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Original paper

On properties of closed sets in the Zariski topology of MV-algebras

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Abstract

This paper presents a localized approach to the Zariski topology by restricting the spectral space to specific subsets of prime ideals within an MV-algebra. We investigate a particular class of Zariski-closed sets and demonstrate that they form a lattice under set inclusion. A distinguished filter within this lattice is then examined, and its algebraic properties are analyzed in detail. Building on this framework, we introduce the concept of v_X-ideals, a new type of ideal defined in terms of these closed sets. We explore their algebraic behavior, including interactions with minimal prime ideals and their stability under homomorphisms. The study reveals new structural insights into Zariski-closed sets and their connections to broader ideal-theoretic constructs. The final diagram synthesizes these findings, offering a unified perspective and laying the foundation for further exploration of topological and algebraic properties in MV-algebras.

Keywords: MV-algebra, v_X-ideal, lattice.

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1 Introduction

MV-algebras, introduced by Chang in 1958 [7, 8], serve as the algebraic foundation for Łukasiewicz's infinite-valued propositional logic. Commonly referred to as many-valued algebras, they play a central role in fuzzy logic—a Mathematical system designed to handle reasoning under uncertainty and vagueness. Unlike classical logic, which relies on binary truth values (true or false), fuzzy logic allows for a continuum of truth values, capturing nuanced degrees of truth or membership. MV-algebras provide the formal tools needed to represent and manipulate these gradations, forming the theoretical core of fuzzy logic and fuzzy set theory. Their algebraic structure enables precise modeling of fuzzy concepts and supports essential reasoning tasks such as fuzzy inference, approximate deduction, and fuzzy control. As such, MV-algebras are indispensable in the development and analysis of systems that require flexible, non-binary logic.

To maintain conciseness, we refer the reader to [7, 8, 9, 24] for foundational results on MV-algebras. Among these, the theory of ideals plays a central role in understanding the structure of MV-algebras [11, 17, 18]. In particular, prime ideals are of special importance: every proper ideal I can be represented as the intersection of prime ideals [9]. Extensive research has explored the properties of ideals and their interactions with various classes of MV-algebras. For a comprehensive overview, we direct the reader to [2, 3, 20, 21, 22].

This property highlights the foundational significance of prime ideals in decomposing and understanding the internal organization of MV-algebras. Over the years, substantial research has focused on the behavior of ideals, their algebraic properties, and their relationships with various subclasses of MV-algebras. These investigations have deepened our understanding of ideal theory and its applications in fuzzy logic and algebraic systems.

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A considerable body of research has been devoted to the study of ideals in MV-algebras, focusing on their structural properties and interactions with various algebraic classes. These investigations have examined how ideals behave under different operations, how they relate to prime and maximal ideals, and how they influence the overall architecture of MV-algebras. For an in-depth treatment of these topics, we refer the reader to [2, 3, 20, 21, 22], which provide comprehensive analyses and classifications of ideal-related phenomena across diverse MV-algebraic frameworks

The prime spectrum of MV-algebras has been studied by Belluce et al. [5]. Additionally, Forouzesh et.al examined the structure of closed sets within the Zariski topology and introduced the inverse topology as a special case of it. By presenting this topology, they offered a novel and distinct perspective on algebraic systems [15].

The closed sets in the Zariski topology have been extensively studied, and many of their topological properties have been established. However, in much of the existing literature, the connection between these sets and specific algebraic structures such as MV-algebras has received limited attention. This study aims to examine the closed sets of the Zariski topology from the perspective of ideals, focusing on identifying relationships between various types of ideals defined in MV-algebras and the closed sets themselves. The ultimate goal is to provide a deeper understanding of the interplay between algebra and topology in the context of many-valued logic.

We begin by assuming that X is a subset of the set of all prime ideals, namely $\mathrm{Spec}(A)$. By restricting the domain to X, we define closed sets of the form $v_X(a)$ and $v_X(Y)$ for each $a \in A$ and each subset $Y \subseteq A$ as follows:

$$v_X(Y) = \{P \in X : Y \subseteq P\}, \quad v_X(a) = \{P \in X : a \in P\}.$$

These definitions provide a localized version of the classical Zariski topology, where the spectral space is limited to a specific subset of prime ideals.

The motivation behind this restriction is to enable a focused study of particular cases, especially those involving the set of minimal prime ideals and the set of maximal ideals, denoted respectively by $\mathrm{Min}(A)$ and $\mathrm{Max}(A)$. This approach allows for a more refined analysis of the topological behavior of these subsets and their algebraic significance within the structure of MV-algebras.

Based on the conducted analysis, it was shown that the collection of all sets $v_X(I)$, where I is a principal ideal of A, forms a lattice. Furthermore, the subcollection $\{v_X(J) : J \text{ is a finite subset of } I\}$ defines a filter within this lattice, as it is upward closed and closed under finite intersections.

Additional results were obtained by examining specific choices of X. If X contains $\mathrm{Max}(A)$, then the filter is proper. On the other hand, if $X = \mathrm{Min}(A)$ and the ideal I contains at least one non-zero-divisor, then this filter coincides with the entire lattice.

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In the continuation of the study, the concept of v_X-ideals is introduced. A proper ideal I of A is called a v_X-ideal if it satisfies the following condition: whenever $v_X(a) \subseteq v_X(b)$ and $a \in I$, then $b \in I$. Alongside this definition, we also determine the minimal and maximal v_X-ideals, providing a clearer understanding of their structural boundaries.

It was shown that every minimal prime ideal over a v_X-ideal is itself a v_X-ideal. As a consequence, any v_X-ideal can be expressed as the intersection of the minimal prime ideals over it. By providing an example, it was shown that the zero ideal is not necessarily a v_X-ideal. Moreover, it was proven that the zero ideal is a v_X-ideal if and only if every minimal prime ideal is a v_X-ideal.

The behavior of v-ideals under homomorphisms is also examined. In particular, we show that for a homomorphism $f : A \to B$ between MV-algebras, every v_X-ideal of B contracts to a v_X-ideal of A if and only if every prime ideal in $Y \subseteq \mathrm{Spec}(B)$ contracts to a v_X-ideal.

Finally, a diagram is presented to illustrate the relationship between the introduced v_X-ideals and other types of ideals defined in [1, 2, 13, 14, 20, 23]. This diagram reflects the connection between Zariski-closed sets and various ideal-theoretic structures, offering a unified perspective on their interplay.

2 Preliminaries

We recollect some definitions and results that will be used in the sequel. In the beginning, we recall the notion of an MV-algebra, which forms the algebraic background for the results developed in this paper.

Definition 2.1. [7] *An MV-algebra is a structure* $(A, \oplus, ^*, 0)$ *where* $\oplus$ *is a binary operation,* * *is a unary operation, and* 0 *is a constant, satisfying the following axioms for all* $x, y \in A$*:*
(MV1) $(A, \oplus, 0)$ *is an Abelian monoid.*
(MV2) $(x^*)^* = x$.

(MV3) $0^* \oplus x = 0^*$.
(MV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

From now on, let A be an MV-algebra. We denote $1 = 0^*$ and define the operation $\odot$ by

$$x \odot y = (x^* \oplus y^*)^*.$$

An order relation $\leq$ on A is defined by: $x \leq y$ if and only if $x^* \oplus y = 1$, equivalently $x \odot y^* = 0$. This is called the natural order, under which A becomes a bounded distributive lattice. The lattice operations are:

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*), \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

An MV-algebra A is called an MV-chain if it is linearly ordered.

An element $a \in A$ is called Boolean if $a \oplus a = a$. The set of all Boolean elements is denoted by $B(A)$ and forms a Boolean algebra. Now, the definition of an ideal is recalled, along with some basic properties of generated ideals. Furthermore, the types of ideals required in the sequel are introduced.

Definition 2.2. [9] *A non-empty subset $I \subseteq A$ is called an ideal if*

(I1) *If $x \in I$, $y \in A$, and $y \leq x$, then $y \in I$.*

(I2) *If $x, y \in I$, then $x \oplus y \in I$.*

We denote the set of all ideals of A by $\mathrm{Id}(A)$.

Proposition 2.3. [9]

(1) *Let X be a non-empty subset of A. Then the ideal generated by X is*

$$(X] = \{a \in A \mid a \leq x_1 \oplus \cdots \oplus x_n \text{ for some } n \in \mathbb{N},\ x_i \in X\}.$$

In particular, for $a \in A$, the principal ideal generated by a is

$$(a] = \{x \in A \mid x \leq na \text{ for some } n \in \mathbb{N}\}.$$

(2) *The set $\mathrm{Id}(A)$ forms a lattice under inclusion, with*

$$I \wedge J = I \cap J, \quad I \vee J = (I \cup J], \quad \forall I, J \in \mathrm{Id}(A).$$

Moreover, $(\mathrm{Id}(A), \vee, \wedge, \{0\}, A)$ is a complete Brouwerian lattice, i.e., $I \wedge \bigvee_j J_j = \bigvee_j (I \wedge J_j)$.

(3) *For all $a, b \in A$, $(a] \wedge (b] = (a \wedge b]$, $(a] \vee (b] = (a \oplus b]$.*

Definition 2.4. [4, 9, 14] *Let I be an ideal of A.*

- *I is proper if $I \neq A$.*
- *A proper ideal P is prime if $x \wedge y \in P$ implies $x \in P$ or $y \in P$.*
- *A prime ideal P is minimal if $Q \subseteq P$ for some prime ideal Q implies $Q = P$.*
- *I is implicative if $x^n \in I$ for some $n \geq 1$ implies $x \in I$.*
- *M is maximal if there is no proper ideal J such that $M \subsetneq J$.*
- *I is obstinate if $a, b \notin I$ implies $a \odot b^* \in I$ and $b \odot a^* \in I$.*

We define $\mathrm{Spec}(A)$ as the set of all prime ideals of A, $\mathrm{Min}(A)$ as the set of all minimal prime ideals, and $\mathrm{Max}(A)$ as the set of all maximal ideals of A. Next, some established properties of prime ideals are presented, and these properties will be used in the proofs of the theorems developed in this paper.

Theorem 2.5. [10] *For a proper ideal P of A, the following conditions are equivalent*

(1) *$P \in \mathrm{Spec}(A)$.*

(2) *If* $I \cap J \subseteq P$*, then either* $I \subseteq P$ *or* $J \subseteq P$*, for all* $I, J \in \mathrm{Id}(A)$*.*

Proposition 2.6. [9] *Let* I *be a proper ideal of* A*.*

(1) *If* $x \in A \setminus I$*, then there exists* $P \in \mathrm{Spec}(A)$ *such that* $I \subseteq P$ *and* $x \notin P$*.*

(2) $I = \bigcap\{P \in \mathrm{Spec}(A) \mid I \subseteq P\}$*.*

Definition 2.7. [15, 16] *Let* X *be a non-empty subset of* A*.*

(1) *The set of all zero divisors of* X*, denoted by* $Z_X(A)$*, is defined as*

$$Z_X(A) = \{a \in A \mid \exists x \in X,\ x \neq 0 \text{ and } x \wedge a = 0\}.$$

The set of all zero divisors of A *is denoted by* $Z(A)$*.*

(2) *For an ideal* I *of* A*, define*

$$(I : X) = \{a \in A \mid a \wedge x \in I \text{ for all } x \in X\}.$$

Clearly, $(I : X)$ *is an ideal of* A*. In particular, if* $I = 0$*, then* $(0 : X) = \mathrm{Ann}(X)$*. Also, for every* $a \in A$*, we have* $(I : a) = (I : (a])$*.*

Theorem 2.8. [15]

(1) *Let* $P \in \mathrm{Min}(A)$ *and let* I *be a finitely generated ideal. Then* $I \subseteq P$ *if and only if* $\mathrm{Ann}(I) \nsubseteq P$*.*

(2) $\displaystyle\bigcap_{P \in \mathrm{Min}(A)} P = \{0\}$*.*

(3) *If* $x \in A \setminus \{0\}$*, then there exists* $P \in \mathrm{Min}(A)$ *such that* $x \notin P$*.*

Proposition 2.9. [21] *Let* I *be a proper ideal of* A *and* P *a prime ideal such that* $I \subseteq P$*. Then* $P \in \mathrm{Min}(I)$ *if and only if for each* $a \in P$ *there exists* $b \in A \setminus P$ *such that* $a \wedge b \in I$*.*

Theorem 2.10. [2] *Let* I *be a proper ideal of* A*.*

(1) *If* $P \in \mathrm{Spec}(A)$ *and* $I \subseteq P$*, then there exists* $P^* \in \mathrm{Min}(I)$ *such that* $P^* \subseteq P$*.*

(2) *If* $P_1, P_2, \ldots, P_n$ *are prime ideals of* A *such that* $I \subseteq \displaystyle\bigcup_{i=1}^{n} P_i$*, then* $I \subseteq P_i$ *for some* $i \in \{1, 2, \ldots, n\}$*.*

In what follows, the notion of $\wedge$-closed systems is recalled, as these systems play a central role in the interaction between prime ideals and the algebraic structure of A. The following definition and lemmas will be used repeatedly in the sequel.

Definition 2.11. [24] *A non-empty subset* $S \subseteq A$ *is called a* $\wedge$*-closed system if* $1 \in S$ *and* $x, y \in S$ *imply* $x \wedge y \in S$*. The set of all* $\wedge$*-closed systems of* A *is denoted by* $S(A)$ *(clearly* $\{1\}, A \in S(A)$*).*

Lemma 2.12. [24] *Let* S *be a* $\wedge$*-closed system in* A*, and let* $I \in \mathrm{Id}(A)$ *be such that* $I \cap S = \emptyset$*. Then there exists a prime ideal* P *of* A *such that* $I \subseteq P$ *and* $P \cap S = \emptyset$*.*

Lemma 2.13. [24] *If* P *is a prime ideal, then* $A \setminus P$ *is a* $\wedge$*-closed system in* A*.*

In what follows, several notions concerning the structure of ideals are recalled, including the constructions $\sigma(I)$ and $O(Q)$, as well as various classes of ideals such as pure ideals, Z°-ideals, Z-ideals, $MaxP$-ideals, and $MinP$-ideals. These definitions and results provide essential tools for the developments that follow.
Consider the set $\sigma(I)$ defined by

$$\sigma(I) = \{x \in A \mid z \vee y = 1 \text{ for some } z \in I \text{ and } y \in \mathrm{Ann}(x)\}.$$

If $\sigma(I) = I$, then I is called a *pure ideal* of A [6].
If Q is a prime ideal of A, define

$$O(Q) = \{a \in A \mid \mathrm{Ann}(a) \nsubseteq Q\}.$$

Theorem 2.14. [1] *Let* $Q \in \mathrm{Spec}(A)$ *and define* $\beta(Q) = \{P \in \mathrm{Min}(A) \mid P \subseteq Q\}$. *Then*

(1) $\sigma(Q) \subseteq O(Q) \subseteq \bigcap\limits_{P \in \beta(Q)} P$.

(2) *If* Q *is a pure ideal, then* $Q \in \mathrm{Min}(A)$.

(3) *If* Q *is a maximal ideal, then* $\sigma(Q) = O(Q)$.

(4) *A maximal ideal* Q *is pure if and only if* $Q \in \mathrm{Min}(A)$.

Define

$$P_a = \{b \in A \mid \text{for all } P \in \mathrm{Min}(A),\ a \in P \Rightarrow b \in P\}, \quad M_a = \bigcap\{M \in \mathrm{Max}(A) \mid a \in M\},$$

$$P(a) = \{P \in \mathrm{Min}(A) \mid a \in P\}, \quad M(a) = \{M \in \mathrm{Max}(A) \mid a \in M\}.$$

Definition 2.15. [1, 2, 13, 20, 23] *Let* I *be a proper ideal of* A.

- I *is a* Z°*-ideal (resp.* Z*-ideal) if* $P_a \subseteq I$ *(resp.* $M_a \subseteq I$*) for all* $a \in I$.
- I *is a* $MaxP$*-ideal if every prime ideal* $P \in r(I)$ *is maximal. If every* $P \in r(I)$ *is minimal, then* I *is a* $MinP$*-ideal.*
- *Let* I, J *be ideals of* A. *Then* I *is a* Z_J*-ideal (resp.* Z°_J*-ideal) if* $M_a \cap J \subseteq I$ *(resp.* $P_a \cap J \subseteq I$*) for all* $a \in I$. *If* $J \nsubseteq I$, *then* I *is a* Z_{NJ}*-ideal (resp.* Z°_{NJ}*-ideal).*
- I *is a* weak Z°*-ideal if for all* $a, b \in A$ *with* $a \wedge b \in I$ *and* $\mathrm{Ann}(a) = \{0\}$, *it follows that* $b \in I$.

Theorem 2.16. [2] *Let* I *be a proper ideal of* A. *Then*

- I *is a* Z°*-ideal if and only if* $P(a) = P(b)$ *and* $a \in I$ *imply* $b \in I$.
- I *is a* Z*-ideal if and only if* $M(a) = M(b)$ *and* $a \in I$ *imply* $b \in I$.

Proposition 2.17. [2] *A is an $MV-$chain if and only if $\{0\}$ is the only $Z^\circ-$ideal of A.*

At the end of this section, some notions from the Zariski topology are recalled.
For any $Y \subseteq A$ and $a \in A$, define

$$r(Y) = \{P \in \mathrm{Spec}(A) \mid Y \nsubseteq P\}, \quad v(Y) = \mathrm{Spec}(A) \setminus r(Y),$$

$$r(a) = \{P \in \mathrm{Spec}(A) \mid a \notin P\}, \quad v(a) = \mathrm{Spec}(A) \setminus r(a).$$

Clearly, $r(A) = \mathrm{Spec}(A)$ and $r(\{0\}) = \emptyset$ [10].
Let I be an ideal of the MV-algebra A. Then the sets $r(I)$ and $v(I)$ form, respectively, the open and closed subsets of the Zariski topology on $\mathrm{Spec}(A)$.

3 Results on closed sets in the Zariski topology

3.1 A lattice of closed sets in the Zariski topology

In this section, we examine the conditions under which the closed sets of the Zariski topology form a distributive lattice. We also investigate filters within this lattice.

Throughout this paper, let X be a subset of $\mathrm{Spec}(A)$. For any subset $Y \subseteq A$ and any element $a \in A$, we define

$$v_X(Y) = \{P \in X : Y \subseteq P\} \quad \text{and} \quad v_X(a) = \{P \in X : a \in P\}.$$

Clearly, $v_X(Y) \subseteq v(Y)$ and $v_X(Y) = v_X((Y])$.

Proposition 3.1. *The set* $L_v = \{v_X(I) : I \text{ is an ideal of } A\}$ *forms a distributive lattice.*

Proof. It is evident that $(L_v, \subseteq)$ is a partially ordered set. Let I_1 and I_2 be two ideals of A. Then

$$v_X(I_1) \cup v_X(I_2) = v_X(I_1 \wedge I_2), \quad v_X(I_1) \cap v_X(I_2) = v_X(I_1 \vee I_2).$$

Moreover, since $A = (1]$, we have $v_X(A) = \emptyset$ and $v_X((0]) = X$. It follows that L_v is a lattice. We now show that L_v is distributive. Let I, J, and K be ideals of A. Then

$$\begin{aligned} v_X(I) \cup (v_X(J) \cap v_X(K)) &= v_X(I) \cup v_X(J \vee K) \\ &= v_X(I \wedge (J \vee K)) \\ &= v_X((I \wedge J) \vee (I \wedge K)) \quad \text{(by Proposition 2.3(2))} \\ &= v_X(I \wedge J) \cap v_X(I \wedge K) \\ &= (v_X(I) \cup v_X(J)) \cap (v_X(I) \cup v_X(K)). \end{aligned}$$

□

Let I be an ideal of A, and let F be a filter of the lattice L_v introduced in the previous proposition. Define

$$\mathcal{V}_X(I) = \{v_X(J) : J \text{ is a finite subset of } I\}, \quad v_X^{-1}(F) = \{a \in A : v_X(a) \in F\}.$$

Clearly, for any finite subset $Y \subseteq A$, we have $v_X(Y) \in F$ if and only if $Y \subseteq v_X^{-1}(F)$.

Proposition 3.2. *Let I be an ideal of A, and let F be a filter of L_v. Then*

(1) *If $v_X^{-1}(F)$ is non-empty, then it is an ideal of A.*

(2) *$v_X^{-1}(F) = A$ if and only if $F = L_v$.*

(3) *$I \subseteq v_X^{-1}(\mathcal{V}_X(I))$ and $v_X(v_X^{-1}(F)) = F$.*

(4) *$\mathcal{V}_X(I)$ is a filter of L_v.*

(5) *If $\mathrm{Max}(A) \subseteq X$, then $\mathcal{V}_X(I)$ is a proper filter of L_v.*

(6) *Let $X = \mathrm{Min}(A)$ and suppose $I \nsubseteq Z_A$. Then $\mathcal{V}_X(I) = L_v$.*

Proof. (1) Let $a, b \in v_X^{-1}(F)$. Then $v_X(a), v_X(b) \in F$, so $v_X(a) \cap v_X(b) \in F$. It follows that $v_X(a \oplus b) \in F$, hence $a \oplus b \in v_X^{-1}(F)$. Now, let $c \in A$ such that $c \leq a$. Then $v_X(a) \subseteq v_X(c)$ implies $v_X(c) \in F$, so $c \in v_X^{-1}(F)$. Thus, $v_X^{-1}(F)$ is an ideal of A.

(2) $v_X^{-1}(F) = A$ if and only if $1 \in v_X^{-1}(F)$, which holds if and only if $v_X(1) \in F$, i.e., $\emptyset \in F$, which implies $F = \mathcal{L}_v$.

(3) Clearly, $I \subseteq v_X^{-1}(\mathcal{V}_X(I))$. For any finite subset $Y \subseteq A$, we have

$$v_X(Y) \in F \iff Y \subseteq v_X^{-1}(F) \iff v_X(Y) \in v_X(v_X^{-1}(F)).$$

(4) Let $v_X(I_1), v_X(I_2) \in \mathcal{V}_X(I)$. Then $v_X(I_1) \cap v_X(I_2) = v_X(I_1 \cup I_2)$. Now, let I_3 be a finite subset of I such that $v_X(I_1) \subseteq v_X(I_3)$. Then

$$v_X(I_3) = v_X(I_1) \cup v_X(I_3) = v_X(I_1 \cap I_3),$$

so $v_X(I_3) \in \mathcal{V}_X(I)$.

(5) Suppose, for contradiction, that $\emptyset \in \mathcal{V}_X(I)$. Then there exists a finite subset $I_1 \subseteq I$ such that $v_X(I_1) = \emptyset$, which implies (I1] = A, a contradiction.

(6) By hypothesis, there exists $a \in I$ such that $\mathrm{Ann}(a) = \{0\}$. Hence $\mathrm{Ann}(a) = \mathrm{Ann}(1)$, and by Theorem 2.8, we have $v_X(a) = v_X(1) = \emptyset$. Thus, $\emptyset \in \mathcal{V}_X(I)$, and by part (4), $\mathcal{V}_X(I) = \mathcal{L}_v$. □

Theorem 3.3. *Let $X = \mathrm{Spec}(A)$, and define $Id_P = \{K : K \text{ is a principal ideal}\}$ and $\Gamma = \{\mathcal{V}_X(I) : I \text{ is a principal ideal}\}$. Then there exists a bijective correspondence between Id_P and Γ.*

Proof. Define the mapping $\varphi : Id_P \to \Gamma$ by $(a] \mapsto v_X((a])$ for each $a \in A$. This mapping is clearly well-defined and surjective. To prove injectivity, suppose $v_X((a]) = v_X((b])$ but $(a] \neq (b]$. Without loss of generality, assume $b \notin (a]$. Let $S = \{1, b\}$, which is a $\wedge$-closed subset of A and satisfies $S \cap (a] = \emptyset$. By Lemma 2.12, there exists $P \in \mathrm{Spec}(A)$ such that $(a] \subseteq P$ and $S \cap P = \emptyset$, hence $b \notin P$. Therefore, $P \in v_X((a])$ and $P \notin v_X((b])$, which contradicts the assumption that $v_X((a]) = v_X((b])$. Thus, the mapping is injective, and the correspondence is bijective. □

In the following, we demonstrate the correspondence established in the previous theorem within an MV-algebra framework.

Example 3.4. *Let $l_2 = \{0,1\}$. Consider the MV-algebra $A = l_2 \times l_2$ with operations defined by $(a,b) \oplus (c,d) = (1,1) \wedge (a+c, b+d) = (1 \wedge (a+c), 1 \wedge (b+d))$ and $(a,b)^* = (1-a, 1-b)$ for every $(a,b), (c,d) \in A$, as in* [19]. *Then $A = \{(0,0),(0,1),(1,0),(1,1)\}$ and has two prime ideals, $P_1 = \{(0,0),(1,0)\}$, $P_2 = \{(0,0),(0,1)\}$. Let $X = \{P_1, P_2\}$. Then $v_X((0,0)) = X$, $v_X((1,1)) = \emptyset$, $v_X((0,1)) = P_2$, $v_X((1,0)) = P_1$.*

3.2 v_X-ideals and $maxv_X$-ideals

In this section, we define v_X-ideals and $maxv_X$-ideals based on the closed sets of the Zariski topology. We explore their interrelationship and investigate the structural properties that characterize these ideals.

Throughout this paper, we use the notation $\mathcal{P}_X = \bigcap\limits_{Q \in X} Q$.

Definition 3.5. *Let I be a proper ideal of A.*

(1) *The ideal I is called a v_X-ideal if $\mathcal{P}_{v_X(a)} \subseteq I$ for every $a \in I$.*

(2) *If $\mathcal{P}_{v_X(I)} = I$, then I is called a $maxv_X$-ideal.*

In what follows, several properties of the previously defined ideals are presented, some of which are immediate and whose proofs will therefore not be discussed.

Lemma 3.6. *Let $a, b \in A$ and $I \in \mathrm{Id}(A)$.*

(1) *$b \in \mathcal{P}_{v_X(a)}$ if and only if $v_X(a) \subseteq v_X(b)$.*

(2) *$\mathcal{P}_X$ is a v_X-ideal and is the smallest such ideal.*

(3) *The intersection of any family of v_X-ideals is a v_X-ideal.*

(4) *Let $\{I_i\}_{i \in \Gamma}$ be a family of v_X-ideals such that $\bigvee\limits_{i \in \Gamma} I_i = \bigcup\limits_{i \in \Gamma} I_i$. Then $\bigcup\limits_{i \in \Gamma} I_i$ is a v_X-ideal.*

(5) *Every element of X is a v_X-ideal.*

(6) *If $\mathcal{P}_X = \{0\}$, then the zero ideal is a v_X-ideal.*

(7) *Every $maxv_X$-ideal is a v_X-ideal.*

(8) *Let $X = \mathrm{Spec}(A)$. Then every proper ideal of A is both a $maxv_X$-ideal and a v_X-ideal.*

Proof. The proofs of parts (1) through (6) follow directly from the definition of v_X-ideals and are straightforward; hence, they are omitted.

(7) Let I be a $maxv_X$-ideal and let $a \in I$. Suppose, for contradiction, that there exists $\alpha \in \mathcal{P}_{v_X(a)}$ such that $\alpha \notin I$. Then $\alpha \notin \mathcal{P}_{v_X(I)}$. Hence, there exists $P \in X$ such that $I \subseteq P$ and $\alpha \notin P$. Since $a \in P$ and $\alpha \in \mathcal{P}_{v_X(a)}$, it follows that $\alpha \in P$, which is a contradiction. Therefore, every $maxv_X$-ideal is a v_X-ideal.

(8) Let I be a proper ideal of A. By Proposition 2.6, we have

$$I = \bigcap \{P \in \mathrm{Spec}(A) : I \subseteq P\} = \mathcal{P}_{v_X(I)}.$$

Hence, I is a $maxv_X$-ideal. It then follows from part (6) that I is also a v_X-ideal. □

We now provide an equivalent definition of v_X-ideals and examine their relationship with $maxv_X$-ideals.

Proposition 3.7. *Let I be an ideal of A. Then the following statements are equivalent*

(1) *I is a v_X-ideal.*

(2) *For all $a \in I$, if $v_X(a) \subseteq v_X(b)$, then $b \in I$.*

Proof. ($1 \Rightarrow 2$) Let $a \in I$ and suppose $v_X(a) \subseteq v_X(b)$. By hypothesis and Lemma 3.6(1), we have $\mathcal{P}_{v_X(a)} \subseteq I$ and $b \in \mathcal{P}_{v_X(a)}$, hence $b \in I$.

($2 \Rightarrow 1$) Let $b \in \mathcal{P}_{v_X(a)}$. By Lemma 3.6(1), this implies $v_X(a) \subseteq v_X(b)$. By hypothesis, since $a \in I$, it follows that $b \in I$. Thus, $\mathcal{P}_{v_X(a)} \subseteq I$. □

In the following example, we examine several ideals with respect to the definition of v_X-ideals, and demonstrate that not every v_X-ideal is necessarily a $maxv_X$-ideal.

Example 3.8. (1) *Let* $l_3 = \{0, 1/2, 1\}$ *and* $l_2 = \{0, 1\}$. *Consider the MV-algebra* $A = l_3 \times l_2$ *with operations defined by*

$$(a, b) \oplus (c, d) = (1, 1) \wedge (a + c, b + d) = (1 \wedge (a + c), 1 \wedge (b + d)), \quad (a, b)^* = (1 - a, 1 - b),$$

for all $a, c \in l_3$ *and* $b, d \in l_2$, *as described in* [19]. *The algebra* A *has four ideals*

$$I_0 = \{(0, 0)\} = ((0, 0)], \quad I_1 = A = ((1, 1)], \quad I_2 = \{(0, 0), (0, 1)\} = ((0, 1)],$$
$$I_3 = \{(0, 0), (1/2, 0), (1, 0)\} = ((1, 0)] = ((1/2, 0)].$$

Clearly, $\mathrm{Spec}(A) = \{I_2, I_3\}$. *Let* $X = \{I_2\}$. *Then*

$$v_X((0, 1)) = v_X((0, 0)) = I_2, \quad v_X((1/2, 0)) = v_X((1, 0)) = v_X((1/2, 1)) = v_X((1, 1)) = \emptyset.$$

It follows that I_2 *is a* v_X*-ideal of* A, *while* I_0 *and* I_3 *are not.*
(2) *Consider the set* $C = \{0, c, 2c, 3c, \ldots, 1 - 2c, 1 - c, 1\}$ *as an MV-algebra with the following operations*

- *If* $a = nc$ *and* $b = mc$, *then* $a \oplus b := (m + n)c$.
- *If* $a = 1 - nc$ *and* $b = 1 - mc$, *then* $a \oplus b := 1$.
- *If* $a = nc$ *and* $b = 1 - mc$ *with* $m \leq n$, *then* $a \oplus b := 1$.
- *If* $a = nc$ *and* $b = 1 - mc$ *with* $n < m$, *then* $a \oplus b := 1 - (m - n)c$.
- *If* $a = 1 - mc$ *and* $b = nc$ *with* $m \leq n$, *then* $a \oplus b := 1$.
- *If* $a = 1 - mc$ *and* $b = nc$ *with* $n < m$, *then* $a \oplus b := 1 - (m - n)c$.
- *If* $a = nc$, *then* $a^* := 1 - nc$.
- *If* $a = 1 - nc$, *then* $a^* := nc$.

The structure $(C, \oplus, ^*, 0)$ *is known as Chang's MV-algebra* [24]. *It has three ideals* $I_0 = \{0\}$, $I_1 = \{0, c, 2c, 3c, \ldots\} = (c]$, *and* $I_2 = C$. *Let* $X_1 = \{I_0\}$ *and* $X_2 = \{I_1\}$. *Clearly,* I_0 *is a* v_{X_1}*-ideal but not a* v_{X_2}*-ideal, while* I_1 *is a* v_{X_2}*-ideal but not a* v_{X_1}*-ideal.*
(3) *Let* A *be the subalgebra of* $[0, 1]^*$ *generated by a sequence of infinitesimals* $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots$. *Let* I *be the ideal generated by* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ *and let* X *be the set of all the ideals generated by* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$
Obviously, $P_{v_X(a)} \subseteq I$, *for every* $a \in I$. *On the other hand , no prime ideal of* X *contains* I. *Hence* $P_{v_X(I)}$ *is the empty intersection and is* A, *which is not included in* I. *Thus,* I *is a vX-ideal but not a maxvX-ideal.*

In the subsequent propositions, the connection between v_X-ideals and $maxv_X$-ideals with principal ideals will be examined.

Proposition 3.9. *Let* $a \in A$.

(1) $\mathcal{P}_{v_X(a)}$ *is a* v_X*-ideal.*

(2) *The principal ideal* $(a]$ *is a* v_X*-ideal if and only if it is a* $maxv_X$*-ideal.*

(3) *If every principal ideal of* A *is a* $maxv_X$*-ideal, then every ideal* I *of* A *is a* v_X*-ideal.*

Proof. (1) Suppose $v_X(\alpha) \subseteq v_X(\beta)$ and $\alpha \in \mathcal{P}_{v_X(a)}$ but $\beta \notin \mathcal{P}_{v_X(a)}$. Then there exists $P \in X$ such that $a \in P$ and $\beta \notin P$. Since $P \notin v_X(\beta)$, it follows that $P \notin v_X(\alpha)$, hence $\alpha \notin P$, contradicting $\alpha \in \mathcal{P}_{v_X(a)}$.
(2) This follows directly from the definitions.
(3) Let $a \in I$. Clearly, $\mathcal{P}_{v_X(a)} = \mathcal{P}_{v_X((a])}$. By hypothesis, $\mathcal{P}_{v_X(a)} = (a]$, so $\mathcal{P}_{v_X(a)} \subseteq I$. ☐

Proposition 3.10. *Let* X *be a finite subset of* $\mathrm{Spec}(A)$. *Then every* v_X*-ideal is a* $maxv_X$*-ideal.*

Proof. Let I be a v_X-ideal. Suppose, for contradiction, that $I = \bigcup_{P \in r_X(I)} P$. By Theorem 2.10(2), there exists $P \in r_X(I)$ such that $I \subseteq P$, contradicting the properness of I. So there exists $\alpha \in I \setminus \bigcup_{P \in r_X(I)} P$.

Now assume, for contradiction, that $v_X(\alpha) \nsubseteq v_X(I)$. Then there exists $Q \in X$ such that $\alpha \in Q$ but $I \nsubseteq Q$, implying $Q \in r_X(I)$ and hence $\alpha \in \bigcup_{P \in r_X(I)} P$, a contradiction. Thus $v_X(\alpha) \subseteq v_X(I)$, so $\mathcal{P}_{v_X(I)} \subseteq \mathcal{P}_{v_X(\alpha)} \subseteq I$ (by hypothesis). Since $I \subseteq \mathcal{P}_{v_X(I)}$, we conclude $I = \mathcal{P}_{v_X(I)}$, i.e., I is a $maxv_X$-ideal. ☐

Corollary 3.11. *In every finite MV-algebra, v_X-ideals and $maxv_X$-ideals coincide.*

Proposition 3.12. *Let I be an ideal and let $P \in \mathrm{Min}(I)$. If I is a v_X-ideal, then P is also a v_X-ideal.*

Proof. Let $a \in P$. By Proposition 2.9, there exists $b \in A \setminus P$ such that $a \wedge b \in I$. Since I is a v_X-ideal, we have $\mathcal{P}_{v_X(a \wedge b)} \subseteq I$. Note that $\mathcal{P}v_X(a) \cap \mathcal{P}v_X(b) \subseteq \mathcal{P}v_X(a) \cup v_X(b) = \mathcal{P}v_X(a \wedge b) \subseteq I$. This implies $\mathcal{P}v_X(a) \cap \mathcal{P}v_X(b) \subseteq P$. But $b \notin P$, so $\mathcal{P}v_X(b) \nsubseteq P$. By Theorem 2.5, it follows that $\mathcal{P}v_X(a) \subseteq P$, hence P is a v_X-ideal. ☐

In Parts 1 and 2 of Example 3.8, we observe that the zero ideal is not necessarily a v_X-ideal.

Corollary 3.13. (1) *The zero ideal is a v_X-ideal if and only if every minimal prime ideal is a v_X-ideal.*

(2) *An ideal I is a v_X-ideal if and only if it is the intersection of minimal prime v_X-ideals over I.*

Proof. (1) If the zero ideal is a v_X-ideal, then by Proposition 3.12, every minimal prime ideal must be a v_X-ideal. Conversely, if all minimal prime ideals are v_X-ideals, then by Theorem 2.8(2) and Lemma 3.6(3), the zero ideal is also a v_X-ideal.

(2) Clearly, I is contained in the intersection of minimal prime v_X-ideals over I. Suppose a belongs to this intersection but $a \notin I$. Then by Proposition 2.6 and Theorem 2.10(1), there exists $P \in \mathrm{Min}(I)$ such that $a \notin P$, contradicting the assumption. The converse is immediate. ☐

Theorem 3.14. *If X is non-empty, then a maximal v_X-ideal exists and it is a prime ideal.*

Proof. Let $\Gamma = \{I : I \text{ is a } v_X\text{-ideal}\}$. By Lemma 3.6(5), $\Gamma \neq \emptyset$, and clearly $(\Gamma, \subseteq)$ is partially ordered. Let $\{I_i\}_{i \in \Omega}$ be a chain in Γ. By Lemma 3.6(4), the union $\bigcup_{i \in \Omega} I_i$ is a v_X-ideal and serves as an upper bound. By Zorn's lemma, Γ has a maximal element J. By Proposition 3.12, any $P \in \mathrm{Min}(J)$ is a v_X-ideal, hence $P \in \Gamma$. On the other hand, since the maximal v_X-ideal is J, it follows that $P = J$. Thus, the maximal v_X-ideal is a prime ideal. ☐

Proposition 3.15. *Let I and J be two ideals of A such that I is a v_X-ideal and $J \nsubseteq I$. Then $(I : J)$ is a v_X-ideal.*

Proof. Let $(I : J) = A$. We get

$$\begin{aligned}(I : J) = A &\Leftrightarrow 1 \in (I : J)\\ &\Leftrightarrow 1 \wedge J \subseteq I\\ &\Leftrightarrow J \subseteq I,\end{aligned}$$

which is a contradiction. Hence $(I : J)$ is a proper ideal of A. Now, let $v_X(a) \subseteq v_X(b)$ and let $a \in (I : J)$. Then $a \wedge x \in I$, for every $x \in J$. Also, we have $v_X(a \wedge x) = v_X(a) \cup v_X(x) \subseteq v_X(b) \cup v_X(x) = v_X(b \wedge x)$, for every $x \in J$. It follows that $b \wedge x \in I$, for every $x \in J$. Hence $b \in (I : J)$. ☐

By the previous proposition, if the zero ideal is a v_X-ideal, then $(0 : J)$ is a v_X-ideal for every non-zero ideal J. Additionally, we conclude that if the zero ideal is a v_X-ideal, then $Ann(J)$ is also a v_X-ideal for every non-zero ideal J. Subsequently, v_X-ideals are examined based on $\wedge$-closed systems.

Let S be a $\wedge$-closed system and let I be an ideal of A. We denote

$$I_S = \{a \in A : a \wedge s \in I \textit{ for some } s \in S\}.$$

Proposition 3.16. *Let S be a $\wedge$-closed system and let I be an ideal of A such that $I \cap S = \emptyset$. Then*

(1) *I_S is a proper ideal of A.*

(2) $I_S = \bigcup_{s \in S} (I : s)$.

(3) *If I is a v_X-ideal, then I_S is also a v_X-ideal.*

Proof. (1) Since $1 \in S$, we have $0 = 0 \wedge 1 \in I$, so $0 \in I_S$. If $1 \in I_S$, then for some $s \in S$, $1 \wedge s = s \in I$, contradicting $I \cap S = \emptyset$. Let $x, y \in I_S$, with $x \wedge s_1, y \wedge s_2 \in I$ for some $s_1, s_2 \in S$. Then

$$(x \oplus y) \wedge (s_1 \wedge s_2) \leq (x \wedge s_1 \wedge s_2) \oplus (y \wedge s_1 \wedge s_2) \in I,$$

so $x \oplus y \in I_S$. If $x \in I_S$ and $y \leq x$, then $x \wedge s \in I$ for some $s \in S$, and since $y \wedge s \leq x \wedge s$, we get $y \wedge s \in I$, hence $y \in I_S$. Thus I_S is a proper ideal.

(2) Immediate from the definition.

(3) Let $\alpha \in I_S$ and suppose $v_X(\alpha) \subseteq v_X(\beta)$. By (2), there exists $s \in S$ such that $\alpha \in (I : s)$. Then by Proposition 3.15, $\beta \in (I : s)$, so $\beta \in I_S$. Hence I_S is a v_X-ideal. □

The following theorem investigates how v_X-ideals relate to specific types of ideals.

Theorem 3.17. (1) *If the zero ideal is a v_X-ideal, then $O(P)$ is a v_X-ideal for every $P \in \mathrm{Spec}(A)$.*

(2) *If the zero ideal is a v_X-ideal, then every maximal and pure ideal is a v_X-ideal.*

(3) *If I is a v_X-ideal such that $I \subseteq B(A)$, then I is an implicative ideal.*

(4) *If $X = \mathrm{Min}(A)$ (resp. $X = \mathrm{Max}(A)$), then Z°-ideals (resp. Z-ideals) and v_X-ideals coincide.*

Proof. (1) Let P be a prime ideal. By Lemma 2.13, $A \setminus P$ is a $\wedge$-closed system, and clearly $O(P) = 0_{A\setminus P}$. By Proposition 3.16, $O(P)$ is a v_X-ideal.

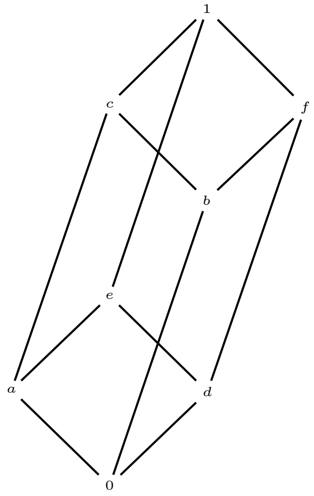
(2) Follows directly from Theorem 2.14(3) and part (1).

(3) Let $a \in A$ and suppose $a^n \in I$ for some $n \in \mathbb{N}$. Since $v_X(a) = v_X(a^n)$ and I is a v_X-ideal, it follows that $a \in I$.

(4) Follows from Theorem 2.16. □

In the following example, we show that the converse of part (3) in the previous theorem does not necessarily hold; that is, every implicative ideal is not necessarily a v_X-ideal. Furthermore, in another part of the example, we will analyze the ideals of a Boolean algebra.

Example 3.18. *1) Let ${}^*\mathbb{R}$ be a non-standard model of real numbers with natural order and ε be a positive infinitesimal element of ${}^*\mathbb{R}$. Let $\varepsilon^2 = \varepsilon \cdot \varepsilon, ..., \varepsilon^n = \varepsilon \cdot \varepsilon \cdot ... \cdot \varepsilon (n - times)$, where $\cdot$ is the usual product in the field ${}^*\mathbb{R}$; then $\varepsilon^i > 0$ for any $i \in \mathbb{N}$ and $\varepsilon^i < \varepsilon^j$, for $i > j$.*
The unit interval ${}^*[0,1] \subseteq^* \mathbb{R}$ is an MV-algebra under the the operations $x \oplus y = min\{1, x + y\}$, $x^* = 1 - x$.
Let $\mathbb{N}$ be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of ${}^*[0,1]$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n\}$ and E be the subalgebra $\bigcup_{n\in\mathbb{N}} E_n$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n, \ldots\}$ [12]. *Obviously, E is not a Boolean algebra.*
The ideals of E are $\{0\}, (\varepsilon], ..., (\varepsilon^i]...$, where $i \in \mathbb{N}$ and $(\varepsilon^i] \subseteq (\varepsilon^j]$, for any $i > j$.
In this example, $(\varepsilon^2]$ is not an implicative ideal because $\varepsilon^2 \in (\varepsilon^2]$ but $\varepsilon \notin (\varepsilon^2]$. Now, put $X = \{(\varepsilon^2]\}$. Obviously, $\mathcal{P}v_X(a) = (\varepsilon^2] \subseteq (\varepsilon^2]$, for every $a \in (\varepsilon^2]$. Then $(\varepsilon^2]$ is a v_X-ideal but is not an implicative ideal. Also, let $X = \{(\varepsilon]\}$. Then $\mathcal{P}v_X(a) = (\varepsilon] \nsubseteq (\varepsilon^2]$ for every $a \in (\varepsilon^2]$. Hence $(\varepsilon^2]$ is not a v_X-ideal and an implicative ideal.
2) Let $A = \{0, a, b, c, d, e, f, 1\}$. Where $0 < a < c < 1$, $0 < a < e < 1$, $0 < b < f < 1$, $0 < d < e < 1$, $0 < d < f < 1$, $a < c, e$ and $b < f, c$.



Define $\oplus$ *and* $*$ *as follows*

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	c	c	e	e	1	1
b	b	c	b	c	f	1	f	1
c	c	c	c	c	1	1	1	1
d	d	e	f	1	d	e	f	1
e	e	e	1	1	e	e	1	1
f	f	1	f	1	f	1	f	1
1	1	1	1	1	1	1	1	1

$*$	0	a	b	c	d	e	f	1
	1	f	e	d	c	b	a	0

Then $(A, \oplus, *, 0)$ *is a Boolean MV-algebra* [19]. *Hence* $Id(A) = \{I_0 = \{0\}, I_1 = \{0, a\}, I_2 = \{0, b\}, I_3 = \{0, d\}, I_4 = \{0, a, b, c\}, I_5 = \{0, a, e, d\}, I_6 = \{0, b, f, d\}\}$ *and* $Spec(A) = \{I_4, I_5, I_6\}$. *If* $X = Spec(A)$, *then every ideal of* A *is a* v_X*-ideal. Let* $X = \{I_4, I_6\}$. *Since* $P_{v_X(0)} = \{0, b\} \not\subseteq I_3$, *it follows that* I_3 *is not a* v_X*-ideal.*

Clearly, from Part 4 of Theorem 3.17 and Proposition 2.17, it follows that if $X = \mathrm{Min}(A)$, then A is an MV-chain if and only if the zero ideal is the only v_X-ideal of A.
In the following, v_X-ideals are analyzed in the context of isomorphisms.

Theorem 3.19. *Let* $f : A \to B$ *be an MV-algebra homomorphism, and let* $X \subseteq \mathrm{Spec}(A)$ *and* $Y \subseteq \mathrm{Spec}(B)$. *Then every* v_X*-ideal of* B *contracts to a* v_X*-ideal of* A *if and only if every* $P \in Y$ *contracts to a* v_X*-ideal.*

Proof. Let I be a v_X-ideal of B, and suppose $v_X(a) \subseteq v_X(b)$ with $a \in f^{-1}(I)$ and $b \in A$. Then $f(a) \in I$. We claim that $v_Y(f(a)) \subseteq v_Y(f(b))$. Let $P \in v_Y(f(a))$, so $P \in Y$ and $f(a) \in P$, hence $a \in f^{-1}(P)$. By hypothesis, $f^{-1}(P) \in v_X(a)$, and since $v_X(a) \subseteq v_X(b)$, it follows that $f^{-1}(P) \in v_X(b)$, i.e., $b \in f^{-1}(P)$, so $f(b) \in P$. Thus $P \in v_Y(f(b))$, and since $f(b) \in I$, we conclude $b \in f^{-1}(I)$.

The converse is straightforward. □

Corollary 3.20. *Let* J *be an ideal containing* I, *and define* $X/I = \{P/I : P \in v_X(I)\}$. *Then* J/I *is a* $v_{X/I}$*-ideal of* A/I *if and only if* J *is a* v_X*-ideal of* A.

Theorem 3.21. *Let* $X, Y \subseteq \mathrm{Spec}(A)$. *Then*

(1) *Every* v_X*-ideal is a* v_Y*-ideal if and only if every* $P \in X$ *is a* v_Y*-ideal.*

(2) *If* $\mathcal{P}_X \subseteq \mathcal{P}_Y$ *and* $X \subseteq \mathrm{Min}(\mathcal{P}_X)$, *then every* v_X*-ideal is a* v_Y*-ideal if and only if* $\mathcal{P}_X = \mathcal{P}_Y$.

(3) *If* $X \subseteq Y$, *then every* v_Y*-ideal is a* v_X*-ideal.*

(4) *If* $X \subseteq Y$ *and every* $P \in Y$ *is a* v_X*-ideal, then every* v_X*-ideal is a* v_Y*-ideal.*

Proof. (1) Let $P \in X$. Since P is a v_X-ideal and by hypothesis also a v_Y-ideal, the claim follows. Conversely, define $f : (A, Y) \to (A, X)$ as an MV-algebra homomorphism. By Theorem 3.19, every v_X-ideal is a v_Y-ideal.

(2) If $\mathcal{P}_X$ is both a v_X-ideal and a v_Y-ideal, and $\mathcal{P}_Y$ is the smallest v_Y-ideal, then $\mathcal{P}_X = \mathcal{P}_Y$. Conversely, if $\mathcal{P}_X = \mathcal{P}_Y$, then $\mathcal{P}_X$ is a v_Y-ideal. By Proposition 3.12, every element of $\mathrm{Min}(\mathcal{P}_X) = \mathrm{Min}(\mathcal{P}_Y)$ is a v_Y-ideal. Since $X \subseteq \mathrm{Min}(\mathcal{P}_X)$, part (1) implies every v_X-ideal is a v_Y-ideal.

(3) Let I be a v_Y-ideal and $a \in I$. Since $v_Y(a) \subseteq v_X(a)$, we have $\mathcal{P}_{v_X(a)} \subseteq \mathcal{P}_{v_Y(a)} \subseteq I$, so I is a v_X-ideal.

(4) Define $f : A \to A$ as the identity homomorphism. By hypothesis and Theorem 3.19, every v_X-ideal is a v_Y-ideal. □

Let (T, τ) be a topological space. For any $M \subseteq T$, we denote by M° and $\overline{M}$ the interior and closure of M, respectively. We recall that in every topological space the closure of the complement of a set coincides with the complement of its interior:

$$\overline{T \setminus S} = T \setminus S^\circ.$$

Proposition 3.22. *Let $Y \subseteq A$. Then*

1. $(\mathcal{P}_X : Y) = \mathcal{P}_{r_X(Y)}$.
2. *If* $\mathcal{P}_X = \{0\}$, *then* $\mathcal{P}_{r_X(Y)} = \mathrm{Ann}(Y)$ *and* $\mathcal{P}_{v_X(Y)} \subseteq \mathrm{Ann}(\mathrm{Ann}(Y))$.

Proof. (1) Let $\alpha \in (\mathcal{P}_X : Y)$, so $\alpha \wedge y \in \mathcal{P}_X$ for all $y \in Y$. If $P \in r_X(Y)$, then $P \in X$ and $Y \nsubseteq P$, so there exists $y \in Y$ with $y \notin P$. Since $\alpha \wedge y \in P$, it follows that $\alpha \in P$. Thus $\alpha \in \mathcal{P}_{r_X(Y)}$, and hence $(\mathcal{P}_X : Y) \subseteq \mathcal{P}_{r_X(Y)}$.

Conversely, let $\alpha \in \mathcal{P}_{r_X(Y)}$ and $y \in Y$. Then $\alpha \in P$ for all $P \in r_X(Y)$, so $\alpha \wedge y \in P$ for all such P. Since $y \in P$ for all $P \in v_X(Y)$, it follows that $\alpha \wedge y \in P$ for all $P \in X$, hence $\alpha \in (\mathcal{P}_X : Y)$.

(2) By part (1), we have $\mathrm{Ann}(Y) = (\{0\} : Y) = \mathcal{P}_{r_X(Y)}$. Now observe

$$\mathcal{P}_{v_X(Y)} = \mathcal{P}_{X \setminus r_X(Y)} \subseteq \mathcal{P}_{X \setminus \overline{r_X(Y)}} = \mathcal{P}_{X \setminus v_X(\mathcal{P}_{r_X(Y)})} = \mathcal{P}_{X \setminus v_X(\mathrm{Ann}(Y))} = \mathcal{P}_{r_X(\mathrm{Ann}(Y))} = \mathrm{Ann}(\mathrm{Ann}(Y)).$$

□

Corollary 3.23. *Let $X = \mathrm{Min}(A)$ and $a \in A$. Then*

1. $\mathcal{P}_{r_X(Y)} = \mathrm{Ann}(Y)$ *for every* $Y \subseteq A$.
2. $\mathrm{Ann}(a)$ *is a* $maxv_X$*-ideal.*
3. $(v_X(a))^\circ = r_X(\mathrm{Ann}(a))$.
4. $v_X(a) = (v_X(a))^\circ$.

Proof. (1) Follows directly from Theorem 2.8(2) and Proposition 3.22.

(2) By Theorem 2.8, $\mathcal{P}_{r_X(a)} = \mathcal{P}_{v_X(\mathrm{Ann}(a))}$ and $\mathcal{P}_X = \{0\}$. By Proposition 3.22, $\mathcal{P}_{v_X(\mathrm{Ann}(a))} = \mathrm{Ann}(a)$, so $\mathrm{Ann}(a)$ is a $maxv_X$-ideal.

(3) From part (1)

$$v_X(\mathrm{Ann}(a)) = v_X(\mathcal{P}_{r_X(a)}) = \overline{r_X(a)} = \overline{\mathrm{Min}(A) \setminus v_X(a)} = \mathrm{Min}(A) \setminus (v_X(a))^\circ,$$

hence $r_X(\mathrm{Ann}(a)) = (v_X(a))^\circ$.

(4) Follows from part (3) and Theorem 2.8(1). □

Note: For any ideal I of A, we denote by I_{v_X} the intersection of all v_X-ideals containing I.

Proposition 3.24. *Let I and J be two ideals of A. Then*

1. *The smallest* v_X*-ideal containing* I *exists.*
2. $I_{v_X} = \{a \in A : \exists b \in I$ *such that* $v_X(b) \subseteq v_X(a)\}$.
3. $(I \cap J)_{v_X} = I_{v_X} \cap J_{v_X}$.
4. *If* $\mathcal{P}_X = \{0\}$ *and every element of* I *is a zero divisor, then* I *is contained in a proper* v_X*-ideal.*

Proof. (1) By definition, I_{v_X} is the smallest v_X-ideal containing I.

(2) Let $K = \{a \in A : \exists b \in I \text{ such that } v_X(b) \subseteq v_X(a)\}$. Clearly, $K \subseteq I_{v_X}$. We show that K is a v_X-ideal containing I. Let $a_1, a_2 \in K$ with $b_1, b_2 \in I$ such that $v_X(b_1) \subseteq v_X(a_1)$ and $v_X(b_2) \subseteq v_X(a_2)$. Then

$$v_X(b_1 \vee b_2) = v_X(b_1) \cap v_X(b_2) \subseteq v_X(a_1) \cap v_X(a_2) \subseteq v_X(a_1 \oplus a_2),$$

so $a_1 \oplus a_2 \in K$. If $\alpha \leq \beta$ and $\beta \in K$, then $v_X(\beta) \subseteq v_X(\alpha)$ and $v_X(b) \subseteq v_X(\beta)$ for some $b \in I$, hence $\alpha \in K$. Thus K is a v_X-ideal containing I, and by minimality, $I_{v_X} = K$.

(3) Let $\alpha \in I_{v_X} \cap J_{v_X}$, with $a \in I$ and $b \in J$ such that $v_X(a), v_X(b) \subseteq v_X(\alpha)$. Then $v_X(a \wedge b) = v_X(a) \cup v_X(b) \subseteq v_X(\alpha)$, so $\alpha \in (I \cap J)_{v_X}$. Hence $I_{v_X} \cap J_{v_X} \subseteq (I \cap J)_{v_X}$, and the reverse inclusion is obvious.

(4) It suffices to show every element of I_{v_X} is a zero divisor. Let $a \in I_{v_X}$, with $b \in I$ such that $v_X(b) \subseteq v_X(a)$. Then $\mathcal{P}_{r_X(b)} \subseteq \mathcal{P}_{r_X(a)}$, and by Proposition 3.22, $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)$. Since $\mathrm{Ann}(b) \neq \{0\}$, it follows that $\mathrm{Ann}(a) \neq \{0\}$, so a is a zero divisor. □

At the end of this section, the relationships between v_X-ideals and other types of ideals are illustrated in the following diagram.

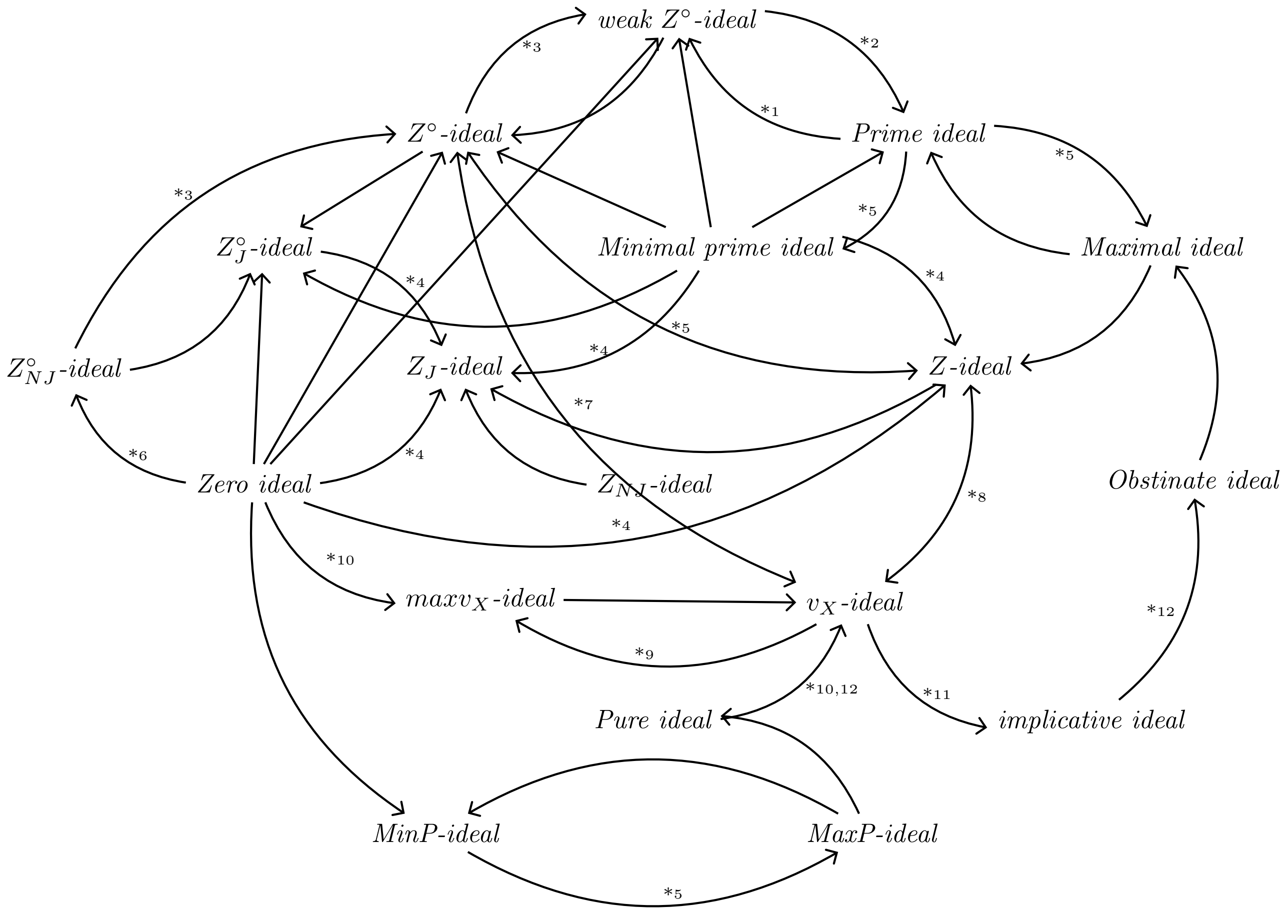


Figure 2

Figure 2: The relationships between v_X-ideals and other ideals
$*_1 :=$ Maximal weak Z°-ideal, $*_2 :=$ Included in complement A_0, $*_3 := MV$-chain
$*_4 :=$ Semisimple, $*_5 :=$ Hyperarchimedean $*_6 := J \neq \{0\}$.
$*_7 := X = Min(A), *_8 := X = Max(A), *_9 := X$ is a finite set.
$*_{10} := P_X = \{0\}, *_{11} := \textit{Included in } B(A), *_{12} := \textit{Maximal ideal}$

Conclusion and future work

This study examined a specific class of Zariski-closed sets and demonstrated that they form a lattice under set inclusion, equipped with a filter possessing distinctive properties. One such property is the existence of a one-to-one correspondence between principal ideals and these filters.

The investigation into v_X-ideals revealed that the intersection of any collection of v_X-ideals results in another v_X-ideal. Moreover, if the union of such ideals forms an ideal, it also qualifies as a v_X-ideal.

It was concluded that if every principal ideal in an MV-algebra A is a maximal v_X-ideal, then every proper ideal of A is also a v_X-ideal. Additionally, it was proven that every maximal v_X-ideal is a v_X-ideal, and in finite MV-algebras, these two notions coincide.

Further analysis of the relationship between v_X-ideals and other types of ideals showed that if the zero ideal is a v_X-ideal, then every maximal and pure ideal is also a v_X-ideal. Moreover, if every element of a v_X-ideal satisfies the Boolean property, then the ideal is an implicative ideal.

In future work, similar investigations can be conducted on open sets in the inverse topology, which represents a special case of the Zariski topology. Ideals can be defined based on these sets, and it is anticipated that further research in this direction will contribute to enriching the diagram presented in this paper by adding new connections and completing its structural framework.

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