Iranian Journal of Fuzzy Systems

Volume 23, Number 2, (2026), pp. 127-139

Original paper

# The modularity condition for bi-uninorms

K. Zhang [1]

[1] *School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou, Jiangsu 215009, China*

zkeyu07@163.com

---

**Abstract**

The modularity equation, viewed as a specialized form of restricted general associative equations, holds important theoretical implications in fuzzy logic and fuzzy theory. In this work, we concentrate on examining the structural properties of two bi-uninorms that satisfy the modularity equation.

*Keywords:* Modularity equation, aggregation function, bi-uninorm, T-uninorm.

---

## 1 Introduction

As part of the architecture of intelligent systems, information aggregation acts as a critical mechanism that integrates data from multiple distributed sources, processes it through evaluation and fusion, and delivers comprehensive outputs tailored to specific objectives. As a mathematical model for processing aggregated information, aggregation functions are extensively employed in various domains, including image processing, statistical modeling, fuzzy logic and decision analysis [2, 3, 5]. The theory of aggregation functions has emerged as a significant methodology across numerous disciplines. Notably, aggregation functions with an annihilator element demonstrate applicability not only in rational decision-making problems, but also within the frameworks of aggregation functions defined over a bipolar scale [10]. In this framework, T-uninorms and S-uninorms form special classes of aggregation functions possessing an annihilator element.

In practice, the selection of aggregation functions is determined by their semantic interpretation, formally characterized through specific mathematical properties or functional equations. Among these, the associativity equation plays a crucial role in multi-input information processing [1]. From a theoretical perspective, the modularity equation serves as a generalized extension of the associativity equation, functioning as an essential element in fuzzy set theory [13]. Concurrently, modularity exhibits fundamental connections with distributivity[7, 18], a property that is axiomatically essential for fuzzy logical connectives.

Recently, many scholars have conducted extensive research on the modularity equation[4, 14, 16]. Qin [11] studied the modularity condition for uninorm continuous in $]0,1[$ and nullnorm. E. Rak [12] evaluated the modularity condition for a particular class of 2-uninorms. Mas et al. [8] investigated the modularity equation in the context of t-operators and uninorms belonging to the classes $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$. Su et al. [13] studied the modularity equation involving two uninorms: one drawn from a widely investigated family of uninorms, while the other represented an arbitrary uninorm without extra axiomatic or structural constraints. Zhao and Liu [17] examined the modularity for T-uninorms and semi-t-operators. Li [6] studied the modularity condition for T-uninorm and S-uninorm and the modularity equation between two T-uninorms or two S-uninorms, where one of its underlying uninorms belongs to $\mathcal{U}_{\max}$ (or $\mathcal{U}_{\min}$).

This paper is focused on investigating the modularity equation for bi-uninorms. As established in the literature, bi-uninorms include the disjunctive uninorm and conjunctive uninorm, which represent specific subclasses of 2-uninorms. However, existing research on the modularity equation for bi-uninorms has primarily focused on special classes, notably

---

Corresponding Author: K. Zhang

Received: September 2025; Revised: April 2026; Accepted: May 2026.

https://doi.org/10.22111/ijfs.2026.52709.9437

when the annihilator elements of two bi-uninorms coincide. Therefore, a comprehensive investigation of more general scenarios is still necessary.

The paper follows the structure outlined below. Section 2 reviews essential definitions and foundational concepts that will be utilized throughout this paper. Section 3 provides a detailed investigation of the modularity condition for bi-uninorms. Finally, Section 4 closes the paper by summarizing our contributions and identifying opportunities for further research.

## 2 Preliminaries

This section reviews fundamental notions and established results on aggregation functions, uninorms, and their known classes, providing a concise reference for the developments that follow.

**Definition 2.1.** [2] *A mapping* $A : [0,1]^2 \to [0,1]$ *is termed an* aggregation function *if it is increasing in each argument and fulfills* $A(0,0) = 0$ *and* $A(1,1) = 1$.

**Definition 2.2.** [15] *An aggregation function* $U : [0,1]^2 \to [0,1]$ *is referred to a* uninorm *if it is commutative, associative and possesses a neutral element* $e \in [0,1]$.

A uninorm is a *t-conorm* when its neutral element $e = 0$, a *t-norm* when $e = 1$, while it is proper if $e \in ]0,1[$. The uninorm $U$ behaves as a t-norm on $[0,e]^2$ and a t-conorm on $[e,1]^2$ if $e \in ]0,1[$, and remains within minimum and maximum on

$$A(e) = [0,e[\times]e,1] \cup ]e,1] \times [0,e[.$$

Denoted by $U \equiv \langle T, e, S\rangle$ the class of all uninorms, $T$ denotes the underlying t-norm, $e$ denotes the neutral element, and $S$ is the underlying t-conorm. Specifically, if $U(1,0) = 0$, the uninorm is referred to as *conjunctive*; whereas it is *disjunctive* if $U(1,0) = 1$.

**Definition 2.3.** [9] *An aggregation operator* $F : [0,1]^2 \to [0,1]$ *is identified as a* bi-uninorm *if it is associative, commutative, and fulfills:*

(i) $F(0,\cdot)$ *and* $F(1,\cdot)$ *are discontinuous;*

(ii) *there exists an annihilator element* $k \in ]0,1[$*, two idempotent elements* $e_0, e_1 \in ]0,1[$*, such that* $F(e_0,\cdot)$ *and* $F(e_1,\cdot)$ *are continuous,* $F(e_0,0) = 0$ *and* $F(e_1,1) = 1$.

Below we present the structural characterization of bi-uninorms.

**Theorem 2.4.** [9] *A function* $F : [0,1]^2 \to [0,1]$ *is a bi-uninorm if and only if there exists* $k \in ]0,1[$*, a conjunctive uninorm* $U_1$ *whose neutral element is* $\frac{e_1-k}{1-k}$*, and a disjunctive uninorm* $U_0$ *is characterized by the neutral element* $\frac{e_0}{k}$ *satisfying*

$$F(x,y) = \begin{cases} kU_0\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_1\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{otherwise.} \end{cases} \tag{1}$$

$U_0$ and $U_1$ are designated as the underlying uninorm. Furthermore, we denote by $\mathcal{U}^k_{e_0,e_1} \equiv \langle U_0, e_0, k, e_1, U_1\rangle$ the class of bi-uninorms characterized by an annihilator element $k$ in conjunction with neutral the elements $e_0 \in ]0,k[$, $e_1 \in ]k,1[$.

Fig. 1 illustrates the structural framework of bi-uninorms as outlined in Theorem 2.4.

**Definition 2.5.** [9] *An aggregation operator* $F : [0,1]^2 \to [0,1]$ *is identified as a* T-uninorm *if it is associative, commutative, and fulfills:*

(i) $F(1,\cdot)$ *is continuous whereas* $F(0,\cdot)$ *is discontinuous;*

(ii) *there exists an annihilator element* $k \in ]0,1]$*,* $F(e,\cdot)$ *is continuous,* $F(e,0) = 0$*, and* $e \in ]0,1[$ *denotes the idempotent element.*

The T-uninorm would be a disjunctive uninorm when its annihilator element $k = 1$, and its neutral element is $e$. We call a T-uninorm proper when $k \in ]0,1[$. The theorem below delivers a comprehensive characterization of the structural properties of T-uninorm.

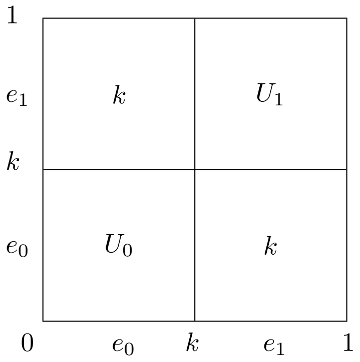

Figure 1: The characterization of bi-uninorms given by Theorem 2.4.

**Theorem 2.6.** [9] *A function* $F : [0,1]^2 \to [0,1]$ *is a T-uninorm if and only if there exists some* $k \in ]0,1]$, $e \in ]0,k[$, *a disjunctive uninorm* $U$ *characterized by the neutral element* $\frac{e}{k}$, *and a t-norm* $T$ *satisfying*

$$F(x,y) = \begin{cases} kU\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)T\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{otherwise.} \end{cases} \tag{2}$$

A T-uninorm is constructed by designating the disjunctive uninorm $U$ as its underlying uninorm, t-norm $T$ as its underlying t-norm. Additionally, we denote by $\mathcal{U}_{e,k}^{T} \equiv \langle U, e, k, T\rangle$ the class of T-uninorms characterized by an annihilator element $k$ in conjunction with neutral element $e \in ]0,k[$.

As described in Theorem 2.6, the characterization of T-uninorms is shown in Fig. 2

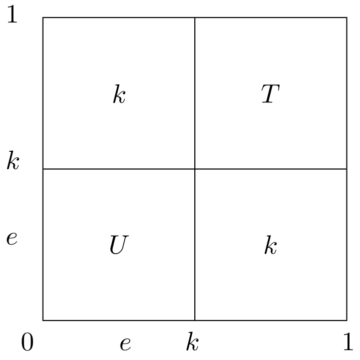

Figure 2: The characterization of T-uninorms from Theorem 2.6.

We begin by recalling the concept of modularity within the context of commutative binary operators.

**Definition 2.7.** [8] *Suppose* $F, G$ *are two commutative binary operations on* $[0,1]^2$. *Then* $F$ *is modular over* $G$ *if for any* $x, y, z \in [0,1]$ *with* $z \leq x$, *it holds*

$$F(x,\, G(y,\, z)) = G(F(x,\, y),\, z)\,. \tag{3}$$

Below is a description of the modularity condition between commutative aggregation with an annihilator element:

**Proposition 2.8.** [6] *Given two commutative aggregation operations* $F, G$ *on* $[0,1]^2$, *their annihilator elements are* $k_1$ *and* $k_2$, *respectively. If* $F$ *and* $G$ *satisfy the modularity condition, then* $k_1 \leq k_2$.

## 3 The modularity condition between bi-uninorms

A detailed investigation of the modularity conditions for bi-uninorms is presented in this section.

We begin by establishing the fundamental relationship between the annihilator and neutral element of bi-uninorm.

**Lemma 3.1.** *Let* $F \equiv \langle U_0, e_0, k, e_1, U_1\rangle$, $G \equiv \langle U_0', e_0', k', e_1', U_1'\rangle$ *be two bi-uninorms that satisfy the modularity condition, then* $0 < e_0' < k < 1$.

*Proof.* Consider $F$ and $G$ fulfilling Eq. (3). If $k \leq e_0'$, substituting $y = z = 0, x = k$ in Eq. (3), one obtains

$$F(k, G(0,0)) = F(k,0) = k, G(F(k,0),0) = G(k,0) = 0,$$

which constitutes an irreconcilable contradiction. Then we can get $0 < e_0' < k < 1$. □

**Lemma 3.2.** *Suppose* $F \equiv \langle U_0, e_0, k, e_1, U_1\rangle$, $G \equiv \langle U_0', e_0', k', e_1', U_1'\rangle$ *are bi-uninorms that satisfy the modularity condition, then* $0 < k' < e_1 < 1$.

*Proof.* Provide that $F$ and $G$ meet the conditions of Eq. (3). If $e_1 \leq k'$, taking $z = k', y = x = 1$ in Eq. (3), one can get

$$F(1, G(1,k')) = F(1,k') = 1, G(F(1,1),k') = G(1,k') = k',$$

thereby establishing a desired contradiction. Then we obtain that $0 < k' < e_1 < 1$. □

According to the definition of bi-uninorm, we have $0 < e_0 < k < e_1 < 1$ and $0 < e_0' < k' < e_1' < 1$. Based on Lemmas 3.1 and 3.2, and considering the relationship between $k$ and $k'$, we start our work with the case that $0 < e_0' < k \leq k' < e_1 < 1$.

**Lemma 3.3.** *Let* $F \equiv \langle U_0, e_0, k, e_1, U_1\rangle$, $G \equiv \langle U_0', e_0', k', e_1', U_1'\rangle$ *be two bi-uninorms that satisfy the modularity condition, suppose* $0 < e_0' < k \leq k' < e_1 < 1$, *then the subsequent conditions hold:*

(i) *there exists a T-uninorm* $F'$ *characterized by the idempotent element* $\frac{e_0}{k'}$ *and annihilator element* $\frac{k}{k'}$; *a conjunctive uninorm* $U_2$ *possessing neutral element* $\frac{e_1-k'}{1-k'}$; *a disjunctive uninorm* $U_0'$ *whose neutral element is* $\frac{e_0'}{k'}$ *and a conjunctive uninorm* $U_1'$ *which is characterized by the neutral element* $\frac{e_1'-k'}{1-k'}$ *satisfying*

$$F(x,y) = \begin{cases} k'F'\left(\frac{x}{k'}, \frac{y}{k'}\right) & \text{if } (x,y) \in [0,k']^2, \\ k' + (1-k')U_2\left(\frac{x-k'}{1-k'}, \frac{y-k'}{1-k'}\right) & \text{if } (x,y) \in [k',1]^2, \\ min(x,y) & \text{if } (x,y) \in [k,k'] \times [k',1] \cup [k',1] \times [k,k'], \\ k & \text{otherwise,} \end{cases} \tag{4}$$

$$G(x,y) = \begin{cases} k'U_0'\left(\frac{x}{k'}, \frac{y}{k'}\right) & \text{if } (x,y) \in [0,k']^2, \\ k' + (1-k')U_1'\left(\frac{x-k'}{1-k'}, \frac{y-k'}{1-k'}\right) & \text{if } (x,y) \in [k',1]^2, \\ k' & \text{otherwise.} \end{cases} \tag{5}$$

(ii) $F'$ *is modular over* $U_0'$.

(iii) $U_2$ *is modular over* $U_1'$.

*Proof.* Consider $F$ and $G$ fulfill Eq. (3). Assume $e_0' < k \leq k' < e_1$.

(i) Consider $z \in [k,k']$ and $x \in [k',1]$ subject to the constraint $z \leq x$. By setting $y = e_0'$ in Eq. (3), one obtains

$$F(x, G(e_0', z)) = F(x,z) = G(k,z) = G(F(x,e_0'),z),$$

taking $x = e_1$, we can get $G(k,z) = F(e_1,z) = z$, implies $F(x,z) = z$. Consequently, one obtains $\forall (x,y) \in [k,k'] \times [k',1] \cup [k',1] \times [k,k']$, $F(x,y) = \min(\text{x, y})$ since $F$ is commutative. Therefore, we can get $F(k',k') = k'$.

Consider $F', U_2 : [0,1]^2 \to [0,1]$ defined by

$$F'(x,y) = \frac{F(k'x, k'y)}{k'},$$

$$U_2(x,y) = \frac{F(k' + (1-k')x, k' + (1-k')y) - k'}{1-k'}.$$

Then $F'$ is a T-uninorm with idempotent element $\frac{e_0}{k'}$ and annihilator element $\frac{k}{k'}$, $U_2$ is a conjunctive uninorm characterized by the neutral element $\frac{e_1 - k'}{1-k'}$.

Bringing all the above discussions together, we can see that $F$ and $G$ admit the explicit representations given in (4) and (5), respectively.

(ii) Given any $x, y, z \in [0, k']$ satisfying $z \le x$, due to $F$ is modular over $G$, one implies that $F'$ exhibits modularity over $U_0'$ .

(iii) Similar to Lemma 3 (ii), the modularity condition between $F$ and $G$ implies that $U_2$ is modular over $U_1'$. □

Fig. 3 illustrates the structural forms of $F$ and $G$ as established by Lemma 3.3.

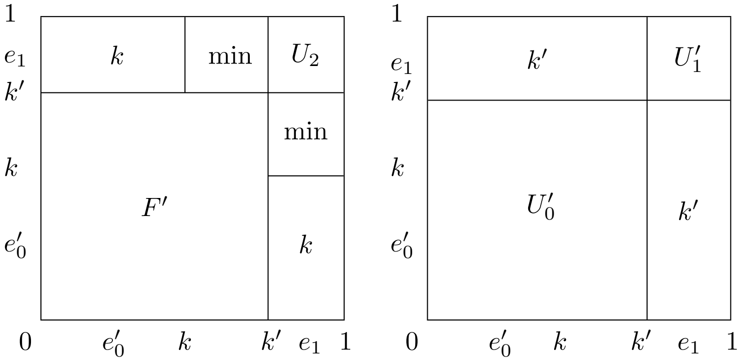

Figure 3: The characterization of $F$ and $G$ from Lemma 3.3.

From Lemma 3.3, to investigate the modularity condition between bi-uninorms, we should study the modularity condition between T-uninorm and uninorm. First let us present some results that were proved in [6].

**Proposition 3.4.** [6] *Suppose $F \equiv \langle U_1, e_1, k_1, T_1\rangle$ is T-uninorm and $U \equiv \langle T_2, e_2, S_2\rangle$ is uninorm, then*

(i) *U is not modular over F.*

(ii) *U is a disjunctive uninorm if F is modular over U.*

Similar to Lemmas 3.1 and 3.2, we provide a detailed characterization between the annihilator of T-uninorm and the neutral element of uninorm in what follows.

**Lemma 3.5.** *Let $F \equiv \langle U_1, e_1, k_1, T_1\rangle$ be a T-uninorm and $U \equiv \langle T_2, e_2, S_2\rangle$ be a uninorm, if F and U satisfy the modularity condition, then $0 < e_2 < k_1 < 1$.*

According to the definition of T-uninorm, we have $0 < e_1 < k_1 < 1$. Based on Lemma 3.5, we start our work with the case where $0 < e_2 < k_1 < 1$.

**Theorem 3.6.** *Let $F \equiv \langle U_1, e_1, k_1, T_1\rangle$ be a T-uninorm and $U \equiv \langle T_2, e_2, S_2\rangle$ be an uninorm, suppose $0 < e_2 < k_1 < 1$, then F is modular over U if and only if the following conditions hold:*

(i) *there exists a disjunctive uninorm $U'$ possessing a neutral element $\frac{e_2}{k_1}$ satisfying*

$$F(x,y) = \begin{cases} k_1 U_1\left(\frac{x}{k_1}, \frac{y}{k_1}\right) & \text{if } (x,y) \in [0,k_1]^2, \\ min(x,y) & \text{if } (x,y) \in [k_1,1]^2, \\ k_1 & \text{otherwise,} \end{cases} \tag{6}$$

$$U(x,y) = \begin{cases} k_1 U'\left(\frac{x}{k_1}, \frac{y}{k_1}\right) & \text{if } (x,y) \in [0,k_1]^2, \\ max(x,y) & \text{otherwise.} \end{cases} \tag{7}$$

(ii) $U_1$ *is modular over* $U'$.

*Proof.* ($\Rightarrow$) Suppose $F$ and $G$ fulfill Eq. (3). Assume $0 < e_2 < k_1 < 1$, from Proposition 3.4 one can get that $U$ is disjunctive.

(i) Setting $y = 1, z \in [0,1], x \in [k_1, 1]$, z $\leq x$ in Eq. (3), one implies

$$U(x,z) = U(F(x,1),z) = F(x,U(1,z)) = F(x,1) = x\,.$$

Consequently, due to $U$ is commutative, we deduce $\forall(x,y) \in [0,1] \times [k_1,1] \cup [k_1,1] \times [0,1]$, $U(x,y) = \max(\text{x, y})$.

Given $z, x \in [k_1, 1]$ where $z \leq x$. Substituting $y = k_1$ into Eq. (3), we imply

$$F(x,z) = F(x,U(k_1,z)) = U(F(x,k_1),z) = U(k_1,z) = z. ==$$

Therefore, Since $F$ is commutative, we have $\forall(x,y) \in [k_1,1]^2$, $F(x,y) = min(x,y)$.

Consider $U' : [0,1]^2 \to [0,1]$ defined by

$$U'(x,y) = \frac{U(k_1x, k_1y)}{k_1}.$$

Then $U'$ is a disjunctive uninorm and $\frac{e_2}{k_1}$ is the neutral element. Summing up, it follows that $F$ and $U$ have the formula (6) and (7), respectively, then Item (i) holds.

(ii) Consider $x, y, z \in [0, k_1]$ subject to the constraint $z \leq x$, we can see that $U_1$ is modular over $U'$ since the modularity condition between $F$ and $G$, i.e., Item (ii) holds.

($\Leftarrow$) Conversely, assuming that $F$ and $U$ exhibit the forms outlined in (6) and (7), respectively, the following cases must be taken into consideration.

Let $y \in [0, k_1]$, $z \leq x$.

(1) If $z, x \in [0, k_1]$, we deduce $F(x, U(y,z)) = U(F(x,y), z)$ because $U_1$ is modularity over $U'$ .

(2) If $z \in [0, k_1]$, $x \in [k_1, 1]$, one can obtain that $U(y,z) \leq U(k_1,k_1) = k_1$ since $U$ is increasing, we consequently obtain the equalities $F(x,U(y,z)) = k_1 = U(k_1,z) = U(F(x,y),z)$.

(3) If $z, x \in [k_1, 1]$, then $F(x,U(y,z)) = F(x,z) = z = U(k_1,z) = U(F(x,y),z)$.

Let $y \in [k_1, 1]$, $z \leq x$.

(4) If $x, z \in [0, k_1]$, we obtain $F(x,U(y,z)) = F(x,y) = k_1 = U(k_1,z) = U(F(x,y),z)$.

(5) If $x \in [k_1, 1]$, $z \in [0, k_1]$, we have $k_1 = F(k_1, k_1) \leq F(x,y)$ because of the monotonicity of $F$, which implies that $F(x,U(y,z)) = F(x,y) = max(F(x,y),z) = U(F(x,y),z)$.

(6) If $z, x \in [k_1, 1]$, one implies that $F(x,y)|_{[k_1,1]^2} = min(x,y)$, $U(x,y)|_{[k_1,1]^2} = max(x,y)$. Based on [8], we obtain $F(x,U(y,z)) = U(F(x,y),z)$. $\square$

The configurations of $F$ and $U$ described in Theorem 3.6 are illustrated in Fig. 4.

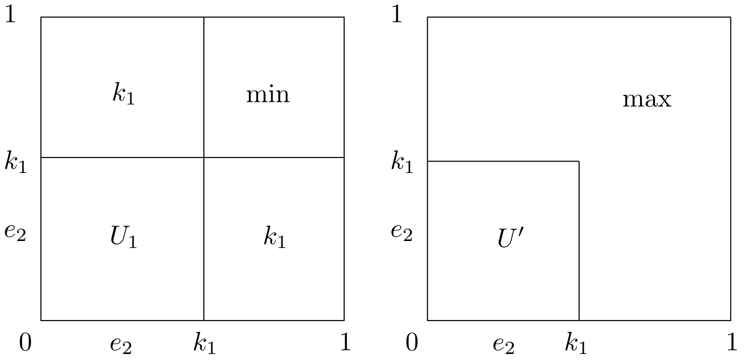

Figure 4: The characterization of $F$ and $U$ from Theorem 3.6.

**Example 3.7.** *Let $F \equiv \langle U_1, e_1, k_1, T_1 \rangle$ be a T-uninorm and $U \equiv \langle T_2, e_2, S_2 \rangle$ be an uninorm, assume $k_1 = \frac{4}{5}$, $e_1 = \frac{1}{3}$, $e_2 = \frac{1}{4}$, and the structures of $F$ and $U$ are given by:*

$$F(x,y) = \begin{cases} 4xy & \text{if } (x,y) \in [0, \frac{1}{4}]^2, \\ \frac{4}{5} & \text{if } (x,y) \in [0, \frac{4}{5}] \times [\frac{4}{5}, 1] \cup [\frac{4}{5}, 1] \times [0, \frac{4}{5}], \\ max(x,y) & \text{if } (x,y) \in [0, \frac{4}{5}] \times ]\frac{1}{3}, \frac{4}{5}] \cup ]\frac{1}{3}, \frac{4}{5}] \times [0, \frac{4}{5}], \\ min(x,y) & \text{otherwise,} \end{cases}$$

$$U(x,y) = \begin{cases} 4xy & \text{if } (x,y) \in [0, \frac{1}{4}]^2, \\ min(x,y) & \text{if } (x,y) \in [0, \frac{1}{4}[ \times [\frac{1}{4}, \frac{1}{3}] \cup [\frac{1}{4}, \frac{1}{3}] \times [0, \frac{1}{4}[, \\ max(x,y) & \text{otherwise.} \end{cases}$$

It follows from a routine calculation that $F$ is modular over $U$.

We have known the necessary condition for two bi-uninorms that satisfy the modularity equation as stated in Lemma 3.3. From Theorem 3.6, it has been established the structures of T-uninorm and uninorm that satisfy the modularity equation. The subsequent result will be induced.

**Theorem 3.8.** *Let $F \equiv \langle U_0, e_0, k, e_1, U_1 \rangle$ and $G \equiv \langle U_0', e_0', k', e_1', U_1' \rangle$ be two bi-uninorms, suppose $0 < e_0' < k \le k' < e_1 < 1$, then $F$ is modular over $G$ if and only if the following conditions hold:*

(i) *there exists a disjunctive uninorm $U_0$ characterized by the neutral element $\frac{e_0}{k}$; a conjunctive uninorm $U_2$ possessing neutral element $\frac{e_1 - k'}{1-k'}$; a disjunctive uninorm $U_0''$ which is characterized by the neutral element $\frac{e_0'}{k}$ and a conjunctive uninorm $U_1'$ possessing neutral element $\frac{e_1'-k'}{1-k'}$ satisfying*

$$F(x,y) = \begin{cases} kU_0\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k' + (1-k')U_2\left(\frac{x-k'}{1-k'}, \frac{y-k'}{1-k'}\right) & \text{if } (x,y) \in [k',1]^2, \\ min(x,y) & \text{if } (x,y) \in [k,k'] \times [k,1] \cup [k,1] \times [k,k'], \\ k & \text{otherwise,} \end{cases} \tag{8}$$

$$G(x,y) = \begin{cases} kU_0''\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k' + (1-k')U_1'\left(\frac{x-k'}{1-k'}, \frac{y-k'}{1-k'}\right) & \text{if } (x,y) \in [k',1]^2, \\ max(x,y) & \text{if } (x,y) \in [0,k'] \times [k,k'] \cup [k,k'] \times [0,k'], \\ k' & \text{otherwise.} \end{cases} \tag{9}$$

(ii) *$U_0$ is modular over $U_0''$.*

(iii) *$U_2$ is modular over $U_1'$.*

*Proof.* ($\Rightarrow$) Let $F$ and $G$ fulfill Eq. (3). Assume $e_0' < k \le k' < e_1$. From Lemma 3.3 and Theorem 3.6, we deduce that $F$ and $G$ possess the explicit forms given by (8) and (9), respectively, i.e., Item (i), Item (ii), and Item (iii) holds.

($\Leftarrow$) Conversely, when $F$ and $G$ exhibit the forms outlined in (8) and (9), respectively, the subsequent cases must be analyzed.

Let $y \in [0,k]$, $z \le x$.

1. If $z, x \in [0,k]$, then the modularity condition of $U_0$ and $U_0''$ yields that $F(x, G(y,z)) = G(F(x,y), z)$.
2. Suppose $z \in [0,k]$, $x \in [k,1]$. Since $G$ is non-decreasing in each argument we have $k = G(k,k) \ge G(y,z)$, thus we obtain the equalities $F(x, G(y,z)) = k = G(k,z) = G(F(x,y), z)$.
3. If $z \in [k,k']$, $x \in [k,1]$, then $F(x, G(y,z)) = F(x,z) = z = G(k,z) = G(F(x,y), z)$.
4. If $z, x \in [k',1]$, then $F(x, G(y,z)) = F(x,k') = k' = G(k,z) = G(F(x,y), z)$.

Let $y \in [k,k']$ and $z \le x$.

(5) If $z, x \in [0, k]$, we obtain that $F(x, G(y, z)) = F(x, y) = k = G(k, z) = G(F(x, y), z)$.

(6) If $z \in [0, k]$, $x \in [k, k']$, then because $F$ is increasing, we obtain $k \leq F(x, y) \leq k'$, consequently $F(x, G(y, z)) = F(x, y) = G(F(x, y), z)$.

(7) If $z \in [0, k]$, $x \in [k', 1]$, we get $F(x, G(y, z)) = F(x, y) = y = G(y, z) = G(F(x, y), z)$.

(8) If $z, x \in [k, k']$, one deduces $F(x, y)|_{[k,k']^2} = min(x, y)$ and $G(x, y)|_{[k,k']^2} = max(x, y)$, according to [8] this yields $F(x, G(y, z)) = G(F(x, y), z)$.

(9) Suppose $z \in [k, k']$, $x \in [k', 1]$. Since $G$ is increasing, it follows that $k \leq G(y, z) \leq k'$. Consequently, we obtain $F(x, G(y, z)) = G(y, z) = G(F(x, y), z)$.

(10) If $z, x \in [k', 1]$, then we obtain $F(x, G(y, z)) = F(x, k') = k' = G(y, z) = G(F(x, y), z)$.

Let $y \in [k', 1]$, $z \leq x$.

(11) If $x, z \in [0, k]$, we can get $F(x, G(y, z)) = F(x, k') = k = G(k, z) = G(F(x, y), z)$.

(12) If $x \in [k, k']$, $z \in [0, k']$, then it follows that $F(x, G(y, z)) = F(x, k') = x = G(x, z) = G(F(x, y), z)$.

(13) Suppose $x \in [k', 1]$, and $z \in [0, k']$. Since $F$ is increasing, it follows that $k' = F(k', k') \leq F(x, y)$. Cosequently, $F(x, G(y, z)) = F(x, k') = k' = G(F(x, y), z)$.

(14) Since $U_2$ is modular over $U_1'$, it follows for all $x, z \in [k', 1]$, we have $F(x, G(y, z)) = G(F(x, y), z)$.

□

**Remark 3.9.** *Theorem 3.8 presents the structural forms of bi-uninorms that satisfy the modularity condition.*

(i) *Theorem 11* [12] *presented the modularity condition for some subclasses of 2-uninorms (which cover bi-uninorms), and the author investigated the case where the annihilator elements of two 2-uninorms are equal. Based on Theorem 10* [12], *the underlying uninorm of 2-uninorms in Theorem 11 belongs to* $\mathcal{U}_{\min}$ *and* $\mathcal{U}_{\max}$. *In this paper, Theorem 3.8 presents the case where the annihilator elements of two bi-uninorms satisfy* $0 < k \leq k' < 1$, *and the underlying uninorm of the bi-uninorm is given without any further assumptions.*

(ii) *A detailed characterization of the modularity condition for uninorms is given in* [13].

The configurations of $F$ and $G$ from Theorem 3.8 are illustrated in Fig. 5.

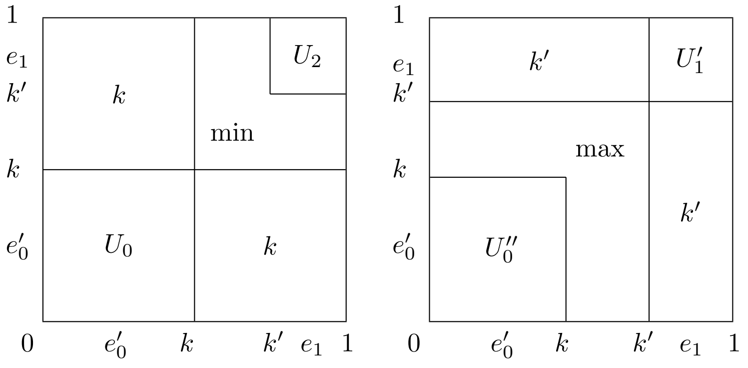

Figure 5: The characterization of $F$ and $G$ from Theorem 3.8.

**Example 3.10.** *Let* $F \equiv \langle U_0, e_0, k, e_1, U_1 \rangle$ *and* $G \equiv \langle U_0', e_0', k', e_1', U_1' \rangle$ *be two bi-uninorms. Assume* $e_0' = \frac{1}{8}$, $e_0 = \frac{1}{4}$, $k = \frac{3}{8}$, $k' = \frac{1}{2}$, $e_1' = \frac{5}{8}$, $e_1 = \frac{3}{4}$, *and the structures of* $F$ *and* $G$ *are given by:*

$$F(x, y) = \begin{cases} max(x + y - \frac{1}{4}, 0) & \text{if } (x, y) \in [0, \frac{1}{8}]^2, \\ \frac{3}{8} & \text{if } (x, y) \in [0, \frac{3}{8}] \times [\frac{3}{8}, 1] \cup [\frac{3}{8}, 1] \times [0, \frac{3}{8}], \\ max(x, y) & \text{if } (x, y) \in [0, \frac{3}{8}] \times ]\frac{1}{4}, \frac{3}{8}] \cup ]\frac{1}{4}, \frac{3}{8}] \times [0, \frac{3}{8}] \cup [\frac{5}{8}, 1] \times ]\frac{3}{4}, 1] \cup ]\frac{3}{4}, 1] \times [\frac{5}{8}, 1], \\ min(x, y) & \text{otherwise}, \end{cases}$$

$$G(x,y)=\begin{cases} max(x+y-\frac{1}{4},0) & \text{if } (x,y)\in[0,\frac{1}{8}]^2,\\ \frac{1}{2} & \text{if } (x,y)\in[0,\frac{1}{2}]\times[\frac{1}{2},1]\cup[\frac{1}{2},1]\times[0,\frac{1}{2}],\\ min(x,y) & \text{if } (x,y)\in[0,\frac{1}{8}[\times[\frac{1}{8},\frac{1}{4}]\cup[\frac{1}{8},\frac{1}{4}]\times[0,\frac{1}{8}[\cup[\frac{1}{2},\frac{5}{8}[\times[\frac{1}{2},1]\cup[\frac{1}{2},1]\times[\frac{1}{2},\frac{5}{8}[,\\ max(x,y) & \text{otherwise.}\end{cases}$$

*Then $F$ is modular over $G$.*

*Proof.* Let $y\in[0,\frac{1}{8}]$, $z\le x$.

(1) Suppose $z\in[0,\frac{1}{8}]$, then $0\le G(y,z)\le\frac{1}{8}$.

- If $x\in[0,\frac{1}{8}]$, then we have $F(x,G(y,z))=G(F(x,y),z)$ since $F=G$.
- If $x\in[\frac{1}{8},\frac{1}{4}]$, then $F(x,G(y,z))=min(x,G(y,z))=G(y,z)=G(F(x,y),z)$.
- If $x\in[\frac{1}{4},\frac{3}{8}]$, then $F(x,G(y,z))=max(x,G(y,z))=x=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=\frac{3}{8}=G(\frac{3}{8},z)=G(F(x,y),z)$.

(2) Suppose $z\in[\frac{1}{8},\frac{1}{4}]$, then $G(y,z)=y$.

- If $x\in[\frac{1}{8},\frac{1}{4}]$, then $F(x,G(y,z))=F(x,y)=y=G(y,z)=G(F(x,y),z)$.
- If $x\in[\frac{1}{4},\frac{3}{8}]$, then $F(x,G(y,z))=F(x,y)=x=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=F(x,y)=\frac{3}{8}=G(\frac{3}{8},z)=G(F(x,y),z)$.

(3) Suppose $z\in[\frac{1}{4},\frac{3}{8}]$, then $G(y,z)=z$.

- If $x\in[\frac{1}{4},\frac{3}{8}]$, then $F(x,G(y,z))=F(x,z)=x=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=F(x,z)=\frac{3}{8}=G(\frac{3}{8},z)=G(F(x,y),z)$.

(4) Suppose $z\in[\frac{3}{8},\frac{1}{2}]$, then $G(y,z)=z$.

- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=F(x,z)=z=G(\frac{3}{8},z)=G(F(x,y),z)$.

(5) Suppose $z,x\in[\frac{1}{2},1]$, then $F(x,G(y,z))=F(x,\frac{1}{2})=\frac{1}{2}=G(\frac{3}{8},z)=G(F(x,y),z)$.

Let $y\in[\frac{1}{8},\frac{1}{4}]$, $z\le x$.

(1) Suppose $z\in[0,\frac{1}{8}]$, then $G(y,z)=z$.

- If $x\in[0,\frac{1}{8}]$, then $F(x,G(y,z))=F(x,z)=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{1}{8},\frac{1}{4}]$, then $\frac{1}{8}\le F(x,y)\le\frac{1}{4}$. Consequently, we obtain the chain of equalities $F(x,G(y,z))=F(x,z)=z=min(F(x,y),z)=G(F(x,y),z)$.
- If $x\in[\frac{1}{4},\frac{3}{8}]$, then $F(x,G(y,z))=F(x,z)=x=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=F(x,z)=\frac{3}{8}=G(\frac{3}{8},z)=G(F(x,y),z)$.

(2) Suppose $z\in[\frac{1}{8},\frac{1}{4}]$, then we have $\frac{1}{8}\le G(y,z)\le\frac{1}{4}$.

- If $x\in[\frac{1}{8},\frac{1}{4}]$, then $F(x,y)|_{[\frac{1}{8},\frac{1}{4}]^2}=min(x,y)$ and $G(x,y)|_{[\frac{1}{8},\frac{1}{4}]^2}=max(x,y)$. According to [8] one can get $F(x,G(y,z))=G(F(x,y),z)$.
- If $x\in[\frac{1}{4},\frac{3}{8}]$, then $F(x,G(y,z))=max(x,G(y,z))=x=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=\frac{3}{8}=G(\frac{3}{8},z)=G(F(x,y),z)$.

(3) Suppose $z\in[\frac{1}{4},\frac{3}{8}]$, then $G(y,z)=z$.

- If $x\in[\frac{1}{4},\frac{3}{8}]$, then $F(x,G(y,z))=F(x,z)=x=G(x,z)=G(F(x,y),z)$.
- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=F(x,z)=\frac{3}{8}=G(\frac{3}{8},z)=G(F(x,y),z)$.

(4) Suppose $z\in[\frac{3}{8},\frac{1}{2}]$, then $G(y,z)=z$.

- If $x\in[\frac{3}{8},1]$, then $F(x,G(y,z))=F(x,z)=z=G(\frac{3}{8},z)=G(F(x,y),z)$.

(5) Suppose $z, x \in [\frac{1}{2}, 1]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(\frac{3}{8}, z) = G(F(x, y), z)$.

Let $y \in [\frac{1}{4}, \frac{3}{8}]$, $z \leq x$.

(1) Suppose $z \in [0, \frac{1}{4}]$, then $G(y, z) = y$.

- If $x \in [0, \frac{1}{4}]$, then $F(x, G(y, z)) = F(x, y) = y = G(y, z) = G(F(x, y), z)$.
- If $x \in [\frac{1}{4}, \frac{3}{8}]$, then we have $\frac{1}{4} \leq F(x, y) \leq \frac{3}{8}$. Consequently, we obtain the chain of equalities $F(x, G(y, z)) = F(x, y) = max(F(x, y), z) = G(F(x, y), z)$.
- If $x \in [\frac{3}{8}, 1]$, then $F(x, G(y, z)) = F(x, y) = \frac{3}{8} = G(\frac{3}{8}, z) = G(F(x, y), z)$.

(2) Suppose $z \in [\frac{1}{4}, \frac{3}{8}]$, then we have $\frac{1}{4} \leq G(y, z) \leq \frac{3}{8}$.

- If $x \in [\frac{1}{4}, \frac{3}{8}]$, then $F(x, y)|_{[\frac{1}{4}, \frac{3}{8}]^2} = max(x, y)$ and $G(x, y)|_{[\frac{1}{4}, \frac{3}{8}]^2} = max(x, y)$. Consequently, we have $F(x, G(y, z)) = G(F(x, y), z)$ since $F = G$.
- If $x \in [\frac{3}{8}, 1]$, then $F(x, G(y, z)) = \frac{3}{8} = G(\frac{3}{8}, z) = G(F(x, y), z)$.

(3) Suppose $z \in [\frac{3}{8}, \frac{1}{2}]$. Then $G(y, z) = z$.

- If $x \in [\frac{3}{8}, 1]$, then $F(x, G(y, z)) = F(x, z) = z = G(\frac{3}{8}, z) = G(F(x, y), z)$.

(4) Suppose $z, x \in [\frac{1}{2}, 1]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(\frac{3}{8}, z) = G(F(x, y), z)$.

Let $y \in [\frac{3}{8}, \frac{1}{2}]$, $z \leq x$.

(1) Suppose $z \in [0, \frac{3}{8}]$. Then $G(y, z) = y$.

- If $x \in [0, \frac{3}{8}]$, then $F(x, G(y, z)) = F(x, y) = \frac{3}{8} = G(\frac{3}{8}, z) = G(F(x, y), z)$.
- If $x \in [\frac{3}{8}, \frac{1}{2}]$, then we have $\frac{3}{8} \leq F(x, y) \leq \frac{1}{2}$. Therefore, we establish the equalities $F(x, G(y, z)) = F(x, y) = max(F(x, y), z) = G(F(x, y), z)$.
- If $x \in [\frac{1}{2}, 1]$, then $F(x, G(y, z)) = F(x, y) = y = G(y, z) = G(F(x, y), z)$.

(2) Suppose $z \in [\frac{3}{8}, \frac{1}{2}]$. Then we have $\frac{3}{8} \leq G(y, z) \leq \frac{1}{2}$.

- If $x \in [\frac{3}{8}, \frac{1}{2}]$, then $F(x, y)|_{[\frac{3}{8}, \frac{1}{2}]^2} = min(x, y)$ and $G(x, y)|_{[\frac{3}{8}, \frac{1}{2}]^2} = max(x, y)$. According to [8] one can get $F(x, G(y, z)) = G(F(x, y), z)$.
- If $x \in [\frac{1}{2}, 1]$, then $F(x, G(y, z)) = min(x, G(y, z)) = G(y, z) = G(F(x, y), z)$.

(3) Suppose $z, x \in [\frac{1}{2}, 1]$. Then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(y, z) = G(F(x, y), z)$.

Let $y \in [\frac{1}{2}, \frac{5}{8}]$, $z \leq x$.

(1) Suppose $z \in [0, \frac{1}{2}]$. Then $G(y, z) = \frac{1}{2}$.

- If $x \in [0, \frac{3}{8}]$, then $z \in [0, \frac{3}{8}]$. Therefore, we establish the equalities $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{3}{8} = G(\frac{3}{8}, z) = G(F(x, y), z)$.
- If $x \in [\frac{3}{8}, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = x = G(x, z) = G(F(x, y), z)$.
- If $x \in [\frac{1}{2}, 1]$, then we have $\frac{1}{2} \leq F(x, y) \leq 1$. Therefore, we establish the equalities $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(F(x, y), z)$.

(2) Suppose $z \in [\frac{1}{2}, \frac{5}{8}]$. Then we have $\frac{1}{2} \leq G(y, z) \leq \frac{5}{8}$.

- If $x \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, y)|_{[\frac{1}{2}, \frac{5}{8}]^2} = min(x, y)$ and $G(x, y)|_{[\frac{1}{2}, \frac{5}{8}]^2} = min(x, y)$. Therefore, we have $F(x, G(y, z)) = G(F(x, y), z)$ since $F = G$.
- If $x \in [\frac{5}{8}, 1]$, then $F(x, G(y, z)) = min(x, G(y, z)) = G(y, z) = G(F(x, y), z)$.

(3) Suppose $z, x \in [\frac{5}{8}, 1]$. Then $F(x, G(y, z)) = F(x, y) = y = G(y, z) = G(F(x, y), z)$.

Let $y \in [\frac{5}{8}, \frac{3}{4}]$, $z \leq x$.

(1) Suppose $x, z \in [0, \frac{3}{8}]$. Then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{3}{8} = G(\frac{3}{8}, z) = G(F(x, y), z)$.

(2) Suppose $x \in [\frac{3}{8}, \frac{1}{2}]$. Then $F(x, y) = x$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = x = G(x, z) = G(F(x, y), z)$.

(3) Suppose $x \in [\frac{1}{2}, \frac{5}{8}]$. Then $F(x, y) = x$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(x, z) = G(F(x, y), z)$.
- If $z \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, G(y, z)) = F(x, z) = z = G(x, z) = G(F(x, y), z)$.

(4) Suppose $x \in [\frac{5}{8}, \frac{3}{4}]$. Then we have $\frac{5}{8} \leq F(x, y) \leq \frac{3}{4}$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(F(x, y), z)$.
- If $z \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, G(y, z)) = F(x, z) = z = min(F(x, y), z) = G(F(x, y), z)$.
- If $z \in [\frac{5}{8}, \frac{3}{4}]$, then $F(x, y)|_{[\frac{5}{8}, \frac{3}{4}]^2} = min(x, y)$ and $G(x, y)|_{[\frac{5}{8}, \frac{3}{4}]^2} = max(x, y)$. According to [8] one can get $F(x, G(y, z)) = G(F(x, y), z)$.

(5) Suppose $x \in [\frac{3}{4}, 1]$. Then $F(x, y) = x$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(x, z) = G(F(x, y), z)$.
- If $z \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, G(y, z)) = F(x, z) = z = G(x, z) = G(F(x, y), z)$.
- If $z \in [\frac{5}{8}, \frac{3}{4}]$, then we have $\frac{5}{8} \leq G(y, z) \leq \frac{3}{4}$. Consequently, we obtain the chain of equalities $F(x, G(y, z)) = max(x, G(y, z)) = x = G(x, z) = G(F(x, y), z)$.
- If $z \in [\frac{3}{4}, 1]$, then $F(x, G(y, z)) = F(x, z) = x = G(x, z) = G(F(x, y), z)$.

Let $y \in [\frac{3}{4}, 1]$, $z \leq x$.

(1) Suppose $x, z \in [0, \frac{3}{8}]$. Then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{3}{8} = G(\frac{3}{8}, z) = G(F(x, y), z)$.

(2) Suppose $x \in [\frac{3}{8}, \frac{1}{2}]$. Then $F(x, y) = x$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = x = G(x, z) = G(F(x, y), z)$.

(3) Suppose $x \in [\frac{1}{2}, \frac{5}{8}]$. Then $F(x, y) = x$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(x, z) = G(F(x, y), z)$.
- If $z \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, G(y, z)) = F(x, z) = z = G(x, z) = G(F(x, y), z)$.

(4) Suppose $x \in [\frac{5}{8}, \frac{3}{4}]$. Then $F(x, y) = y$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(y, z) = G(F(x, y), z)$.
- If $z \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, G(y, z)) = F(x, z) = z = G(y, z) = G(F(x, y), z)$.
- If $z \in [\frac{5}{8}, \frac{3}{4}]$, then $F(x, G(y, z)) = F(x, y) = y = G(y, z) = G(F(x, y), z)$.

(5) Suppose $x \in [\frac{3}{4}, 1]$. Then we have $\frac{3}{4} \leq F(x, y) \leq 1$.

- If $z \in [0, \frac{1}{2}]$, then $F(x, G(y, z)) = F(x, \frac{1}{2}) = \frac{1}{2} = G(F(x, y), z)$.
- If $z \in [\frac{1}{2}, \frac{5}{8}]$, then $F(x, G(y, z)) = F(x, z) = z = min(F(x, y), z) = G(F(x, y), z)$.
- If $z \in [\frac{5}{8}, \frac{3}{4}]$, then $F(x, G(y, z)) = F(x, y) = max(F(x, y), z) = G(F(x, y), z)$.
- If $z \in [\frac{3}{4}, 1]$, then $F(x, y)|_{[\frac{3}{4}, 1]^2} = max(x, y)$ and $G(x, y)|_{[\frac{3}{4}, 1]^2} = max(x, y)$. Therefore, we have $F(x, G(y, z)) = G(F(x, y), z)$ since $F = G$. □

**Remark 3.11.** *Theorem 3.8, characterizes the structures of bi-uninorms satisfying the modularity condition. Therefore, we present Example 3.10 , in which we construct uninorms satisfying the modularity condition:*

- *Since* $U_0$ *and* $U_0''$ *are disjunctive uninorms, their structures are constructed according to Theorem 11 in* [13].
- *Since* $U_2$ *and* $U_1'$ *are conjunctive uninorms, their structures are constructed according to Theorem 15 in* [13].

## 4 Conclusions

This paper investigate the modularity condition for a pair of bi-uninorms. Our results were derived by analyzing the modularity equation involving T-uninorms and uninorms. Future work will extend this study by investigating modularity conditions for bi-uninorms and other associative aggregation operators.

## A The proof of Proposition 2.8

*Proof.* Suppose that $F$ and $G$ satisfy Eq. (3). Let $k_1 > k_2$, substituting $x = k_1, z = k_2$ in Eq. (3), then we have

$$k_1 = F(k_1, k_2) = F(k_1, G(y, k_2)) = G(F(k_1, y), k_2) = G(k_1, k_2) = k_2,$$

which is a contradiction. Thus $k_1 \leq k_2$. □

## B The proof of Proposition 3.4

*Proof.* (i) Let $U$ and $F$ satisfy Eq. (3) Substituting $z \in [k_1, 1], x = y = 1$ in Eq. (3). Substituting

$$U(1, F(1, z)) = U(1, z) = z = F(1, z) = F(U(1, 1), z)),$$

suppose $a = max(k_1, e_2)$, then $U(1, a) \geq U(1, e_2) = 1$. Given that of $a \geq k_1$, we have $U(1, a) = a$, which implies $a = 1$. Then we obtain $k_1 = 1$ or $e_2 = 1$, which immediately yields a contradiction. Consequently, $U$ is not modular over $F$.

(ii) Assume that $U$ is conjunctive. Then $U(0, 1) = 0$, $U(0, x) = 0$, for all $x \in [0, 1]$. Now let $x \in [k_1, 1]$, substituting $z = 0, y = 1$ in Eq. (3), yields

$$0 = U(x, 0) = U(F(x, 1), 0)) = F(x, U(1, 0)) = F(x, 0) = k_1,$$

this leads to a contradiction. Therefore, $U$ is disjunctive. □

## References

[1] J. Aczél, V. D. Belousov, M. Hosszú, *Generalized associativity and bisymmetry on quasigroups*, Mathematica Academiae Scientiarum Hungaricae, **11** (1963), 127-136. https://doi.org/10.1007/BF02020630

[2] G. Beliakov, A. Pradera, T. Calvo, *Aggregation functions: A guide for practitioners*, Springer Berlin, Heidelberg, 2007. https://doi.org/10.1007/978-3-540-73721-6

[3] T. Calvo, G. Mayor, R. Mesiar, *Aggregation operators: New trends and applications*, Physica Heidelberg, 2002. https://doi.org/10.1007/978-3-7908-1787-4

[4] W. Fechner, E. Rak, L. Zedam, *The modularity law in some classes of aggregation operators*, Fuzzy Sets and Systems, **332** (2018), 56-73. https://doi.org/10.1016/j.fss.2017.03.010

[5] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Cambridge University Press, 2009. https://doi.org/10.1017/CBO9781139644150

[6] Y. Li, *The study of modularity equation for some aggregations(in Chinese)*, Shaanxi Normal University Press, Shanxi, (2021). https://doi.org/10.27292/d.cnki.gsxfu.2021.002036

[7] R. Li, Y. Su, W. Zong, H. Liu, *Distributivity and conditional distributivity of bi-uninorms over uninorms*, Aequationes Mathematicae, **99** (2025), 1441-1454. https://doi.org/10.1007/s00010-025-01168-3

[8] M. Mas, G. Mayor, J. Torrens, *The modularity condition for uninorms and t-operators*, Fuzzy Sets and Systems, **126**(2) (2002), 207-218. http://doi.org/10.1016/S0165-0114(01)00055-0

[9] M. Mas, R. Mesiar, M. Monserrat, J. Torrens, *Aggregation operators with annihilator*, International Journal of General Systems, **34**(1) (2005), 17-38. https://doi.org/10.1080/03081070512331318347

[10] R. D. Neves, E. Raufaste, *A psychological study of bipolarity in the possibilistic framework*, Proceedings of IPMU-04, Perugia, (2004), 975-981. `https://api.semanticscholar.org/CorpusID:18686363`

[11] F. Qin, *Uninorm solutions and (or) nullnorm solutions to the modularity condition equations*, Fuzzy Sets and Systems, **148**(2) (2004), 231-242. `https://doi.org/10.1016/j.fss.2004.04.012`

[12] E. Rak, *The modularity equation in the class of 2-uninorms*, Advances in Intelligent Systems and Computing, **322** (2015), 45-54. `https://doi.org/10.1007/978-3-319-11313-5_5`

[13] Y. Su, J. V. Riera, D. R. Aguilera, J. Torrens, *The modularity condition for uninorms revisited*, Fuzzy Sets and Systems, **357** (2019), 27-46. `https://doi.org/10.1016/j.fss.2018.02.008`

[14] K. Xiao, Y. Su, W. Zong, *The modularity condition for semi-t-operators and Bi-uninorms*, Fuzzy Sets and Systems, **528** (2026). `https://doi.org/10.1016/j.fss.2025.109729`

[15] R. R. Yager, A. Rybalov, *Uninorm aggregation operators*, Fuzzy Sets and Systems, **80**(1) (1996), 111-120. `https://doi.org/10.1016/0165-0114(95)00133-6`

[16] Y. Zhao, H. Liu, *The modularity equation for Mayor's aggregation operators and uninorms*, Iranian Journal of Fuzzy Systems, **19**(4) (2022), 137-145. `https://doi.org/10.22111/ijfs.2022.7092`

[17] Y. Zhao, H. Liu, *The modularity equation for semi-t-operators and T-uninorms*, International Journal of Approximate Reasoning, **146** (2022), 106-118. `https://doi.org/10.1016/j.ijar.2022.04.005`

[18] W. Zong, Y. Su, J. V. Riera, D. Ruiz-Aguilera, *An insight into the conditional distributivity of nullnorms over uninorms*, Fuzzy Sets and Systems, **441** (2022), 215-223. `https://doi.org/10.1016/j.fss.2021.08.025`