

Fuzzy geometric spaces and their application to Persian medicine as a multicriteria decision-making problem

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Abstract

In this article, we introduce fuzzy geometric spaces as a generalization of broader concepts of fuzzy sets, such as q-rung orthopair, picture, spherical, and multi-fuzzy sets. The primary objective of this study is to develop a comprehensive framework capable of handling diverse variable types, thereby overcoming the limitations of existing methods that are typically restricted to specific categories (e.g., discrete, continuous, qualitative, or quantitative). We define distance and similarity measures for these spaces and present an algorithm for their application in multicriteria decision-making problems. A key advantage of this framework is its ability to accommodate diverse fuzzification methods within a single problem, a capability absent in conventional fuzzy methodologies. Finally, this framework is applied to analyze Mizaj questionnaires in Persian medicine, yielding more accurate and reliable results.

Keywords: Fuzzy geometric space, multi-degree of membership, multicriteria decision-making problems.

1 Introduction

In the real life such as in engineering, medical, management, economy and social sciences, there are many vague and uncertain problems which need to make a suitable decision with respect to some criteria, properties, limitation, etc. The theory of fuzzy sets proposed by Zadeh in 1965 [47], has shown successful applications in various fields. For example analyzing the answers of the questionnaires is a famous and applicable method to identify and modeling the phenomena and problems in many sciences such as psychology, medicine, economy, social sciences, etc. A beneficial way to analyze the answers in the questionnaire method is fuzzyfying them.

In this article, we present a generalization of the multi-fuzzy set by using the concept of geometric space; then, by using the concept of fuzzy geometric space propose a method for modeling multicriteria decision-making problems. Finally, we model the problem of Mizaj (temperament) diagnosis in Persian medicine as an example of multicriteria decision-making problems; therefore, we continue the literature and motivation of this article in two separate topics: fuzzy sets and their extensions and multicriteria decision-making problems.

1.1 Fuzzy sets and their extensions

In the theory of fuzzy sets, there are two values in interval $[0, 1]$, which show the degree of membership and nonmembership of each element in a fuzzy set. The sum of these values is equal to 1. However, in reality, this sum may not always be exactly 1. In such cases, there is said some degree of indeterminacy. Atanassov in 1986 [10] introduced the concept of intuitionistic fuzzy set as a generalization of fuzzy sets. Afterwards Yager in 2013 [44], Senapati and Yager in 2019 [35], Yager in 2017 [45] and Alcantud in 2023 [5], defined the concept of Pythagorean fuzzy set, Fermatean fuzzy set, q-rung orthopair fuzzy set and complementary fuzzy set, respectively, which develop the domain of degree

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of membership and nonmembership of each element. Another development of the domain of degree of membership and nonmembership occurred through the use of complex values, which is suitable for modeling problems involving both uncertainty and periodicity simultaneously, such as wave functions in quantum mechanics, impedance in electrical engineering, and complex amplitude [3, 8]. Picture and spherical fuzzy sets were introduced in 2014 and 2018 [9, 13], respectively, as extensions of intuitionistic and Pythagorean fuzzy sets, resp, by adding a neutral membership degree, effectively modeling yes, abstain, no, and refusal voting scenarios. Sebastian and Ramakrishnan in 2011 in based on ordered sequences of membership function introduced another generalization of intuitionistic fuzzy sets [34]. In this method, the number of the parameters that determine the multi-membership degree, is the same for all elements. The applicability of this concept in real-world scenarios surpasses that of others. However, in many situation in real life such as in medical diagnosis and management, the parameters used to identify the degree of multi-membership for each element is not same. For example the number of stool types with respect to the Bristol stool chart is not equal to the types of abdominal pains (see Example 2.2 and Example 2.13). In the table of the risk factors of energy management, the number for parameters of the policy, financial, operational, efficiency and market risks as considered in [42] are 3, 4, 6, 4, 3. To address this, this article generalizes the concepts of fuzzy set, intuitionistic, Pythagorean, Fermatean, and q-rung orthopair, as well as picture, spherical and finite dimensional multi-fuzzy set to the fuzzy geometric space.

The geometric space was defined by Freni in 2004 [17]. The category of geometric spaces as a generalization of the category of algebraic hyperstructures, category of topological spaces and the category of hypergraphs was introduced by Porhaghani and Torabi between 2021 and 2024 [25, 26, 27, 38]. In the first part of this article we present the concepts of distance and similarity measures for fuzzy geometric spaces, along with some examples, and fuzzy mean of a set of fuzzy geometric spaces. Subsequently, we describe an algorithm for applying fuzzy geometric spaces to multicriteria decision-making problems.

1.2 Multicriteria decision making problems

Medical diagnosis is an important multicriteria decision making problem. In fact, to diagnose the disease and syndromes, the experts compare the patient's symptoms with known diseases and syndromes with similar symptoms; thus, for each disease and syndromes, if a suitable questionnaire is designed and analyzing the answers by a suitable method, then the experts can diagnose them more carefully. The methods in traditional medicines enable specialists to diagnose diseases and syndromes through symptoms without the need for modern tools. Self-administered questionnaires are widely used in traditional medicine to identify certain syndromes , Numerous examples can be found in the literature and on the Internet concerning Traditional Chinese medicine (TCM), traditional Indian medicine (Ayurveda) and traditional Iranian medicine (Persian medicine). However, the methods used to design and analyze the responses to these questionnaires are often unsuitable, leading to unreliable results. For instance, for example Naisal and Reji Kumar in their work utilized [23] mathematical modeling of "Ayurveda Doshas" through fuzzy soft topology. They demonstrated that many online tests are not valid and consulting an expert is necessary to determine one's "Doshas". Similarly, in some researches on Persian medicine, scholars. Prefer to personally interview all volunteers to determine their Mizaj rather than relying on self-administered Mizaj questionnaire (see for an example [15]). Wu in [43] introduced a method based on intuitionistic fuzzy set to realize the diagnostic decision making for TCM which make it possible to achieve intelligent diagnostics reasonably and easily in favor of the differentiation of TCM syndrome for the whole diagnosis.

One of the oldest traditional medicine is Persian medicine. Mizaj (temperament) is a core concept in Persian medicine that plays key role in defining health and illness. In this view, each person has specific Mizaj and in many diseases, characteristic changes, that can be recognize based on a set of well known and classified concepts, occur in the person's Mizaj [24]. According to "The Canon of Medicine" ([18]), Mizaj is a quality that arises from the interaction of four elements (warmness, coldness, wetness and dryness) and creates the physical and mental characteristics of human beings. Avicenna classified the general population into nine main groups based on their mizaj which results from the combination of two sets "cold, moderate and warm", and "wet, moderate and dry". He described ten criteria to cluster individuals into these nine groups. Until now, some researchers have tried to design and validate self-administered Mizaj questionnaires to identify individuals" Mizaj. Three well-known self-administered Mizaj questionnaires have been introduced in [2, 22, 33]. These questionnaires, like the most questionnaires in Persian medicine, use a similar method to analyze answers, which is not infallible. In the last section of this article, we provide the conditions to compare the results of these three questionnaires. Moreover, we present a method that utilises fuzzy geometric space, to provide more accurate results of these questionnaires. In our method each parameter is compared against its corresponding criterion. The resulting similarity scores, calculated using the measure introduced in the first part of this article, are then aggregated into a weighted sum. Additionally, our method identifies the of warmth, coldness, wetness, dryness and moderation for each case. Considering the capabilities of fuzzy geometric spaces, the finding of this study offer valuable applications for multicriteria decision-making in medicine. In particular, they can support the development of

more precise questionnaires for assessing Mizaj and Sue-Mizaj (distemperament) in Persian medicine.

1.3 Outline of this article

One of the primary motivations of this study is to develop a unified and comprehensive framework applicable to all types of variables. In many existing studies, [19, 31, 39, 46] methods are often restricted to specific categories of variables, such as discrete, continuous, qualitative, or quantitative types. This limitation reduces the flexibility and practical applicability of these methods in real-world applications, where diverse variable types frequently coexist within the same dataset. To address this gap, the present study proposes a generalized methodology applicable regardless of variable type. The proposed approach is designed to handle discrete and continuous variables, as well as qualitative and quantitative data, without requiring structural modifications to the core framework. This generality enhances the robustness and adaptability of the method across diverse applications. Accordingly, the primary motivation of this article is to provide a unified solution that overcomes the limitation of variable-specific methods and ensures broad applicability across different data structures. By establishing a framework independent of variable type, this study advances a more flexible and comprehensive analytical tool suitable for a wide range of theoretical and practical problems.

The rest of this article is organized as follows. In Section 2, we first recall some well-known concepts; then, introduce the fuzzy geometric space, its properties, related concepts, and propositions. In Section 3, we describe an algorithm to apply fuzzy geometric space in multicriteria decision-making problems. In Section 4, first, we discuss the history of Persian medicine and articles related to modern Mizaj determination based on the Canon of Medicine. Then, we prepare a self-report questionnaire for Mizaj determination by analyzing three well-known questionnaires under the supervision of an experienced specialist in this field (one of the authors). Then we provide the inputs for the algorithm described in Section 3. Using an algorithm implemented in Maple (and Python) software (Appendix B (and Appendix C)), we compare the method of analyzing those three questionnaires with our method through several challenging examples under the supervision of the specialist. In Section 5, we will introduce the main difference between our multicriteria decision-making method and well-known fuzzy multicriteria decision-making methods, the practical applications of the proposed framework, along with directions for future research.

2 Fuzzy geometric space

In this section we present the concepts of fuzzy geometric space as a generalization of (intuitionistic, Pythagorean, Fermatean, q-rung orthopair, picture, spherical and finite dimensional multi-, resp.) fuzzy set, the concepts of distance and similarity measure of fuzzy geometric spaces, some example of them, and fuzzy mean of the set of fuzzy geometric spaces. First, we recall some concepts of fuzzy sets and its generalization and the concept of geometric space.

A *fuzzy set* $A = (X, \mu)$ on the non-empty set X is given by

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A : X \rightarrow [0, 1]$ is the *membership function* of the fuzzy set A , and $\mu_A(x)$ is called the degree of membership of x in A . The family of all fuzzy sets on X is denoted by $\text{FS}(X)$, for more information, see [47].

Recall from [10] that, an *intuitionistic fuzzy set* $A = (X, \mu, \nu)$ on the non-empty set X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ are functions called *membership function* and *nonmembership function* respectively, satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ are called the *degree of membership* and the *degree of nonmembership* of x in A . $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the *degree of indeterminacy* of x . The family of all intuitionistic fuzzy sets on X is denoted by $\text{IFS}(X)$. Clearly, if $A \in \text{FS}(X)$, then $\{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in X\} \in \text{IFS}(X)$, in this case $\pi_A(x) = 0$ for all $x \in X$.

An *Pythagorean fuzzy set* $A = (x, \mu, \nu)$ on the non-empty set X is given by $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ are functions called *membership function* and *nonmembership function* respectively, satisfying $0 \leq \mu_A^2(x) + \nu_A^2(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ are called the *degree of membership* and the *degree of nonmembership* of x in A . $\pi_A(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$ is called the *degree of indeterminacy* of x . The family of all Pythagorean fuzzy sets on X is denoted by $\text{PFS}(X)$, for more information, see [14, 44].

Recall from [35] that, a *Fermatean fuzzy set* $A = (x, \mu, \nu)$ on the non-empty set X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ are function called *membership function* and *nonmembership function* respectively, satisfying $0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ are called the *degree of membership* and the *degree of nonmembership* of x in A . $\pi_A(x) = \sqrt[3]{1 - \mu_A^3(x) - \nu_A^3(x)}$ is called the *degree of indeterminacy* of x . The family of all Fermatean fuzzy sets on X is denoted by $\text{FFS}(X)$.

A *q-rung orthopair fuzzy set* $A = (x, \mu, \nu)$ on the non-empty set X is given by $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ are called *membership function* and *nonmembership function* respectively, satisfying $0 \leq \mu_A^q(x) + \nu_A^q(x) \leq 1$ for all $x \in X$. The family of all q-rung orthopair fuzzy sets on X is denoted by $\text{qROFS}(X)$ see [5, 45].

Clearly, intuitionistic (Pythagorean and Fermatean, resp.) fuzzy set is an special class of q-rung orthopair fuzzy set for $q = 1$ ($q = 2$ and $q = 3$, resp.).

A multi-fuzzy set A on the non-empty set X is given by ordered sequence $A = \{(x, \mu_1(x), \mu_2(x), \dots) \mid x \in X\}$, where $\mu_i : X \rightarrow [0, 1]$ satisfying $\sum_1^\infty \mu_i(x) = 1$. The function $\mu_A = (\mu_1, \mu_2, \dots)$ is called a multi-membership function multi-fuzzy set A . If the sequences of the membership functions have only n -terms, for some finite number n , then n is called the dimension of A . Let $L_i = [0, 1]$ for $i = 1, \dots, k$; then, the family of all multi-fuzzy sets on X of dimension n is denoted by $\text{M}^n \text{FS}(X)$, see [34].

Recall from [13] that, a *picture fuzzy set* $A = (x, \mu, \nu)$ on the non-empty set X is given by $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A, \eta_A, \nu_A : X \rightarrow [0, 1]$ satisfying $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x)$ are, called the *degree of positive membership*, *degree of neutral membership* and the *degree of negative membership* of x in A . $\pi_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called the *degree of refusal membership* of x .

Obviously, (intuitionistic or picture, resp.) fuzzy set is an special class of multi-fuzzy set of dimension $n = 2$ ($n = 3$ or $n = 4$, resp.).

An *spherical fuzzy set* $A = (x, \mu, \nu)$ on the non-empty set X is given by $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A, \eta_A, \nu_A : X \rightarrow [0, 1]$ satisfying $0 \leq \mu_A^2(x) + \eta_A^2(x) + \nu_A^2(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x)$ are, called the *degree of positive membership*, *degree of neutral membership* and the *degree of negative membership* of x in A , see [9].

A *geometric space* is a pair (S, \mathcal{B}) , where S is a non-empty its elements are called *points*, \mathcal{B} is a non-empty family of subsets of S , its elements are called *blocks*. If C is a subset of S , then it is called a *B-part* of S if $B \cap C \neq \emptyset$ implies $B \subseteq C$ for every $B \in \mathcal{B}$. For a subset $X \subseteq S$, the intersection of all \mathcal{B} -parts of S containing X is denoted by $\Gamma(X)$, for more information, see [17].

In the following, we present the concept of fuzzy geometric space.

Definition 2.1. Let (S, \mathcal{B}) be a non-empty geometric space, such that \mathcal{B} is a partition of S . Let $n_{\mathcal{B}} = (n_B)_{B \in \mathcal{B}}$ be a family of natural number, $k \in \mathbb{N}$. Let $\mu^B : B \rightarrow [0, 1]^{n_B+1}$ be a vector-valued function for all $B \in \mathcal{B}$, with $\mu^B(x) = (\mu_0^B(x), \dots, \mu_{n_B}^B(x))$ for all $x \in B$ and $\mu = \bigcup_{B \in \mathcal{B}} \mu^B$; then, (S, \mathcal{B}, μ) is called $(n_{\mathcal{B}}, k)$ -fuzzy geometric space if $\sum_{i=0}^{n_B} (\mu_i^B(x))^k \leq 1$, for all $B \in \mathcal{B}$ and $x \in B$. $\mu_0^B(x)$ is called the *degree of nonmembership of x in S* and $(\mu_1^B(x), \dots, \mu_{n_B}^B(x))$ is called the *multi-degree of membership of x in S* . $\pi_{\mu}^B(x) = \sqrt[k]{1 - \sum_{i=0}^{n_B} (\mu_i^B(x))^k}$ is called the *degree of indeterminacy of x in S* . The set of all $(n_{\mathcal{B}}, k)$ -fuzzy geometric spaces (S, \mathcal{B}, μ) is denoted by $\text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$. If no ambiguity the member $(S, \mathcal{B}, \mu) \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$ is identified with $\mu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$.

In above definition, since \mathcal{B} is a partition of S , for each $x \in S$ there exists a unique block $B \in \mathcal{B}$. such that $x \in B$, To simplify the notation, we denote the tuple $\mu^B(x) = (\mu_0^B(x), \dots, \mu_{n_B}^B(x))$ by $\mu(x)$. If there is no ambiguity we identify the single family (n) by its element (i.e. n). When $n_B = n$ for all $B \in \mathcal{B}$ and there is no ambiguity, we identify the family $n_{\mathcal{B}} = \underbrace{(n, \dots, n)}_{|\mathcal{B}| \text{ times}}$ by n .

For example, if (X, μ) is a fuzzy set, then $(X, \{X\}, (1 - \mu, \mu))$ is a $(1, 1)$ -fuzzy geometric space, in this case $\pi(x) = 0$ for all $x \in X$. If (X, μ, ν) is an intuitionistic (Pythagorean, Fermatean, q-rung orthopair, picture, spherical or n -dimensional multi-, resp.) fuzzy set, then $(X, \{X\}, (\nu, \mu))$ is $(1, 1)$ -fuzzy $((1, 2)$ -fuzzy, $(1, 3)$ -fuzzy, $(1, q)$ -fuzzy, $(2, 1)$ -fuzzy, $(2, 2)$ -fuzzy or $(n - 1, 1)$ -fuzzy, resp.) geometric space; hence, the (Pythagorean, Fermatean, q-rung orthopair, picture, spherical or n -dimensional multi-, resp.) fuzzy set is an special class of fuzzy geometric spaces.

In the following example, we describe some methods of fuzzification for a geometric space in medicine.

Example 2.2. Let an expert diagnoses a person's disease by checking some symptoms such as abdominal pains, his stool type with respect to the Bristol stool chart (Figure 1) for the past 14 days and his Body Mass Index (BMI); then, the geometric space of symptoms is $(S, \mathcal{B}) = (\{a, s, b\}, \{\{a\}, \{s\}, \{b\}\})$, where a, s and b are abdominal pain, stool type and BMI respectively. Let acute, chronic and progressive pain be the types of abdominal pains which are checked by the

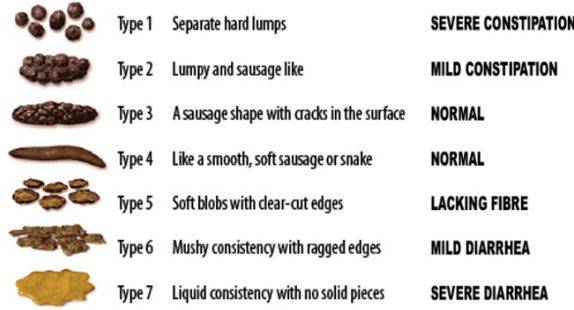


Figure 1: Bristol Stool Chart

expert. We defined the $\mu^{\{a\}} : \{a\} \rightarrow [0, 1]^4$ with

$$\mu^{\{a\}}(a) = \begin{cases} (1, 0, 0, 0) & \text{if the patient has not any pain,} \\ (0, 1, 0, 0) & \text{if the patient has the acute pain,} \\ (0, 0, 1, 0) & \text{if the patient has the chronic pain,} \\ (0, 0, 0, 1) & \text{if the patient has the progressive pain.} \end{cases}$$

If a patient had no stool in x_0 days of 14 days and had it x_i days of type i , then we define the $\mu^{\{s\}} : \{s\} \rightarrow [0, 1]^8$ with $\mu^{\{s\}}(s) = (\frac{x_0}{14}, \dots, \frac{x_7}{14})$. To fuzzify BMI, we define the vector-valued function $\mu^{\{b\}} : \{b\} \rightarrow [0, 1]^3$ by

$$\mu^{\{b\}}(b) = \begin{cases} (1, 0, 0) & \text{if } BMI < 18.5, \\ (\frac{1}{2}, \frac{1}{2}, 0) & \text{if } 18.5 < BMI \leq 20, \\ (0, 1, 0) & \text{if } 20 < BMI \leq 25, \\ (0, \frac{1}{2}, \frac{1}{2}) & \text{if } 25 < BMI \leq 30, \\ (0, \frac{1}{3}, \frac{2}{3}) & \text{if } 30 < BMI \leq 35, \\ (0, \frac{1}{4}, \frac{3}{4}) & \text{if } 35 < BMI \leq 40, \\ (0, 0, 1) & \text{if } BMI > 40 \end{cases}$$

Then (S, \mathcal{B}, μ) is a $(n_B, 1)$ -fuzzy geometric space for each patient, where $n_B = (n_{\{a\}}, n_{\{s\}}, n_{\{b\}})$, and $n_{\{a\}} = 3$, $n_{\{s\}} = 7$, $n_{\{b\}} = 2$ and $\mu = \bigcup_{B \in \mathcal{B}} \mu_B$. In this case, the nonmembership functions of abdominal pain and stool type is corresponded to the option of nothing in the respective table and the nonmembership of BMI is corresponded to the normal case of BMI (i.e. 20-25).

For example, let a patient had no stool in 2 days of 14 days and had it five days of type 1, two days of type 2, four days of type 4, respectively, and forgot a day of 14 days; then we have the following table.

Stool Type	nothing	1	2	3	4	5	6	7
days	2	5	2	0	4	0	0	0

Let for abdominal pain and BMI we have the following tables.

Abdominal Pain Type	nothing	Acute	Chronic	Progressive
			✓	

BMI	< 18.5	18.5 – 20	20 – 25	25 – 30	30 – 35	35 – 40	≥ 40
			✓				

Then (S, \mathcal{B}, μ_p) is a $(n_B, 1)$ -fuzzy geometric space where $\mu_p = \mu^{\{a\}} \cup \mu^{\{s\}} \cup \mu^{\{b\}}$, and $\mu^{\{a\}}(a_p) = (0, 0, 1, 0)$, $\mu^{\{s\}}(s_p) = (\frac{2}{14}, \frac{5}{14}, \frac{2}{14}, 0, \frac{4}{14}, 0, 0, 0)$ and $\mu^{\{b\}}(b_p) = (0, 1, 0)$, respectively.

Let the symptoms of diseases A, B, C and D be given in terms of abdominal pain, BMI, and stool type, as presented in Table 1. By comparing the patient’s symptoms over the past 14 days with the symptoms of the diseases, the type of disease can be diagnosed (Example 2.13). To compare these symptoms introduce several mapping functions in the remainder of this article.”

Table 1: Symptoms of diseases

Disease \ Symptoms	Abdominal pain type	Stool type - Days	BMI
A	Chronic	$x_1 = 4, x_2 = 3, x_6 = 3, x_7 = 4$	20 – 25
B	Progressive	$x_6 = 14$	< 18.5
C	Chronic	$x_0 = 4, x_1 = 5, x_2 = 5$	25 – 30
D	Chroniv	$x_3 = 7, x_4 = 7$	30 – 35

Table 2: The fuzzy maps of diseases

Disease \ Fuzzy-values	$\mu^{\{a\}}$	$\mu^{\{s\}}$	$\mu^{\{b\}}$
μ_A	(0, 0, 1, 0)	$(0, \frac{4}{14}, \frac{3}{14}, 0, 0, 0, \frac{3}{14}, \frac{4}{14})$	(0, 1, 0)
μ_B	(0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 1, 0)	(1, 0, 0)
μ_C	(0, 0, 1, 0)	$(\frac{4}{14}, \frac{5}{14}, \frac{5}{14}, 0, 0, 0, 0, 0)$	$(0, \frac{1}{2}, \frac{1}{2})$
μ_D	(0, 1, 0, 0)	$(0, 0, 0, \frac{7}{14}, \frac{7}{14}, 0, 0, 0)$	$(0, \frac{1}{3}, \frac{2}{3})$

The fuzzification of diseases A, B, C and D with respect to $\mu^{\{a\}}$, $\mu^{\{s\}}$ and $\mu^{\{b\}}$ is presented in the Table 2.

In this example, we calculate $\mu_p, \mu_A, \mu_B, \mu_C, \mu_D \in \text{FGS}_{(3,7,2)}^1$. Moreover, $\pi_{\mu_p}^{\{s\}}(s_p) = \frac{1}{14}$ and $\pi_{\mu_i}^{\{j\}}(y_j) = 0$ for all $i \in \{p, A, B, C, D\}$, $j \in \{\{a\}, \{b\}\}$ and $y \in \{a, b\}$

Definition 2.3. Let (S, \mathcal{B}) be a non-empty geometric space such that \mathcal{B} is a partition of S , and let $\mu, \nu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$; then,

1. $\mu \leq \nu \iff \mu_i^B(x) \leq \nu_i^B(x)$, $\mu_0^B(x) \geq \nu_0^B(x)$ and $\pi_{\mu}^B(x) \geq \pi_{\nu}^B(x)$ for all $B \in \mathcal{B}, x \in B$ and $1 \leq i \leq n_{\mathcal{B}}$;
2. $\mu = \nu \iff \mu \leq \nu$ and $\nu \leq \mu$;

In For example, to model expert consensus or to combine the conditions required for a phenomenon to occur, we employ intersection and union operators, respectively. These operators are examined in the following section. fuzzy geometric spaces.

Proposition 2.4. Let (S, \mathcal{B}) be a non-empty geometric space, where \mathcal{B} is a partition of S , let $\mu, \nu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$, and let

$$\mu \cap \nu = \left\{ (x, \max\{\mu_0^B(x), \nu_0^B(x)\}, \min\{\mu_1^B(x), \nu_1^B(x)\}, \dots, \min\{\mu_{n_{\mathcal{B}}}^B(x), \nu_{n_{\mathcal{B}}}^B(x)\}) \mid B \in \mathcal{B}, x \in B \right\},$$

$$\mu \cup \nu = \left\{ (x, \min\{\mu_0^B(x), \nu_0^B(x)\}, \max\{\mu_1^B(x), \nu_1^B(x)\}, \dots, \max\{\mu_{n_{\mathcal{B}}}^B(x), \nu_{n_{\mathcal{B}}}^B(x)\}) \mid B \in \mathcal{B}, x \in B \right\}.$$

then $\mu \cap \nu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$, and if $n_{\mathcal{B}} = (1)_{\mathcal{B}}$, then $\mu \cup \nu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$.

Proof. Let $x \in S$; then, there exists a unique block $B \in \mathcal{B}$ such that $x \in B$;

$$(\mu \cap \nu)(x) = (\max\{\mu_0^B(x), \nu_0^B(x)\}, \min\{\mu_1^B(x), \nu_1^B(x)\}, \dots, \min\{\mu_{n_{\mathcal{B}}}^B(x), \nu_{n_{\mathcal{B}}}^B(x)\});$$

Since $\min\{\mu_i^B(x), \nu_i^B(x)\} \leq \mu_i^B(x)$ and $\min\{\mu_i^B(x), \nu_i^B(x)\} \leq \nu_i^B(x)$ for all $1 \leq i \leq n_{\mathcal{B}}$, so there are two cases; If $\max\{\mu_0^B(x), \nu_0^B(x)\} = \mu_0^B(x)$, then

$$(\mu_0^B(x))^k + \sum_{i=1}^{n_{\mathcal{B}}} (\min\{\mu_i^B(x), \nu_i^B(x)\})^k \leq \sum_{i=0}^{n_{\mathcal{B}}} (\mu_i^B(x))^k \leq 1;$$

If $\max\{\mu_0^B(x), \nu_0^B(x)\} = \nu_0^B(x)$, then

$$(\nu_0^B(x))^k + \sum_{i=1}^{n_{\mathcal{B}}} (\min\{\mu_i^B(x), \nu_i^B(x)\})^k \leq \sum_{i=0}^{n_{\mathcal{B}}} (\nu_i^B(x))^k \leq 1;$$

hence, $\mu \cap \nu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$.

If $n_{\mathcal{B}} = (1)_{\mathcal{B}}$, then $\mu_0^B(x) \leq \sqrt[k]{1 - (\mu_1^B(x))^k}$ and $\nu_0^B(x) \leq \sqrt[k]{1 - (\nu_1^B(x))^k}$ for all $B \in \mathcal{B}$ and $x \in B$; thus, $\max\{\mu_1^B(x), \nu_1^B(x)\} = \mu_1^B(x)$ yields $\min\{\mu_0^B(x), \nu_0^B(x)\} = \mu_0^B(x)$ and $\max\{\mu_1^B(x), \nu_1^B(x)\} = \nu_1^B(x)$ results in $\min\{\mu_0^B(x), \nu_0^B(x)\} = \nu_0^B(x)$, for all $B \in \mathcal{B}$ and $x \in B$, this completes the proof. \square

Proposition 2.5. Let (S, \mathcal{B}) be a non-empty geometric space such that \mathcal{B} is a partition of S and $\mu, \nu, v \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$; then,

1. $\mu \cap \mu = \mu$
2. $\mu \cup \mu = \mu$
3. $\mu \cap \nu = \nu \cap \mu$;
4. $\mu \cup \nu = \nu \cup \mu$;
5. $\mu \cap (\nu \cap v) = (\mu \cap \nu) \cap v$;
6. $\mu \cup (\nu \cup v) = (\mu \cup \nu) \cup v$.

Proof. The items (1) and (2) are obvious. Since min and max are symmetric operators, so the proof of (3) and (4) are clear. Since $\min\{a, \min\{b, c\}\} = \min\{a, b, c\} = \min\{\min\{a, b\}, c\}$ and $\max\{a, \max\{b, c\}\} = \max\{a, b, c\} = \max\{\max\{a, b\}, c\}$ for all $a, b, c \in [0, 1]$, so the proof of (5) and (6) are also obvious. \square

To compare fuzzy geometric spaces and determine their difference or similarity, distance or similarity measures is a suitable tool that will be discussed in the following.

Definition 2.6. Let (S, \mathcal{B}) be a non-empty geometric space, where \mathcal{B} is a partition of S ; then, a map $d : \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \times \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \rightarrow [0, 1]$ is called a fuzzy distance measure if the following conditions satisfy for all $\mu, \nu, v \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$:

1. $0 \leq d(\mu, \nu) \leq 1$;
2. $d(\mu, \nu) = 0 \iff \mu = \nu$;
3. $d(\mu, \nu) = d(\nu, \mu)$;
4. If $\mu \leq \nu \leq v$, then $d(\mu, \nu) \leq d(\mu, v)$ and $d(\nu, v) \leq d(\mu, v)$.

We now introduce some distance measures.

Theorem 2.7. Let (S, \mathcal{B}) be a finite geometric space (i.e. $|S| < \infty$, where $|S|$ is the cardinal number of S), and \mathcal{B} is a partition of S . Let $d_B : \text{FGS}_{n_{\mathcal{B}}}^k(B, \{B\}) \times \text{FGS}_{n_{\mathcal{B}}}^k(B, \{B\}) \rightarrow [0, 1]$ be a fuzzy distance measure for all $B \in \mathcal{B}$.

1. The maps $D_a, D_g : \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \times \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \rightarrow [0, 1]$ with

$$D_a(\mu, \nu) = \frac{1}{|\mathcal{B}|} \sum_{B \in \mathcal{B}} d_B(\mu^B, \nu^B) \quad \text{and} \quad D_g(\mu, \nu) = \sqrt[|\mathcal{B}|]{\prod_{B \in \mathcal{B}} d_B(\mu^B, \nu^B)},$$

are fuzzy distance measures.

2. The map $D_b : \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \times \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \rightarrow [0, 1]$ with

$$D_b(\mu, \nu) = \sqrt{\frac{1}{|\mathcal{B}|(|\mathcal{B}| - 1)} \sum_{\substack{B_1, B_2 \in \mathcal{B}, \\ B_2 \neq B_1}} d_{B_1}(\mu^{B_1}, \nu^{B_1}) d_{B_2}(\mu^{B_2}, \nu^{B_2})},$$

is a fuzzy distance measures.

1. If $w_B \in \mathbb{N}$ be a weight of the block B for all $B \in \mathcal{B}$, then the maps $D_a^w, D_g^w : \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \times \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \rightarrow [0, 1]$ with

$$D_a^w(\mu, \nu) = \frac{1}{W} \sum_{B \in \mathcal{B}} w_B d_B(\mu^B, \nu^B) \quad \text{and} \quad D_g^w(\mu, \nu) = \sqrt[w]{\prod_{B \in \mathcal{B}} (d_B(\mu^B, \nu^B))^{w_B}},$$

are fuzzy distance measures where $W = \sum_{B \in \mathcal{B}} w_B$.

Proof. Let $\mu, \nu, v \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$;

1. (a) Since $0 \leq d_B(\mu^B, \nu^B) \leq 1$ for all $B \in \mathcal{B}$, so $0 \leq D_a \leq 1$, $0 \leq D_g \leq 1$.

- (b) Since $d_B(\mu^B, \nu^B) = 0$ if and only if $\mu^B = \nu^B$ for all $B \in \mathcal{B}$, so $D_a(\mu, \nu) = 0$ if and only if $\mu = \nu$.
 Since $d_B(\mu^B, \nu^B) = 0$ if and only if $\mu^B = \nu^B$ for all $B \in \mathcal{B}$, so $D_g(\mu, \nu) = 0$ if and only if $\mu = \nu$.
- (c) Since $d_B(\mu^B, \nu^B) = d_B(\nu^B, \mu^B)$ for all $B \in \mathcal{B}$ so $D_a(\mu, \nu) = D_a(\nu, \mu)$ and $D_g(\mu, \nu) = D_g(\nu, \mu)$.
- (d) Let $\mu \leq \nu \leq \nu$; then, $\mu_j^B(x) \leq \nu_j^B(x) \leq \nu_j^B(x)$, $\mu_0^B(x) \geq \nu_0^B(x) \geq \nu_0^B(x)$ and $\pi_\mu^B(x) \geq \pi_\nu^B(x) \geq \pi_\nu^B(x)$, for all $B \in \mathcal{B}$, $x \in B$ and $1 \leq j \leq n_B$; thus, $\mu^B \leq \nu^B \leq \nu^B$ for all $B \in \mathcal{B}$; therefor, $d_B(\mu^B, \nu^B) \leq d_B(\mu^B, \nu^B)$ and $d_B(\nu^B, \nu^B) \leq d_B(\mu^B, \nu^B)$ for all $B \in \mathcal{B}$; since d_B is a fuzzy distance measure for all $B \in \mathcal{B}$, so

$$\sum_{B \in \mathcal{B}} d_B(\mu^B, \nu^B) \leq \sum_{B \in \mathcal{B}} d_B(\mu^B, \nu^B) \quad , \quad \sum_{B \in \mathcal{B}} d_B(\nu^B, \nu^B) \leq \sum_{B \in \mathcal{B}} d_B(\mu^B, \nu^B),$$

$$\prod_{B \in \mathcal{B}} d_B(\mu^B, \nu^B) \leq \prod_{B \in \mathcal{B}} d_B(\mu^B, \nu^B) \quad \text{and} \quad \prod_{B \in \mathcal{B}} d_B(\nu^B, \nu^B) \leq \prod_{B \in \mathcal{B}} d_B(\mu^B, \nu^B);$$

hence, $D_a(\mu, \nu) \leq D_a(\mu, \nu)$, $D_a(\nu, \nu) \leq D_a(\mu, \nu)$, $D_g(\mu, \nu) \leq D_g(\mu, \nu)$ and $D_g(\nu, \nu) \leq D_g(\mu, \nu)$;
 therefore, D_a and D_g are fuzzy distance measures.

2. Obviously, we have

$$D_b(\mu, \nu) = \sqrt{\frac{1}{|\mathcal{B}|} \sum_{B_1 \in \mathcal{B}} d_{B_1}(\mu^{B_1}, \nu^{B_1}) \left(\frac{1}{|\mathcal{B}|-1} \sum_{\substack{B_2 \in \mathcal{B}, \\ B_2 \neq B_1}} d_{B_2}(\mu^{B_2}, \nu^{B_2}) \right)};$$

Since D_a is a fuzzy distance measure, therefore $D_{B_1}(\mu, \nu) = \frac{1}{|\mathcal{B}|-1} \sum_{\substack{B_2 \in \mathcal{B}, \\ B_2 \neq B_1}} d_{B_2}(\mu^{B_2}, \nu^{B_2})$ is a fuzzy distance measure for all $B_1 \in \mathcal{B}$. By similar argument as for D_g in last case, $d_{B_1}(\mu^{B_1}, \nu^{B_1})D_{B_1}(\mu, \nu)$ is a fuzzy distance measure, similarly, $\frac{1}{|\mathcal{B}|} \sum_{B_1 \in \mathcal{B}} d_{B_1}(\mu^{B_1}, \nu^{B_1})D_{B_1}(\mu, \nu)$ is a fuzzy distance measure; and so, D_b is a fuzzy distance measure.

3. It is similar to the proof of fuzzy distance measures D_a and D_g . □

Theorem 2.8. Let (S, \mathcal{B}) be a finite geometric space, where \mathcal{B} is a partition of S , let $t \in \mathbb{N}$ and let $D_t : \text{FGS}_{n_B}^k(S, \mathcal{B}) \times \text{FGS}_{n_B}^k(S, \mathcal{B}) \rightarrow [0, 1]$ be a map with $D_t(\mu, \nu) = \frac{1}{|\mathcal{B}|} \sum_{B \in \mathcal{B}} d_t(\mu^B, \nu^B)$, where $d_t : \text{FGS}_{n_B}^k(B, \{B\}) \times \text{FGS}_{n_B}^k(B, \{B\}) \rightarrow [0, 1]$ is a map for all $B \in \mathcal{B}$ with

$$d_t(\mu^B, \nu^B) = \frac{1}{2|B|} \sum_{i=1}^{|B|} \left(\sum_{j=0}^{n_B} |\mu_j^B(x_i) - \nu_j^B(x_i)|^t + |\pi_\mu^B(x_i) - \pi_\nu^B(x_i)|^t \right),$$

for all $\mu, \nu \in \text{FGS}_{n_B}^k(B, \{B\})$. If $t \geq k$, then D_t is a fuzzy distance measure.

Proof. We show that $d_t : \text{FGS}_{n_B}^k(B, \{B\}) \times \text{FGS}_{n_B}^k(B, \{B\}) \rightarrow [0, 1]$ is a fuzzy distance measure for all $B \in \mathcal{B}$, and so by Theorem 2.7 the proof. Let $B \in \mathcal{B}$ and $\mu^B, \nu^B, \nu^B \in \text{FGS}_{n_B}^k(B, \{B\})$.

1. Clearly, $d_t(\mu^B, \nu^B) \geq 0$. Since $|a - b|^n \leq a^n + b^n$ and $a^n \leq a^m$ if $m \leq n$, for all $0 \leq a, b \leq 1$ and $m, n \in \mathbb{N}$, so

$$\begin{aligned} |\mu_j^B(x_i) - \nu_j^B(x_i)|^t + |\pi_\mu^B(x_i) - \pi_\nu^B(x_i)|^t &\leq \left((\mu_j^B(x_i))^t + (\nu_j^B(x_i))^t \right) + (\pi_\mu^B(x_i))^t + (\pi_\nu^B(x_i))^t \\ &\leq \left((\mu_j^B(x_i))^k + (\nu_j^B(x_i))^k \right) + (\pi_\mu^B(x_i))^k + (\pi_\nu^B(x_i))^k, \end{aligned}$$

for all $x_i \in B$ and $0 \leq j \leq n_B$; hence,

$$\begin{aligned} \sum_{j=0}^{n_B} |\mu_j^B(x_i) - \nu_j^B(x_i)|^t + |\pi_\mu^B(x_i) - \pi_\nu^B(x_i)|^t &\leq \sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^t + (\nu_j^B(x_i))^t \right) + (\pi_\mu^B(x_i))^t + (\pi_\nu^B(x_i))^t \\ &\leq \sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k + (\nu_j^B(x_i))^k \right) + (\pi_\mu^B(x_i))^k + (\pi_\nu^B(x_i))^k = 2, \end{aligned}$$

for all $x_i \in B$; thus,

$$\frac{1}{2} \sum_{j=0}^{n_B} |\mu_j^B(x_i) - \nu_j^B(x_i)|^t + |\pi_\mu^B(x_i) - \pi_\nu^B(x_i)|^t \leq 1,$$

for all $x_i \in B$; therefore, $d_t(\mu^B, \nu^B) \leq 1$.

2. Clearly, $d_t(\mu^B, \nu^B) = 0 \iff \mu^B = \nu^B$.
3. Obviously, $d_t(\mu^B, \nu^B) = d_t(\nu^B, \mu^B)$.
4. Let $\mu^B \leq \nu^B \leq v^B$; then, $\mu_j^B(x) \leq \nu_j^B(x) \leq v_j^B(x)$, $\mu_0^B(x) \geq \nu_0^B(x) \geq v_0^B(x)$ and $\pi_\mu^B(x) \geq \pi_\nu^B(x) \geq \pi_v^B(x)$, for all $x \in B$ and $1 \leq j \leq n_B$; thus, $|\mu_j^B(x) - \nu_j^B(x)| \leq |\mu_j^B(x) - v_j^B(x)|$ and $|\pi_\mu^B(x) - \pi_\nu^B(x)| \leq |\pi_\mu^B(x) - \pi_v^B(x)|$, for all $x \in B$ and $0 \leq j \leq n_B$; therefore,

$$\sum_{j=0}^{n_B} |\mu_j^B(x) - \nu_j^B(x)|^t + |\pi_\mu^B(x) - \pi_\nu^B(x)|^t \leq \sum_{j=0}^{n_B} |\mu_j^B(x) - v_j^B(x)|^t + |\pi_\mu^B(x) - \pi_v^B(x)|^t,$$

for all $x \in B$ and $0 \leq j \leq n_B$; hence, $d_t(\mu^B, \nu^B) \leq d_t(\mu^B, v^B)$; similarly, $d_t(\nu^B, v^B) \leq d_t(\mu^B, v^B)$;

therefore, d_t is a fuzzy distance measure. \square

In above theorem the weight of all points of S are the same; if each point $x \in S$ have an special weight, then we have the following theorem.

Theorem 2.9. Let (S, \mathcal{B}) be a finite geometric space, where \mathcal{B} is a partition of S , let $t \in \mathbb{N}$ and let $w_x \in \mathbb{N}$ be a weight of the point x for all $x \in S$. Let $D_t^w : \text{FGS}_{n_B}^k(S, \mathcal{B}) \times \text{FGS}_{n_B}^k(S, \mathcal{B}) \rightarrow [0, 1]$ be a map with $D_t^w(\mu, \nu) = \frac{1}{|\mathcal{B}|} \sum_{B \in \mathcal{B}} d_t^w(\mu^B, \nu^B)$, where $d_t^w : \text{FGS}_{n_B}^k(B, \{B\}) \times \text{FGS}_{n_B}^k(B, \{B\}) \rightarrow [0, 1]$ is a map for all $B \in \mathcal{B}$ with

$$d_t^w(\mu^B, \nu^B) = \frac{1}{2 \sum_{i=1}^{|\mathcal{B}|} w_{x_i}} \sum_{i=1}^{|\mathcal{B}|} w_{x_i} \left(\sum_{j=0}^{n_B} |\mu_j^B(x_i) - \nu_j^B(x_i)|^t + |\pi_\mu^B(x_i) - \pi_\nu^B(x_i)|^t \right),$$

for all $\mu, \nu \in \text{FGS}_{n_B}^k(B, \{B\})$. If $t \geq k$, then D_t^w is a fuzzy distance measure that is called a fuzzy weighted distance measure of D_t .

Proof. It is similar to that of Theorem 2.8. \square

Theorem 2.10. Let (S, \mathcal{B}) be a finite geometric space, where \mathcal{B} is a partition of S , and let $w_x \in \mathbb{N}$ is a weight of the point x for all $x \in S$. Let $D_e^w : \text{FGS}_{n_B}^k(S, \mathcal{B}) \times \text{FGS}_{n_B}^k(S, \mathcal{B}) \rightarrow [0, 1]$ be a map with $D_e^w(\mu, \nu) = \frac{1}{|\mathcal{B}|} \sum_{B \in \mathcal{B}} d_e^w(\mu^B, \nu^B)$, where $d_e^w : \text{FGS}_{n_B}^k(B, \{B\}) \times \text{FGS}_{n_B}^k(B, \{B\}) \rightarrow [0, 1]$ is a map for all $B \in \mathcal{B}$ with

$$d_e^w(\mu^B, \nu^B) = \frac{1}{2 \sum_{i=1}^{|\mathcal{B}|} w_{x_i}} \sum_{i=1}^{|\mathcal{B}|} w_{x_i} \sqrt{\sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k - (\nu_j^B(x_i))^k \right)^2 + \left((\pi_\mu^B(x_i))^k - (\pi_\nu^B(x_i))^k \right)^2},$$

for all $\mu^B, \nu^B \in \text{FGS}_{n_B}^k(B, \{B\})$; then, D_e^w is a fuzzy distance measure.

Proof. We show that $d_e^w : \text{FGS}_{n_B}^k(B, \{B\}) \times \text{FGS}_{n_B}^k(B, \{B\}) \rightarrow [0, 1]$ is a fuzzy distance measure for all $B \in \mathcal{B}$, and so by Theorem 2.7 the proof. Is completed let $B \in \mathcal{B}$ and $\mu^B, \nu^B, v^B \in \text{FGS}_{n_B}^k(B, \{B\})$;

1. Clearly, $d_e^w(\mu^B, \nu^B) \geq 0$. Since $(a^k - b^k)^2 \leq a^{2k} + b^{2k} \leq a^k + b^k$ for all $0 \leq a, b \leq 1$, we have

$$\left((\mu_j^B(x_i))^k - (\nu_j^B(x_i))^k \right)^2 + \left((\pi_\mu^B(x_i))^k - (\pi_\nu^B(x_i))^k \right)^2 \leq (\mu_j^B(x_i))^k + (\nu_j^B(x_i))^k + (\pi_\mu^B(x_i))^k + (\pi_\nu^B(x_i))^k,$$

for all $x_i \in B$; hence,

$$\begin{aligned} & \sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k - (\nu_j^B(x_i))^k \right)^2 + \left((\pi_\mu^B(x_i))^k - (\pi_\nu^B(x_i))^k \right)^2 \leq \\ & \sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k + (\nu_j^B(x_i))^k \right) + (\pi_\mu^B(x_i))^k + (\pi_\nu^B(x_i))^k \leq 2, \end{aligned}$$

for all $x_i \in B$; thus,

$$\frac{1}{2} \sqrt{\sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k - (\nu_j^B(x_i))^k \right)^2 + \left((\pi_\mu^B(x_i))^k - (\pi_\nu^B(x_i))^k \right)^2} \leq \frac{1}{2} \sqrt{2} \leq 1,$$

for all $x_i \in B$; therefore, $d_e^w(\mu^B, \nu^B) \leq 1$.

2. Clearly, $d_e^w(\mu^B, \nu^B) = 0 \iff \mu^B = \nu^B$.
3. Obviously, $d_e^w(\mu^B, \nu^B) = d_e^w(\nu^B, \mu^B)$.
4. Let $\mu^B \leq \nu^B \leq v^B$; then, $\mu_j^B(x) \leq \nu_j^B(x) \leq v_j^B(x)$, $\mu_0^B(x) \geq \nu_0^B(x) \geq v_0^B(x)$ and $\pi_\mu^B(x) \geq \pi_\nu^B(x) \geq \pi_v^B(x)$, for all $x \in B$ and $1 \leq j \leq n_B$; thus, $\left((\mu_j^B(x))^k - (\nu_j^B(x))^k\right)^2 \leq \left((\mu_j^B(x))^k - (v_j^B(x))^k\right)^2$ and $\left((\pi_\mu^B(x))^k - (\pi_\nu^B(x))^k\right)^2 \leq \left((\pi_\mu^B(x))^k - (\pi_v^B(x))^k\right)^2$, for all $x \in B$ and $0 \leq j \leq n_B$; hence,

$$\sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k - (\nu_j^B(x_i))^k \right)^2 + \left((\pi_\mu^B(x_i))^k - (\pi_\nu^B(x_i))^k \right)^2 \leq \sum_{j=0}^{n_B} \left((\mu_j^B(x_i))^k - (v_j^B(x_i))^k \right)^2 + \left((\pi_\mu^B(x_i))^k - (\pi_v^B(x_i))^k \right)^2,$$

for all $x \in B$ and $0 \leq j \leq n_B$; therefore, $d_t^w(\mu^B, \nu^B) \leq d_e^w(\mu^B, v^B)$; similarly, $d_e^w(\nu^B, v^B) \leq d_e^w(\mu^B, v^B)$;

and so, d_e^w is a fuzzy distance measure. \square

Definition 2.11. Let (S, \mathcal{B}) be a non-empty geometric space, where \mathcal{B} is a partition of S ; then, the map $s : \text{FGS}_{n_B}^k(S, \mathcal{B}) \times \text{FGS}_{n_B}^k(S, \mathcal{B}) \rightarrow [0, 1]$ is called a fuzzy similarity measure if the following conditions satisfy for all $\mu, \nu, v \in \text{FGS}_{n_B}^k(S, \mathcal{B})$:

1. $0 \leq s(\mu, \nu) \leq 1$;
2. $s(\mu, \nu) = 1 \iff \mu = \nu$;
3. $s(\mu, \nu) = s(\nu, \mu)$;
4. If $\mu \leq \nu \leq v$, then $s(\mu, \nu) \leq s(\mu, v)$ and $s(\mu, v) \leq s(\nu, v)$.

Obviously, we have the following proposition.

Proposition 2.12. If $d : \text{FGS}_{n_B}^k(S, \mathcal{B}) \times \text{FGS}_{n_B}^k(S, \mathcal{B}) \rightarrow [0, 1]$ is a fuzzy (weighted) distance measure, then $1 - d$ is a fuzzy (weighted) similarity measure and vice versa.

Example 2.13. Let $(S, \mathcal{B}) = (\{a, s, b\}, \{\{a\}, \{s\}, \{b\}\})$ and $\mu_p, \mu_A, \mu_B, \mu_C, \mu_D \in \text{FGS}_{(3,7,2)}^1(S, \mathcal{B})$ be the $(3, 7, 2, 1)$ -fuzzy geometric spaces of the patient and diseases in Example 2.2. Let $D_t : \text{FGS}_{(3,7,2)}^1(S, \mathcal{B}) \times \text{FGS}_{(3,7,2)}^1(S, \mathcal{B}) \rightarrow [0, 1]$ be the distance measure defined in Theorem 2.8. Using the similarity measure $s = 1 - D_t$ for $t = 1$, we determine the similarity between the patient's symptoms and the four diseases A, B, C , and D . Since physicians usually compare symptoms linearly in their mind, we take $t = 1$. In practice, the precise determination of parameter values in each problem should be based on expert opinions and relevant researches. In traditional Persian medicine problems, this is usually achieved through a consensus of an expert panel. Now we calculate the similarity between μ_p and each diseases μ_A, μ_B, μ_C and μ_D with respect to similarity measure s .

$$d_1(\mu_p^{\{a\}}, \mu_A^{\{a\}}) = \frac{1}{2} \left((|0 - 0| + |0 - 0| + |1 - 1| + |0 - 0|) + |0 - 0| \right) = 0,$$

$$d_1(\mu_p^{\{s\}}, \mu_A^{\{s\}}) = \frac{1}{2} \frac{1}{14} \left((|2 - 0| + |5 - 4| + |2 - 3| + |0 - 0| + |4 - 0| + |0 - 0| + |0 - 3| + |0 - 4|) + |1 - 0| \right) = \frac{1}{2} \frac{16}{14} = \frac{4}{7},$$

$$d_1(\mu_p^{\{a\}}, \mu_A^{\{a\}}) = \frac{1}{2} \left((|0 - 0| + |1 - 1| + |0 - 0|) + |0 - 0| \right) = 0,$$

$$D_1(\mu_p, \mu_A) = \frac{1}{3} \left(0 + \frac{4}{7} + 0 \right) = \frac{4}{21},$$

$$s(\mu_p, \mu_A) = 1 - D_1(\mu_p, \mu_A) = 1 - \frac{4}{21} = \frac{17}{21} = 0.81.$$

Similarly,

$$s(\mu_p, \mu_B) = D_1(\mu_p, \mu_B) = 1 - 1 = 0,$$

$$s(\mu_p, \mu_C) = D_1(\mu_p, \mu_C) = 1 - \frac{2}{7} = \frac{5}{7} = 0.714,$$

$$s(\mu_p, \mu_D) = D_1(\mu_p, \mu_D) = 1 - \frac{31}{42} = \frac{11}{42} = 0.261.$$

Therefore, the patient's symptoms are more similar to disease C .

In Examples 2.2 and 2.13, we used the assignment method for fuzzifying each of the disease symptoms. Employing different fuzzification methods within a single problem is not feasible in any existing fuzzy methodology; this example clearly demonstrates the superiority of fuzzy geometric spaces over current methods. To appropriately select fuzzifying methods and similarity measure maps for each specific problem, it is necessary to consult relevant experts and review related research. For instance, a method similar to the Delphi technique could be utilized for this purpose.

In many multicriteria decision making problems, a suitable method to determine the average of a set of data such as the average of experts' opinions is important. Now we state and prove the concept of fuzzy mean of the set of fuzzy geometric spaces.

Definition 2.14. Let (S, \mathcal{B}) be a finite geometric space, where \mathcal{B} is a partition of S , and let $w_x \in \mathbb{N}$ be a weight of the point x for all $x \in S$. Let $m \in \mathbb{N}$ and $\mu_1, \dots, \mu_m \in \text{FGS}_{n_B}^k(S, \mathcal{B})$; then, the fuzzy weighted mean of $\{\mu_1, \dots, \mu_m\}$ is given by

$$\mathfrak{M}^w(\mu_1, \dots, \mu_m)(x) = (\mathfrak{m}^w(\mu_{10}^B, \dots, \mu_{m0}^B)(x), \dots, \mathfrak{m}^w(\mu_{1n_B}^B, \dots, \mu_{mn_B}^B)(x)),$$

for all $B \in \mathcal{B}$ and $x \in B$, where

$$\mathfrak{m}^w(\mu_{1j}^B, \dots, \mu_{mj}^B)(x) = \sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k},$$

for all $0 \leq j \leq n_B$, where $W = \sum_{i=1}^m w_i$.

Theorem 2.15. Let (S, \mathcal{B}) be a finite geometric space, where \mathcal{B} is a partition of S , and let $w_x \in \mathbb{N}$ be a weight of the point x for all $x \in S$. Let $m \in \mathbb{N}$ and $\mu_1, \dots, \mu_m \in \text{FGS}_{n_B}^k(S, \mathcal{B})$; then, $\mathfrak{M}^w(\mu_1, \dots, \mu_m) \in \text{FGS}_{n_B}^k(S, \mathcal{B})$ and $\pi_{\mathfrak{M}^w}^B(x) = \mathfrak{m}^w(\pi_{\mu_1}^B(x), \dots, \pi_{\mu_m}^B(x))$, for all $B \in \mathcal{B}$ and $x \in B$.

Proof. To show that $\mathfrak{M}^w(\mu_1, \dots, \mu_m)$ is a (n_B, k) -fuzzy geometric space, we show that $0 \leq \mathfrak{m}^w(\mu_{1j}^B, \dots, \mu_{mj}^B)(x) \leq 1$, for all $0 \leq j \leq m$, $B \in \mathcal{B}$ and $x \in B$, and $\sum_{j=1}^{n_B} (\mathfrak{m}^w(\mu_{1j}^B, \dots, \mu_{mj}^B)(x))^k \leq 1$, for all $0 \leq j \leq m$, $B \in \mathcal{B}$ and $x \in B$. Let $B \in \mathcal{B}$ and $x \in B$; since $0 \leq \mu_{ij}^B(x) \leq 1$ for all $1 \leq i \leq m$ and $0 \leq j \leq m$, so $0 \leq \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k \leq w$ for all $0 \leq j \leq m$; then, $0 \leq \sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k} \leq 1$, for all $0 \leq j \leq m$. Since $\mu_1, \dots, \mu_m \in \text{FGS}_{n_B}^k(S, \mathcal{B})$, so $\sum_{j=1}^{n_B} (\mu_{ij}^B(x))^k \leq 1$ for all $1 \leq i \leq m$; then,

$$\sum_{j=1}^{n_B} \left(\sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k} \right)^k = \frac{1}{W} \sum_{j=1}^{n_B} \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k = \frac{1}{W} \sum_{i=1}^m w_i \sum_{j=1}^{n_B} (\mu_{ij}^B(x))^k \leq 1.$$

We now compute the degree of indeterminacy of \mathfrak{M}^w in $x \in S$. We have

$$\begin{aligned} (\pi_{\mathfrak{M}^w}^B(x))^k &= 1 - \sum_{j=1}^{n_B} \left(\sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k} \right)^k = 1 - \frac{1}{W} \sum_{i=1}^m w_i \sum_{j=1}^{n_B} (\mu_{ij}^B(x))^k \\ &= \frac{1}{W} \left(\sum_{i=1}^m w_i - \sum_{i=1}^m w_i \sum_{j=1}^{n_B} (\mu_{ij}^B(x))^k \right) = \frac{1}{W} \left(\sum_{i=1}^m w_i \left(1 - \sum_{j=1}^{n_B} (\mu_{ij}^B(x))^k \right) \right) = \frac{1}{W} \sum_{i=1}^m w_i (\pi_{\mu_i}^B(x))^k; \end{aligned}$$

thus, $\pi_{\mathfrak{M}^w}^B(x) = \mathfrak{m}^w(\pi_{\mu_1}^B(x), \dots, \pi_{\mu_m}^B(x))$. □

For intuitionistic fuzzy set (i.e., when $k = 1$) the preceding theorem yields the arithmetic mean and generalizes the results of [6] within $(1, 1)$ -fuzzy geometric spaces.

Theorem 2.16. Let (S, \mathcal{B}) be a finite geometric space, where \mathcal{B} is a partition of S , and let $w_x \in \mathbb{N}$ be a weight of the point x for all $x \in S$. Let $m \in \mathbb{N}$ and $\mu_1, \dots, \mu_m \in \text{FGS}_{n_B}^k(S, \mathcal{B})$; then, $\mathfrak{M}^w(\mu_1, \dots, \mu_m)(x)$ is a mean operator for all $x \in S$.

Proof. Let $x \in S$; then, there exists a unique $B \in \mathcal{B}$ such that $x \in B$. Let $\nu \in \text{FGS}_{n_B}^k(S, \mathcal{B})$.

Idempotency: Let $\mu_1(x) = \mu_2(x) = \dots = \mu_m(x) = \mu(x)$; clearly, $\mathfrak{M}^w(\mu_1, \dots, \mu_m)(x) = \mu(x)$.

Monotonicity: Let $\mu_1(x) \leq \nu(x)$; then, $\mu_{1j}^B(x) \leq \nu_j^B(x)$ for all $1 \leq j \leq n_B$, and $\mu_{10}^B(x) \geq \nu_0^B(x)$ and $\pi_{\mu_1}^B(x) \geq \pi_\nu^B(x)$; thus,

$$\begin{aligned} \sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\mu_{ij}^B(x))^k} &\leq \sqrt[k]{\frac{1}{W} \left(w_1 (\nu_j^B(x))^k + \sum_{i=2}^m w_i (\mu_{ij}^B(x))^k \right)} \quad \text{for all } 1 \leq j \leq n_B, \\ \sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\mu_{i0}^B(x))^k} &\geq \sqrt[k]{\frac{1}{W} \left(w_1 (\nu_0^B(x))^k + \sum_{i=2}^m w_i (\nu_{i0}^B(x))^k \right)}, \\ \sqrt[k]{\frac{1}{W} \sum_{i=1}^m w_i (\pi_{\mu_i}^B(x))^k} &\geq \sqrt[k]{\frac{1}{W} \left(w_1 (\pi_\nu^B(x))^k + \sum_{i=2}^m w_i (\pi_{\mu_i}^B(x))^k \right)}; \end{aligned}$$

hence, $\mathfrak{M}^w(\mu_1, \dots, \mu_m)(x) \leq \mathfrak{M}^w(\nu, \dots, \mu_m)(x)$.

Commutativity: Obviously, $\mathfrak{M}^w(\mu_1, \dots, \mu_m)(x) = \mathfrak{M}^w(\mu_{\sigma(1)}, \dots, \mu_{\sigma(m)})(x)$ for all permutation σ on $\{1, \dots, m\}$.

Boundedness: It follows from by the properties of idempotency and monotonicity;

therefore, \mathfrak{M}^w is a mean operator. \square

3 Methodology and algorithm

A multicriteria decision-making problem exists when a set of alternative must be evaluated against multiple criteria. Medical diagnosis represents a prominent application domain of multicriteria decision-making (MCDM) problems. In these problems, signs and symptoms serve as the alternatives, while diseases function as the criteria, with each disease characterized by a specific set of these clinical features. we describe an algorithm for applying fuzzy geometric space to multicriteria decision-making problems.

Let (S, \mathcal{B}) be a finite geometric space. Let $n_{\mathcal{B}} = (n_B)_{B \in \mathcal{B}}$ be a family of natural number, $k \in \mathbb{N}$ and $C \subseteq \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$ be a non-empty and finite set. In this section we consider the multicriteria decision making problem, denoted by $\text{FDM}(S, n_{\mathcal{B}}, k, C)$, where S , and C . Finding a solution for $\nu \in \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$ in multicriteria decision-making problem $\text{FDM}(S, n_{\mathcal{B}}, k, C)$ with respect to a similarity measure $s : \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \times \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \rightarrow [0, 1]$ means finding a criterion $\mu_1 \in C$ such that $s(\mu_1, \nu) = \max\{s(\mu, \nu) | \mu \in C\}$. To introduce our algorithm for solveing multicriteria decision-making problems, we first define the necessary foundational concepts below.

Definition 3.1. Let $\text{FDM}(S, n_{\mathcal{B}}, k, C)$ be a multicriteria decision making problem and $A \subseteq \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$ be a non-empty and finite set; then, the relation matrix $\mathcal{R}_S(A)$ with respect to alternative set S is defined as follows.

$$\mathcal{R}_S(A) = \begin{pmatrix} \nu_1(x_1) & \dots & \nu_1(x_m) \\ \vdots & \ddots & \vdots \\ \nu_l(x_1) & \dots & \nu_l(x_m) \end{pmatrix},$$

where $l = |A|$, $m = |S|$, $x_i \in S$ and $\nu_j \in A$ for all $1 \leq i \leq m$ and $1 \leq j \leq l$. The transpose of the relation matrix $\mathcal{R}_S(C)$, i.e.

$$\mathcal{C}(S) = (\mathcal{R}_S(C))^T = \begin{pmatrix} \mu_1(x_1) & \dots & \mu_n(x_1) \\ \vdots & \ddots & \vdots \\ \mu_1(x_m) & \dots & \mu_n(x_m) \end{pmatrix},$$

is called the criterion matrix of $\text{FDM}(S, n_{\mathcal{B}}, k, C)$, where $n = |C|$, $x_i \in S$ and $\mu_j \in C$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 3.2. Let $\text{FDM}(S, n_{\mathcal{B}}, k, C)$ be a multicriteria decision making problem and $A \subseteq \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$ be a non-empty and finite set. Let $f : \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \times \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B}) \rightarrow [0, 1]$ be a similarity measure or distance measure; then, the operation $*_f$ with respect to f between relation matrix $\mathcal{R}_S(A)$ and criterion matrix $D = (\mathcal{R}_S(C))^T$, is defined as

$$\mathcal{R}_S(A) *_f \mathcal{C}(S) = (f(\nu_i, \mu_j))_{l \times n},$$

where $l = |A|$, $n = |C|$, $\nu_i \in A$ and $\mu_j \in C$ for all $1 \leq i \leq l$ and $1 \leq j \leq n$. $\mathcal{R}_S(A) *_f D$ is called decision matrix.

In the above definition if f is a similarity (distance, resp.) measure, a solution of A in $\text{FDM}(S, n_{\mathcal{B}}, k, C)$ with respect to f is $\{\max\{f(\nu_i, \mu_j) | 1 \leq j \leq n\} | 1 \leq i \leq l\}$ ($\{\min\{f(\nu_i, \mu_j) | 1 \leq j \leq n\} | 1 \leq i \leq l\}$, resp.).

Building on the theoretical framework established in Sections 2 and 3, this section introduces an algorithm to solve the multicriteria decision making problem presented earlier. The algorithm takes the multicriteria decision making problem $\text{FDM}(S, n_{\mathcal{B}}, k, C)$, a non-empty and finite set $A \subseteq \text{FGS}_{n_{\mathcal{B}}}^k(S, \mathcal{B})$ (the set of samples whose similarity degree to the criteria is to be quantified) and similarity measure s , and proceeds through the following steps to ensure the output satisfies the solution conditions of A in $\text{FDM}(S, n_{\mathcal{B}}, k, C)$ with respect to s .

Algorithm 1 Multicriteria Decision Making Algorithm

Input: $(\text{FDM}(S, n_{\mathcal{B}}, k, C), A, s)$.

Output: The solution of A in $\text{FDM}(S, n_{\mathcal{B}}, k, C)$ with respect to similarity measure s .

- Compute the decision matrix $Decision := \mathcal{R}_S(A) *_s \mathcal{C}(S)$.
- Determine the maximum element of each rows of $Decision$ and put it in $Result$.
- Return $Result$.

End.

The flowchart of Algorithm 1 is shown in Figure 2.

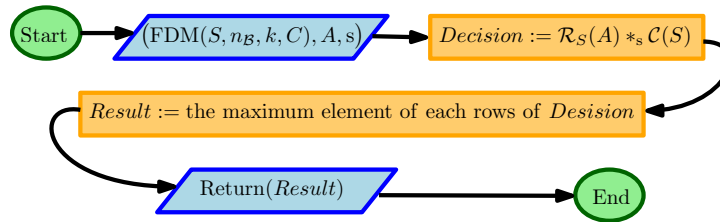


Figure 2: Flowchart of Algorithm 1

4 Application of fuzzy geometric space in Persian medicine

Persian medicine, is one of the oldest traditional medicine. Mizaj (temperament) is a main concept in Persian medicine which plays a fundamental role in defining health and illness. In this view, each person has a specific Mizaj, just as he each a specific fingerprint. In many diseases, special changes, which are recognizable with respect to a set of well known and classified concepts, occur in the person's Mizaj [24].

According to "The Canon of Medicine" ([18]), Mizaj is a quality that yields from the interaction between the four elements (warmness, coldness, wetness and dryness) and creates the physical and mental characteristics of human beings. Avicenna has classified general population to nine main groups with respect to their mizaj: four simple Mizajes (warm, cold, wet, and dry), four complex Mizajes (warm-dry, warm-wet, cold-wet, and cold-dry) and one moderate Mizaj. He described ten criteria which is called "Ajnas-e-Ashara" to cluster them into these nine groups. These criteria are: characteristics of touch, muscle and fat mass, hair condition, skin color, physique, impressibility speed, sleep and wakefulness, physical functions, quality of waste matter (stool, urine and sweat) and the psychic functions. Moreover, he has introduced different subjective and objective indices for each criterion to identify warmness-coldness and wetness-dryness respectively.

Research from the Mizaj framework has extensively explored the relationship between temperament and individuals' physical and mental traits. of [1, 4, 15, 21, 28, 32, 37, 40, 41, 48, 49]; therefore, a suitable tool to accurately determine Mizaj is necessary. To date, several researchers have attempted to design and validate self-administered Mizaj questionnaire to assess individuals temperament. "Three widely recognized self-administered Mizaj (temperament) questionnaires in Persian Medicine are those developed by Akhtari et al. [2], Mojahedi et al. [22], and Salmannezhad et al. [33], commonly referred to as the Akhtari, Mojahedi, and Salmannezhad questionnaires, respectively. The sensitivity and specificity of these questionnaires at the specified cut-off points are summarized in Table 3. Sensitivity or true positive rate (TPR) and specificity or true negative rate (TNR) refer to the probability of a positive test, result given the condition is truly present, and the probability of a negative test, result given the condition is truly absent, respectively. They are computed by the following formulas.

$$TPR = \frac{TP}{TP + FN} \quad \text{and} \quad TNR = \frac{TN}{TN + FP},$$

Table 3: The sensitivity and specificity of the questionnaires.

Questionnaire		Mojahedi (M)	Salmannezad (S)	Akhtari	
Sensitivity for	warmness	65%	71%	> M	
	moderate in warmness-coldness		63%		< S
	coldness	52%	80%	> M	< S
	wetness	53%	74%	> M	< S
	moderate in wetness-dryness		71%		< S
	dryness	53%	78%	> M	
Specificity for	warmness	93%	68%	\simeq M	> S
	moderate in warmness-coldness		71%		> S
	coldness	97%	57%	\simeq M	> S
	wetness	76%	58%	\simeq M	> S
	moderate in wetness-dryness		69%		> S
	dryness	67%	85%	\simeq M	

Table 4: The guide table of the three questionnaires

Questionnaire	Coldness-Warmness			Wetness-Dryness		
	Cold	Moderate	Warm	Wet	Moderate	Dry
Mojahedi	$A_m < 15$	$15 \leq A_m \leq 18$	$A_m > 18$	$B_m < 4$	$B_m = 4$	$B_m > 4$
Salmannezhad	$A < 47$	$47 \leq A \leq 49$	$A > 49$	$B < 15$	$15 \leq B \leq 16$	$B > 16$
Akhtari	$A < 38$	$38 \leq A \leq 44$	$A > 44$	$B < 13$	$13 \leq B \leq 14$	$B > 14$

where TP , TN , FP and FN are the numbers of true positive, true negative, false positive and false negative of the results of the test, respectively.

In this section, drawing on the three existing questionnaires, we first design a Mizaj questionnaire (Table 7, Appendix A) that conceptually integrates elements from all three. We then compare the relative weights assigned to each Ajnas-e-Ashara criterion across these instruments and evaluate the findings against the results produced by our proposed method. Afterwards we introduce a method to fuzzify the questionnaire and identify the input parameters of Algorithm 1. Finally we compare our method with the three questionnaire methods through several challenging examples. Unless otherwise specified, references to 'questionnaires' in the remainder of this section pertain exclusively to those developed by Mojahedi, Salmannezhad, and Akhtari.

In Table 7, we constructed a unified questionnaire by taking the union of the three source questionnaires and removing items with redundant or overlapping content. For instance, items addressing constitutional body type, tendencies toward obesity or thinness, and obesity predisposition serve the same conceptual purpose as scales measuring wetness-dryness. In Table 6, we compare the questions of the three questionnaire based on Table 7. In the Mojahedi-questionnaire, for each question there exists three options. In fact the option 1 is combined with the option 2 and the option 4 is combined with the option 5 for each question.

To determine the result of the filled out questionnaire of each three questionnaires, we have to multiply the number of options in each column by the number of options and add the results together, then according to the questionnaire guide (Table 4). For example, if a person chooses option 1 for all questions in Akhtari questionnaire, then the addition is 15 for coldness-warmness and 5 for wetness-dryness; then, the Mizaj is cold and wet since $15 < 38$ and $5 < 13$. To formulate the methods of these three questionnaires, let $x_{ij} = 1$, where the person select the option j for question i and $x_{ik} = 0$ for all $k \neq j$ and for all meaningful i and j in each questionnaire. Let

$$A_m = \sum_{\substack{1 \leq i \leq 8 \\ 1 \leq j \leq 3}} jx_{ij}, \quad B_m = \sum_{\substack{i \in \{a,b\} \\ 1 \leq j \leq 3}} jx_{ij} \quad \text{and} \quad A = \sum_{\substack{1 \leq i \leq 15 \\ 1 \leq j \leq 5}} jx_{ij}, \quad B = \sum_{\substack{i \in \{a,b,c,d,e\} \\ 1 \leq j \leq 5}} jx_{ij}.$$

The Table 4 shows the guide table of Mizaj determination of the three questionnaires.

In each questionnaire of coldness-warmness (wetness-dryness, resp.), the middle option for moderate Mizaj, the left option for cold (wet, resp.) Mizaj, and the right option for warm (dry, resp.) Mizaj are design in each question. In the questionnaire with five options (Salmannezhad and Akhtari), the option 2 (the option 4, resp.) is between the middle option and the left (right, resp.) option; thus, it has half the value of the left (right, resp.) option in each question. For

a better understanding of such questionnaires, due to the oppositeness of coldness and warmth, wetness and dryness, it is enough to consider the moderate point as the origin of the coordinate axis such that two opposite elements are located on both sides of it. In this case the left and the right option in each question has the value -2 and 2 (or -1 and 1 , resp.), respectively, in the five-options (three-options, resp.) questionnaires. Let

$$A'_m = \sum_{\substack{1 \leq i \leq 8 \\ 1 \leq j \leq 3}} (j-2)x_{ij}, \quad B'_m = \sum_{\substack{i \in \{a,b\} \\ 1 \leq j \leq 3}} (j-2)x_{ij} \quad \text{and} \quad A' = \sum_{\substack{1 \leq i \leq 15 \\ 1 \leq j \leq 5}} (j-3)x_{ij}, \quad B' = \sum_{\substack{i \in \{a,b,c,d,e\} \\ 1 \leq j \leq 5}} (j-3)x_{ij},$$

where x_{ij} is defined as above, for all meaningful i and j in each questionnaire. So

$$A_m = A'_m + 16, \quad B_m = B'_m + 4, \quad A = A' + 45 \quad \text{and} \quad B = B' + 15;$$

thus, the moderate points of coldness-warmness and wetness-dryness for each three questionnaires and the percents of them with respect to the number of question are summarized in the following table, where the values of the right, the middle and the left option in coldness-warmness (wetness-dryness, resp.) is denoted by C, M and W (W, M and D , resp.), respectively.

Questionnaire	Coldness-Warmness	Wetness-Dryness
Mojahedi	$-1 \leq A'_m \leq 2$ $12.5\%C \leq A'_m \leq 25\%W$	$B'_m = 0$ $B'_m = 100\%M$
Salmanezhad	$2 \leq A' \leq 4$ $6.7\%W \leq A' \leq 13.3\%W$	$0 \leq B' \leq 1$ $100\%M \leq B' \leq 10\%D$
Akhtari	$-7 \leq A' \leq -1$ $23.3\%C \leq A' \leq 3.3\%C$	$-2 \leq B' \leq -1$ $20\%W \leq B' \leq 10\%W$

According to the method of these three questionnaires, clearly, in conditions that Mizaj is very cold, very warm, very wet or very dry, they show the correct results, but when the person's Mizaj is close to moderate, the result is not necessarily correct. For example in Mojahedi-questionnaire, if a person is obese and has dry skin, or he is thin and has soft skin, respectively, then $x_a = (1, 0, 0)$ and $x_b = (0, 0, 1)$, or $x_a = (0, 0, 1)$ and $x_b = (1, 0, 0)$, respectively; thus, for both conditions, $A_m = 4$ that means the Mizaj is moderate in wetness-dryness, while with respect to [11, 29], the obese person has wet Mizaj and the thin person has dry Mizaj. Note that in [11, 29], the research data related to the Mojahedi-questionnaire are used. It seems that by changing the weights of criteria, we will have the better results.

The weight of each of the criteria of Ajnas-e-Asharah according to the number of questions of each criterion in each of the three questionnaires and the weight of each of the criteria with respect to the opinion of the specialist in Persian medicine (one of the authors) is specified in Table 5.

In the following, we describe the method of determining the Mizaj by Algorithm1. First we identify the input parameters of Algorithm 1. Let $S = \{Q_i | i \in \Lambda\}$, where $\Lambda = \{1, \dots, 13\} \cup \{a, b, c, d\}$ and Q_i is the i -th question in Table 7 for all $i \in \Lambda$; then, (S, \mathcal{B}) is a geometric space where

$$\mathcal{B} = \{\{Q_1\}, \{Q_2\}, \{Q_3, Q_4, Q_5, Q_6\}, \{Q_7, Q_8\}, \{Q_9, Q_{10}\}, \{Q_{11}, Q_{12}, Q_{13}\}, \{Q_a\}, \{Q_b\}, \{Q_c\}, \{Q_d\}\}.$$

To fuzzify the Geometric space (S, \mathcal{B}) , let $i \in \Lambda$; then, $\nu(Q_i) = (x_{i1}, x_{i2}, x_{i3})$, where $x_{i1} = 1$, $x_{i2} = 1$ or $x_{i3} = 1$ if the options 1, 3 or 5 are selected respectively, for the question i , and $x_{i1} = x_{i2} = \frac{1}{2}$ or $x_{i3} = x_{i2} = \frac{1}{2}$ if the options 2 or 4 are selected respectively, for the question i . The wight of each block $B \in \mathcal{B}$ (w_B) is equal to the corresponding weight of B in Table 5 and the wight of each $Q \in B$ is $\frac{w_B}{|B|}$. For example the block $\{Q_3, Q_4, Q_5, Q_6\}$ is corresponded to the physical functions criterion in Ajnas-e-Asharah; thus, $w_B = 3$ and $w_i = \frac{3}{4}$ for all $3 \leq i \leq 6$, where w_i is the weight of Q_i . Let $S_1 = \{Q_i | 1 \leq i \leq 13\}$, $\mathcal{B}_1 = \{\{Q_1\}, \{Q_2\}, \{Q_3, Q_4, Q_5, Q_6\}, \{Q_7, Q_8\}, \{Q_9, Q_{10}\}, \{Q_{11}, Q_{12}, Q_{13}\}\}$, $S_2 = \{Q_i | i \in \{a, b, c, d\}\}$ and $\mathcal{B}_2 = \{\{Q_a\}, \{Q_b\}, \{Q_c\}, \{Q_d\}\}$; then, (S_i, \mathcal{B}_i) is a geometric space for $i = 1, 2$. The set of criteria is $C_i = \{\mu_1, \mu_2, \mu_3\} \subseteq \text{FGS}_2^1(S_i, \mathcal{B}_i)$, for $i = 1, 2$, where $\mu_1(Q_i) = (1, 0, 0)$, $\mu_2(Q_i) = (0, 1, 0)$, $\mu_3(Q_i) = (0, 0, 1)$, for all $i \in \Lambda$; thus, $\mathcal{C}(S_1) = [\mu_j(Q_i)]_{13 \times 3}$ and $\mathcal{C}(S_2) = [\mu_j(Q_i)]_{4 \times 3}$. The systems $\text{FDM}(S_1, 2, 1, C_1)$ and $\text{FDM}(S_2, 2, 1, C_2)$ are defined for coldness-warmness and wetness-dryness questionnaire, respectively. A suitable similarity measure for $\text{FDM}(S_i, 2, 1, C_i)$ is $s_i = 1 - D_1^{W_i}$ for $i = 1, 2$, where $W_1 = \{w_i | 1 \leq i \leq 13\}$, $W_2 = \{w_i | i \in \{a, b, c, d\}\}$ and $D_1^{W_i}$ is the distance measure defined in Theorem 2.9. Let A_1 and A_2 be the sets of fuzzified filled out questionnaires for coldness-warmness and wetness-dryness respectively. In such a questionnaires the nonmembership function is corresponded to the option of moderate for each question.

Example 4.1. Let $\text{FDM}(S_i, 2, 1, C_i)$, $\mathcal{C}(S_i)$, $s_i = 1 - D_1^{W_i}$ and A_i are defined as above for $i = 1, 2$. Let five persons selected the options of the questionnaires as follows.

Table 5: The weight of each of the ten criteria. (The number in each cell shows the number of questions for each criterion of Ajnas-e-Asharah in the related questionnaire.)

Questionnaire	Coldness Warmness			Mazaheri	Wetness Dryness		
	Mojahedi	Salmannezhad	Akhtari		Mojahedi	Salmannezhad	Akhtari
Ajnas-e-Asharah							
Touch	1	1	2	1	1		1
Muscle and fat mass				2			
Hair condition				1		1	
Skin color			1	1		1	
Sleep and wakefulness				2		1	1
Physical functions	3	5	4	3			
Quality of waste matter				1			
Physique	1	2	2	2			
Impressibility speed	2	2	2	3			
Psychic function	1	5	4	3			
Muscle and fat mass + Physique				4	1	2	3
The number of questions	8	15	15		2	5	5

Question Person	Coldness-Warmness												
	Q ₁	Q ₂	Q ₃	Q ₄	Q ₅	Q ₆	Q ₇	Q ₈	Q ₉	Q ₁₀	Q ₁₁	Q ₁₂	Q ₁₃
1	2	3	4	3	4	3	4	4	2	2	4	3	4
2	2	4	5	3	5	3	5	5	2	2	5	5	5
3	3	1	4	4	4	4	5	4	1	1	5	1	4
4	1	2	3	3	4	4	4	4	1	1	2	2	2
5	1	4	2	2	2	5	2	3	1	3	5	2	4

Question Person	Wetness-Dryness			
	Q _a	Q _b	Q _c	Q _d
1	3	2	4	4
2	4	2	4	4
3	2	1	5	4
4	1	4	1	4
5	5	4	5	1

then we have,

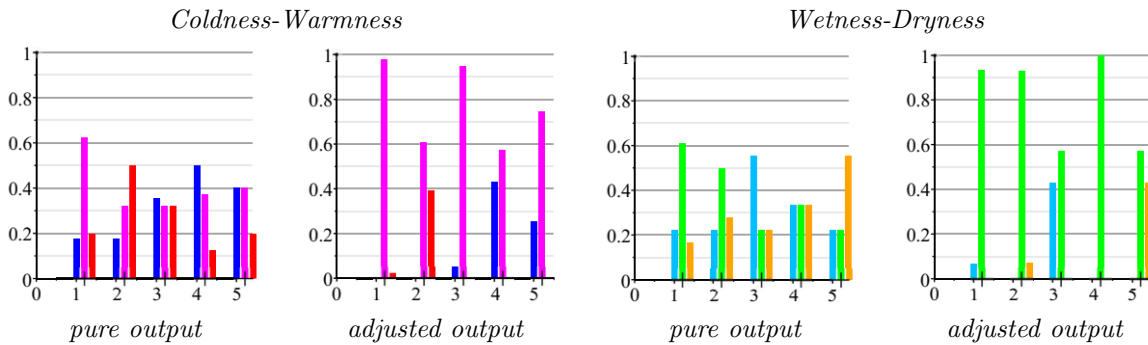
$$(\mathcal{R}_{S_1}(A_1))^T = \begin{pmatrix} (\frac{1}{2}, \frac{1}{2}, 0) & (\frac{1}{2}, \frac{1}{2}, 0) & (0, 1, 0) & (1, 0, 0) & (1, 0, 0) \\ (0, 1, 0) & (0, \frac{1}{2}, \frac{1}{2}) & (1, 0, 0) & (\frac{1}{2}, \frac{1}{2}, 0) & (0, \frac{1}{2}, \frac{1}{2}) \\ (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) & (0, 1, 0) & (\frac{1}{2}, \frac{1}{2}, 0) \\ (0, 1, 0) & (0, 1, 0) & (0, \frac{1}{2}, \frac{1}{2}) & (0, 1, 0) & (\frac{1}{2}, \frac{1}{2}, 0) \\ (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) & (0, \frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, 0) \\ (0, 1, 0) & (0, 1, 0) & (0, \frac{1}{2}, \frac{1}{2}) & (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) \\ (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, 0) \\ (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) & (0, \frac{1}{2}, \frac{1}{2}) & (0, 1, 0) \\ (\frac{1}{2}, \frac{1}{2}, 0) & (\frac{1}{2}, \frac{1}{2}, 0) & (1, 0, 0) & (1, 0, 0) & (1, 0, 0) \\ (\frac{1}{2}, \frac{1}{2}, 0) & (\frac{1}{2}, \frac{1}{2}, 0) & (1, 0, 0) & (1, 0, 0) & (0, 1, 0) \\ (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (0, 0, 1) & (\frac{1}{2}, \frac{1}{2}, 0) & (0, 0, 1) \\ (0, 1, 0) & (0, 0, 1) & (1, 0, 0) & (\frac{1}{2}, \frac{1}{2}, 0) & (\frac{1}{2}, \frac{1}{2}, 0) \\ (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, 0) & (0, \frac{1}{2}, \frac{1}{2}) \end{pmatrix}$$

and

$$\mathcal{R}_{S_2}(A_2) = \begin{pmatrix} (0, 1, 0) & (\frac{1}{2}, \frac{1}{2}, 0) & (0, \frac{1}{2}, \frac{1}{2}) & (0, \frac{1}{2}, \frac{1}{2}) \\ (0, \frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, 0) & (0, \frac{1}{2}, \frac{1}{2}) & (0, \frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}, 0) & (1, 0, 0) & (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) \\ (1, 0, 0) & (0, \frac{1}{2}, \frac{1}{2}) & (1, 0, 0) & (0, \frac{1}{2}, \frac{1}{2}) \\ (0, 0, 1) & (0, \frac{1}{2}, \frac{1}{2}) & (0, 0, 1) & (1, 0, 0) \end{pmatrix}.$$

Since coldness, warmth, wetness and dryness neutralize each other, respectively, we have to adjust the outputs of Algorithm 1 to have more accurate results. We implemented the algorithm in both Maple and Python B and C), respectively, and used these implementation to compute the examples. The algorithm's outputs are described below. According to a Persian medicine specialist (one of the authors), a Mizaj is classified as warm, cold, wet, or dry if the adjusted algorithmic output indicates at least 20

Person	1	2	3	4	5	Person	1	2	3	4	5
pure output											
Cold	0.18	0.18	0.36	0.50	0.40	Wet	0.22	0.22	0.56	0.33	0.22
Moderate	0.62	0.32	0.32	0.38	0.40	Moderate	0.61	0.50	0.22	0.33	0.22
Warm	0.20	0.50	0.32	0.12	0.20	Dry	0.17	0.28	0.22	0.33	0.56
adjusted output											
Cold	0	0	0.053	0.43	0.26	Wet	0.067	0	0.43	0	0
Moderate	0.98	0.61	0.95	0.57	0.74	Moderate	0.93	0.93	0.57	1	0.57
Warm	0.022	0.39	0	0	0	Dry	0	0.071	0	0	0.43



The results of these examples, obtained using the three questionnaire methods, are described below.

Person \ Questionnaire	Coldness-Warmness					Wetness-Dryness				
	1	2	3	4	5	1	2	3	4	5
Mojahedi	16	17	17	15	12	4	4	4	6	4
	M	M	M	M	C	M	M	M	D	M
Salmannezhad	50	60	54	39	47	14	16	11	11	22
	W	W	W	C	M	W	M	W	W	D
Akhtari	47	56	49	36	45	13	14	9	17	18
	W	W	W	C	W	M	M	W	D	D

As we can see in the above table, the persons 1, 2, 3 and 4 are moderate in coldness-warmness with respect to Mojahedi-questionnaire while by our method, the persons 2 and 4 are 61% cold and 57% warm respectively. The persons 1, 2 and 3 are warm with respect to Salmannezhad and Akhtari questionnaires while the persons 1 and 3 are 98% and 95% moderate in coldness-warmness respectively. The person 5 is 26% cold with respect to our methods while it is moderate and warm with respect to Salmannezhad and Akhtari questionnaires respectively. The results of the three questionnaire in wetness-dryness have significant differences with each other and with the results of our method too. All the results of our method is confirmed by the specialist in Persian medicine.

Based on the aforementioned, we propose that specialist in Persian medicine to for assessing Mizaj (temperament) and Sue-Mizaj (distemperament) of vital organs, leveraging the findings and methodological insights from the reviewed articles. For instance, to enhance the validity and quantifiability of the questionnaire presented in Table 7, response options could be refined with objective, measurable criteria. Specifically, subjective descriptors such as "obesity" or "thinness" may be replaced with Body Mass Index (BMI).

Table 6: The relation between the questions of the questionnaires

Questionnaire The union of the questionnaires	Mojahedi	Salmanezhad	Akhtari
Coldness-Warmness			
Q_1	Q_1	Q_3	Q_1, Q_{10}
Q_2		Q_c	Q_7
Q_3	Q_7	Q_6	Q_{12}
Q_4		Q_8	Q_{13}
Q_5	Q_4	Q_{10}	
Q_6	Q_8	Q_1, Q_{14}	Q_{11}, Q_{14}
Q_7		Q_{12}	Q_8
Q_8	Q_2	Q_{15}	Q_{15}
Q_9	Q_3	Q_9	Q_3
Q_{10}	Q_6	Q_{11}	Q_6
Q_{11}		Q_2, Q_4, Q_5	Q_2, Q_5
Q_{12}	Q_5	Q_7	Q_{12}
Q_{13}		Q_{13}	Q_4
Wetness-Dryness			
Q_a		Q_a	Q_e
Q_b	Q_a	Q_c, Q_d	Q_a, Q_c, Q_d
Q_c		Q_e	
Q_d	Q_b		Q_b

5 Compare and conclusion

The multicriteria decision-making method introduced in this article is well suited for modeling problems that assess the degree of similarity between each case and a set of specified criteria. Specifically, the criteria, fuzzification method, and similarity measure must first be established based on relevant literature, empirical data, and expert input, utilizing the tools presented in this article. Subsequently, the algorithm can be applied to each case to determine its degree of similarity to each criterion. Specifically, the criteria, fuzzification method, and similarity measure must first be established based on relevant literature, empirical data, and expert input, utilizing the tools presented in this article. Subsequently, the algorithm can be applied to each case to determine its degree of similarity to each criterion. For example, in this article, a person's Miza,j is identified by determining their degree of similarity to each temperature state (cold, moderate, warm) and moisture state (wet, moderate, dry). This differs from well-known fuzzy decision-making methods such as TOPSIS [12], AHP [36], VIKOR [20], and those described in [3, 6, 30], which typically rank alternatives to identify the optimal solution.

The fuzzy geometric space introduced in this article, that generalizes intuitionistic, Pythagorean, Fermatean, q-rung orthopair, picture, spherical and finite dimensional multi-fuzzy sets, provides a more applicable tool for modeling compared to existing fuzzy sets, because it allows each element to be evaluated with different parameters, and unlike multi-fuzzy sets, the number of parameters does not need to be the same for all elements. In other words, each element can be measured with its own appropriate number of parameters. One of our future works is to present suitable fuzzifying methods for various problems, that provides fuzzy geometric spaces as an applicable tool for modeling medical, social, legal, engineering, and other problems.

In the future, the decision-making method described in this article, particularly that discussed in Section 4, can be used for a more precise analysis of answers in fields such as psychology, medicine, economy, social sciences, etc. that utilize the questionnaire method. In future research, the multicriteria decision-making methods discussed above could be enhanced by replacing intuitionistic fuzzy space with the fuzzy geometric space proposed in this study. Additionally, new aggregation operators and related concepts—building upon the approaches presented in [16, 30]—can be defined and investigated. Furthermore, fuzzy geometric spaces can be integrated with other mathematical frameworks, such as soft set [7], and fuzzy graph theory. As an initial step in this direction, our ongoing work focuses on the development of fuzzy and vague graph models.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This study does not involve ethical issues.

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Data availability Our manuscript has no associated data.

Authorship contributions All authors contributed to the study conception and design. The first draft of the manuscript was written by Asieh Pourhaghani and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Human participants and/or animals This article does not contain any studies with human participants or animals performed by any of the authors.

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A Questionnaire

Ajnas-e-Ashara		Question	1	2	3	4	5
Coldness-Warmness							
Touch	1	When others touch your hand, what do they say about its warmness or coldness?	cold	almost cold	ordinary	almost warm	warm
skin color	2	Your skin color is in what range?	white	almost white	normal	brunette	yellow
Physical functions	3	How is your voice power?	weak	almost weak	ordinary	almost strong	strong
	4	How energetic is your daily routine?	low	almost low	ordinary	almost high	high
	5	How do you pronounce several consequent sentences?	articulate	almost articulate	ordinary	almost continuous	continuous
	6	How is your physical movements?	slow	almost slow	ordinary	almost fast	fast
Physique	7	How big your chest (or extremities)?	small	almost small	middle	almost vast (large)	vast (large)
	8	How big is the palm of your hand?	small	almost small	middle	almost big	big
Impressibility speed	9	Do you tolerate cold weather better or hot weather?	always hot	usually hot	both	usually cold	always cold
	10	How do you get feel by eating warm nature foods (as honey, spices, peper) or cold nature foods (as buttermilk, yogurt and cucumber)?	I get very bad feel with cold nature food.	I get bad feel with cold nature food.	Both are the same for me.	I get bad feel with warm nature food	I get very bad feel with warm nature food.
psychic functions	11	How are your social relationship (level of cheerfulness, silent and talkativeness, resp.)?	almost poor (not cheerful, silent resp.)	poor (not cheerful, silent resp.)	ordinary	almost strong (talkative, cheerful resp.)	strong (talkative, cheerful resp.)
	12	Are you hasty or patient? (speed of decision, speed of anger and rage)	patient	almost patient	ordinary	almost hasty	hasty
	13	How brave (responsible) are you?	coward (very low responsible)	almost coward (low responsible)	ordinary	almost brave (responsible)	brave (high responsible)
Wetness-Dryness							
Sleep-wakefulness	a	Do you usually sleep a lot or sleep less?	lot	almost lot	ordinary	almost less	less
Muscle and fat mass + Physique	b	How is your body (obesity and thinness or obesity talent)?	very large body (obese, high talent)	large (almost obese, almost high talent)	ordinary	small body (almost thin, almost low talent)	very small body (thin, low talent)
Hair condition	c	How is the thickness of your hair?	very thick	thick	ordinary	thin	very thin
Touch	d	How is the condition of your skin's softness and dryness?	very soft	soft	ordinary	dry	very dry

Table 7: The union of the questionnaires

B Maple code implementation

```

.....
dt:=proc(A,B,t)
  local n,j,i,d,a,b;
  n:=nops(A);
  a:=1-sum(A[i],i=1..n);
  b:=1-sum(B[i],i=1..n);
  d:=0;

```

```

for j to n do
  d:=d+abs(A[j]-B[j])^t;
end do;
d:=d+abs(a-b)^t;
return(d/2);
end proc;

dwt:=proc(A,B,t,W)
  local m,i,d,w;
  m:=nops(A);
  d:=0;
  w:=sum(W[i],i=1..m);
  for i to m do
    d:=d+W[i]*dt(A[i],B[i],t);
  end do;
  return(d/w);
end proc;

with(plots): with(plottools):
DWT:=proc(M,N,t,W,C,T)
  local i,j,m,n,D,d,F,H,L,S,s,F1,H1,L1,ss,Sd,SS;
  m:=nops(M);
  n:=nops(N);
  for i to m do
    D[i]:=seq(dwt(M[i],N[j],t,W[j]),j=1..n);
    d[i]:=min[index](D[i]);
    S[i]:=seq(1-D[i][j],j=1..n);
  end do;
  ss:=seq(S[i],i=1..m);
  S:=seq(S[i],i=1..m);
  for i to m do
    if S[i][1] <> 0 and S[i][3] <> 0 then
      s:=min(S[i][1],S[i][3]);
      S[i][1]:=(S[i][1]-s)/(1-s);
      S[i][3]:=(S[i][3]-s)/(1-s);
      S[i][2]:=(S[i][2]+s)/(1-s);
    end if;
    if S[i][2]>=0.8 then d[i]:=2; else
      if S[i][1] <> 0 then d[i]:=1; else d[i]:=3; end if;
    end if;
  end do;
  d:=seq(d[i],i=1..m);
  SS:=seq(seq(S[i][j],j=1..3),i=1..m);
  Sd:=seq([T[d[i]],evalf[2](S[i][d[i]])],i=1..m);
  F:={};
  H:={point([0,1],color=black,symbol=point,symbolsize=1)};
  for i to m do
    F:=F union {seq(line([(j-1)/(n+2)+i,0],[j-1)/(n+2)+i,S[i][j]],color=C[j],style=line,thickness=3),
      j=1..n)};
    H:=H union {seq(point([i+1/(n+2),0],color=black,symbol=cross,symbolsize=40),i=1..m)};
    L:=line([1/2,0],[m+1/2,0],color=black,thickness=1);
  end do;
  F1:={};
  H1:={point([0,1],color=black,symbol=point,symbolsize=1)};
  for i to m do
    F1:=F1 union {seq(line([(j-1)/(n+2)+i,0],[j-1)/(n+2)+i,1-D[i][j]],color=C[j],style=line,thickness=3),
      j=1..n)};
    H1:=H1 union {seq(point([i+1/(n+2),0],color=black,symbol=cross,symbolsize=40),i=1..m)};
    L1:=line([1/2,0],[m,0],color=black,thickness=1);
  end do;
  return(matrix([T,op(evalf[2](SS))]),matrix(Sd),display(F,H,L,axis[2]=[gridlines]),display(F1,H1,L1,

```

```
axis[2]=[gridlines]),matrix([T,op(evalf[2](ss))]);
end proc:
```

```
fuzzy:=proc(M)
local n,A,i;
n:=nops(M);
for i to n do
if M[i]=0 then A[i]:=[0,0,0]; end if;
if M[i]=1 then A[i]:=[1,0,0]; end if;
if M[i]=2 then A[i]:=[1/2,1/2,0]; end if;
if M[i]=3 then A[i]:=[0,1,0]; end if;
if M[i]=4 then A[i]:=[0,1/2,1/2]; end if;
if M[i]=5 then A[i]:=[0,0,1] end if;
end do;
A:=[seq(A[i],i=1..n)];
return(A);
end proc:
```

```
Mizaj:=proc(A,B)
local Nt,Nd,wt,wd,Wt,Wd,Mt,Md,mm,ss,aa,mmt,sst,aat,mmd,ssd,aad,T,D,n,i,j;
n:=nops(A);
Nt:=[[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],[1,0,0],
[1,0,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],
[0,1,0],[0,1,0],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],[0,0,1],
[0,0,1],[0,0,1],[0,0,1]];
Nd:=[[1,0,0],[1,0,0],[1,0,0],[1,0,0],[0,1,0],[0,1,0],[0,1,0],[0,1,0],[0,0,1],[0,0,1],[0,0,1],[0,0,1],
[0,0,1]];
wt:=[2,1,3/4,3/4,3/4,3/4,1,1,3/2,3/2,1,1,1];
Wt:=[wt,wt,wt];
wd:=[2,4,1,2];
Wd:=[wd,wd,wd];
for i to n do Mt[i]:=fuzzy(A[i]); Md[i]:=fuzzy(B[i]) end do;
Mt:=[seq(Mt[i],i=1..n)];
Md:=[seq(Md[i],i=1..n)];
T:=DWT(Mt,Nt,1,Wt,[blue,magenta,red],["Cold","Moderate","Warm"]);
D:=DWT(Md,Nd,1,Wd,["DeepSkyBlue","Lime","Orange"],["Wet","Moderate","Dry"]);
return(T,D);
end proc:
```

.....

C Python code implementation

```
import numpy as np

def dt(A, B, t):
    n = len(A)
    a = 1 - sum(A)
    b = 1 - sum(B)
    d = 0

    for j in range(n):
        d += abs(A[j] - B[j]) ** t

    d += abs(a - b) ** t
    return d / 2

def dwt(A, B, t, W):
    m = len(A)
    d = 0
    w = sum(W)
```

```

for i in range(m):
    d += W[i] * dt(A[i], B[i], t)

return d / w

def DWT(M, N, t, W, T):
    m = len(M)
    n = len(N)

    D = []
    d = []
    S = []

    for i in range(m):
        D_i = [dwt(M[i], N[j], t, W[j]) for j in range(n)]
        D.append(D_i)
        d.append(np.argmax(D_i))
        S.append([1 - D_i[j] for j in range(n)])

    ss = S.copy()
    S_copy = [row.copy() for row in S]

    for i in range(m):
        if S_copy[i][0] != 0 and S_copy[i][2] != 0:
            s = min(S_copy[i][0], S_copy[i][2])
            S_copy[i][0] = (S_copy[i][0] - s) / (1 - s)
            S_copy[i][2] = (S_copy[i][2] - s) / (1 - s)
            S_copy[i][1] = (S_copy[i][1] + s) / (1 - s)

        if S_copy[i][1] >= 0.8:
            d[i] = 1
        else:
            if S_copy[i][0] != 0:
                d[i] = 0
            else:
                d[i] = 2

    SS = [[round(val, 2) for val in row] for row in S_copy]
    Sd = [[T[d[i]], round(S_copy[i][d[i]], 2)] for i in range(m)]

    matrix1 = [T] + SS
    matrix3 = [T] + [[round(val, 2) for val in row] for row in ss]

    return matrix1, Sd, matrix3

def fuzzy(M):
    A = []
    for i in range(len(M)):
        if M[i] == 0:
            A.append([0, 0, 0])
        elif M[i] == 1:
            A.append([1, 0, 0])
        elif M[i] == 2:
            A.append([0.5, 0.5, 0])
        elif M[i] == 3:
            A.append([0, 1, 0])
        elif M[i] == 4:
            A.append([0, 0.5, 0.5])
        elif M[i] == 5:
            A.append([0, 0, 1])

```

