Iranian Journal of Fuzzy Systems

Volume 23, Number 2, (2026), pp. 141-156

Original paper

Implication operators on bounded posets of closed intervals: A new approach

F. Karaçal, [1], K. Karacair, [2], Ü. Ertuğrul, [3] and R. Mesiar, [4]

[1,2,3]*Department of Mathematics, Karadeniz Technical University, 61080 Trabzon, Turkey*
[4]*Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovakia*
[4]*CE IT4Innovations – IRAFM, University of Ostrava, Ostrava, Czech Republic*

fkaracal@yahoo.com, kubrakaracair@ktu.edu.tr, uertugrul@ktu.edu.tr, mesiar@math.sk

Abstract

In this paper, under the inclusion order, we investigate fuzzy implication construction methods on P^S, the poset of closed intervals of a bounded poset P. We first propose some methods for constructing a fuzzy implication on P^S using pre-implications, up-sets, and down-sets on P. Next, we add two new construction methods based only on the relationship between elements of P^S. The methods are supported by propositions, examples, and related results.

Keywords: Fuzzy implication, closed intervals, inclusion order, up-sets, down-sets.

1 Introduction

Fuzzy set theory, first introduced by Zadeh in 1965, extends classical set theory by letting elements have membership degrees. It is a powerful tool for dealing with uncertainty and imprecision. Fuzzy implication functions, fundamental to fuzzy logic and its applications such as edge detection, approximate reasoning, data mining, and pattern recognition [9, 11, 18, 19, 24, 29], serve as fuzzy counterparts to classical implications in many-valued logic. In [31], Zadeh first used implications in his algorithm of fuzzy inference. In these years, the theoretical basis of the fuzzy implication has been strengthened with many studies [1, 26, 27, 28]. Fodor and Roubens developed the formulation of its definition in the known sense [13]. A detailed study can be found in [3]. The theory of fuzzy implication functions has emerged over the decades by exploring their different construction methods, various properties, and interrelationships with other fuzzy connectives such as t-norms (t-conorms) and uninorms, etc. [2, 4, 5, 25, 33].

Fuzzy implication functions are utilized in real-world applications. They are useful for constructing fuzzy rule-based systems, adaptable solutions in medical diagnosis, and also for risk assessment and financial forecasting. Furthermore, they play a vital role in image processing for tasks like edge detection, as well as playing a significant role in approximate reasoning, data mining, and pattern recognition, handling vagueness and ambiguity in complex data.

Implications on the real unit interval $[0, 1]$ with the unique property of decreasing with respect to the first component and increasing with respect to the second component have been presented by Kitainik [17] and have been further studied by many authors such as Fodor [12], Balasubramaniam[6], Baczyński and Jayaram[3]. They are an important type of fuzzy connectives in fuzzy logic, as a natural generalization of the classical Boolean implications. Therefore, their investigation on more general algebraic structures has become an interesting subject for researchers.

The construction of fuzzy implication functions on partially ordered sets (posets) has received considerable attention. Palmeira and Bedregal have introduced fuzzy implications on bounded lattices [22], extending it based on their previous works [20, 21, 23]. Karaçal et al. have investigated some construction methods of implications on a bounded lattice, in

Corresponding Author: Ü. Ertuğrul
Received: December 2025; Revised: April 2026; Accepted: May 2026.
https://doi.org/10.22111/ijfs.2026.54178.9595

which there is no additional condition to construct a new implication. Moreover, they have elaborated some properties of the construction methods such as the neutrality property, the identity principle, and the ordering property [15]. In [16], Kesicioğlu et al. have proposed the concepts of linear and $g-$convex combinations of implications on a bounded lattice, which extend the concept of convex combination of fuzzy implication on the unit interval to bounded lattices. Besides that, in [30], Wang et al. constructed some fuzzy implications on bounded posets by means of homomorphisms between two bounded posets. These papers highlight the ongoing interest in generalizing fuzzy logical operators to more complex domains.

While the majority of the manuscripts on fuzzy implication functions have focused on the real unit interval $[0, 1]$ or on the bounded lattice, the uncertainty of real-world data has necessitated the development of interval-valued fuzzy sets that make it possible to represent the uncertainty of membership degrees by means of intervals instead of single values. As a natural consequence, interval-valued fuzzy implication functions have also become an intensively studied research topic. This extension enables modelling of more complex systems, especially in scenarios where expert opinions may differ, or the data is inherently imprecise.

The literature contains some research on interval-valued implications. In [14], some construction methods of interval-valued implications have been proposed. More recently, Dai et al. explored fuzzy implications and coimplications on the poset of closed intervals, through the use of a fuzzy implication and a fuzzy coimplication on the given bounded poset in [10]. They have, based on the inclusion order, focused on the investigation of fuzzy implications and coimplications on P^S. They have proposed some construction methods for fuzzy implications and fuzzy coimplications, which is the only study in the literature examining the implications on P^S.

In this paper, we first introduce pre-implications, extending the standard notion of implication on a bounded poset P. We explore construction methods of implications on the bounded poset P^S, some of which generalize the existing construction methods besides the methods in [10], using its up-sets (or down-sets). Unlike existing literature, our methods are based on pre-implications, not implications. We then demonstrate how to derive the up-set (and down-set) of P^S from an up-set (or a down-set) of P, highlighting the applicability of our approaches. We also include specific up-sets (and down-sets) obtained by various fuzzy logic operators on P. Furthermore, we examine whether these implication construction methods satisfy properties such as neutrality, the identity principle, and the ordering property. In addition, we discuss several propositions that illustrate the theorems, along with examples and figures. Finally, we end the paper with some conclusions.

2 Preliminaries

We first introduce some basic related notions as follows. A partially order set P (a poset, in short) is a non-empty set equipped with an order relation $\leq$ that is binary, reflexive, antisymmetric and transitive, and denoted as $(P, \leq)$. A poset $(P, \leq)$ is called a chain if a and b are comparable for all $a, b \in P$, i.e., either $a \leq b$ or $b \leq a$ for all $a, b \in P$. If elements a and b are incomparable, the notation $a||b$ is used to denote it. A poset $(P, \leq)$ is called a bounded poset if there exist the greatest element and the smallest element of P, denoted by 1 and 0, respectively. If $\text{Sup}\{a, b\}$ and $\text{Inf}\{a, b\}$ exist for all $a, b \in P$, the poset $(L, \leq)$ is called a lattice. A lattice $(L, \leq)$ is said to be complete if, for any subset A of L, there exist $\text{Sup}A$ and $\text{Inf}A$.

A subset D of P is called a down-set if $x \in D$, $y \in P$ and $y \leq x$ imply $y \in D$. Similarly, a subset U of P is called an up-set if $x \in U$, $y \in P$ and $x \leq y$ imply $y \in U$. The reader is referred to [7] for more details.

We denote the cardinality of a set A by $|A|$.

In the following, unless otherwise stated, we consider P to be a bounded poset.

Definition 2.1. [10, 32] *Let $(P, \leqslant, 0, 1)$ be a bounded poset, and $P^S = \{[x_1, x_2] \mid x_1, x_2 \in P \text{ with } x_1 \leq x_2\} \cup \{\varnothing\}$. Here, $[x_1, x_2] = \{u \in P \mid x_1 \leq u \leq x_2\}$.*

Clearly, P^S becomes a poset under the inclusion order (denoted by $\subseteq$). One can easily observe that the smallest and the greatest elements of P^S are $\varnothing$ and $P = [0, 1]$, respectively. Therefore, $(P^S, \subseteq, \emptyset, [0, 1])$ is a bounded poset.

Definition 2.2. [3] *A function $I : P^2 \to P$ on a bounded lattice $(P, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:*

(I1) *I is a decreasing operation on the first variable, that is, for every $x, z \in P$ with $x \leq z$, $I(z, y) \leq I(x, y)$ for all $y \in P$.*

(I2) *I is an increasing operation on the second variable, that is, for every $y, z \in P$ with $y \leq z$, $I(x, y) \leq I(x, z)$ for all $x \in P$.*

(I3) $I(0, 0) = 1$.

(I4) $I(1, 1) = 1$.

(I5) $I(1,0)=0$.

Throughout the paper, we will denote by $\mathcal{F}$ the set of all implications on a bounded poset P.

Definition 2.3. [3, 8] *A function $C : P^2 \to P$ on a bounded poset $(P, \leq, 0, 1)$ is called a coimplication if it satisfies the following conditions:*

(I1) *C is a decreasing operation on the first variable, that is, for every $x, z \in P$ with $x \leq z$, $C(z, y) \leq C(x, y)$ for all $y \in P$.*

(I2) *C is an increasing operation on the second variable, that is, for every $y, z \in P$ with $y \leq z$, $C(x, y) \leq C(x, z)$ for all $x \in P$.*

(I3) $C(0,0)=0$.

(I4) $C(1,1)=0$.

(I5) $C(0,1)=1$.

Throughout the paper, we will denote by $\mathcal{C}$ the set of all coimplications on a bounded lattice P.

The following are some of possible properties of implication and coimplication operators on bounded lattices.

Definition 2.4. [3] *Let $(P, \leqslant, 0, 1)$ be a bounded poset, and let $I : P^2 \to P$ be an implication on a bounded poset $(P, \leq, 0, 1)$. Then I is said to satisfy:*

(IP) *the identity principle if $I(x, x) = 1$ for all $x \in P$;*

(NP) *the left neutrality property if $I(1, x) = x$ for all $x \in P$;*

(OP) *the ordering property if $I(x, y) = 1$ if and only if $x \leq y$.*

(CB) *the consequent boundary if $x \leq I(y, x)$.*

Definition 2.5. [3] *Let $(P, \leqslant, 0, 1)$ be a bounded poset, and $C : P^2 \to P$ be a coimplication. Then C is said to satisfy:*

(CIP) *the identity principle if $C(x, x) = 0$ for all $x \in P$;*

(CNP) *the left neutrality property if $C(0, x) = x$ for all $x \in P$;*

(COP) *the ordering property if $C(x, y) = 0$ if and only if $y \leq x$.*

(CCB) *the consequent boundary if $C(x, y) \leq y$.*

Example 2.6. [3] *Let $(P, \leq, 0, 1)$ be a bounded poset.*

i. *The binary operation $I_s : P^2 \to P$ given by*

$$I_s(x,y)=\begin{cases}1 & \text{if } x=0 \text{ or } y=1,\\ 0 & \text{otherwise,}\end{cases}$$

is the smallest fuzzy implication on P.

ii. *The binary operation $C_g : P^2 \to P$ given by*

$$C_g(x,y)=\begin{cases}0 & \text{if } x=1 \text{ or } y=0,\\ 1 & \text{otherwise,}\end{cases}$$

is the greatest fuzzy coimplication on P.

Theorem 2.7. [10] *Let $(P, \leq, 0, 1)$ be a bounded poset, $I \in \mathcal{F}$ and $C \in \mathcal{C}$. Then the mapping $\mathcal{I} : P^S \times P^S \to P^S$ defined by, for all $[x_1, x_2], [y_1, y_2] \in P^S$,*

$$\mathcal{I}([x_1,x_2],[y_1,y_2])=\begin{cases}[0,1] & \text{if } [x_1,x_2]=\varnothing\\ \varnothing & \text{if } [x_1,x_2]\neq\varnothing \text{ and } [y_1,y_2]=\varnothing,\\ [C(x_1,y_1),I(x_2,y_2)] & \text{otherwise,}\end{cases} \tag{1}$$

is an implication on P^S.

Theorem 2.8. [10] *Let $(P, \leq, 0, 1)$ be a bounded poset, $I \in \mathcal{F}$ and $C \in \mathcal{C}$. Then the mapping $\mathcal{C} : P^S \times P^S \to P^S$ defined by, for all $[x_1, x_2], [y_1, y_2] \in P^S$,*

$$\mathcal{C}([x_1,x_2],[y_1,y_2])=\begin{cases}\varnothing & \text{if } [y_1,y_2]=\varnothing\\ [0,1] & \text{if } [x_1,x_2]=\varnothing \text{ and } [y_1,y_2]\neq\varnothing,\\ [I(x_1,y_1),C(x_2,y_2)] & \text{otherwise,}\end{cases} \tag{2}$$

is a coimplication on P^S.

3 Construction methods for implications on bounded lattices.

In this section, we first introduce the notation of pre-implications, which are a weaker form of implications on the bounded posets. Then, with the help of up-sets (or down-sets) of the bounded poset P (see Propositions 3.2 and 3.3), and using logical operators on P (see Proposition 3.5), we present some propositions for obtaining up-sets (or down-sets) on P^S. We focus on construction methods for implication operators on P^S via pre-implication operators and up-sets (or down-sets) on the bounded poset P. We concentrate on some properties of these operators and also elaborate on them with the propositions. To better understand our construction methods, we visualize P^S for some bounded poset P and obtain implications by applying some of our construction methods.

Definition 3.1. *A function $PI : P^2 \to P$ on a bounded lattice $(P, \leq, 0, 1)$ is called a pre-implication if it satisfies the following conditions:*

(I1) *PI is a decreasing operation on the first variable, that is, for every $x, z \in P$ with $x \leq z$, $PI(z, y) \leq PI(x, y)$ for all $y \in P$.*

(I2) *PI is an increasing operation on the second variable, that is, for every $y, z \in P$ with $y \leq z$, $PI(x, y) \leq PI(x, z)$ for all $x \in P$.*

Throughout the paper, we will denote the set of all pre-implications on a bounded poset P by $\mathcal{PF}$.

Given the definitions of up-sets and down-sets, the proofs of Propositions 3.2 and 3.3 can be obtained easily.

Proposition 3.2. *Let $D \neq \emptyset$ and $U \neq \emptyset$ be a down-set and an up-set of P, respectively.*

i. $D^{S1} = \{[x_1, x_2] \in P^S \mid x_1 \leq x_2,\ x_1 \in U\} \cup \{\varnothing\}$ *is a down-set of P^S.*

ii. $U^{S1} = \{[x_1, x_2] \in P^S \mid x_1 \leq x_2,\ x_2 \in U\}$ *is an up-set of P^S.*

iii. $D^{S2} = \{[x_1, x_2] \in P^S \mid x_1 \leq x_2,\ x_2 \in D\} \cup \{\varnothing\}$ *is a down-set of P^S.*

iv. $U^{S2} = \{[x_1, x_2] \in P^S \mid x_1 \leq x_2,\ x_1 \in D\}$ *is an up-set of P^S.*

Proof. i. Let $[x_1, x_2] \subseteq [y_1, y_2]$ with $[x_1, x_2] \in P^S$ and $[y_1, y_2] \in D^{S1}$. We need to show that $[x_1, x_2] \in D^{S1}$. If $[x_1, x_2] = \emptyset$, the proof is clear. Then let us suppose that $[x_1, x_2] \neq \emptyset$, and thus $[y_1, y_2] \neq \emptyset$. It follows that $y_1 \leq x_1$ and $x_2 \leq y_2$ from $[x_1, x_2] \subseteq [y_1, y_2]$. We have that $x_1 \in U$ by the fact that U is an up-set and $y_1 \in U$. Hence $[x_1, x_2] \in D^{S1}$. Consequently, D^{S1} is a down-set of P^S. □

Proposition 3.3. *Let $D \neq \varnothing$ and $U \neq \varnothing$ be a down-set and up-set of P,respectively.*

i. $D^{S3} = \{[x_1, x_2] \in P^S \mid x_1 \leq x_2,\ x_1 \in U,\ x_2 \in D\} \cup \{\varnothing\}$ *is a down-set of P^S.*

ii. $U^{S3} = \{[x_1, x_2] \in P^S \mid x_1 \leq y_2,\ x_1 \in D,\ x_2 \in U\}$ *is an up-set of P^S.*

Proof. i. Let $[x_1, x_2] \subseteq [y_1, y_2]$ with $[x_1, x_2] \in P^S$ and $[y_1, y_2] \in D^{S3}$. We need to show that $[x_1, x_2] \in D^{S3}$. If $[x_1, x_2] = \emptyset$, the proof is clear. Then let us suppose that $[x_1, x_2] \neq \emptyset$, and thus $[y_1, y_2] \neq \emptyset$. It follows that $y_1 \leq x_1$ and $x_2 \leq y_2$ from $[x_1, x_2] \subseteq [y_1, y_2]$. We have that $x_1 \in U$ and $x_2 \in D$ using $y_1 \in U$, $y_2 \in D$ by the fact that D and U are down-set and up-set of P, respectively. Hence $[x_1, x_2] \in D^{S3}$. Consequently, $[x_1, x_2] \in D^{S3}$. D^{S3} is a down-set of P^S. □

Corollary 3.4. *Let $a, b \in P$.*

i. $D^{S3} = \{[x_1, x_2] \in P^S : a \leq x_1 \leq x_2 \leq b\} \cup \{\varnothing\}$ *is a down-set of P^S.*

ii. $U^{S3} = \{[x_1, x_2] \in P^S :\ x_1 \leq x_2, x_1 \leq a, b \leq x_2\}$ *is an up-set of P^S.*

Proposition 3.5. *Let $(P, \leq)$ be a poset, $a, b, c, d \in P$, F, $K : P^2 \to P$ be increasing functions in both coordinates, $f : P \to P$ be a decreasing function, and $g : P \to P$ be an increasing function. Then the below introduced sets U^{S4}, U^{S5} and U^{S6} are up-sets of the bounded poset P^S, and D^{S4}, D^{S5} and D^{S6} are down-sets of P^S.*

i. $U^{S4} = \{[x_1, x_2] \mid x_1 \leq x_2,\ F(x_1, a) \leq c,\ K(x_2, b) \geq d\}$.

ii. $U^{S5} = \{[x_1, x_2] \mid x_1 \leq x_2,\ F(x_1, a) \leq K(x_2, b)\}$.

iii. $U^{S6} = \{[x_1, x_2] \mid x_1 \leq x_2,\ f(x_1) \geq a,\ b \leq g(x_2)\}$.

iv. $D^{S4} = \{[x_1, x_2] \mid x_1 \leq x_2,\ F(x_1, a) \geq c,\ K(x_2, b) \leq d\} \cup \{\varnothing\}$.

v. $D^{S5} = \{[x_1, x_2] \mid x_1 \leq x_2,\ F(x_1, a) \geq K(x_2, b)\} \cup \{\varnothing\}$.

vi. $D^{S6} = \{[x_1, x_2] \mid x_1 \leq x_2,\ f(x_1) \leq a,\ b \geq g(x_2)\} \cup \{\varnothing\}$.

Proof. i. Let $[x_1, x_2] \subseteq [y_1, y_2]$ with $[x_1, x_2] \in U^{S4}$ and $[y_1, y_2] \in P^S$. We need to show that $[y_1, y_2] \in U^{S4}$. It follows that $y_1 \leq x_1$ and $x_2 \leq y_2$ from $[x_1, x_2] \subseteq [y_1, y_2]$. It follows $F(y_1, a) \leq F(x_1, a)$ and $K(x_2, b) \leq K(y_2, b)$ using $y_1 \leq x_1$ and $x_2 \leq y_2$ by the fact that the functions F, K are increasing functions in both coordinates. Also, we know $F(x_1, a) \leq c$, $K(x_2, b) \geq d$ from $[x_1, x_2] \in U^{S4}$. It follows immediately that $F(y_1, a) \leq c$ and $K(y_2, b) \geq d$ from the transitivity property of $\leq$. Consequently, $[y_1, y_2] \in U^{S4}$. U^{S4} is an up-set of P^s.

The other claims follow in a similar way. □

Utilizing the fundamental operations on lattices, the following result can be obtained as a direct consequence of the Proposition 3.5.

Corollary 3.6. *In Proposition 3.5, if P is a lattice, F is taken as the meet operation $\wedge$, K as the join operation $\vee$, f is a negation $f = N$(i.e. a decreasing function $N : P \to P$ reversing the boundary $N(0) = 1$ and $N(1) = 0$), and $g = id_P$, where $id(x) = x$ for all $x \in P$, then the following up-sets and down-sets are obtained:*

i. $U^{S4} = \{[x_1, x_2] \mid x_1 \leq x_2,\ x_1 \wedge a \leq c,\ x_2 \vee b \geq d\}$.

ii. $U^{S5} = P^S$.

iii. $U^{S6} = \{[x_1, x_2] \mid x_1 \leq x_2,\ N(x_1) \geq a,\ b \leq x_2\}$.

iv. $D^{S4} = \{[x_1, x_2] \mid x_1 \leq x_2,\ x_1 \wedge a \geq c,\ x_2 \vee b \leq d\} \cup \{\varnothing\}$.

v. $D^{S5} = \{[x_1, x_2] \mid x_1 \leq x_2,\ \ x_1 = x_2\} \cup \{\varnothing\}$.

vi. $D^{S6} = \{[x_1, x_2] \mid x_1 \leq x_2,\ \ N(x_1) \leq a,\ b \geq x_2\} \cup \{\varnothing\}$.

Lemma 3.7. *Let $(P, \leq, 0, 1)$ be a bounded poset, $PI_1, PI_2 \in \mathcal{PI}$, $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in P^S \backslash \{\varnothing\}$. Then, the following claims hold:*

i. *If $[x_1, x_2] \subseteq [y_1, y_2]$, then $[PI_1(y_1, z_1), PI_2(y_2, z_2)] \subseteq [PI_1(x_1, z_1), PI_2(x_2, z_2)]$.*

ii. *If $[y_1, y_2] \subseteq [z_1, z_2]$, then $[PI_1(x_1, y_1), PI_2(x_2, y_2)] \subseteq [PI_1(x_1, z_1), PI_2(x_2, z_2)]$.*

Proof.

i. First, we claim that $PI_1(x_1, z_1) \leq PI_1(y_1, z_1)$ and $PI_2(y_2, z_2) \leq PI_2(x_2, z_2)$. It follows $y_1 \leq x_1$ and $x_2 \leq y_2$ by $[x_1, x_2] \subseteq [y_1, y_2]$. We obtain that $PI_1(x_1, z_1) \leq PI_1(y_1, z_1)$ by $y_1 \leq x_1$ from the fact that PI_1 is a decreasing operation on the first variable. Also, it can be deduced that $PI_2(y_2, z_2) \leq PI_2(x_2, z_2)$ by $x_2 \leq y_2$ from the fact that PI_2 is a decreasing operation on the first variable. If $[PI_1(y_1, z_1), PI_2(y_2, z_2)] = \varnothing$, the proof is clear. Suppose that $[PI_1(y_1, z_1), PI_2(y_2, z_2)] \neq \varnothing$ and $a \in [PI_1(y_1, z_1), PI_2(y_2, z_2)]$. It follows that $PI_1(y_1, z_1) \leq a \leq PI_2(y_2, z_2)$. By the above claim we have that $PI_1(x_1, z_1) \leq PI_1(y_1, z_1) \leq a \leq PI_2(y_2, z_2) \leq PI_2(x_2, z_2)$. Hence, $PI_1(x_1, z_1) \leq a \leq PI_2(y_2, z_2)$, that is $a \in [PI_1(x_1, z_1), PI_2(x_2, z_2)]$. Consequently, it is obtained that $[PI_1(y_1, z_1), PI_2(y_2, z_2)] \subseteq [PI_1(x_1, z_1), PI_2(x_2, z_2)]$.

ii. It can be proven by a similar argument to the previous one from the fact that PI_1, PI_2 are increasing operations on the second variable.

□

Theorem 3.8. *Let $(P, \leq, 0, 1)$ be a bounded poset, $\varnothing \neq U^S$ be an up-set of P^S such that $\varnothing \notin U^S$ and $PI_1, PI_2 \in \mathcal{PF}$. Then the mapping $\mathcal{I}_1 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] \notin U^S \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } [x_1, x_2] \in U^S \text{ and } [y_1, y_2] \notin U^S, \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise,} \end{cases} \tag{3}$$

is an implication on P^S.

Proof. **I1.** We need to validate that $\mathcal{I}_1([y_1, y_2], [z_1, z_2]) \subseteq \mathcal{I}_1([x_1, x_2], [z_1, z_2])$ holds for all $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in P^S$ with $[x_1, x_2] \subseteq [y_1, y_2]$. When $[z_1, z_2] = [0, 1]$, the proof is clear. It remains to consider the following cases.
1. Let $[x_1, x_2] \in U^S$. In this case, it follows that $[y_1, y_2] \in U^S$.
1.1. If $[z_1, z_2] \in U^S$, then $\mathcal{I}_1([y_1, y_2]), [z_1, z_2]) = [PI_1(y_1, z_1), PI_2(y_2, z_2)] \subseteq [PI_1(x_1, y_1), PI_2(x_2, y_2)] = \mathcal{I}_1([x_1, x_2], [z_1, z_2])$ using Lemma 3.7, we have that:
1.2. If $[z_1, z_2] \notin U^S$, then $\mathcal{I}_1([y_1, y_2], [z_1, z_2]) = \varnothing = \mathcal{I}_1([x_1, x_2], [z_1, z_2])$.
2. Let $[x_1, x_2] \notin U^S$. $\mathcal{I}_1([y_1, y_2], [z_1, z_2]) \subseteq [0, 1] = \mathcal{I}_1([x_1, x_2], [z_1, z_2])$.

I2. We need to validate that $\mathcal{I}_1([x_1, x_2], [y_1, y_2]) \subseteq \mathcal{I}_1([x_1, x_2], [z_1, z_2])$ holds for all $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in P^S$ with $[y_1, y_2] \subseteq [z_1, z_2]$. When $[z_1, z_2] = [0, 1]$, the proof is clear. Whenever $[y_1, y_2] = [0, 1]$, so is $[z_1, z_2]$. Thus, the proof is still evident. Taking all the above cases into consideration, we complete the proof by examining the following cases.
1. Let $[x_1, x_2] \in U^S$.
1.1. if $[y_1, y_2] \in U^S$, this implies that $[z_1, z_2] \in U^S$, then using Lemma 3.7
$\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = [PI_1(x_1, y_1), I_2(x_2, y_2)] \subseteq [PI_1(x_1, z_1), I_2(x_2, z_2)] = \mathcal{I}_1([x_1, x_2], [z_1, z_2])$.
1.2. if $[y_1, y_2] \notin U^S$, then
$\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = \varnothing \subseteq \mathcal{I}_1([x_1, x_2], [z_1, z_2])$.
2. Let $[x_1, x_2] \notin U^S$. Then, $\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = [0, 1] = \mathcal{I}_1([x_1, x_2], [z_1, z_2])$.

I3. $\mathcal{I}_1(\varnothing, \varnothing) = [0, 1]$ since $\varnothing \notin U^S$.

I4. $\mathcal{I}_1([0, 1], [0, 1]) = [0, 1]$ from $[y_1, y_2] = [0, 1]$.

I5. $\mathcal{I}_1([0, 1], \varnothing) = \varnothing$ since $[0, 1] \in U^S$ and $\varnothing \notin U^S$.

□

In the case that $U^S = P^S \backslash \{\varnothing\}$, the Theorem 3.8 is as follows.

Corollary 3.9. *Let $(P, \leq, 0, 1)$ be a bounded poset, $PI_1, PI_2 \in \mathcal{PF}$, then the mapping $\mathcal{I}_1 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] = \varnothing \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } [x_1, x_2] \neq \varnothing \text{ and } [y_1, y_2] = \varnothing, \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{4}$$

is an implication on P^S.

Remark 3.10. *It is worth noting that Theorem 3.8 generalizes Theorem 2.7, if we take $PI_1 \in \mathcal{C}$ and $PI_2 \in \mathcal{F}$ in Corollary 3.9.*

Example 3.11. *Consider the bounded poset $P_1 = \{0, x_1, x_2, 1\}$ described in Fig. 1. It is easy to obtain the lattice ${P_1}^S$ as in Fig. 2. Take the coimplication $PI_1 = C$, the implication $PI_2 = I$ as in Tables 1 and 2, and the up-set $U^S = \{[0, x_1], [x_2, 1], [0, 1]\}$ of ${P_1}^S$.*

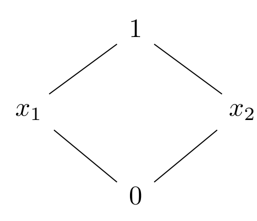

Figure 1: The Hasse diagram of the lattice P_1.

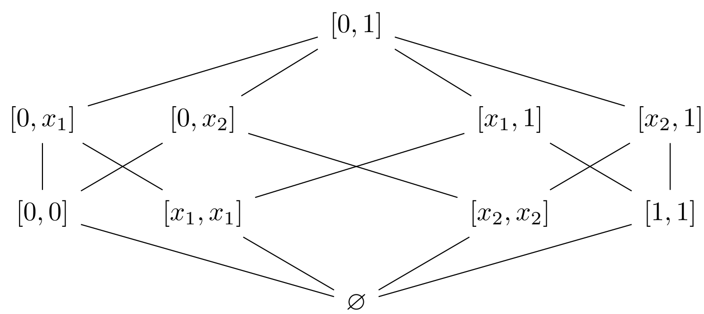

Figure 2: The Hasse diagram of the lattice ${P_1}^S$.

C	0	x_1	x_2	1
0	0	1	1	1
x_1	0	0	x_1	x_1
x_2	0	x_2	0	x_2
1	0	0	0	0

Table 1: The coimplication C on P_1.

I	0	x_1	x_2	1
0	1	1	1	1
x_1	0	1	x_2	1
x_2	0	x_1	1	1
1	0	0	0	1

Table 2: The implication I on P_1.

Applying applying the formula (3) in Theorem 3.8, we obtain the implication $\mathcal{I}_1$ on ${P_1}^S$ as in Table 3.

$\mathcal{I}_1$	$\varnothing$	$[0,0]$	$[0,x_1]$	$[0,x_2]$	$[x_1,x_1]$	$[x_1,1]$	$[x_2,x_2]$	$[x_2,1]$	$[1,1]$	$[0,1]$
$\varnothing$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[0,0]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[0,x_1]$	$\varnothing$	$\varnothing$	$[0,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$[0,1]$
$[0,x_2]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[x_1,x_1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[x_1,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[x_2,x_2]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[x_2,1]$	$\varnothing$	$\varnothing$	$[0,0]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$	$\varnothing$	$[0,1]$
$[1,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[0,1]$	$\varnothing$	$\varnothing$	$[0,0]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$[0,1]$

Table 3: The implication $\mathcal{I}_1$ *on* ${P_1}^S$.

We have that $\mathcal{I}_1([0,0],\varnothing) = [0,1]$ from Table 3 and $\mathcal{I}([0,0],\varnothing) = \varnothing$ from Theorem 2.7. Hence, $\mathcal{I} \neq \mathcal{I}_1$. Moreover, Table 3 shows that $\mathcal{I}_1$ satisfies (IP) and (LOP).

Proposition 3.12. *Let $(P, \leq, 0, 1)$ be bounded poset and $PI_1 \in \mathcal{C}$ and $PI_2 \in \mathcal{F}$. Then the following statements hold:*

i. *If $\mathcal{I}_1$ satisfies (NP), then PI_1 satisfies (CNP) and PI_2 satisfies (NP).*

ii. *If PI_1 satisfies (CIP) and PI_2 satisfies (IP), then $\mathcal{I}_1$ satisfies (IP).*

iii. *If $\mathcal{I}_1$ satisfies (OP), then $U^S = P^S \backslash \{\varnothing\}$.*

iv. *If $\mathcal{I}_1$ satisfies (OP), then PI_1 satisfies (COP) and PI_2 satisfies (OP).*

Proof.

i. Let $\mathcal{I}_1$ satisfy (NP). Then $\mathcal{I}_1([0,1], [x,x]) = [x,x]$ for all $x \in P$. Thus it must be that $[PI_1(0,x), PI_2(1,x)] = [x,x]$ and $PI_1(0,x) = x$ and $PI_2(1,x) = x$ for all $x \in P$. Thus, PI_1 satisfies (CNP) and PI_2 satisfies (NP).

ii. We assume that PI_1 satisfies (CIP) and PI_2 satisfies (IP). If $[x,y] \notin U^S$, then $\mathcal{I}_1([x,y],[x,y]) = [0,1]$. If $[x,y] \in U^S$, then $\mathcal{I}_1([x,y],[x,y]) = [PI_1(x,x), PI_2(y,y)] = [0,1]$. Thus $\mathcal{I}_1$ satisfies (IP).

iii. It is either $P^S \backslash (U^S \cup \{\varnothing\}) \neq \varnothing$ or $P^S \backslash (U^S \cup \{\varnothing\}) = \varnothing$. Consider $L^S \backslash (U^S \cup \{\varnothing\}) \neq \varnothing$. It follows that there exists $[a,b] \in P^S$ such that $[a,b] \notin U^S \cup \{\varnothing\}$. Then, $\mathcal{I}_1([a,b],\varnothing) = [0,1]$. $\mathcal{I}_1$ does not satisfy (OP) from $[a,b] \nsubseteq \varnothing$ and $\mathcal{I}_1([a,b],\varnothing) = [0,1]$. So it must hold that $L^S \backslash (U^S \cup \{\varnothing\}) = \varnothing$, and thus $U^S = P^S \backslash \{\varnothing\}$.

iv. Let us suppose that $PI_1(x,y) = 0$ for any $x, y \in P$. From $[x,x],[y,1] \neq \varnothing$, $\mathcal{I}_1([x,x],[y,1]) = [PI_1(x,y), PI_2(x,1)] = [0,1]$. By the fact that $\mathcal{I}_1$ satisfies (OP), it follows that $[x,x] \subseteq [y,1]$, and thus $y \leq x$. Also, if $y \leq x$, then $[x,x] \subseteq [y,1]$. Since $\mathcal{I}_1$ satisfies (OP), we have that $\mathcal{I}_1([x,x],[y,1]) = [PI_1(x,y), PI_2(x,1)] = [0,1]$ and thus $PI_1(x,y) = 0$. Consequently PI_1 satisfies (COP). In a dual manner, it can be shown that PI_2 satisfies (OP).

□

Remark 3.13. *Consider the set $A = \{a,b\}$. Take the bounded poset $P = P(A)$ of all subsets of A, and the up-set $U^S = \{[\{b\},A],[\varnothing,A]\}$ of $P(A)^S$ in Theorem 3.8.*

i. *If $PI_1 \in \mathcal{C}$ satisfies (CNP), and $PI_2 \in \mathcal{F}$ satisfies (NP), $\mathcal{I}_1$ may not satisfy (NP) since $\mathcal{I}_1([\varnothing,A],[\{a\},\{a\}]) = \varnothing \neq [\{a\},\{a\}]$.*

ii. *If $PI_1 \in \mathcal{C}$ satisfies (CCB) and $PI_2 \in \mathcal{F}$ satisfies (CB), $\mathcal{I}_1$ may not satisfy (CB) since $\mathcal{I}_1([\{b\},A],[\{a\},\{a\}]) = \varnothing$ does not include $[\{a\},\{a\}]$.*

iii. *If $PI_1 \in \mathcal{C}$ satisfies (COP) and $PI_2 \in \mathcal{F}$ satisfies (OP), $\mathcal{I}_1$ may not satisfy (OP). To see this consider the bounded lattice $\mathcal{P}(\mathcal{A})$ for $A = \{a,b\}$ under the inclusion order and $U^S = \{[\{b\},A],[\varnothing,A]\} \subseteq P(A)^S$, $\mathcal{I}_1([\{a\},A],[\{a\},\{a\}]) = [\varnothing,A]$,but we have that $[\{a\},A] \nsubseteq [\{a\},\{a\}]$.*

Proposition 3.14. *Let $(P,\leq,0,1)$ be a bounded poset, $U^S = P^S \backslash \{\varnothing\}$, $PI_1 \in \mathcal{C}$ and $PI_2 \in \mathcal{F}$. If PI_1 satisfies (CNP) and PI_2 satisfies (NP), then $\mathcal{I}_1$ satisfies (NP).*

Proof. Suppose that $[x_1,x_2] \in P^S \backslash \{\varnothing\}$. Then, $\mathcal{I}_1([0,1],[x_1,x_2]) = [PI_1(0,x_1), PI_2(1,x_2)] = [x_1,x_2]$. Also, we have that $\mathcal{I}_1([0,1],\varnothing) = \varnothing$. Thus, $\mathcal{I}_1$ satisfies (NP). □

Remark 3.15. *Let $(P,\leq,0,1)$ be a bounded poset. If $U^S \neq P^S \backslash \{\varnothing\}$, then $\mathcal{I}_1$ does not satisfy (NP).*

Proof. Since $U^S \neq P^S \backslash \{\varnothing\}$, there exists $[x_1,x_2] \in P^S \backslash \{\varnothing\}$ such that $[x_1,x_2] \notin U^S$. Then, we have $\mathcal{I}_1([0,1],[x_1,x_2]) = \varnothing \neq [x_1,x_2]$. Therefore, $\mathcal{I}_1$ does not satisfy (NP). □

Proposition 3.16. *Let $(P,\leq,0,1)$ be an infinite bounded poset and $PI_1, PI_2 \in \mathcal{PF}$, then the mapping $\mathcal{I}_2 : P^S \times P^S \to P^S$ defined, for all $[x_1,x_2],[y_1,y_2] \in P^S$, by*

$$\mathcal{I}_2([x_1,x_2],[y_1,y_2]) = \begin{cases} [0,1] & \text{if } |[x_1,x_2]| \neq \infty \text{ or } [y_1,y_2] = [0,1], \\ \varnothing & \text{if } |[x_1,x_2]| = \infty \text{ and } |[y_1,y_2]| \neq \infty, \\ [PI_1(x_1,y_1), PI_2(x_2,y_2)] & \text{otherwise}, \end{cases} \tag{5}$$

is an implication on P^S.

Proof. If we put $U^S = \{[x_1,x_2] \in P^S \mid |[x_1,x_2]| = \infty\}$ in Theorem 3.8 , the proof is obvious . □

Proposition 3.17. *Let $(P,\leq,0,1)$ be a non-chain bounded poset and $PI_1, PI_2 \in \mathcal{PF}$, then the mapping $\mathcal{I}_3 : P^S \times P^S \to P^S$ defined by, for all $[x_1,x_2],[y_1,y_2] \in P^S$,*

$$\mathcal{I}_3([x_1,x_2],[y_1,y_2]) = \begin{cases} [0,1] & \text{if } [x_1,x_2] \text{ is a chain or } [y_1,y_2] = [0,1], \\ \varnothing & \text{if } [x_1,x_2] \text{ is not a chain and } [y_1,y_2] \text{ is a chain}, \\ [PI_1(x_1,y_1), PI_2(x_2,y_2)] & \text{otherwise}, \end{cases} \tag{6}$$

is an implication on P^S.

Proof. If we put $U^S = \{[x_1,x_2] \in P^S : [x_1,x_2] \text{ is not a chain}\}$ in Theorem 3.8, the proof is obvious. □

Theorem 3.18. *Let $(P, \leq, 0, 1)$ be a bounded poset, $\varnothing \neq D^S$ be a down-set of P^S such that $[0,1] \notin D^S$ and $PI_1, PI_2 \in \mathcal{PF}$. Then the mapping $\mathcal{I}_4 : P^S \times P^S \to L^S$ defined by, for all $[x_1, x_2], [y_1, y_2] \in P^S$,*

$$\mathcal{I}_4([x_1, x_2], [y_1, y_2]) = \begin{cases} [0,1] & \text{if } [x_1, x_2] \in D^S \text{ or } [y_1, y_2] = [0,1] \\ \varnothing & \text{if } [x_1, x_2] \notin D^S \text{and } [y_1, y_2] \in D^S, \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise,} \end{cases} \tag{7}$$

is an implication on P^S.

Proof. The proof follows from Theorem 3.8 and the fact that $U^S = P^S \backslash D^S$ is an up-set set. □

The properties of $\mathcal{I}_4$ in Theorem 3.18, namely (IP), (NP), (CB), and (OP), are investigated as dual counterparts to Proposition 3.12, Remark 3.13.

By defining $D^S = \{[x_1, x_2] \in P^S \mid [x_1, x_2] \subseteq A\}$ for $\varnothing \neq A \subseteq P$, we obtain the following result, whose proof is straight-forward.

Proposition 3.19. *Let $(P, \leq, 0, 1)$ be a bounded poset and $PI_1, PI_2 \in \mathcal{PF}$ and $\varnothing \neq A \subseteq P$, then the mapping $\mathcal{I}_5 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_5([x_1, x_2], [y_1, y_2]) = \begin{cases} [0,1] & \text{if } [x_1, x_2] \subseteq A \text{ or } [y_1, y_2] = [0,1] \\ \varnothing & \text{if } [x_1, x_2] \nsubseteq A \text{ and } [y_1, y_2] \subseteq A, \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise,} \end{cases} \tag{8}$$

is an implication on P^S.

Since $D^{S4} = \{[x_1, x_2] \mid x_1 \leq x_2,\ x_1 \wedge a \geq c,\ x_2 \vee b \leq d\} \cup \{\varnothing\}$ is a down-set by Corolllary 3.6, we have the following proposition if we put $D^S = D^{S4}$ in Theorem 3.18.

Proposition 3.20. *Let $(P, \leq, 0, 1)$ be a bounded lattice, $PI_1, PI_2 \in \mathcal{PF}$ and $a, b, c, d \in P$ be arbitrary fixed elements. Then the mapping $\mathcal{I}_6 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_6([x_1, x_2], [y_1, y_2]) = \begin{cases} [0,1] & \text{if } ([x_1, x_2] = \varnothing \text{ or } (x_1 \wedge a \geq c,\ x_2 \vee b \leq d)) \text{ or} \\ & [y_1, y_2] = [0,1], \\ \varnothing & \text{if } (x_1 \wedge a \ngeq c \text{ or } x_2 \vee b \nleq d) \text{ and} \\ & ([y_1, y_2] = \varnothing \text{ or } (y_1 \wedge a \geq c,\ y_2 \vee b \leq d)), \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise,} \end{cases} \tag{9}$$

is an implication on P^S.

Based on Theorem 3.18, we $D^S = \{[x_1, x_2] \in P^S | [x_1, x_2] \cap A = \varnothing\}$, see the following result.

Proposition 3.21. *Let $(P, \leq, 0, 1)$ bounded poset, $PI_1, PI_2 \in \mathcal{PF}$ and $A \subseteq P$ be arbitrary fixed set. Then the mapping $\mathcal{I}_7 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$,*

$$by \mathcal{I}_7([x_1, x_2], [y_1, y_2]) = \begin{cases} [0,1] & \text{if } [x_1, x_2] \cap A = \varnothing \text{ or } [y_1, y_2] = [0,1] \\ \varnothing & \text{if } [x_1, x_2] \cap A \neq \varnothing \text{ and } [y_1, y_2] \cap A = \varnothing, \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise,} \end{cases} \tag{10}$$

is an implication on P^S.

Example 3.22. *Consider the bounded poset $P_2 = \{0, x_1, x_2, x_3, x_4, 1\}$ described in Fig. 3. It is easy to obtain the lattice ${P_2}^S$ as in Fig. 4. Take PI_1, PI_2 as in Tables 4 and 5, the subset $A = \{x_3, x_4\}$ of P_1.*

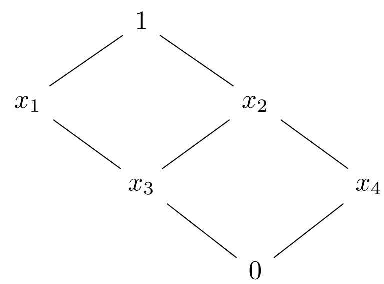

Figure 3: The Hasse diagram of the lattice P_2.

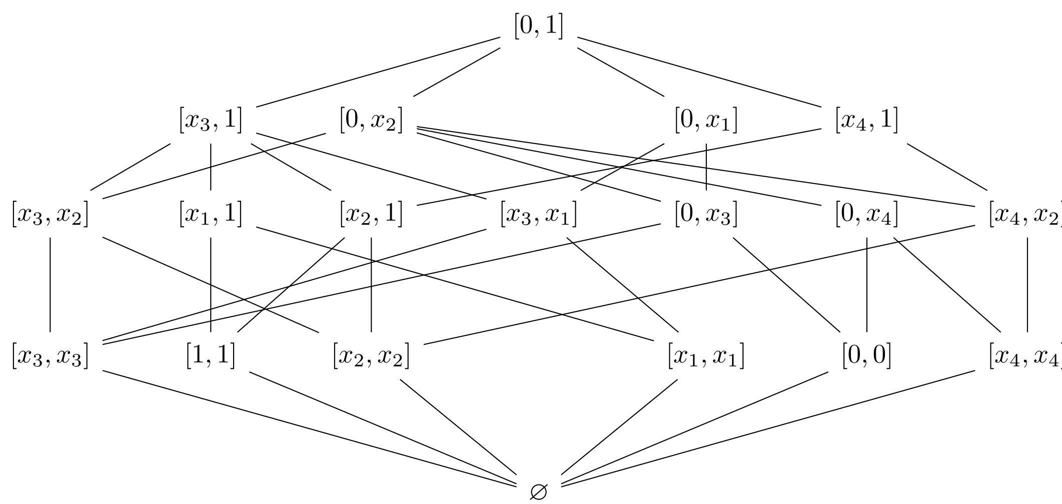

Figure 4: The Hasse diagram of the lattice ${P_2}^S$.

PI_1	0	x_1	x_2	x_3	x_4	1
0	1	1	1	1	1	1
x_1	0	1	x_2	0	0	1
x_2	0	x_2	1	0	x_2	1
x_3	0	1	1	1	x_2	1
x_4	0	1	1	x_2	x_2	1
1	0	0	0	0	0	0

Table 4: The function PI_1 on P_2.

PI_2	0	x_1	x_2	x_3	x_4	1
0	0	1	1	1	1	1
x_1	0	x_3	x_3	x_3	0	1
x_2	0	x_2	1	x_3	x_2	1
x_3	0	1	1	1	1	1
x_4	0	1	1	x_2	1	1
1	0	x_3	x_3	x_3	0	1

Table 5: The function PI_2 on P_2.

Applying Equation 10 from (10) in Proposition 3.21, yields the implication $\mathcal{I}_7$ on ${P_1}^S$, as shown in Table 6.

$\mathcal{I}_7$	$\varnothing$	$[0,0]$	$[0,x_3]$	$[0,x_4]$	$[0,x_2]$	$[0,x_1]$	$[x_3,x_3]$	$[x_3,x_2]$	$[x_3,x_1]$	$[x_3,1]$	$[x_4,x_4]$	$[x_4,x_2]$	$[x_4,1]$	$[x_1,x_1]$	$[x_1,1]$	$[x_2,x_2]$	$[x_2,1]$	$[1,1]$	$[0,1]$
$\varnothing$	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	[0,1]	$[0,1]$
$[0,0]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[0,x_3]$	$\varnothing$	$\varnothing$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[0,x_4]$	$\varnothing$	$\varnothing$	$\varnothing$	$[1,1]$	$[1,1]$	$[1,1]$	$\varnothing$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[0,x_2]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$[1,1]$	$\varnothing$	$[1,1]$	$[1,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[0,x_1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_3,x_3]$	$\varnothing$	$\varnothing$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[x_2,1]$	$[x_2,1]$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_3,x_2]$	$\varnothing$	$\varnothing$	$[0,x_3]$	$[0,x_2]$	$[0,1]$	$[0,x_2]$	$\varnothing$	$[1,1]$	$\varnothing$	$[1,1]$	$[x_2,x_2]$	$[x_2,1]$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_3,x_1]$	$\varnothing$	$\varnothing$	$[0,x_3]$	$[0,0]$	$[0,x_3]$	$[0,x_3]$	$[0,x_3]$	$[1,1]$	$\varnothing$	$[1,1]$	$\varnothing$	$\varnothing$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_3,1]$	$\varnothing$	$\varnothing$	$[0,x_3]$	$[0,0]$	$[0,x_3]$	$[0,x_3]$	$\varnothing$	$\varnothing$	$\varnothing$	$[1,1]$	$\varnothing$	$\varnothing$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_4,x_4]$	$\varnothing$	$\varnothing$	$[0,x_2]$	$[0,1]$	$[0,1]$	$[0,1]$	$[x_2,x_2]$	$[x_2,1]$	$[x_2,1]$	$[x_2,1]$	$[x_2,1]$	$[x_2,1]$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_4,x_2]$	$\varnothing$	$\varnothing$	$[0,x_3]$	$[0,x_2]$	$[0,1]$	$[0,x_2]$	$\varnothing$	$[x_2,1]$	$[x_2,x_2]$	$[x_2,1]$	$[x_2,x_2]$	$[x_2,1]$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_4,1]$	$\varnothing$	$\varnothing$	$[0,x_3]$	$[0,0]$	$[0,x_2]$	$[0,x_3]$	$\varnothing$	$\varnothing$	$\varnothing$	$[x_2,1]$	$\varnothing$	$\varnothing$	$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_1,x_1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_1,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_2,x_2]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[x_2,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[1,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$
$[0,1]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[0,1]$

Table 6: The implication $\mathcal{I}_7$ *on* ${P_2}^S$.

Theorem 3.23. *Let $(P, \leq, 0, 1)$ be a bounded poset, $PI_1, PI_2 \in \mathcal{PF}$, U^S be an up-set of P^S and D^S be a down-set of P^S. Then the mapping $\mathcal{I}_8 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_8([x_1,x_2],[y_1,y_2]) = \begin{cases} [0,1] & \text{if } [x_1,x_2] = \varnothing \text{ or } [y_1,y_2] = [0,1], \\ \varnothing & \text{if } ([x_1,x_2] \in U^S, [y_1,y_2] \in D^S) \text{ or } ([x_1,x_2] \neq \varnothing, [y_1,y_2] = \varnothing), \\ [PI_1(x_1,y_1), PI_2(x_2,y_2)] & \text{otherwise,} \end{cases} \tag{11}$$

is an implication on P^S.

Proof. **I1.** We need to show that $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) \subseteq \mathcal{I}_8([x_1,x_2],[z_1,z_2])$ for all $[x_1,x_2],[z_1,z_2] \in L^S$ with $[x_1,x_2] \subseteq [y_1,y_2]$. It is trivial for $[z_1,z_2] = [0,1]$. If $[x_1,x_2] = \varnothing$, then $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) \subseteq [0,1] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$. It is obvious since $[x_1,x_2] = \varnothing$ when $[y_1,y_2] = \varnothing$. If $[z_1,z_2] = \varnothing$ with $[x_1,x_2],[y_1,y_2] \neq \varnothing$, then $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) = \varnothing = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$. Then, we review the following cases:

1. Let $[z_1,z_2] \notin D^S$.
   $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) = [PI_1(y_1,z_1), PI_2(y_2,z_2)] \subseteq [PI_1(x_1,z_1), PI_2(x_2,z_2)] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.
2. Let $[z_1,z_2] \in D^S$.
   1.1. if $[x_1,x_2] \in U^S$, it follows that $[y_1,y_2] \in U^S$, then
   $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) = \varnothing = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.
   1.2. $[x_1,x_2] \notin U^S$,
   1.2.1. if $[y_1,y_2] \in U^S$, then
   $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) = \varnothing \subseteq [PI_1(x_1,z_1), PI_2(x_2,z_2)] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.
   1.2.2. if $[y_1,y_2] \notin U^S$, then
   $\mathcal{I}_8([y_1,y_2],[z_1,z_2]) = [PI_1(y_1,z_1), PI_2(y_2,z_2)] \subseteq [PI_1(x_1,z_1), PI_2(x_2,z_2)] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.

I2. We need to show that $\mathcal{I}_8([x_1,x_2],[y_1,y_2]) \subseteq \mathcal{I}_8([x_1,x_2],[z_1,z_2])$ for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in P^S$ with $[y_1,y_2] \subseteq [z_1,z_2]$. It follows directly for $[x_1,x_2] = \varnothing$. In both cases, whether $[y_1,y_2]$ or $[z_1,z_2] = [0,1]$, the proof is immediate. If $[y_1,y_2] = \varnothing$ or $[z_1,z_2] = \varnothing$, and $[x_1,x_2] \neq \varnothing$, then $\mathcal{I}_8([x_1,x_2],[y_1,y_2]) = \varnothing \subseteq \mathcal{I}_8([x_1,x_2],[z_1,z_2])$. We distinguish the following cases:

1. Let $[x_1,x_2] \in U^S$.
1.1. if $[y_1,y_2] \notin D^S$, we have $[z_1,z_2] \notin D^S$. Consequently,
   $\mathcal{I}_8([x_1,x_2],[y_1,y_2]) = [PI_1(x_1,y_1), PI_2(x_2,y_2)] \subseteq [PI_1(x_1,z_1), PI_2(x_2,z_2)] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.
1.2. $[y_1,y_2] \in D^S$.
   $\mathcal{I}_8([x_1,x_2],[y_1,y_2]) = \varnothing \subseteq [PI_1(x_1,z_1), PI_2(x_2,z_2)] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.
2. if $[x_1,x_2] \notin U^S$, then
   $\mathcal{I}_8([x_1,x_2],[y_1,y_2]) = [PI_1(x_1,y_1), PI_2(x_2,y_2)] \subseteq [PI_1(x_1,z_1), PI_2(x_2,z_2)] = \mathcal{I}_8([x_1,x_2],[z_1,z_2])$.

Finally, it easy is to check that (I3-I5) hold with the definition of $\mathcal{I}_8$.

□

The following proposition provides an application of previous theorem since $U^S = \{[x_1, x_2] \in P^S \mid a \in [x_1, x_2]\}$ is an up-set and $U^S = \{[x_1, x_2] \in P^S \mid b \notin [x_1, x_2]\}$ is a down -set in P^S, where $a, b \in P$.

Proposition 3.24. *Let $(P, \leq, 0, 1)$ be a bounded lattice, $PI_1, PI_2 \in \mathcal{PF}$ and $a, b \in P$ be arbitrary fixed elements. Then the mapping $\mathcal{I}_9 : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_9([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] = \varnothing \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } (a \in [x_1, x_2], b \notin [y_1, y_2]) \text{ or } ([x_1, x_2] \neq \varnothing, [y_1, y_2] = \varnothing), \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{12}$$

is an implication on P^S.

The following proposition provides an application of previous theorem since $U^S = \{[x_1, x_2] \in P^S \ : \ |[x_1, x_2]| = \infty\}$ is an up-set and $D^S = \{[x_1, x_2] \in P^S \ : \ |[x_1, x_2]| \neq \infty\}$ is a down -set in P^S.

Proposition 3.25. *Let $(P, \leq, 0, 1)$ be a bounded poset and $PI_1, PI_2 \in \mathcal{PF}$. Then the mapping $\mathcal{I}_{10} : P^S \times P^S \to P^S$ defined, for $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_{10}([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] = \varnothing \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } (|[x_1, x_2]| = \infty, |[y_1, y_2]| \neq \infty) \text{ or } ([x_1, x_2] \neq \varnothing, [y_1, y_2] = \varnothing), \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{13}$$

is an implication on P^S.

By defining $U^S = \{[x_1, x_2] \in P^S | [x_1, x_2] \cup B = [0, 1]\}$ and $D^S = \{[x_1, x_2] \in P^S | [x_1, x_2] \cap A = \varnothing]\}$ in Theorem 3.23, we obtain the following result:

Proposition 3.26. *Let $(P, \leq, 0, 1)$ be a bounded poset, $PI_1, IP_2 \in \mathcal{PF}$ and $A, B \subseteq P$ be an arbitrary fixed sets. Then the mapping $\mathcal{I}_{11} : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_{11}([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] = \varnothing \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } ([x_1, x_2] \cup B = [0, 1], [y_1, y_2] \cap A = \varnothing) \text{ or} \\ & \quad ([x_1, x_2] \neq \varnothing, [y_1, y_2] = \varnothing), \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{14}$$

is an implication on P^S.

Theorem 3.27. *Let $(P, \leq, 0, 1)$ be a bounded poset and $PI_1, PI_2 \in \mathcal{PF}$. Then the mapping $\mathcal{I}_{12} : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_{12}([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] = \varnothing \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } [x_1, x_2] \neq \varnothing, \ [y_1, y_2] \neq [0, 1], \ [y_1, y_2] \subseteq [x_1, x_2], \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{15}$$

is an implication on P^S.

Theorem 3.28. *Let $(P, \leq, 0, 1)$ be a bounded poset and $PI_1, PI_2 \in \mathcal{PF}$. Then the mapping $\mathcal{I}_{12} : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_{12}([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] = \varnothing \text{ or } [y_1, y_2] = [0, 1], \\ \varnothing & \text{if } [x_1, x_2] \neq \varnothing, \ [y_1, y_2] \neq [0, 1], \ [y_1, y_2] \subseteq [x_1, x_2], \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{16}$$

is an implication on P^S.

Proof. **I1.** Let $[x_1, x_2], [y_1, y_2], [z_1, z_2]$ be arbitrary elements in L^S. We need to show that if $[x_1, x_2] \subseteq [y_1, y_2]$, then the equality $\mathcal{I}_{12}([y_1, y_2], [z_1, z_2]) \subseteq \mathcal{I}_{12}([x_1, x_2], [z_1, z_2])$ holds. It is immediate when $[x_1, x_2] = \varnothing$ or $[z_1, z_2] = [0, 1]$. Excluding trivial cases, we now consider the remaining ones.

1. Let $[z_1, z_2] \subseteq [y_1, y_2]$.
   $\mathcal{I}_{12}([y_1, y_2], [z_1, z_2]) = \varnothing \subseteq \mathcal{I}_{12}([x_1, x_2], [z_1, z_2])$.
2. Let $[z_1, z_2] \nsubseteq [y_1, y_2]$. It follows that $[z_1, z_2] \nsubseteq [x_1, x_2]$ by using $[x_1, x_2] \subseteq [y_1, y_2]$.
   $\mathcal{I}_{12}([y_1, y_2], [z_1, z_2]) = [PI_1(y_1, z_1), PI_2(y_2, z_2)] \subseteq [PI_1(x_1, z_1), PI_2(x_2, z_2)] = \mathcal{I}_{12}([x_1, x_2], [z_1, z_2])$.

I2. Let $[x_1, x_2], [y_1, y_2], [z_1, z_2]$ be arbitrary elements in P^S. We need to show that the equality $\mathcal{I}_{12}([x_1, x_2], [y_1, y_2]) \subseteq \mathcal{I}_{12}([x_1, x_2], [z_1, z_2])$ holds if $[y_1, y_2] \subseteq [z_1, z_2]$. In the cases $[x_1, x_2] = \varnothing$ or $[z_1, z_2] = [0, 1]$, the proof follows directly. We examine the remaining cases below.

1. Let $[y_1, y_2] \subseteq [x_1, x_2]$.
   $\mathcal{I}_{12}([x_1, x_2], [y_1, y_2]) = \varnothing \subseteq \mathcal{I}_{12}([x_1, x_2], [z_1, z_2])$.
2. Let $[y_1, y_2] \nsubseteq [x_1, x_2]$. It follows that $[z_1, z_2] \nsubseteq [x_1, x_2]$ from $[y_1, y_2] \subseteq [z_1, z_2]$.
   $\mathcal{I}_{12}([x_1, x_2], [y_1, y_2]) = [PI_1(x_1, y_1), PI_2(x_2, y_2)] \subseteq [PI_1(x_1, z_1), PI_2(x_2, z_2)] = \mathcal{I}_{12}([x_1, x_2], [z_1, z_2])$.

Lastly, one can readily verify that (I3)–(I5) are fulfilled by the definition of $\mathcal{I}_{12}$.

□

Remark 3.29. *In Theorem 3.28, $\mathcal{I}_{12}$ does not satisfy (NP), since $\mathcal{I}_{12}([0, 1], [y_1, y_2]) = \varnothing$ for $[y_1, y_2] \subseteq [0, 1]$.*

Example 3.30. *Consider the set $A = \{a, b\}$, $P(A) = \{\varnothing, \{a\}, \{b\}, A\}$. Then the bounded poset $(P(A)^S, \subseteq)$ is characterized by the Hasse diagram shown in Figure 5.*

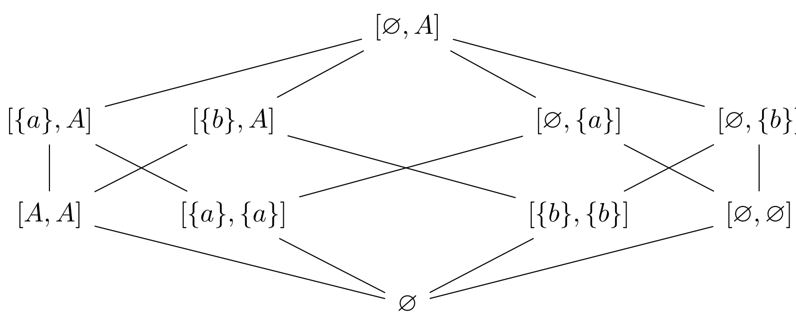

Figure 5: The Hasse diagram of the bounded poset $\mathcal{P}(\mathcal{A})^S$.

PI_1	$\varnothing$	$\{a\}$	$\{b\}$	A
$\varnothing$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
A	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$

Table 7: The function PI_1 on $\mathcal{P}(\mathcal{A})$.

PI_2	$\varnothing$	$\{a\}$	$\{b\}$	A
$\varnothing$	A	A	A	A
$\{a\}$	$\{b\}$	A	$\{b\}$	A
$\{b\}$	$\{a\}$	$\{a\}$	A	A
A	$\varnothing$	$\{a\}$	$\{b\}$	A

Table 8: The function PI_2 on $\mathcal{P}(\mathcal{A})$.

$\mathcal{I}_{12}$	$\varnothing$	$[\varnothing, \varnothing]$	$[\varnothing, \{a\}]$	$[\varnothing, \{b\}]$	$[\{a\}, \{a\}]$	$[\{b\}, \{b\}]$	$[A, A]$	$[\{a\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$\varnothing$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$	$[\varnothing, A]$
$[\varnothing, \varnothing]$	$\varnothing$	$\varnothing$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$[\varnothing, \{a\}]$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, \{b\}]$	$\varnothing$	$[\{b\}, \{b\}]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$[\varnothing, \{b\}]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$[\{a\}, \{a\}]$	$\varnothing$	$[\{b\}, \{b\}]$	$[\{b\}, A]$	$[\{b\}, \{b\}]$	$\varnothing$	$[\{b\}, \{b\}]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$[\{b\}, \{b\}]$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, A]$	$\varnothing$	$\varnothing$	$[\{b\}, A]$	$[\{b\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$[A, A]$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, \{b\}]$	$\varnothing$	$[\{b\}, \{b\}]$	$\varnothing$	$[\{b\}, A]$	$[\{b\}, A]$	$[\varnothing, A]$
$[\{a\}, A]$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, \{b\}]$	$\varnothing$	$[\{b\}, \{b\}]$	$\varnothing$	$\varnothing$	$[\{b\}, A]$	$[\varnothing, A]$
$[\{b\}, A]$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, \{b\}]$	$\varnothing$	$\varnothing$	$\varnothing$	$[\{b\}, A]$	$\varnothing$	$[\varnothing, A]$
$[\varnothing, A]$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$[\varnothing, A]$

Table 9: The implication $\mathcal{I}_{12}$ on $\mathcal{P}(\mathcal{A})^S$.

Theorem 3.31. *Let $(P, \leq, 0, 1)$ be a bounded poset and $PI_1, PI_2 \in \mathcal{PF}$. Then the mapping $\mathcal{I}_{13} : P^S \times P^S \to P^S$ defined, for all $[x_1, x_2], [y_1, y_2] \in P^S$, by*

$$\mathcal{I}_{13}([x_1, x_2], [y_1, y_2]) = \begin{cases} [0, 1] & \text{if } [x_1, x_2] \subseteq [y_1, y_2], \\ \varnothing & \text{if } [x_1, x_2] \neq \varnothing, \ [y_1, y_2] = \varnothing, \\ [PI_1(x_1, y_1), PI_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{17}$$

is an implication on P^S.

Proof. The proof is similar to the proof of Theorem 3.28. □

Remark 3.32. *We take $PI_1 \in \mathcal{C}$ and $PI_2 \in \mathcal{F}$ in Theorem 3.31.*

i. *$\mathcal{I}_{13}$ satisfies (IP) because $\mathcal{I}_{13}([y_1, y_2], [y_1, y_2]) = [0, 1]$ since $[y_1, y_2] \subseteq [y_1, y_2]$ for all $[y_1, y_2] \in P^S$.*

ii. *If PI_1 satisfies (CNP) and PI_2 satisfies (NP), it can be easily show that $\mathcal{I}_{13}$ satisfies (NP).*

4 Conclusions

In this paper, we investigate implication construction methods on P^S, primarily utilizing pre-implication functions on P. This approach generalizes the method introduced in [10] (see Theorem 2.7). Unlike construction methods in the literature that assign specific values to subsets of P^S, using up-sets (or down-sets) yields a wider variety of constructions (see Propositions 3.16, 3.17, 3.19, 3.20, 3.21 and 3.26). Moreover, the properties, such as the neutrality, ordering properties and initiality principle are deeply investigated in Proposition 3.16 for the construction method $\mathcal{I}_1$ given in Theorem 3.8. In Remark 3.13, we investigate whether the converse of the propositions holds. To better illustrate its structure, we visualized P^S, which is derived from a bounded poset P in Figure 3. Finally, we present two new construction methods $\mathcal{I}_{12}$ and $\mathcal{I}_{13}$ in Theorems 3.28 and 3.31, which are derived from the relationship between the intervals $[x_1, x_2]$ and $[y_1, y_2]$ in P^S. Additionally, the proposed methods methods can be interpreted within the framework of coimplications on P^S.

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