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Original paper

Strength of connectedness in fuzzy bunch graphs and fuzzy bunch hypergraphs: A new approach

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Abstract

The existing strength of connectedness in fuzzy graph theory is a max–min quantity. According to this definition, the strength of a path is the membership of its weakest edge, and the connectedness between two vertices is the maximum such bottleneck over all paths. That definition is exact for systems in which the weakest edge is the only controlling factor, but it is too rigid when cumulative route quality matters as well. In this paper we adopt a new notion in which the strength of a simple path is a convex combination of its bottleneck and its average edge membership. The new framework defined in this paper for the strength of connectedness is successfully applicable to systems where the classical bottleneck constraint is significant, as well as to systems where the cumulative effects of all edge constraints are more significant than just the bottleneck constraint. Capacity or bandwidth constraints in a network rely only on the weakest (bottleneck) edge, whereas speed, latency, or smoothness constraints have cumulative effects on the entire path from the source to the destination hub in a network. We develop the corresponding theory for fuzzy bunch graphs and fuzzy bunch hypergraphs, that is, grouped fuzzy structures in which vertices are partitioned into bunches and higher-order relations may occur across bunches.

Keywords: Fuzzy graph, fuzzy bunch graph, fuzzy bunch hypergraph, connectedness, path strength.

1 Introduction

Fuzzy graph theory combines graph structure with membership graded relations [12]. The resulting concept of path strength is one of the central structural notions in the subject. If P is a path, the existing definition of strength of P is the minimum membership of its edges, and the strength of connectedness between two vertices is the maximum such bottleneck over all paths joining them. In 2003, Bhutani and Rosenfeld [4] have studied strong arcs in fuzzy graphs. Based on this definition, Mathew and Sunitha [10] categorized arcs in fuzzy graphs. In [6, 7], Binu et al. have introduced connectivity index and studied connectivity status of fuzzy graphs. This max–min concept is computationally useful, and entirely natural when the weakest step controls everything. Recent developments in graph theory have introduced the concept of bunch graphs [13] to handle complex data structures. Kundu et al. [8] explored the fundamental properties of fuzzy bunch graphs, providing a theoretical framework for their application in intelligent systems.

In terms of practical applications, Bhadoria et al. [3] utilized bunch graphs combined with auto-encoders to achieve dimensionality reduction, specifically for character recognition tasks. Furthermore, the versatility of this structure was

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demonstrated in social contexts by Samanta et al. [13], who analyzed the dynamics of competition and cooperation within COVID-19 research networks.

Many empirical systems, however, are not controlled by a single weakest step. A transport route may contain one narrow segment but otherwise consist of excellent segments; an information route may contain one moderate-quality channel but traverse a collection of otherwise highly reliable links; a collaboration route may involve one weak thematic overlap inside a chain of strong overlaps. In such settings, a path with one moderate bottleneck and many strong edges may be more useful than another path whose every edge is merely average. The classical max–min definition does not distinguish these cases, because it discards all information except the smallest edge membership.

The present paper studies a mathematically conservative way to enrich the definition without abandoning the classical one. For a priority weight parameter $\omega \in [0, 1]$, the *Convex Bottleneck-Average* (CBA) path strength gives weight ω to the weakest edge and weight $1 - \omega$ to the average edge membership. At $\omega = 1$ the existing definition comes true. At intermediate values of ω, the model is still bottleneck-sensitive but no longer blind to aggregate route quality.

A second theme of the paper is grouped network structure. In many datasets the vertex set is grouped and substructured like researchers belong to institutes, channels belong to anatomical regions, employees belong to teams, sensors belong to subnetworks, etc. Formally, the connectivity between a source vertex in a source local block and a destination vertex in a destination local block passes through bunches b_i and b_j in a fuzzy bunch graph $H \odot (F_1, F_2, \ldots, F_p)$ with p bunches. Taking higher-order connections into consideration, the fuzzy bunch hypergraph is being introduced, in which fuzzy hyperedges may be local to one bunch or transversal across several bunches.

Recent studies show that many real networks require higher-order models rather than only pairwise links [9]. Hypergraphs and simplicial complexes are useful for this purpose, but they may lead to different structural and dynamical conclusions [1, 2, 15]. Such differences are important in applications such as synchronization and neuroscience [5, 16]. However, a fuzzy framework for grouped higher-order systems is still needed. This paper fills that gap by introducing CBA connectedness for fuzzy bunch graphs and fuzzy bunch hypergraphs, where path strength balances bottleneck control with average route quality.

The main contributions are given as follows.

- CBA path strength and CBA strength of connectedness for fuzzy graphs using a bottleneck–average convex combination are defined, which reduces to existing max-min notion at $\omega = 1$.
- We proved that for a fixed vertex pair the new connectedness is a convex, nonincreasing function of ω.
- CBA strength of connectedness in fuzzy bunch graphs and fuzzy bunch hypergraphs are investigated.
- We applied the theory to a real Scopus dataset of mathematics institutes, using graph-level network decomposition.

The rest of the paper is organized as follows. Section 2 introduces the new connectedness notion on a fuzzy graph and records its basic properties. Section 3 discusses the grouped fuzzy bunch graph model, while Section 3.2 develops the corresponding CBA connectedness theory. Section 4 introduces fuzzy bunch hypergraphs, and Section 4.1 develops the corresponding CBA strength of connectedness in fuzzy bunch hypergraphs. Section 5 and Section 6 contain the Scopus applications, and Section 7 is the conclusion of the paper.

2 New connectedness notion in fuzzy graphs

A fuzzy graph as introduced by Rosenfeld [12] is a pair $\widetilde{G} = (\sigma, \mu)$ on a finite vertex set V, where $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices and $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation satisfying

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y), \qquad \mu(x, x) = 0.$$

In many applications, σ is fixed to 1, thereby concentrating the uncertainty on the edges. We keep the general vertex-membership function because it is harmless and occasionally convenient.

A *simple path* from x to y is a sequence

$$P : x = x_0, x_1, \ldots, x_k = y,$$

such that all vertices $x_0, \ldots, x_k$ are distinct and

$$\mu(x_{r-1}, x_r) > 0 \qquad (1 \leq r \leq k).$$

We write $\mathcal{P}_{\mathrm{s}}(x, y)$ for the set of all simple x–y paths, where no additional edge weights are present between any two same vertices. Since the graph is finite, $\mathcal{P}_{\mathrm{s}}(x, y)$ is finite.

The existing path strength and strength of connectedness [4] are defined as

$$\mathrm{str}_{\min}(P) = \min_{1 \le r \le k} \mu(x_{r-1}, x_r), \qquad \mathrm{CONN}_{\min}(x, y) = \max_{P \in \mathcal{P}_{\mathrm{s}}(x,y)} \mathrm{str}_{\min}(P).$$

Consider a communication network in which routing decisions must be optimized for efficiency and reliability. network in which routing decisions must be optimized for efficiency and reliability. If there are two potential paths between a source and a destination router—one consisting of only 2 hops with a bottleneck membership value of 0.375, and another consisting of 8 hops with the exact same bottleneck value of 0.375, the existing max-min paradigm evaluates both paths as strictly equal in strength. In physical reality, this mathematical equivalence is highly inaccurate. In telecommunications and other kinetic networks, cumulative friction, such as latency or data transmission delay, is a critical factor. The 8-hop path introduces significantly higher cumulative latency, making it practically inferior to the shorter path. To address this fundamental limitation, this paper introduces a novel, composite framework for the strength of connectedness.

Here is the new definition as proposed:

Definition 2.1 (CBA path strength)**.** Let $P = x_0, \dots, x_k$ be a simple path in a fuzzy graph, and let

$$w_r = \mu(x_{r-1}, x_r) \qquad (1 \le r \le k).$$

For $\omega \in [0, 1]$, the *CBA path strength* of P is

$$\mathrm{str}_\omega(P) = \omega \min_{1 \le r \le k} w_r + (1 - \omega) \frac{1}{k} \sum_{r=1}^{k} w_r.$$

The *CBA strength of connectedness* between x and y is

$$\mathrm{CONN}_\omega(x, y) = \max_{P \in \mathcal{P}_{\mathrm{s}}(x,y)} \mathrm{str}_\omega(P).$$

Any path attaining the maximum is called a *CBA strongest path* for the parameter ω.

The two extreme values of ω have transparent meanings. When $\omega = 1$, only the weakest edge matters and one recovers the existing theory. When $\omega = 0$, the score is the maximum average edge membership over all simple paths. For intermediate ω, the path score is a compromise between the weakest step and the aggregate route quality.

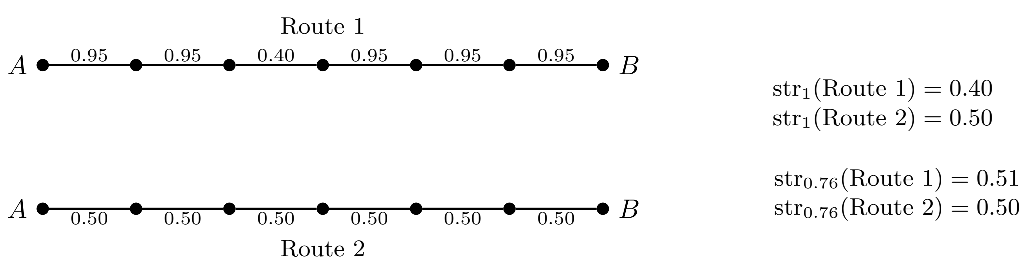


Figure 1: Comparison between two possible routes where CBA strength definition is significant.

The CBA definition distinguishes the purely bottleneck view from the bottleneck–average view. Under the classical rule $\omega = 1$, Route 2 wins because its weakest edge is 0.50, whereas Route 1 has a single edge of weight 0.40. Under the CBA rule with $\omega = 0.76$, Route 1 becomes stronger because its overall route quality is higher.

Proposition 2.2. *For every fuzzy graph and every pair $x, y \in V$,*

$$\mathrm{CONN}_1(x, y) = \mathrm{CONN}_{\min}(x, y).$$

Moreover,

$$\mathrm{CONN}_0(x, y) = \max_{P \in \mathcal{P}_{\mathrm{s}}(x,y)} \frac{1}{|P|} \sum_{e \in P} \mu(e).$$

Proof. At $\omega = 1$, $\mathrm{str}_\omega(P)$ reduces to the bottleneck $\min_{e\in P}\mu(e)$, hence $\mathrm{CONN}_1 = \mathrm{CONN}_{\min}$. At $\omega = 0$, the bottleneck term disappears and only the path average remains. ☐

Proposition 2.3. *Fix $x, y \in V$. The function*

$$\omega \longmapsto \mathrm{CONN}_\omega(x, y),$$

is convex and nonincreasing on $[0, 1]$.

Proof. For a fixed simple path P,

$$\mathrm{str}_\omega(P) = \omega m(P) + (1-\omega)a(P) = a(P) - \omega\big(a(P) - m(P)\big),$$

where

$$m(P) = \min_{e\in P}\mu(e), \qquad a(P) = \frac{1}{|P|}\sum_{e\in P}\mu(e).$$

Since $m(P) \le a(P)$, the slope of $\mathrm{str}_\omega(P)$ is $m(P) - a(P) \le 0$. Hence each path-score function is nonincreasing in ω. The CBA connectedness $\mathrm{CONN}_\omega(x, y)$ is the maximum of finitely many such CBA strengths of paths, so it is convex and nonincreasing. ☐

2.1 Fuzzy hypergraphs and pairwise projections

A *fuzzy hypergraph* newly defined by Pramanik et al. [11] is a triple

$$H = (V, \sigma, \eta),$$

where V is a nonempty vertex set, $\sigma : V \to [0, 1]$ assigns vertex membership grades, and $\eta : \mathcal{E} \to [0, 1]$ assigns hyperedge membership grades to a family $\mathcal{E} = \mathcal{P}(V) \setminus \{\emptyset\}$ of nonempty vertex subsets (hyperedges), such that

$$\eta(E) \le \min_{x\in E}\sigma(x), \qquad \forall\, E \in \mathcal{E}.$$

This constraint ensures that a hyperedge membership cannot exceed the weakest membership of its incident vertices, mirroring the standard fuzzy-graph bound for edges.

Given $\widetilde{\mathcal{H}}$, its *pairwise projection* is the fuzzy graph

$$\Gamma(\widetilde{\mathcal{H}}) = (V, \sigma, \lambda),$$

with

$$\lambda(x, y) = \max_{x,y\in E, E\in\mathcal{E}}\{\eta(E)\} \qquad (x \ne y).$$

3 Fuzzy bunch graphs

We now introduce the grouped pairwise model.

3.1 Definition and basic structure

Definition 3.1 (Bunch graph)**.** Let H be a graph with vertex set $B = \{b_1, \ldots, b_p\}$. For each $i \in \{1, \ldots, p\}$ let F_i be a graph on a vertex set S_i, where the sets $S_1, \ldots, S_p$ are pairwise disjoint and disjoint from B. The *bunch graph* $\mathcal{B} = H \odot (F_1, \ldots, F_p)$ is the graph with vertex set $V(\mathcal{B}) = B \cup \bigcup_{i=1}^{p} S_i$ and edge set $\mathcal{E}(\mathcal{B}) = \mathcal{E}(H) \cup \bigcup_{i=1}^{p}\mathcal{E}(F_i) \cup \bigcup_{i=1}^{p}\{b_i s : s \in S_i\}$. The vertices b_i are the *bunch nodes* and the vertices of S_i are the *simple vertices of bunch* i and $b_i s$ are called the star edges of the graph $\mathcal{B}$.

The work of Samanta et al. [14] defines all the meanings of symbols related to the definition of bunch graphs.

Definition 3.2 (Fuzzy bunch graph)**.** A *fuzzy bunch graph* is a fuzzy graph $\widetilde{G} = (V, \sigma, \mu)$ on the underlying bunch graph $\mathcal{B} = H \odot (F_1, \ldots, F_p)$ such that

$$\mu(x, y) > 0,\ x \in C_i,\ y \in C_j,\ i \neq j \quad \Longrightarrow \quad x = b_i,\ y = b_j.$$

where $C_i = S_i \cup \{b_i\}$ for all $i = 1, 2, \cdots, p$. The induced fuzzy graph on C_i is denoted by $\widetilde{G}[C_i] = F_i$. The quotient fuzzy graph $\mathcal{Q}(\widetilde{G})$ is the fuzzy graph on vertex set $[m] = \{1, \ldots, m\}$ with

$$\widehat{\sigma}(i) = \sigma(b_i), \qquad \widehat{\mu}(i, j) = \mu(b_i, b_j) \quad (i \neq j).$$

Example 3.3. An example of a fuzzy bunch graph is shown in Figure 2. For $\omega = 0.8$, the highlighted route from $x \in C_1$ to $z \in C_3$ through b_2 has score $0.8 \times 0.55 + 0.2 \times 0.70 = 0.58$.

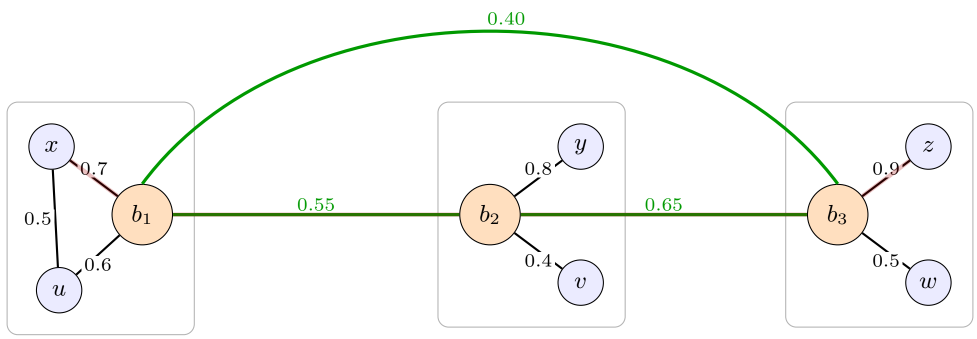


Figure 2: An example of a fuzzy bunch graph.

3.2 CBA strength of connectedness in fuzzy bunch graphs

As we have discussed earlier in this literature, for strength of connectedness in a fuzzy graph, the bottleneck is not enough. We must also record total weight and length.

Definition 3.4 (Path CBA Triplet)**.** Let P be a simple path in a collection of all paths $\mathcal{P}_{\mathrm{s}}(\widetilde{G}; x, y)$ from source node x to destination node y in a fuzzy graph. Its *CBA triplet* is the triple

$$\pi(P) = \big(s(P), t(P), \ell(P)\big),$$

where

$$s(P) = \min_{e \in P} \mu(e), \qquad t(P) = \sum_{e \in P} \mu(e), \qquad \ell(P) = |P|.$$

Then

$$\mathrm{str}_\omega(P) = \omega \cdot s(P) + (1 - \omega) \frac{t(P)}{\ell(P)}.$$

Let $x \in C_i$ and $y \in C_j$, $i \neq j$. We define the three CBA Triplets as follows:

$$\Pi_i(x) = \{\pi(P) : P \in \mathcal{P}_{\mathrm{s}}(\widetilde{G}[C_i]; x, b_i)\},$$

$$\Pi_{ij} = \{\pi(Q) : Q \in \mathcal{P}_{\mathrm{s}}(\mathcal{Q}(\widetilde{G}); b_i, b_j)\},$$

$$\Pi_j(y) = \{\pi(R) : R \in \mathcal{P}_{\mathrm{s}}(\widetilde{G}[C_j]; b_j, y)\}.$$

If $p_1 = (s_1, t_1, \ell_1) \in \Pi_i(x)$, $p_2 = (s_2, t_2, \ell_2) \in \Pi_{ij}$ and $p_3 = (s_3, t_3, \ell_3) \in \Pi_j(y)$, write

$$\Phi_\omega(p_1, p_2, p_3) = \omega \min\{s_1, s_2, s_3\} + (1 - \omega) \frac{t_1 + t_2 + t_3}{\ell_1 + \ell_2 + \ell_3}.$$

Theorem 3.5. *Let $\widetilde{G}$ be a fuzzy bunch graph, let $x \in C_i$, $y \in C_j$, and assume $i \neq j$. Then*

$$\mathrm{CONN}_\omega(x, y) = \max_{p_1 \in \Pi_i(x),\, p_2 \in \Pi_{ij},\, p_3 \in \Pi_j(y)} \Phi_\omega(p_1, p_2, p_3).$$

Proof. Let

$$P : x = v_0, v_1, \ldots, v_k = y,$$

be a simple path from x to y. Because $x \in C_i$ and $y \in C_j$ with $i \neq j$, the path must leave C_i and later enter C_j. By the defining property of a bunch graph, the only way to leave C_i is through the distinguished vertex b_i; similarly, the only way to enter C_j from another bunch is through b_j. Since P is simple, it cannot leave a bunch and then re-enter it without repeating the corresponding distinguished vertex. Hence P decomposes uniquely into three contiguous simple segments:

1. a local path P_1 from x to b_i in $\widetilde{G}[C_i]$;
2. a core path P_2 from b_i to b_j in the quotient graph $\mathcal{Q}(\widetilde{G})$;
3. a local path P_3 from b_j to y in $\widetilde{G}[C_j]$.

Write

$$\pi(P_r) = (s_r, t_r, \ell_r) \qquad (r = 1, 2, 3).$$

Then the concatenated path $P = P_1 \cup P_2 \cup P_3$ has

$$s(P) = \min\{s_1, s_2, s_3\}, \qquad t(P) = t_1 + t_2 + t_3, \qquad \ell(P) = \ell_1 + \ell_2 + \ell_3.$$

Therefore

$$\mathrm{str}_\omega(P) = \Phi_\omega\big(\pi(P_1), \pi(P_2), \pi(P_3)\big).$$

Since every simple x–y path determines exactly one triple of admissible profiles, we obtain

$$\mathrm{CONN}_\omega(x, y) \leq \max_{p_1 \in \Pi_i(x),\, p_2 \in \Pi_{ij},\, p_3 \in \Pi_j(y)} \Phi_\omega(p_1, p_2, p_3).$$

Conversely, pick arbitrary CBA triplets $p_1 \in \Pi_i(x)$, $p_2 \in \Pi_{ij}$, $p_3 \in \Pi_j(y)$. By definition, there exist simple paths P_1, P_2, P_3 realizing these triplet values. Their concatenation is again a simple x–y path because the local segments lie in disjoint bunches and the quotient segment uses only distinguished vertices. Its CBA score is exactly $\Phi_\omega(p_1, p_2, p_3)$. Therefore the right-hand side is bounded above by $\mathrm{CONN}_\omega(x, y)$. The two bounds give equality. □

Corollary 3.6. *For $x \in C_i$, $y \in C_j$, $i \neq j$, define*

$$\alpha_i(x) = \max_{(s,t,\ell) \in \Pi_i(x)} s, \qquad K_{ij} = \max_{(s,t,\ell) \in \Pi_{ij}} s, \qquad \alpha_j(y) = \max_{(s,t,\ell) \in \Pi_j(y)} s.$$

Then

$$\mathrm{CONN}_1(x, y) = \min\{\alpha_i(x), K_{ij}, \alpha_j(y)\}.$$

Proof. Set $\omega = 1$ in Theorem 3.5. The optimization becomes

$$\max_{p_1, p_2, p_3} \min\{s_1, s_2, s_3\},$$

which equals the minimum of the independently attainable maxima $\alpha_i(x), K_{ij}, \alpha_j(y)$. □

Corollary 3.7. *Assume that every local bunch is a star centered at b_i, and let $p_i(u) = \mu(u, b_i)$ for $u \in C_i$. Then for $f \in C_i$, $g \in C_j$, $i \neq j$,*

$$\mathrm{CONN}_\omega(f, g) = \max_{Q \in \mathcal{P}_s(\mathcal{Q}(\widetilde{G}); b_i, b_j)} \left[\omega \min\{p_i(f), s(Q), p_j(g)\} + (1 - \omega) \frac{p_i(f) + t(Q) + p_j(g)}{|Q| + 2} \right].$$

Proof. In a star local graph the only simple path from f to b_i is the single edge $f b_i$, so its CBA triplet is $(p_i(f), p_i(f), 1)$; similarly on the target side. Substitute these singleton CBA triplet sets into Theorem 3.5. □

When evaluating the CBA strength of connectedness in a fuzzy graph, if a path has no contributing role, it is convenient to identify and eliminate it beforehand for subsequent calculations. In this context, finding such paths is the challenging part. Here is the way to find those paths.

Definition 3.8 (Dominating Path)**.** Let $p = (s, t, \ell)$ and $q = (s', t', \ell')$ be CBA triplets. We say that p *dominates* q or q is dominated by p, written $q \preceq p$, if

$$s \geq s', \qquad t \geq t', \qquad \ell \leq \ell'.$$

The intuition is clear. The dominated path has worse bottleneck, smaller total weight, and greater length. Hence it will not contribute in calculation of CBA strength of connectedness.

Theorem 3.9. *Let $p = (s, t, \ell)$ and $q = (s', t', \ell')$ be two CBA path triplets in one of the sets $\Pi_i(x)$, Π_{ij}, or $\Pi_j(y)$. If $q \preceq p$, then for every pair of external extensions having CBA path triplets $r = (s_r, t_r, \ell_r)$ and $u = (s_u, t_u, \ell_u)$,*

$$\Phi_\omega(p, r, u) \geq \Phi_\omega(q, r, u) \qquad \textit{for every } \omega \in [0, 1].$$

Consequently, dominated profiles may be discarded without changing $\mathrm{CONN}_\omega(x, y)$.

Proof. Since $s \geq s'$,

$$\min\{s, s_r, s_u\} \geq \min\{s', s_r, s_u\}.$$

It remains to compare the average terms. Set $T = t_r + t_u$ and $L = \ell_r + \ell_u$. We must show

$$\frac{t + T}{\ell + L} \geq \frac{t' + T}{\ell' + L}.$$

After cross-multiplication,

$$(t + T)(\ell' + L) - (t' + T)(\ell + L) = (t\ell' - t'\ell) + L(t - t') + T(\ell' - \ell).$$

Because $t \geq t'$ and $\ell' \geq \ell$, all three terms on the right are nonnegative:

$$t\ell' - t'\ell = t(\ell' - \ell) + \ell(t - t') \geq 0.$$

Hence the average term for p is at least the average term for q. Combining the bottleneck and average parts proves the inequality for Φ_ω. □

4 Fuzzy bunch hypergraphs

Definition 4.1 (Fuzzy bunch hypergraph)**.** Let $\widetilde{G} = (V, \sigma, \mu)$ be a fuzzy bunch graph on the underlying bunch graph $\mathcal{B} = H \odot (F_1, \ldots, F_p)$. A *fuzzy bunch hypergraph* (Figure 3) is a fuzzy hypergraph $\widetilde{\mathcal{H}} = (V, \sigma, \theta)$ such that θ satisfies the following:

$$\theta(E) > 0 \Longrightarrow \text{if } x \in C_i, y \in C_j (i \neq j) \text{ and } x, y \in E \text{ then } b_i, b_j \in E.$$

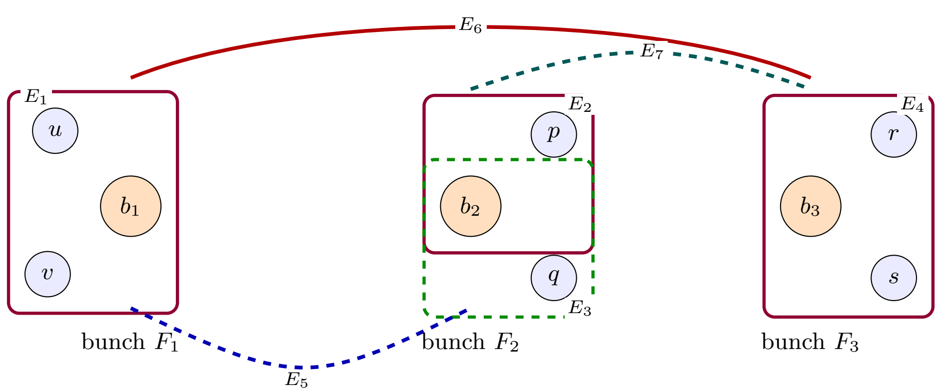


Figure 3: A fuzzy bunch hypergraph.

4.1 CBA strength of connectedness in fuzzy bunch hypergraphs

Definition 4.2 (CBA hyperpath strength)**.** A *simple hyperpath* from x to y in $\tilde{\mathcal{H}}$ is an alternating sequence of vertices and hyperedges

$$P : x = v_1, E_1, v_2, E_2, \ldots, v_k, E_k, v_{k+1} = y,$$

such that $v_r, v_{r+1} \in E_r$ for all $r = 1, 2, \ldots, k$.

For each step, define $\theta(E_r) > 0$, and let

$$\rho_r(P) = \theta(E_r) \quad (1 \leq r \leq k).$$

Then

$$\mathrm{str}_\omega(P) = \omega \min_{1 \leq r \leq k} \rho_r(P) + (1-\omega)\frac{1}{k}\sum_{r=1}^{k} \rho_r(P).$$

The CBA strength of connectedness in $\tilde{\mathcal{H}}$ is

$$\mathrm{CONN}_\omega^{\tilde{\mathcal{H}}}(x, y) = \max\{\mathrm{str}_\omega(P) : P \text{ is a simple hyperpath from } x \text{ to } y\}.$$

This is the natural higher-order network of the graph definition. Each step uses a hyperedge to move from one vertex to another, and the step weight is the weaker participation grade of the two vertices inside that hyperedge. Here, the hyperpath is called *simple* in the sense that one may use so many hyperedges containing the same set of vertices.

Theorem 4.3. *For every fuzzy hypergraph $\widetilde{\mathcal{H}}$, every $\omega \in [0,1]$, and every $x, y \in V$,*

$$\mathrm{CONN}_\omega^{\widetilde{\mathcal{H}}}(x, y) = \mathrm{CONN}_\omega^{\Gamma(\widetilde{\mathcal{H}})}(x, y).$$

Proof. We prove the two inequalities separately.

Hypergraph to graph. Let

$$P : x = v_1, E_1, v_2, E_2, \ldots, v_k, E_k, v_{k+1} = y,$$

be a simple hyperpath in $\tilde{\mathcal{H}}$. For each step, set $\rho_r = \theta(E_r)$. By definition of the pairwise projection, for the consecutive vertices $v_r, v_{r+1} \in E_r$, we have

$$\lambda(v_r, v_{r+1}) \geq \theta(E_r) = \rho_r \quad (1 \leq r \leq k).$$

Hence, the graph path $v_1, v_2, \ldots, v_{k+1}$ in $\Gamma(\tilde{\mathcal{H}})$ consists of exactly k edges, with a bottleneck at least $\min_r \rho_r$ and an average edge weight of at least $k^{-1}\sum_r \rho_r$. Therefore,

$$\mathrm{str}_\omega^{\Gamma(\tilde{\mathcal{H}})}(v_1, \ldots, v_{k+1}) \geq \mathrm{str}_\omega^{\tilde{\mathcal{H}}}(P).$$

Taking the maximum over all simple hyperpaths gives

$$\mathrm{CONN}_\omega^{\Gamma(\tilde{\mathcal{H}})}(x, y) \geq \mathrm{CONN}_\omega^{\tilde{\mathcal{H}}}(x, y).$$

Graph to hypergraph. Let $Q : x = v_1, v_2, \ldots, v_{k+1} = y$ be a simple path in $\Gamma(\tilde{\mathcal{H}})$. For each edge $v_r v_{r+1}$ $(1 \leq r \leq k)$, choose a hyperedge $E_r \in \mathcal{E}$ that attains the maximum in the definition of $\lambda(v_r, v_{r+1})$; this is possible because $\mathcal{E}$ is finite. Then

$$\lambda(v_r, v_{r+1}) = \theta(E_r).$$

Thus

$$x = v_1, E_1, v_2, \ldots, v_k, E_k, v_{k+1} = y,$$

is a simple hyperpath in $\tilde{\mathcal{H}}$ whose step weights are exactly the graph edge weights of Q. Hence

$$\mathrm{str}_\omega^{\tilde{\mathcal{H}}}(x, E_1, \ldots, E_k, y) = \mathrm{str}_\omega^{\Gamma(\tilde{\mathcal{H}})}(Q).$$

Taking the maximum over all simple graph paths yields

$$\mathrm{CONN}_\omega^{\tilde{\mathcal{H}}}(x, y) \geq \mathrm{CONN}_\omega^{\Gamma(\tilde{\mathcal{H}})}(x, y).$$

Combining the two inequalities proves the theorem. □

5 Application of CBA strength of connectedness

5.1 Data construction

We use the Scopus-derived dataset. It contains nine institutes and 484 faculty records with complete publication, citation, and h-index information. The institutes become bunches. The local fuzzy graph F_i is a star. For a faculty member $f \in F_i$, the edge weight $p_i(f)$ is the prominence score already defined from normalized publication, citation, and h-index ranks. Thus every local profile is trivial:

$$\pi(f, b_i) = \big(p_i(f),\, p_i(f),\, 1\big).$$

For the CBA analysis we take $\omega = 0.8$, so that the bottleneck remains dominant, while the average path quality still contributes visibly.

5.2 Local prominence structure

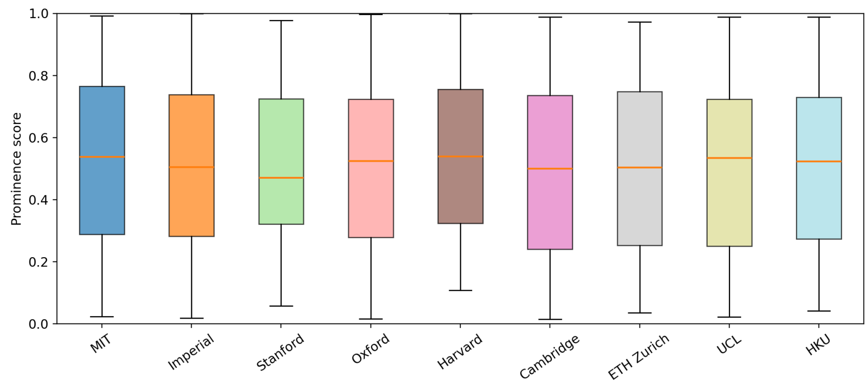


Figure 4: Distribution of faculty prominence scores by institute.

The mean prominence values remain close to 0.5 (Figure 4) because the scores are percentile-based within each institute, but the spreads differ. This matters in the CBA exactly as it did in the classical one: the local edge weights are the first and last terms that enter every cross-institute faculty path.

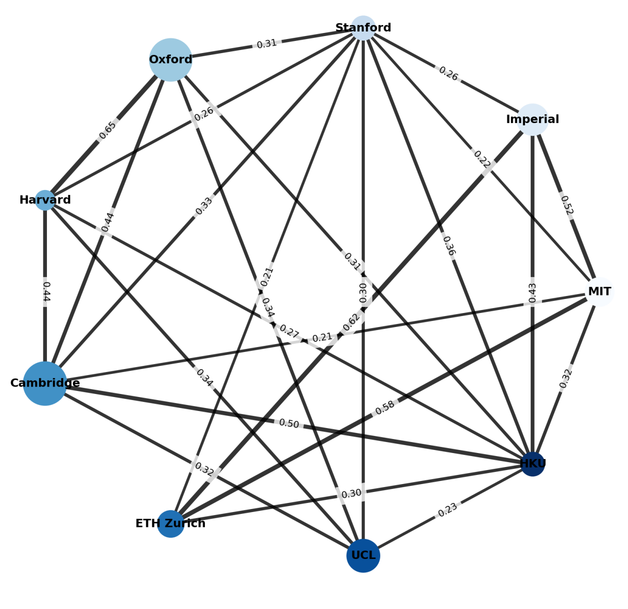


Figure 5: Derived fuzzy graph from fuzzy hypergraph

The derived graph (Figure 5) is only the starting point. CBA strength of connectedness between institute cores is obtained by maximizing the CBA path score over all simple core paths. The resulting matrix is shown in Figure 6.

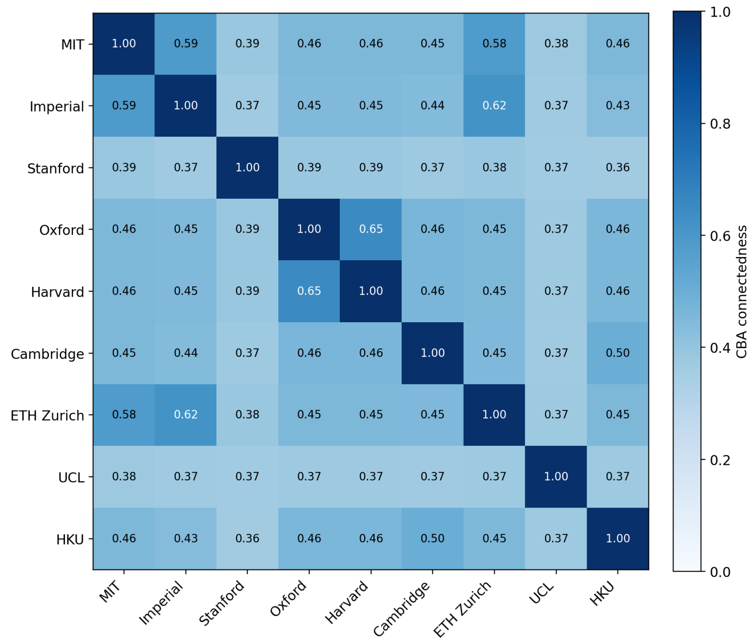


Figure 6: CBA institute-core connectedness matrix on the quotient graph for $\omega = 0.8$.

The off-diagonal mean direct edge weight is 0.28. Under classical max–min composition the mean institute-core connectedness rises to 0.42; under the CBA definition it rises further to 0.44. This is the expected direction: the CBA rule never penalizes a path relative to its bottleneck, because the average term can only improve the score.

The strongest institute pairs are listed in Table 1. The first five remain the same as under the classical rule, but several medium-strength pairs gain because they can exploit long routes whose worst edge is moderate but whose average edge quality is high.

Table 1: Strongest institute-core pairs at $\omega = 0.8$.

Institute A	Institute B	CBA connectedness
Oxford	Harvard	0.65
Imperial	ETH Zurich	0.62
MIT	Imperial	0.59
MIT	ETH Zurich	0.58
Cambridge	HKU	0.50
Oxford	Cambridge	0.46
Harvard	Cambridge	0.46
Oxford	HKU	0.46

5.3 Comparison with direct and classical values

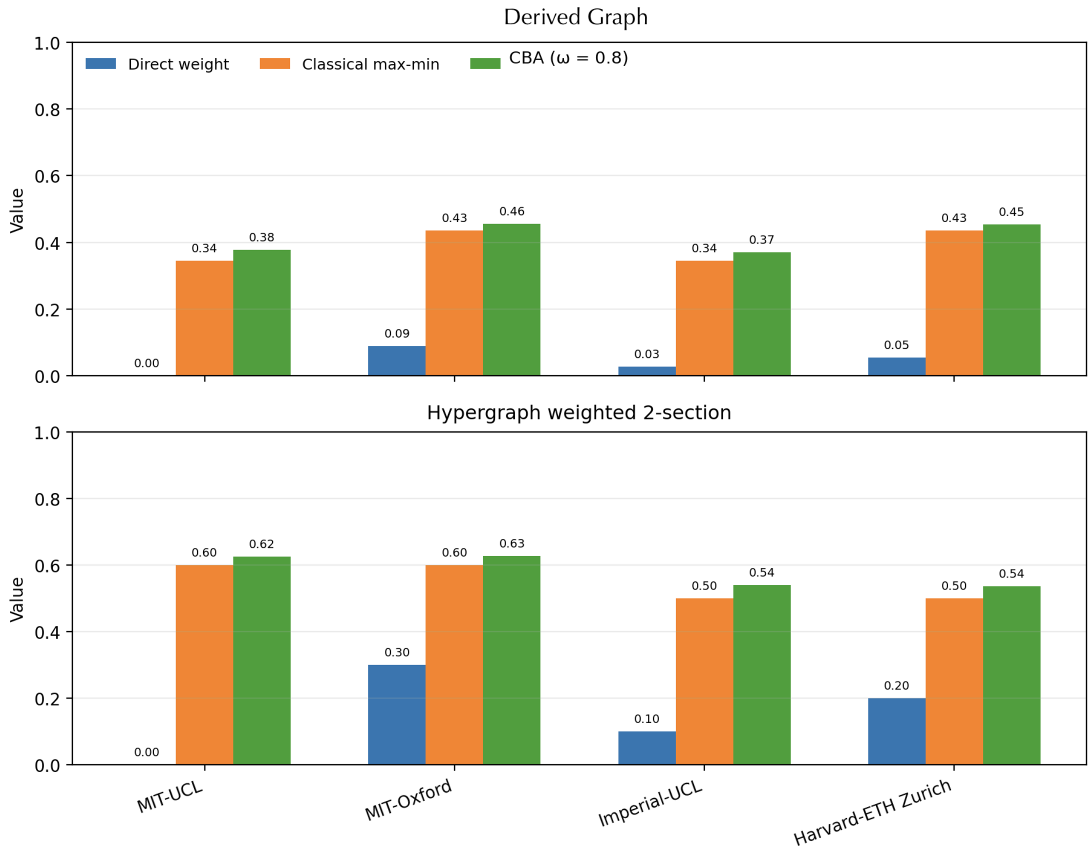


Figure 7: Selected institute pairs comparing direct edge weight, classical max–min connectedness, and CBA connectedness.

For the derived graph in Figure 7, the pair MIT–UCL is a representative example. Its direct edge weight is 0, because the selected restricted-topic sets do not overlap directly. Its classical connectedness is 0.34, and its CBA connectedness rises to 0.38. The CBA strongest path is

$$\text{MIT} \to \text{ETH Zurich} \to \text{Imperial} \to \text{HKU} \to \text{Cambridge} \to \text{Oxford} \to \text{Harvard} \to \text{UCL},$$

with bottleneck 0.34, average edge weight 0.51, and seven steps. Under the classical rule only the bottleneck 0.34 is retained. Under the CBA rule with $\omega = 0.8$,

$$0.8 \times 0.34 + 0.2 \times 0.51 = 0.37.$$

5.4 Sensitivity with respect to ω

The sensitivity curves in Figure 8 illustrate two theoretical points. First, every curve is nonincreasing, exactly as predicted by Theorem 2.3. Second, the gains from moving away from $\omega = 1$ are pair-dependent.

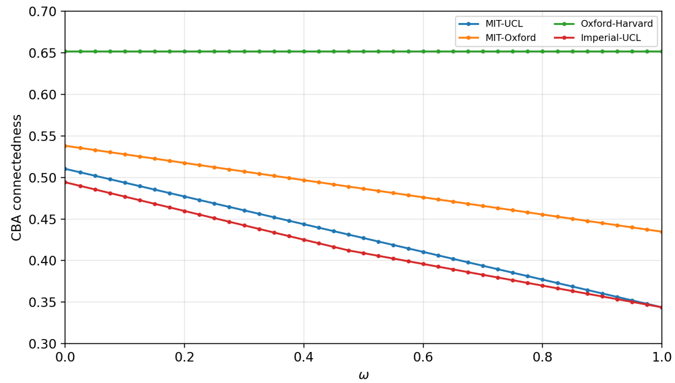


Figure 8: CBA institute-core connectedness as a function of ω for four representative pairs. The curves are nonincreasing, in agreement with Theorem 2.3.

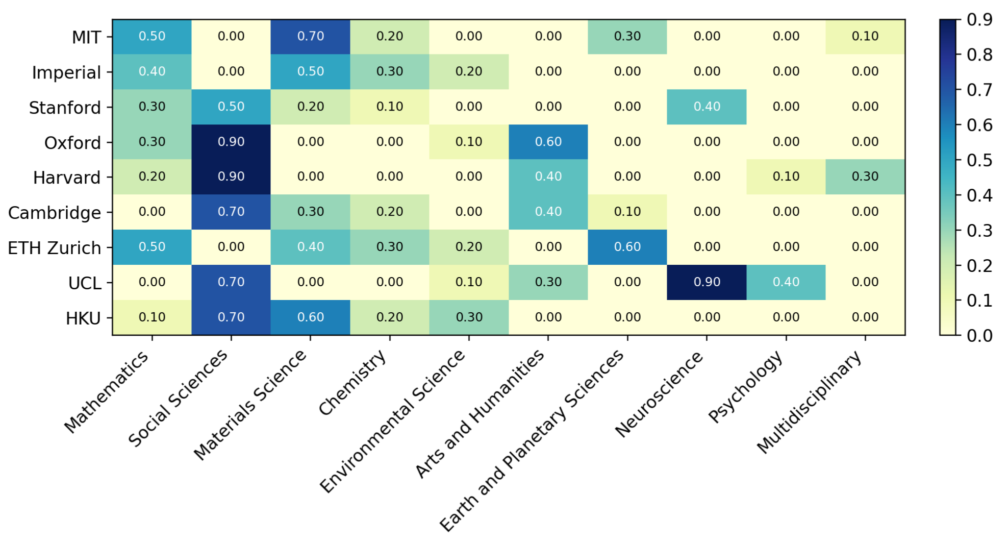
Pairs such as MIT–UCL and MIT–Oxford gain substantially because their strongest indirect routes have averages far above their bottlenecks. In contrast, pairs such as Oxford–Harvard are already realized by very short, high-quality routes, so the CBA adjustment is modest.

6 Application of dealing relationships on topics of universities as fuzzy bunch hypergraph

6.1 Higher-order construction

The vertex set is the same nine-bunch family used in the graph application. In each university, topic-wise membership weights are shown in Figure 9. We now add higher-order structure in two ways.

First, each institute carries local hyperedges extracted from the faculty data: high-h-index, high-document, high-citation, and triple-elite groups. Their sizes are summarized in Figure 10. Second, each selected topic defines a transversal fuzzy hyperedge on institute cores, with the membership of a core equal to the topic weight in the cleaned topic-membership matrix.



	Mathematics	Social Sciences	Materials Science	Chemistry	Environmental Science	Arts and Humanities	Earth and Planetary Sciences	Neuroscience	Psychology	Multidisciplinary
MIT	0.50	0.00	0.70	0.20	0.00	0.00	0.30	0.00	0.00	0.10
Imperial	0.40	0.00	0.50	0.30	0.20	0.00	0.00	0.00	0.00	0.00
Stanford	0.30	0.50	0.20	0.10	0.00	0.00	0.00	0.40	0.00	0.00
Oxford	0.30	0.90	0.00	0.00	0.10	0.60	0.00	0.00	0.00	0.00
Harvard	0.20	0.90	0.00	0.00	0.00	0.40	0.00	0.00	0.10	0.30
Cambridge	0.00	0.70	0.30	0.20	0.00	0.40	0.10	0.00	0.00	0.00
ETH Zurich	0.50	0.00	0.40	0.30	0.20	0.00	0.60	0.00	0.00	0.00
UCL	0.00	0.70	0.00	0.00	0.10	0.30	0.00	0.90	0.40	0.00
HKU	0.10	0.70	0.60	0.20	0.30	0.00	0.00	0.00	0.00	0.00

Figure 9: Selected-topic membership weights used to define transversal fuzzy hyperedges on institute cores. Each column corresponds to one transversal topic hyperedge.

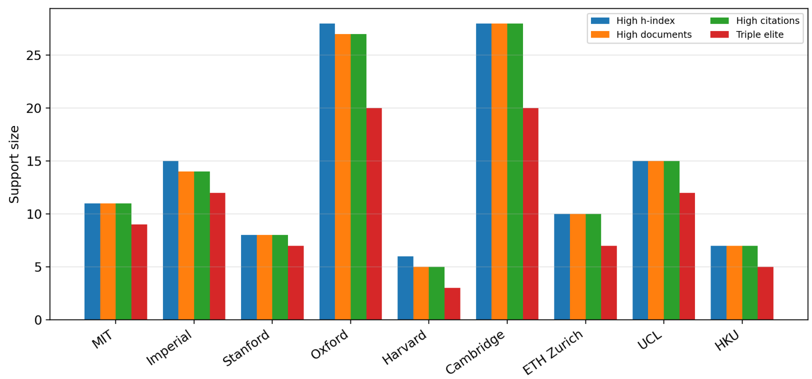


Figure 10: Sizes of the local elite-cohort hyperedges in the institute bunches.

The pairwise projection of this fuzzy bunch hypergraph has direct pairwise weights given by the strongest topic-wise co-membership between institute cores. By Theorem 4.3, CBA connectedness in the hypergraph is exactly CBA connectedness in this pairwise projection.

6.2 CBA strength of connectedness

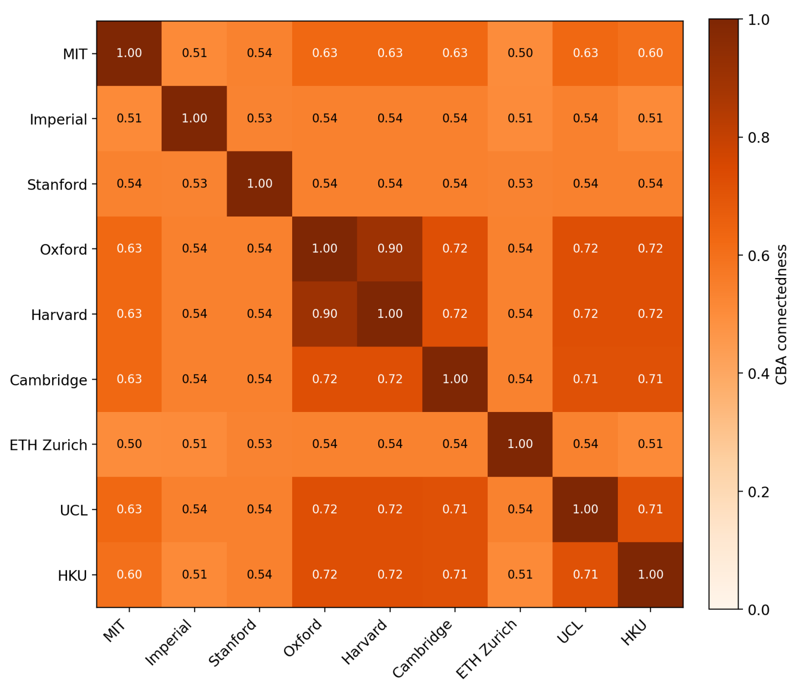


Figure 11: CBA institute-core connectedness in the fuzzy bunch hypergraph, computed on the pairwise projection for $\omega = 0.8$.

The CBA institute-core connectedness in the fuzzy bunch hypergraph is shown in Figure 11. The off-diagonal mean direct projected weight is 0.45. Under classical max–min composition the mean rises to 0.58; under the CBA notion it rises further to 0.60. The hypergraph values are systematically larger than the graph-skeleton values because the topic hyperedges preserve structured co-participation that the direct fuzzy-Jaccard graph flattens.

Table 2: Strongest institute-core pairs in the CBA pairwise projection of the fuzzy bunch hypergraph at $\omega = 0.8$.

Institute A	Institute B	CBA connectedness
Oxford	Harvard	0.90
Oxford	UCL	0.72
Oxford	Cambridge	0.72
Oxford	HKU	0.72
Harvard	UCL	0.72
Harvard	Cambridge	0.72
Harvard	HKU	0.72
Cambridge	UCL	0.71

A useful example is again MIT–UCL. In the hypergraph model, its direct projected weight is 0, its classical connectedness is 0.60, and its CBA connectedness rises to 0.63. The CBA strongest path

$$\text{MIT} \to \text{HKU} \to \text{Oxford} \to \text{Harvard} \to \text{UCL}$$

has bottleneck 0.60 and average 0.73. The higher-order topic hyperedges therefore produce a stronger and more homogeneous route than the graph model.

6.3 Sensitivity with respect to ω

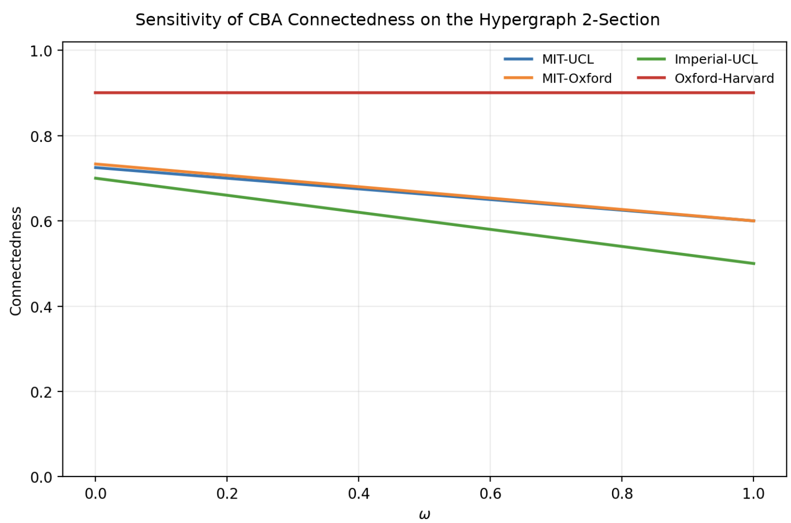


Figure 12: CBA institute-core connectedness in the pairwise projection as a function of ω for four representative pairs.

The curves in Figure 12 again decrease monotonically with ω. Compared with the graph application, the higher-order curves start from markedly larger values near $\omega = 0$ and remain larger across most of the interval. This reflects a genuine higher-order effect: topic hyperedges produce routes whose bottlenecks are already respectable and whose mean step qualities are often even better.

7 Discussion and conclusion

The main purpose of the paper was to examine whether the new CBA notion of strength of connectedness can be incorporated rigorously into bunch-based fuzzy models. For simple paths, it is seen that it does. Once that refinement is made, the CBA rule becomes mathematically workable in both fuzzy bunch graphs and fuzzy bunch hypergraphs.

For fuzzy bunch graphs, the CBA strength of a path is obtained by considering source local CBA strength, bunch-based CBA strength and destination local CBA strength. Cross-bunch connectedness is not a simple minimum anymore, because the average term forces one to keep track of total weight and length as well as the bottleneck. Nevertheless, the grouped structure still yields an exact formula: every cross-bunch path factors into source-local, bunch-based, and target-local segments, and the CBA score of the whole path is obtained by combining their profiles. The old classical factorization theorem is then recovered cleanly at $\omega = 1$.

For fuzzy bunch hypergraphs, the crucial result is the pairwise projection theorem. The higher-order network structure does not require a new connectedness computation as we have proved both are the same by Theorem 4.3. The correct pairwise projection preserves the CBA path scores exactly, so the graph theory transfers to the hypergraph theory without approximation. Under the distinguished-vertex restriction, the bunch-hypergraph case reduces to the bunch-graph case on the pairwise projection.

The Scopus applications illustrate how the new notion changes empirical conclusions. In the graph structure, CBA connectedness is slightly but consistently larger than the classical max–min connectedness because routes with good overall quality receive extra credit. In the hypergraph model the increase is stronger, because higher-order topic participation generates paths that are not only bottleneck-robust but also average-strong. The parameter ω therefore becomes a meaningful modeling choice rather than a cosmetic one: it controls how much route quality beyond the weakest step should matter in the application at hand.

In summary, this new CBA definition is not a replacement for the classical max–min concept in every application. Rather, it is a parametric extension that retains the classical theory as the case $\omega = 1$ while permitting route-average information to matter when the application justifies it. Fuzzy bunch graphs and fuzzy bunch hypergraphs offer a natural framework in which this extension can be rigorously formulated and applied to real-world grouped data.

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