$. Obviously, $\bigcap_{P \in Y} (P : s_P) \subseteq (P^{'} : s_{P^{'}})$. It follows that $s_{P^{'}} \wedge (\bigcap_{P \in Y} (P : s_P)) \subseteq P^{'}$. By Lemma 4.9(1), we obtain $(s_{P^{'}}] \wedge (\bigcap_{P \in Y} (P : s_P)) \subseteq P^{'}$. This gives $P^{'} \in V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_P))$. We get $Y \subseteq V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_P))$. Hence $\overline{Y} \subseteq V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_P))$. We aim to show that $V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_P))$ is the smallest closed set containing Y. Now, let there exist the ideal I such that $Y \subseteq V_{\mathbb{S}}(I)$. For this purpose, it must be shown that $V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_P)) \subseteq V_{\mathbb{S}}(I)$.
Let $P \in Y$. It follows from $Y \subseteq V_{\mathbb{S}}(I)$ that there exists $s \in \mathbb{S}$ such that $(s] \wedge I \subseteq P$. Hence $I \subseteq (P : (s]) \subseteq (P : s) \subseteq (P : s_P)$, for every $P \in Y$. We obtain $I \subseteq \bigcap_{P \in Y} (P : s_P)$. It follows from Lemma 4.2 part (1) that $V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_P)) \subseteq V_{\mathbb{S}}(I)$. Therefore $\overline{Y} = V_{\mathbb{S}}(\bigcap_{P \in Y} (P : s_p))$. □

Theorem 4.11. *Let $P, Q \in Spec_{\mathbb{S}}(A)$. Then the following statements hold:*
(1) $V_{\mathbb{S}}(P) = V_{\mathbb{S}}((P : s))$, *for all* $s \in \mathbb{S}$.
(2) $\overline{\{P\}} = V_{\mathbb{S}}(P)$.
(3) $\overline{\{P\}} = \overline{\{Q\}}$ *if and only if there exists* $s \in \mathbb{S}$ *such that* $(P : s) = (Q : s)$.

Proof. (1) Since $P \subseteq (P : s)$, by Lemma 4.2 (1), we obtain $V_{\mathbb{S}}((P : s)) \subseteq V_{\mathbb{S}}(P)$, for all $s \in \mathbb{S}$.
Now, let $P^{'} \in V_{\mathbb{S}}(P)$. Then there exists $s^{'} \in \mathbb{S}$ such that $(s^{'}] \wedge P \subseteq P^{'}$. It follows from Lemma 4.9(4) that $(s^{'} \wedge s] \wedge (P : s) \subseteq (s^{'}] \wedge P \subseteq P^{'}$, for all $s \in \mathbb{S}$. Hence $P^{'} \in V_{\mathbb{S}}((P : s))$, for all $s \in \mathbb{S}$. Thus $V_{\mathbb{S}}(P) \subseteq V_{\mathbb{S}}((P : s))$, for all $s \in \mathbb{S}$. Therefore $V_{\mathbb{S}}(P) = V_{\mathbb{S}}((P : s))$, for all $s \in \mathbb{S}$.
(2) From part 1 of this theorem and Proposition 4.10, it follows.
(3) It follows from (1) and (2) that if there exists $s \in \mathbb{S}$ such that $(P : s) = (Q : s)$, then $\overline{\{P\}} = \overline{\{Q\}}$.
Conversely, let $\overline{\{P\}} = \overline{\{Q\}}$. It follows from (2) that $V_{\mathbb{S}}(P) = V_{\mathbb{S}}(Q)$. On the other hand $P \in V_{\mathbb{S}}(P)$, hence there exists $s_1 \in \mathbb{S}$ such that $(s_1] \wedge Q \subseteq P$. So $Q \subseteq (P : (s_1])$ and it follows from Lemma 4.9 that $Q \subseteq (P : (s_1]) \subseteq (P : s_1)$. Then by Proposition 3.17, $Q \subseteq (P : s_1) \subseteq (P : s_P)$. Similarly, from Lemma 4.9 and Proposition 3.17, it follows that there exist s_2 and s_Q such that $P \subseteq (Q : (s_2]) \subseteq (Q : s_Q)$. Now, put $s = s_P \wedge s_Q$ and let $x \in (Q : s)$. Then $x \wedge s = x \wedge s_P \wedge s_Q \in Q$ hence $x \wedge s_P \wedge s_Q \in (P : s_P)$. Thus $x \wedge s_P \wedge s_Q \wedge s_P \in P$ implies that $x \in (P : s)$. Therefore $(Q : s) \subseteq (P : s)$. Similarly, it can be shown that $(P : s) \subseteq (Q : s)$. Hence there exists $s \in \mathbb{S}$ such that $(P : s) = (Q : s)$. □

Proposition 4.12. *Assume that $e \in B(A)$. Then the following statements hold:*
(1) $V_{\mathbb{S}}((e]) = Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}((e^*])$.
(2) $U_{\mathbb{S}}(e)$ *is clopen.*
(3) *If $Spec_{\mathbb{S}}(A)$ is a connected space, then either $(e] \cap \mathbb{S} \neq \emptyset$ or $(s \odot e^*] \cap \mathbb{S} \neq \emptyset$, for some $s \in \mathbb{S}$.*

Proof. Let $P \in V_{\mathbb{S}}((e])$. Then $P \in Spec_{\mathbb{S}}(A)$ and $(s_1] \wedge (e] \subseteq P$ for some $s_1 \in \mathbb{S}$. We want to show that $P \notin V_{\mathbb{S}}((e^*])$. By the contrary, let $P \in V_{\mathbb{S}}((e^*])$. Then $P \in Spec_{\mathbb{S}}(A)$ and $(s_2] \wedge (e] \subseteq P$ for some $s_2 \in \mathbb{S}$. Put $s = s_1 \wedge s_2$. It follows that $(s] \wedge (e] = (s_1] \wedge (s_2] \wedge (e] \subseteq (s_1] \wedge (e] \subseteq P$ and $(s] \wedge (e^*] = (s_1] \wedge (s_2] \wedge (e^*] \subseteq (s_2] \wedge (e^*] \subseteq P$. Hence

$$\begin{aligned} ((s] \wedge (e]) \vee ((s] \wedge (e^*]) \subseteq P &\Rightarrow (s] \wedge ((e] \vee (e^*]) \subseteq P \\ &\Rightarrow (s] \wedge (e \oplus e^*] \subseteq P \\ &\Rightarrow (s] \wedge (1] \subseteq P \\ &\Rightarrow s \in P. \end{aligned}$$

which is a contradiction. So $V_{\mathbb{S}}((e]) \subseteq Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}((e^*])$. Now, let $P \in Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}((e^*])$. Then $P \in Spec_{\mathbb{S}}(A)$ and $(s] \wedge (e^*] \nsubseteq P$, for all $s \in \mathbb{S}$. It follows that $s \wedge e^* \notin P$, for all $s \in \mathbb{S}$. On the other hand $0 = e \wedge e^* \in P$ hence there exit $s' \in \mathbb{S}$ such that $s' \wedge e \in P$. Then $(s'] \wedge (e] \subseteq P$. We obtain $P \in V_{\mathbb{S}}((e])$ thus $Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}((e^*]) \subseteq V_{\mathbb{S}}((e])$. Therefore $V_{\mathbb{S}}((e]) = Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}((e^*])$, for every $e \in B(A)$.
(2) It is clear from part (1).
(3) Let $e \in B(A)$. We claim that $V_{\mathbb{S}}((e]) \cap V_{\mathbb{S}}((s \odot e^*]) = \emptyset$ and $V_{\mathbb{S}}((e]) \cup V_{\mathbb{S}}((s \odot e^*]) = Spec_{\mathbb{S}}(A)$ for some $s \in \mathbb{S}$. First, let $P \in V_{\mathbb{S}}((e]) \cap V_{\mathbb{S}}((s \odot e^*])$. Then there exist $s_1, s_2 \in \mathbb{S}$ such that $(s_1] \wedge (e] \subseteq P$ and $(s_2] \wedge (s \odot e^*] \subseteq P$. Now, put $s' = s_1 \wedge s_2$. Thus $s' \wedge e \in P$ and $s' \wedge (s \odot e^*) \in P$. By Theorem 2.5, we obtain $s' \wedge s \leq s' \wedge (e \vee s) = s' \wedge (e \oplus (e^* \odot s) \leq (s' \wedge e) \oplus (s' \wedge (s \odot e^*))$. Hence $s \wedge s' \in P$, which is a contradiction. Thus $V_{\mathbb{S}}((e]) \cap V_{\mathbb{S}}((s \odot e^*]) = \emptyset$. Also, it follows from Lemma 4.2 and Theorems 2.5 and 2.4 that

$$\begin{aligned} V_{\mathbb{S}}(e) \cup V_{\mathbb{S}}(s \odot e^*) &= V_{\mathbb{S}}((e]) \cup V_{\mathbb{S}}((s \odot e^*]) \\ &= V_{\mathbb{S}}((e] \cap (s \odot e^*]) \\ &= V_{\mathbb{S}}((e \wedge (s \odot e^*)]) \\ &= V_{\mathbb{S}}((e \wedge (s \wedge e^*))) \\ &= V_{\mathbb{S}}(0) \\ &= Spec_{\mathbb{S}}(A). \end{aligned}$$

Hence $U_{\mathbb{S}}(e) \cup U_{\mathbb{S}}(s \odot e^*) = Spec_{\mathbb{S}}(A)$ and $U_{\mathbb{S}}(e) \cap U_{\mathbb{S}}(s \odot e^*) = \emptyset$. By hypothesis, we get either $U_{\mathbb{S}}(e) = Spec_{\mathbb{S}}(A)$ or $U_{\mathbb{S}}(s \odot e^*) = Spec_{\mathbb{S}}(A)$. Then $V_{\mathbb{S}}(e) = \emptyset$ or $V_{\mathbb{S}}(s \odot e^*) = \emptyset$. It follows Lemma 4.2 (7) that either $(e] \cap \mathbb{S} \neq \emptyset$ or $(s \odot e^*] \cap \mathbb{S} \neq \emptyset$, for some $s \in \mathbb{S}$. □

Proposition 4.13. *Let X be a compact open subset of $Spec_{\mathbb{S}}(A)$. Then there exists $a \in A$ such that $X = U_{\mathbb{S}}(a)$.*

Proof. Since X is an open subset of $Spec_{\mathbb{S}}(A)$ we obtain that there exists a subset B of A such that $X = \bigcup_{b \in B} U_{\mathbb{S}}(b)$. Hence there exists $C = \{b_i : b_i \in B, 1 \leq i \leq n\}$ such that $X = \bigcup_{i=1}^{n} U_{\mathbb{S}}(b_i)$. Put $a = \bigvee_{i=1}^{n} b_i$, then it follows from Lemma 4.3 that $X = U_{\mathbb{S}}(a)$. □

Example 4.14. *Let $A = [0,1]^*$ be an MV-algebra ultrapower of $[0,1]$ with the infinities ε. Let $S = \{1, s\}$ and $s = \varepsilon$. Obviously, S is a $\wedge$-closed system. Take $X = \{1, \frac{1}{2}\}$. Note that in A, which is a chain, every ideal is prime, so the zero ideal is an S-prime ideal. Obviously, $V_{\mathbb{S}}(X) = \{\{0\}\}$ and $V_S((X]) = \emptyset$.*

Note: • It follows from Lemma 4.2 (1) that $V_S((X]) \subseteq V_S(X)$, for every subset X of A.
• The subset X is called down-set, if $x \in X$ and $y \in A$ such that $y \leq x$, then $y \in X$.

Lemma 4.15. *Let X be a down-set of A. Then $V_S((X]) = V_S(X)$.*

Proof. By the previous note, it suffices to prove that $V_S(X) \subseteq V_S((X])$.
Let $P \in V_S(X)$. Then there exists $s \in S$ such that $(s] \cap X \subseteq P$. We claim that $(s] \wedge (X] \subseteq P$. Now, let $\alpha \in (s] \wedge (X]$. Then $\alpha \leq ns$ and $\alpha \leq x_1 \oplus ... \oplus x_m$ for some $m, n \in \mathbb{N}$ and $x_i \in X$. Hence $\alpha \leq ns \wedge (x_1 \oplus ... \oplus x_m) \leq (ns \wedge x_1) \oplus ... \oplus (ns \wedge x_m)$. Obviously, $ns \wedge x_i \leq x_i$ for every $1 \leq i \leq m$. We have $ns \wedge x_i \in X$ so $ns \wedge x_i \in (X]$. On the other hand $ns \wedge x_i \in (s]$ implies that $ns \wedge x_i \in (s] \cap X$. Hence $ns \wedge x_i \in P$, for every $1 \leq i \leq m$. we get $\alpha \in P$. Hence $P \in V_S((X])$. □

Theorem 4.16. *Let $V_{\mathbb{S}}(X) = V_{\mathbb{S}}((X])$, for every subset X of A. Then the following sentences hold:*
(1) *$Spec_{\mathbb{S}}(A)$ is a compact topological space.*
(2) *$Spec_{\mathbb{S}}(A)$ is a connected topological space.*

Proof. (1) Assume that $X = \{a_i : a_i \in A\}$ and $Spec_{\mathbb{S}}(A) = \bigcup_{a_i \in X} U_{\mathbb{S}}(a_i)$, which is an open cover of $Spec_{\mathbb{S}}(A)$. It follows from Lemma 4.2 that

$$\begin{aligned} Spec_{\mathbb{S}}(A) &= \bigcup_{a_i \in X} (Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}(a_i)) = Spec_{\mathbb{S}}(A) \setminus \bigcap_{a_i \in X} V_{\mathbb{S}}(a_i) \\ &= Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}(\bigcup_{a_i \in X} (a_i)) = Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}(X) \\ &= Spec_{\mathbb{S}}(A) \setminus V_{\mathbb{S}}((X]). \end{aligned}$$

Hence $V_{\mathbb{S}}((X]) = \emptyset$. It follows from Lemma 4.2 that $(X] \cap \mathbb{S} \neq \emptyset$. Then there exists $s \in \mathbb{S}$ and $a_{i_1}, ..., a_{i_k} \in X$ such that $s \leq a_{i_1} \oplus ... \oplus a_{i_k}$. Put $Y = \{a_{i_1}, ..., a_{i_k}\}$. We obtain $\mathbb{S} \cap (Y] = \emptyset$. By Lemma 4.2, we get $V_{\mathbb{S}}((Y]) \neq \emptyset$. On the other hand, $\emptyset = V_{\mathbb{S}}((Y]) = V_{\mathbb{S}}(Y) = V_{\mathbb{S}}(\bigcup_{j=1}^{k} (a_{i_j}) = \bigcap_{j=1}^{k} V_{\mathbb{S}}(a_{i_j})$. Hence

$$Spec_{\mathbb{S}}(A) = Spec_{\mathbb{S}}(A) \setminus \bigcap_{j=1}^{k} V_{\mathbb{S}}(a_{i_j}) = \bigcup_{j=1}^{k} U_{\mathbb{S}}(a_{i_j}).$$

Therefore $Spec_{\mathbb{S}}(A)$ is a compact topological space.
(2) Let I and J be two ideals of A such that $V_{\mathbb{S}}(I) \cup V_{\mathbb{S}}(J) = Spec_{\mathbb{S}}(A)$ and $V_{\mathbb{S}}(I) \cap V_{\mathbb{S}}(J) = \emptyset$. By Lemma 4.2, we get $V_{\mathbb{S}}(I \cap J) = Spec_{\mathbb{S}}(A)$ and $V_{\mathbb{S}}(I \cup J) = \emptyset$. Again, by using Lemma 4.2 we have $(I \cup J) \cap \mathbb{S} \neq \emptyset$. So $(I \cap \mathbb{S}) \cup (J \cap \mathbb{S}) \neq \emptyset$ hence $I \cap \mathbb{S} \neq \emptyset$ or $J \cap \mathbb{S} \neq \emptyset$. It follows from Lemma 4.2 that $V_{\mathbb{S}}(I) = \emptyset$ or $V_{\mathbb{S}}(J) = \emptyset$. Therefore $Spec_{\mathbb{S}}(A)$ is a connected topological space. □

Theorem 4.17. *$Spec_{\mathbb{S}}(A)$ is a T_0-space if and only if every $\mathbb{S}$-prime ideal is a prime ideal of A.*

Proof. Let $Spec_{\mathbb{S}}(A)$ be a T_0-space and let P be an $\mathbb{S}$-prime ideal. Then by Proposition 3.18, there exists $s \in \mathbb{S}$ such that $(P : s)$ is a prime ideal. By Lemma 4.9(5), we have $(P : s) = ((P : s) : s)$. It follows from Theorem 4.11 that $\overline{\{P\}} = \overline{\{(P : s)\}}$. By Theorem 2.19, we obtain that $P = (P : s)$. Hence P is a prime ideal of A.
Conversely, suppose that every $\mathbb{S}$-prime ideal is a prime ideal. Let P_1 and P_2 be two $\mathbb{S}$-prime ideal such that $P_1 \neq P_2$ and $\overline{\{P_1\}} = \overline{\{P_2\}}$. By Theorem 4.9, we get $(P_1 : s) = (P_2 : s)$ for some $s \in \mathbb{S}$. It follows from $P_1 \cap S = P_2 \cap S = \emptyset$ that $s \notin P_1 \cup P_2$ and $P_1 = (P_1 : s) = (P_2 : s) = P_2$. By Theorem 2.19, we obtain $Spec_{\mathbb{S}}(A)$ is a T_0-space. □

Theorem 4.18. *The following statements are equivalent:*
(1) *$Spec_{\mathbb{S}}(A)$ is a T_1-space.*
(2) *Every $\mathbb{S}$-prime ideal is a prime ideal and a maximal element of the set $\Gamma = \{I \in Spec(A) : I \cap \mathbb{S} = \emptyset\}$.*
(3) *$Dim_{\mathbb{S}}(A) = 0$.*

Proof. $1 \Rightarrow 2$) Let P be an $\mathbb{S}$-prime ideal. It follows from Proposition 3.18 that there exists $s \in \mathbb{S}$ such that $(P : s)$ is a prime ideal. Also, by Theorems 2.19 and 4.11, we get $\overline{\{P\}} = \{P\} = V_{\mathbb{S}}(P)$.
Obviously, $(P : s) \cap \mathbb{S} = \emptyset$ and $(s] \wedge P \subseteq P \subseteq (P : s)$. Then $(P : s) \in V_{\mathbb{S}}(P) = \overline{\{P\}} = \{P\}$. Hence $(P : s) = P$ thus P is a prime ideal. Now, we want to show that P is a maximal element of Γ. By contrary, let Q be a prime ideal such that $P \subseteq Q$ and $Q \cap \mathbb{S} = \emptyset$. Hence $(s] \wedge P \subseteq Q$, for every $s \in \mathbb{S}$. We obtain $Q \in V_{\mathbb{S}}(P) = \overline{\{P\}} = \{P\}$. Thus $P = Q$. Therefore P is a maximal element of Γ.
$2 \Rightarrow 3$) It is clear.
$3 \Rightarrow 1$) By hypothesis and Proposition 3.4, we get every $\mathbb{S}$-prime ideal is a prime ideal. Let P be a prime ideal. We want to show that $\overline{\{P\}} = \{P\}$. Let $Q \in \overline{\{P\}}$. It follows from Theorem 4.11 that $Q \in V_{\mathbb{S}}(P)$. Then there exists $s \in \mathbb{S}$ such that $(s] \wedge P \subseteq Q$. Hence $P \subseteq (Q : s)$. On the other hand, since Q is a prime ideal, we get $Q = (Q : s)$, thus $P \subseteq Q$. By hypothesis, we obtain $P = Q$. Hence $Q \in \{P\}$, so $\overline{\{P\}} \subseteq \{P\}$. Obviously, $\{P\} \subseteq \overline{\{P\}}$. Thus $\overline{\{P\}} = \{P\}$. It follows from Theorem 2.19 that $Spec_{\mathbb{S}}(A)$ is a T_1-space. □

Theorem 4.19. (1) *If the zero ideal is an $\mathbb{S}$-prime ideal but not an $\mathbb{S}$-maximal ideal, then $Spec_{\mathbb{S}}(A)$ is not a Hausdorff space.*
(2) *If the zero ideal is an $\mathbb{S}$-maximal ideal, then $Spec_{\mathbb{S}}(A)$ is a T_1-space.*

Proof. By the contrary, let $Spec_{\mathbb{S}}(A)$ be a Hausdorff space. Then $Spec_{\mathbb{S}}(A)$ is a T_1-space. It follows from Theorem 4.18 that $Dim_{\mathbb{S}}(A) = 0$. We claim that the zero ideal is an $\mathbb{S}$-maximal ideal. Let Q be an ideal of A. We consider two cases:

Case 1: if $Q \cap \mathbb{S} \neq \emptyset$, then the zero ideal is an $\mathbb{S}$-maximal ideal.
Case 2: if $Q \cap \mathbb{S} = \emptyset$, then by Theorem 2.12, there exists $P \in Spec(A)$ such that $P \cap \mathbb{S} = \emptyset$ and $Q \subseteq P$. Thus P is an $\mathbb{S}$-prime ideal. Obviously, $\{0\} \subseteq P$ implies that $P = \{0\}$(*since* $Dim_{\mathbb{S}}(A) = 0$). Hence $Q = \{0\}$ and $(s] \wedge Q = \{0\}$, for every $s \in \mathbb{S}$. We have the zero ideal is an $\mathbb{S}$-maximal ideal, which is a contradiction. Therefore $Spec_{\mathbb{S}}(A)$ is not a Hausdorff space.
2) By hypothesis $Dim_{\mathbb{S}}(A) = 0$. It follows from Theorem 4.18 that $Spec_{\mathbb{S}}(A)$ is a T_1-space. □

5 Conclusion and future work

The concept of an S-prime ideal has been introduced for every proper ideal of an MV-algebra, with the set of all S-prime ideals are denoted as $Spec_S(A)$. The study of these ideals has led to the conclusion that any prime ideal disjoint from the $\wedge$-closed system S is an S-prime ideal. An example has illustrated that the intersection of two S-prime ideals is not necessarily an S-prime ideal. Furthermore, it has been proven that if an ideal J contains at least one element from S, the intersection of J with any S-prime ideal is also an S-prime ideal. We have also demonstrated that the intersection of all prime ideals containing a proper ideal I is equivalent to the intersection of S-prime ideals that, for each $s \in S$, contain $I \wedge (s]$. This equivalence has implied that every proper ideal is expressible as an intersection of S-prime ideals. Finally, the study of S-prime ideals under the isomorphism of an MV-algebra f has led to the conclusion that the inverse image of any $f(S)$-prime ideal is an S-prime ideal. In addition, we have defined S-maximal ideals and proven that every S-maximal ideal is an S-prime ideal. Also, an equivalent definition for S-maximal ideals has also been provided.
Using the results from the study of S-prime ideals and the definitions of the sets $V_S(I)$ and $U_S(I)$ for every ideal I, we have proved that the set $\tau = \{V_{\mathbb{S}}(I) : I \in Id(A)\}$ forms a topology on $Spec_{\mathbb{S}}(A)$, with the set $\Gamma = \{U_{\mathbb{S}}(a) : a \in A\}$ serving as a basis for this topology. This topology has been referred to as the $\mathbb{S}$-spectral topology for the complete $\wedge$-closed system $\mathbb{S}$. It has been proven that the $\mathbb{S}$-spectral topology on $Spec_{\mathbb{S}}(A)$ is a T_0-space if and only if the sets $Spec_{\mathbb{S}}(A)$ and $Spec(A)$ are equivalent. It has been concluded that every open and compact subset is equivalent to one of the basic open sets. Finally, it has been proven that $Spec_{\mathbb{S}}(A)$ is a T_1-space if and only if $Dim_{\mathbb{S}}(A) = 0$.

S-prime ideals have exhibited features similar to those of prime ideals. By examining these features, we aim to define S-minimal ideals and explore the inverse topology on the set of all S-minimal ideals in MV-algebras.

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