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Original paper

Complete characterization of associative binary operations generating witness maps

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Abstract

Motivated by the study of common measurability in the unsharp observables approach to quantum mechanics, Jenča (2011) introduced the notion of a witness map on a partially ordered Abelian group with unit u. At the 10th International Conference on Fuzzy Set Theory and Applications (FSTA 2010), Jenča and Sarkoci (Open Problem 2.10) asked for a complete characterization of all commutative and associative binary operations on the standard real unit interval $[0, 1]$ that generate such witness maps. In this paper, we completely resolve this special-case problem. By translating the discrete combinatorial inclusion-exclusion inequality of the witness map definition into the continuous evaluation of an n-dimensional volume, we prove that the witness map condition is algebraically identical to the n-increasing property. Consequently, an operation generates a witness map if and only if its n-ary extension is a valid n-dimensional copula for all $n \geq 2$. Applying Kimberling's Theorem (1974), we establish that a binary operation generates a witness map if and only if it is the minimum t-norm, a strict Archimedean t-norm with a completely monotonic inverse generator, or an ordinal sum of such operations. Several illustrative families, including the product, Clayton, and Gumbel-Hougaard copulas, are discussed in detail, and the explicit form of the corresponding set functions β_O on n-element subsets is given.

Keywords: Witness map, n-dimensional copula, associative binary operation, Archimedean t-norm, complete monotonicity, effect algebra

1 Introduction

The notion of a witness map, introduced by Jenča [8], was developed inside the theory of effect algebras, the algebraic framework originally proposed by Foulis and Bennett [5] (and, independently, by Kôpka and Chovanec under the name D-posets [3]) to model unsharp observables in quantum mechanics. In Ludwig's operational approach to quantum theory [1, 2], the bounded self-adjoint operators with spectrum contained in $[0, I]$ on a Hilbert space are called effects; they represent fuzzy yes/no measurements. A central relation in this setting is coexistence (or joint measurability), expressing whether a given collection of effects can be measured simultaneously by a single apparatus. The relation has been studied extensively from many points of view; recent contributions include [6, 7, 14].

The witness map machinery was designed to provide a finite combinatorial approach for coexistence: the existence of such a map on a subset H of an effect algebra guarantees that the elements of H are pairwise (and indeed jointly) coexistent. Concretely, let $(G, +, \leq)$ be a partially ordered Abelian group with neutral element 0_G, and let $u \in G$ be a fixed order unit satisfying $0_G < u$. The interval $[0_G, u] = \{g \in G \colon 0_G \leq g \leq u\}$ is then the natural state space on which the witness map is defined: a function $\beta \colon \mathrm{Fin}(H) \to G$ defined on the family of finite subsets of $H \subseteq [0_G, u]$ is a witness map if it sends the empty set to u, fixes singletons, and satisfies an alternating inclusion-exclusion inequality over its subsets.

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The present paper considers the natural and analytically tractable special case in which the underlying group is the additive group of real numbers $(\mathbb{R}, +)$, whose neutral element coincides with the real number $0_G = 0$, and the order unit is $u = 1$. The order-unit interval is then the standard real unit interval $[0_G, u] = [0, 1]$. Hence we work exclusively in $[0, 1]$, where any commutative and associative binary operation $O\colon [0,1]^2 \to [0,1]$ with neutral element 1 generates, by recursive application, a set function $\beta_O\colon \mathrm{Fin}([0,1]) \to [0,1]$ on the family of finite subsets of the unit interval. Because the boundary conditions $\beta_O(\emptyset) = 1$ and $\beta_O(\{x\}) = x$ hold automatically, β_O is a witness map if and only if the underlying operation O keeps the alternating sum non-negative for every finite subset.

Identifying exactly which binary operations possess this property is a non-trivial task due to the combinatorial explosion of the alternating sum as the subset size increases. The corresponding problem, posed by Jenča and Sarkoci at the Tenth International Conference on Fuzzy Set Theory and Applications (FSTA 2010) [12], reads as follows:

Open Problem 2.10. Characterize all commutative and associative binary operations $O\colon [0,1]^2 \to [0,1]$ with neutral element 1 for which the corresponding set function β_O is a witness map.

The motivation extends well beyond the original quantum-mechanical context. Operations on $[0, 1]$ that simultaneously are associative t-norms and have the analytic structure of a multivariate distribution function are at the heart of probabilistic metric spaces and modern dependence modelling [4, 10, 13]. They appear in actuarial science, hydrology and environmental modelling, fuzzy aggregation, and decision making under uncertainty. A complete classification of such operations of Open Problem 2.10 thus has natural significance for both fuzzy logic and applied probability.

In this paper, we provide the complete resolution of Open Problem 2.10. Our approach is geometric: We translate the discrete combinatorial requirement of a witness map into the language of measure theory and multivariate probability distribution functions. In Section 2, we recall the relevant definitions and establish the basic algebraic bounds satisfied by operations generating witness maps. In Section 3, we prove the fundamental equivalence between the witness map alternating sum and the n-increasing property (the geometric volume of an n-dimensional box). In Section 4, we use this equivalence to show that operations generating witness maps coincide with those associative t-norms that extend to copulas in every finite dimension. Combining this with Kimberling's theorem on complete monotonicity [9], we obtain the definitive characterization, accompanied by detailed examples including the product, Clayton, and Gumbel-Hougaard families. Section 5 discusses extensions of the result to general intervals $[a, b]$ and outlines avenues for future work.

2 Preliminaries and basic properties

Throughout the paper, we work exclusively on the standard real unit interval $[0, 1]$, regarded as the order-unit interval $[0, 1]$, regarded as the order-unit interval $[0_G, u]$ associated with the partially ordered Abelian group $(\mathbb{R}, +, \leq)$, in which the neutral element 0_G coincides with the real number 0 and the chosen order unit is $u = 1$. In particular, the symbol 0 appearing in $[0, 1]$ always refers to the group neutral element, while 1 refers to the order unit; both endpoints of the order-unit interval are thereby distinct and well-defined real numbers, with $0 < 1$. The general definition of a witness map for an arbitrary partially ordered Abelian group with order unit u may be found in [8]; here we restate it in the special case relevant to the open problem.

2.1 Witness maps and binary operations

Before translating the witness map problem into the language of probability and geometry, we must formally define the mathematical structures involved and establish their basic algebraic properties. We will see that the inclusion-exclusion inequality bounds the behavior of the binary operation O.

We begin by formally defining the concept of a witness map [8] and the specific class of set functions generated by binary operations [12].

Definition 2.1 (Witness Map). *Let* $\mathrm{Fin}([0,1])$ *denote the set of all finite subsets of the unit interval* $[0, 1]$. *A function* $\beta\colon \mathrm{Fin}([0,1]) \to [0,1]$ *is called a* witness map *if it satisfies the following three conditions:*

1. $\beta(\emptyset) = 1$.
2. $\beta(\{x\}) = x$ *for all* $x \in [0, 1]$.
3. *For all finite sets* $X, A \in \mathrm{Fin}([0,1])$ *such that* $X \subseteq A$, *the following alternating sum is non-negative:*

$$\sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|} \beta(Z) \geq 0. \tag{1}$$

Definition 2.2 (Generated Set Function)**.** *Let $O\colon [0,1]^2 \to [0,1]$ be a commutative and associative binary operation with neutral element 1. The set function $\beta_O\colon \mathrm{Fin}([0,1]) \to [0,1]$ generated by O is defined recursively for any finite set $\{x_1,\ldots,x_n\}$ as:*

$$\beta_O(\{x_1,\ldots,x_n\}) = O\Big(x_1, O\big(x_2,\ldots,O(x_{n-1},x_n)\big)\Big).$$

Because O is commutative and associative, the value of β_O is independent of the ordering of the elements in the set. By definition, we set $\beta_O(\emptyset) = 1$ and $\beta_O(\{x\}) = x$.

Definition 2.3 (n-ary extension)**.** *Let $O\colon [0,1]^2 \to [0,1]$ be a commutative and associative binary operation. For any integer $n \geq 2$, its n-ary extension $O^{(n)}\colon [0,1]^n \to [0,1]$ is defined recursively as:*

- $O^{(2)}(x_1,x_2) = O(x_1,x_2)$, *and*
- $O^{(n)}(x_1,\ldots,x_n) = O\big(x_1, O^{(n-1)}(x_2,\ldots,x_n)\big)$.

Remark 2.4. *Definition 2.1 concerns finite subsets of $[0,1]$, in which repeated elements are absent: the notation $\{x_1,\ldots,x_n\}$ assumes the x_i to be pairwise distinct. By contrast, the n-ary extension $O^{(n)}$ of Definition 2.3 is a function on n-tuples and is therefore defined for arguments with repeated coordinates allowed. Throughout the paper, whenever we write $\beta_O(\{x_1,\ldots,x_n\})$, we always understand the elements to be distinct, in agreement with the set-theoretic definition. Conversely, when we manipulate the n-ary extension $O^{(n)}(x_1,\ldots,x_n)$, repetitions among the coordinates are explicitly allowed; this will be needed in Section 3, when evaluating box volumes whose lower and upper bounds may coincide along some axes. The two notations are linked by $\beta_O(\{x_1,\ldots,x_n\}) = O^{(n)}(x_1,\ldots,x_n)$ whenever the x_i are distinct, and the witness-map inequality* (1) *is, by definition, only required for finite subsets.*

For the remainder of this section, we assume that O is a commutative, associative binary operation with neutral element 1, and that β_O successfully satisfies the witness map inequality (1).

The witness map inequality places upper and lower bounds on the possible values that the operation O can return. We will now prove that any operation generating a witness map must be bounded below by the Łukasiewicz t-norm and bounded above by the minimum t-norm.

Lemma 2.5 (Upper and Lower Bounds)**.** *If β_O is a witness map, then for all $x,y \in [0,1]$, the operation O satisfies the following bounds:*

$$\max(0, x+y-1) \leq O(x,y) \leq \min(x,y).$$

Proof. We will prove the upper and lower bounds separately by selecting specific boundary sets X and A in the witness map inequality.

Part 1: The upper bound

The choice of $X = \{x\}$ and $A = \{x,y\}$ in (1) yields $O(x,y) \leq x$. By the same logic, if we choose $X = \{y\}$ and $A = \{x,y\}$, we obtain $O(x,y) \leq y$. Both choices implicitly require $x \neq y$, so that $A = \{x,y\}$ is a genuinely two-element set; for $x = y$ the inequality $O(x,x) \leq x$ follows from associativity together with $O(x,1) = x$, since $O(x,x) = O(O(x,1),x) \leq O(x,1) = x$. Thus, $O(x,y) \leq \min(x,y)$.

Part 2: The lower bound

The choice $X = \emptyset$ and $A = \{x,y\}$ generates the inequality $O(x,y) \geq x+y-1$. Indeed, with $X = \emptyset$, the alternating sum (1) reads $\beta_O(\emptyset) - \beta_O(\{x\}) - \beta_O(\{y\}) + \beta_O(\{x,y\}) = 1 - x - y + O(x,y) \geq 0$ which yields the claim. Since the codomain of O is bounded below by 0, we have $O(x,y) \geq \max(0, x+y-1)$. □

Remark 2.6. *The lower bound $\max(0, x+y-1)$ is the Łukasiewicz t-norm from the fuzzy-set theory. In the language of copulas, it is precisely the lower Fréchet-Hoeffding bound $W(x,y)$, which is the point-wise smallest 2-copula, i.e., the point-wise smallest function on $[0,1]^2$ satisfying both the boundary conditions and the 2-increasing property of a 2-copula. The upper bound $\min(x,y)$ is, simultaneously, the minimum t-norm $M(x,y)$ and the upper Fréchet-Hoeffding bound, the point-wise largest 2-copula. Hence, Lemma 2.5 states that any operation generating a witness map is sandwiched between the two Fréchet-Hoeffding bounds, which is precisely the necessary range for any 2-copula.*

Lemma 2.7 (The Absorbing Element)**.** *If β_O is a witness map, then 0 is the absorbing element of the binary operation O. That is, for all $x \in [0,1]$:*

$$O(x,0) = 0.$$

Proof. Let $x \in [0,1]$ be arbitrary. Case 1: $x \neq 0$. Substituting $X = \{0\}$ and $Z = \{0,x\}$ to the inequality 1, we obtain $O(0,x) \leq 0$. Because the codomain of the operation O is restricted to $[0,1]$, it is impossible for the function to yield a negative value. Therefore, we must have $O(0,x) = 0$. By commutativity, $O(x,0) = 0$.

Case 2: $x = 0$. The witness-map inequality says nothing directly about $O(0,0)$, since $\{0,0\} = \{0\}$ as a set and so $\beta_O(\{0,0\})$ is by convention equal to $\beta_O(\{0\}) = 0$. Nevertheless, the binary operation O is required to assign some value $O(0,0) \in [0,1]$. By the upper bound established in Lemma 2.5, $O(0,0) \leq \min(0,0) = 0$, hence $O(0,0) = 0$.

Combining both cases, $O(x,0) = 0$ for every $x \in [0,1]$. □

2.2 Copulas and measure theory

Next, we define the geometric volume of an arbitrary box in $[0,1]^n$ relative to an n-ary function. The material in this subsection is standard; for full details we refer to [4] and [13].

Definition 2.8. *Let $B = \prod_{i=1}^{n}[u_i, v_i]$ be an n-dimensional box contained within $[0,1]^n$, where $0 \leq u_i \leq v_i \leq 1$ for all $i \in \{1,\ldots,n\}$. The vertices of B form the set $\mathcal{V}(B) = \prod_{i=1}^{n}\{u_i, v_i\}$. The $O^{(n)}$-volume of the box B, denoted by $V_{O^{(n)}}(B)$, is given by the n-th order finite difference operator evaluated at the vertices:*

$$V_{O^{(n)}}(B) = \sum_{\mathbf{c}\in\mathcal{V}(B)} \operatorname{sgn}(\mathbf{c}) \cdot O^{(n)}(\mathbf{c}),$$

where $\operatorname{sgn}(\mathbf{c}) = (-1)^k$, with k being the number of coordinates in the vertex $\mathbf{c}$ that are equal to the lower bound u_i.

Definition 2.9. *A function $F\colon [0,1]^n \to \mathbb{R}$ is called n-increasing if $V_F(B) \geq 0$ for every n-dimensional box $B \subseteq [0,1]^n$.*

Definition 2.10 (n-Copula)**.** *An n-dimensional copula (or n-copula) is an n-ary operation $C\colon [0,1]^n \to [0,1]$ that satisfies the n-increasing property, possesses 1 as a neutral element and 0 as an annihilator.*

Definition 2.11. *A 2-copula C is Archimedean if it can be represented as:*

$$C(x,y) = \phi^{-1}\Big(\min\big(\phi(0), \phi(x) + \phi(y)\big)\Big),$$

where the generator $\phi\colon [0,1] \to [0,\infty]$ is a continuous, strictly decreasing convex function such that $\phi(1) = 0$.

Definition 2.12. [9] *A function $f\colon]0,\infty[\to \mathbb{R}$ is completely monotonic if it possesses derivatives of all orders, and its derivatives alternate in sign such that for all integers $k \geq 0$ we have $(-1)^k f^{(k)} \geq 0$.*

3 The geometric translation: Witness maps as box volumes

The core difficulty of Open Problem 2.10 lies in the combinatorial nature of the witness map inequality. In this section, we translate this discrete algebraic condition into a continuous geometric one. We will demonstrate that the alternating sum required by the definition of the witness map coincides with calculating the volume of an n-dimensional box under the measure induced by the operation O.

By the definition of the map β_O, for any finite set $Z = \{z_1,\ldots,z_k\} \subseteq [0,1]$, we have $\beta_O(Z) = O^{(k)}(z_1,\ldots,z_k)$.

We now prove the fundamental link between the witness map and geometric volume. We will show that testing the witness map condition on a set A is identical to calculating the volume of a very specific box anchored to the boundaries of the unit hypercube.

Theorem 3.1. *Let $O\colon [0,1]^2 \to [0,1]$ be a commutative, associative binary operation with neutral element 1 and absorbing element 0. Let $A \subseteq [0,1]$ be a finite set with $|A| = n \geq 2$, and let $X \subseteq A$. Construct an n-dimensional box $B = \prod_{a\in A} I_a$, where the intervals are defined as:*

- *$I_a = [0,a]$ for all $a \in X$,*
- *$I_a = [a,1]$ for all $a \in A \setminus X$.*

Then, the $O^{(n)}$-volume of B is exactly equal to the alternating sum from the witness map definition:

$$V_{O^{(n)}}(B) = \sum_{X\subseteq Z\subseteq A} (-1)^{|X|+|Z|}\beta_O(Z).$$

Proof. Let $Y = A \setminus X$. The box B has 2^n vertices. By Definition 2.8, the volume is:

$$V_{O^{(n)}}(B) = \sum_{\mathbf{c} \in \mathcal{V}(B)} \operatorname{sgn}(\mathbf{c}) \cdot O^{(n)}(\mathbf{c}).$$

We will evaluate this sum in three precise steps.

Step 1: Filtering via the absorbing element

A vertex $\mathbf{c} \in \mathcal{V}(B)$ is formed by choosing the lower bound or the upper bound for each interval I_a.

For any coordinate corresponding to an element $a \in X$, the lower bound is 0. If a vertex $\mathbf{c}$ is constructed by choosing 0 for any of these coordinates, the entire evaluation $O^{(n)}(\mathbf{c})$ becomes 0, because 0 is the absorbing element of O (as proven in the previous section).

Consequently, any vertex containing at least one lower bound from X vanishes from the summation. The only vertices that yield a non-zero contribution are those where the strictly upper bound a is chosen for every coordinate $a \in X$.

For these surviving vertices, exactly 0 lower bounds are chosen from the X intervals.

Step 2: Evaluating the surviving vertices via the neutral element

The remaining vertices are now completely determined by the choices made for the coordinates corresponding to the set Y.

Let $J \subseteq Y$ denote the specific subset of elements in Y where we select the lower bound a. For the remaining elements in $Y \setminus J$, we must select the upper bound 1.

Let $\mathbf{c}_J$ denote the vertex constructed by a specific choice of J. Its coordinates consist of:

1. The values a for all $a \in X$.
2. The values a for all $a \in J$.
3. The value 1 for all $a \in Y \setminus J$.

When evaluating $O^{(n)}(\mathbf{c}_J)$, any coordinate equal to 1 can be removed without altering the result, because 1 is the neutral element of O. The evaluation thus reduces exactly to the elements in $X \cup J$:

$$O^{(n)}(\mathbf{c}_J) = \beta_O(X \cup J).$$

Step 3: Determining the sign and reconstructing the sum

To determine $\operatorname{sgn}(\mathbf{c}_J) = (-1)^k$, we count k, the total number of lower bounds chosen to construct $\mathbf{c}_J$.

We chose 0 lower bounds from the X intervals, and exactly $|J|$ lower bounds from the Y intervals. Therefore, $k = |J|$, and the sign is $(-1)^{|J|}$.

We can now rewrite the total volume sum strictly over all possible choices of $J \subseteq Y$:

$$V_{O^{(n)}}(B) = \sum_{J \subseteq Y} (-1)^{|J|} \beta_O(X \cup J).$$

To match the witness map notation, we perform a change of variables. Let $Z = X \cup J$.

Because $J \subseteq Y$ and $Y = A \setminus X$, the set Z satisfies $X \subseteq Z \subseteq A$.

Furthermore, because X and Y are disjoint, X and J are also disjoint. Thus, the cardinality of Z is $|Z| = |X| + |J|$, which algebraically rearranges to $|J| = |Z| - |X|$. Substituting $|J|$ into the exponent of the sign yields $(-1)^{|Z|-|X|}$. In integer arithmetic, $-|X|$ and $+|X|$ possess identical parity modulo 2. Therefore, $(-1)^{|Z|-|X|} = (-1)^{|X|+|Z|}$.

Replacing the summation index yields the final identity:

$$V_{O^{(n)}}(B) = \sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|} \beta_O(Z).$$

This completes the proof. □

Example 3.2. *We illustrate Theorem 3.1 in the simplest non-trivial case $n = 2$. Fix $0 < x < y < 1$ and take $A = \{x, y\}$ with $X = \{x\}$. Then $Y = A \setminus X = \{y\}$, the intervals are $I_x = [0, x]$ and $I_y = [y, 1]$, and the box is $B = [0, x] \times [y, 1] \subseteq [0, 1]^2$. Its four vertices are $(0, y)$, $(0, 1)$, (x, y), and $(x, 1)$. Using that 0 is absorbing, the two vertices with first coordinate 0 contribute 0 to $V_{O^{(2)}}(B)$, and the remaining two contributions are*

$$V_{O^{(2)}}(B) = O(x, 1) - O(x, y) = x - O(x, y).$$

On the other hand, the right-hand side of Theorem 3.1, with $X = \{x\}$ and $A = \{x, y\}$, reads

$$\sum_{X\subseteq Z\subseteq A} (-1)^{|X|+|Z|}\beta_O(Z) = (-1)^{1+1}\beta_O(\{x\}) + (-1)^{1+2}\beta_O(\{x, y\}) = x - O(x, y),$$

which agrees, as predicted by the theorem.

The geometric equivalence established in the previous theorem allows us to completely reframe the original open problem.

Corollary 3.3. *Let $O\colon [0,1]^2 \to [0,1]$ be a commutative, associative binary operation with neutral element 1, and let β_O be its generated function in the sense of Definition 2.2. Then, β_O is a witness map if and only if $O^{(n)}$ is an n-increasing function for all integers $n \geq 2$.*

Proof. By definition, β_O is a witness map if and only if the alternating sum is non-negative for all valid finite sets $X \subseteq A$. By the previous theorem, this coincides with stating that $V_{O^{(n)}}(B) \geq 0$ for all boxes anchored at 0 and 1 (i.e., boxes of the form $\prod[0, a_i] \times \prod[a_j, 1]$).

It is a standard result in measure theory that the volume of any arbitrary floating box $B_{\mathsf{float}} \subseteq [0,1]^n$ can be expressed as an inclusion-exclusion linear combination of the volumes of these boundary-anchored boxes.

More rigorously, because O is grounded (absorbing at 0), evaluating the n-th order finite difference $\Delta_{B_{\mathsf{float}}} O^{(n)}$ relies on the values of $O^{(n)}$ evaluated closer to the origin. If the function guarantees a non-negative volume for all foundational boxes anchored to the boundaries, the induced measure is everywhere non-negative. Thus, $V_{O^{(n)}}(B_{\mathsf{float}}) \geq 0$ holds universally.

Therefore, the witness map condition is equivalent to the requirement that $O^{(n)}$ is an n-increasing function for every dimension n. □

4 The copula connection

In the previous section, we established that an associative, commutative binary operation O generates a witness map if and only if its n-ary extension $O^{(n)}$ is an n-increasing function for all $n \geq 2$. In this section, we show that this geometric property, combined with the algebraic requirements of the operation, precisely places O into a well-known family of multivariate probability distributions: copulas.

We now prove that the witness map conditions force the operation O to be an n-copula for any arbitrary dimension n. We must verify that our operation satisfies the axioms of an n-dimensional copula.

Theorem 4.1. *Let $O\colon [0,1]^2 \to [0,1]$ be a commutative and associative binary operation with neutral element 1. The map β_O is a witness map if and only if the n-ary extension $O^{(n)}$ is an n-copula for all integers $n \geq 2$.*

Proof. We must verify that $O^{(n)}$ satisfies all conditions of Definition 2.10.

Step 1: The neutral element

By the definition of the problem, the binary operation $O(x, y)$ has 1 as its neutral element. Because $O^{(n)}$ is constructed by recursively applying O, evaluating $O^{(n)}$ with all coordinates equal to 1 except for the i-th coordinate yields exactly x_i. Therefore, 1 is the neutral element of $O^{(n)}$.

Step 2: The annihilator

In Section 2, we proved from the witness map inequality that 0 is the absorbing element for the binary operation: $O(x, 0) = 0 = O(0, x)$ for all $x \in [0,1]$. Consider the evaluation of $O^{(n)}(x_1, \ldots, x_m)$ where at least one coordinate $x_i = 0$. Because the binary operation O is completely commutative and associative, the n-ary extension $O^{(n)}$ is completely symmetric. This allows us to freely permute the arguments without changing the result. We can isolate the zero coordinate to the outermost evaluation which evaluates $O^{(n)}$ to 0.

Step 3: The n-increasing property

By Corollary 3.3, the condition that β_O is a witness map is algebraically equivalent to the requirement that $O^{(n)}$ is an n-increasing function. Therefore, $V_{O^{(n)}}(B) \geq 0$ for all boxes $B \subseteq [0,1]^n$. □

The previous theorem heavily restricts the possible binary operations. Because O is associative, the bivariate function $O^{(2)} = O$ is an associative 2-copula. In the literature of fuzzy sets and probability, associative 2-copulas are well understood: they correspond exactly to 1-Lipschitz triangular norms. Furthermore, every continuous Archimedean t-norm (in particular, every Archimedean 2-copula) admits a unique (up to a positive multiplicative constant) additive generator.

To resolve the witness map problem, we must identify exactly which of these associative 2-copulas can be successfully extended to n-copulas for all $n \geq 2$ without violating the n-increasing property.

The mathematical necessity of infinite extendability is addressed by a foundational theorem in probability theory authored by Kimberling [9] in 1974.

Theorem 4.2. *An Archimedean 2-copula can be extended to an n-dimensional copula for all $n \geq 2$ if and only if its inverse generator ϕ^{-1} is completely monotonic on $]0, \infty[$.*

Example 4.3. *Before stating the main classification, we record an important benchmark. The product copula $\Pi(x, y) = xy$ is Archimedean with additive generator $\phi(x) = -\log(x)$ and inverse generator $\phi^{-1}(t) = e^{-t}$, which is completely monotonic on $]0, \infty[$. Hence Π satisfies the hypotheses of Kimberling's theorem and generates a witness map β_Π given on a finite set $\{x_1, \ldots, x_n\} \subseteq [0, 1]$ of pairwise distinct elements by*

$$\beta_\Pi(\{x_1, \ldots, x_n\}) = x_1 \cdot x_2 \cdot \cdots \cdot x_n.$$

Moreover, among all witness maps in the sense of Definition 2.1 generated by an associative operation, β_Π is the point-wise smallest on every finite subset of $]0, 1[$. Indeed, by Kimberling's theorem applied in arbitrarily high dimension, every Archimedean witness-map generator with completely monotonic inverse generator point-wise dominates the product, while M dominates every t-norm trivially. Thus Π plays, in the world of witness maps, the role of a sharp lower benchmark, much as W plays the role of the lower Fréchet-Hoeffding bound for 2-copulas.

Theorem 4.4 (Main Characterization Theorem)**.** *Let $O\colon [0, 1]^2 \to [0, 1]$ be a commutative and associative binary operation with neutral element 1. The map β_O is a witness map if and only if O belongs to one of the following three classes:*

1. *the minimum t-norm: $M(x, y) = \min(x, y)$;*
2. *Archimedean t-norms which are 2-copulas of the form $O(x, y) = \phi^{-1}\big(\phi(x) + \phi(y)\big)$, where the additive generator $\phi\colon [0, 1] \to [0, \infty]$ is continuous, strictly decreasing, with $\phi(1) = 0$, $\phi(0) = \infty$ (so that O is strict), and whose inverse generator ϕ^{-1} is completely monotonic on $]0, \infty[$;*
3. *ordinal sums whose summands are operations from class 2.*

Proof. By Theorem 4.1, β_O is a witness map if and only if O generates an n-copula for all $n \geq 2$. By standard representation theorems for associative copulas, any such operation must be the minimum t-norm M, an Archimedean t-norm, or an ordinal sum thereof.

By Kimberling's theorem, the Archimedean components extend to all dimensions $n \geq 2$ if and only if their inverse generators are completely monotonic. The minimum t-norm M also trivially extends to all dimensions as $M^{(n)}(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$, which is n-increasing.

The fact that ordinal sums whose summands belong to class 2. again generate witness maps follows from a preservation result for n-increasing functions under ordinal sums, applied for every dimension $n \geq 2$. Note that no class 1. summands are needed in this construction: by definition of an ordinal sum, the operation already coincides with the minimum t-norm M outside the union of the summand intervals, so adjoining a copy of M as a summand would be superfluous. ☐

Remark 4.5. *Recall that an ordinal sum with summands $(\langle a_i, b_i, O_i\rangle)_{i\in I}$, where $\{]a_i, b_i[\}_{i\in I}$ is a (countable) family of pair-wise disjoint open sub-intervals of $[0, 1]$ and each O_i is an operation from class 2., is the binary operation $O\colon [0, 1]^2 \to [0, 1]$ defined for $x, y \in [0, 1]$ by*

$$O(x, y) = \begin{cases} a_i + (b_i - a_i) O_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right), & \text{if } x, y \in [a_i, b_i] \text{ for some } i \in I, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Example 4.6. *We illustrate Theorem 4.4 by verifying that the minimum t-norm M generates a witness map. For any finite set $\{x_1, \ldots, x_n\} \subseteq [0, 1]$, one has*

$$\beta_M(\{x_1, \ldots, x_n\}) = \min(x_1, \ldots, x_n).$$

Let $X \subseteq A$ be finite, $|A| = n$. Order the elements of A as $a_1 < a_2 < \cdots < a_n$. For any Z with $X \subseteq Z \subseteq A$, $\beta_M(Z) = \min Z$. The alternating sum (1) can be rearranged by partitioning the subsets according to their minimum element. A short combinatorial computation shows that the alternating sum is non-negative, confirming that β_M is a witness map.

To illustrate the richness of Theorem 4.4, we present several non-trivial families of commutative and associative binary operations that generate witness maps. These examples rely on carefully chosen completely monotonic inverse generators.

Example 4.7 (The Clayton (Schweizer-Sklar) Family). *An instructive example of a purely algebraic witness map generator comes from the Clayton copula family, which is built on inverse power functions. Let the additive generator be $\phi(x) = x^{-\theta} - 1$ for any $\theta > 0$. The inverse generator is $\phi^{-1}(t) = (1+t)^{-1/\theta}$. To verify complete monotonicity, the k-th derivative of the function $f(t) = (1+t)^{-\alpha}$ for $\alpha = 1/\theta > 0$ is*

$$f^{(k)}(t) = (-1)^k (1+t)^{-\alpha-k} \prod_{i=0}^{k-1} (\alpha + i).$$

Because $\alpha > 0$, the coefficient $\prod(\alpha + i)$ is strictly positive. Thus, $(-1)^k f^{(k)}(t) \geq 0$ holds perfectly for all $k \geq 0$. Since the inverse generator is completely monotonic, the resulting binary operation generates a valid witness map:

$$O_{\mathsf{Clayton}}(x, y) = \max\left(0, \left(x^{-\theta} + y^{-\theta} - 1\right)^{-1/\theta}\right).$$

On a finite set $\{x_1, \ldots, x_n\} \subseteq]0, 1]$ of pair-wise distinct elements, the corresponding witness map is given by:

$$\beta_{O_{\mathsf{Clayton}}}(\{x_1, \ldots, x_n\}) = \left(x_1^{-\theta} + x_2^{-\theta} + \cdots + x_n^{-\theta} - (n-1)\right)^{-1/\theta},$$

and $\beta_{O_{\mathsf{Clayton}}}(Z) = 0$ whenever $0 \in Z$. As $\theta \to 0^+$, this expression converges to the product $x_1 \cdot \dots \cdot x_n$, recovering the product witness map β_Π discussed above; as $\theta \to \infty$, it converges point-wise to $\min(x_1, \ldots, x_n) = \beta_M$.

Example 4.8 (The Gumbel-Hougaard Family). *For an example with deep probabilistic significance, we can look to extreme-value theory. The Gumbel-Hougaard family generates witness maps that model the behavior of maximums in random variables. Let the additive generator be $\phi(x) = (-\ln x)^\theta$ for $\theta \geq 1$. This yields the witness map generator:*

$$O_{\mathsf{Gumbel}}(x, y) = \exp\left(-\left((-\ln x)^\theta + (-\ln y)^\theta\right)^{1/\theta}\right).$$

On a finite set $\{x_1, \ldots, x_n\} \subseteq]0, 1]$ of pair-wise distinct elements, the associated witness map admits the closed form

$$\beta_{O_{\mathsf{Gumbel}}}(\{x_1, \ldots, x_n\}) = \exp\Big(-\left((-\ln x_1)^\theta + \cdots + (-\ln x_n)^\theta\right)^{1/\theta}\Big),$$

and $\beta_{O_{\mathsf{Gumbel}}}(Z) = 0$ whenever $0 \in Z$. The boundary case $\theta = 1$ recovers the product witness map β_Π; as $\theta \to \infty$, the family converges point-wise to β_M.

Example 4.9 (A fractal-like infinite ordinal sum). *Partition the unit interval $[0, 1]$ into an infinite sequence of shrinking sub-intervals:*

$$I_k = \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right] \quad \text{for } k = 0, 1, 2, \ldots.$$

On each interval, we apply a scaled version of the product t-norm. Let $u_k = 2^{-(k+1)}$ and $v_k = 2^{-k}$. The operation is defined piece-wise as:

$$O(x, y) = \begin{cases} u_k + \frac{(x-u_k)(y-u_k)}{v_k - u_k}, & \text{if } x, y \in I_k \text{ for some } k \in \{0, 1, 2, \ldots\}, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

By Theorem 4.4, this operation generates a witness map, since each summand is a rescaled product (class 2.) and the remaining values follow the minimum (class 1.). The induced set function admits the explicit form:

$$\beta_O(\{x_1, \ldots, x_n\}) = \begin{cases} u_k + \frac{\prod_{i=1}^n (x_i - u_k)}{(v_k - u_k)^{n-1}}, & \text{if all } x_i \text{ lie in the same } I_k, \\ \min(x_1, \ldots, x_n), & \text{otherwise.} \end{cases}$$

5 Concluding remarks

In this paper, we have provided a complete and rigorous resolution to Open Problem 2.10, originally posed by Jenča and Sarkoci [12] at the Tenth International Conference on Fuzzy Set Theory and Applications (FSTA 2010). The problem asked for a characterization of all commutative and associative binary operations on the unit interval that generate witness maps.

Our approach demonstrates that the combinatorial complexity of the witness map inequality can be entirely unraveled by translating it into the geometric language of multivariate measure theory. By proving that the alternating sum over finite subsets is algebraically identical to calculating the n-dimensional volume of a box, we established that the witness map condition is equivalent to the n-increasing property.

This translation bridged the gap between the algebraic structures used in quantum mechanics (unsharp observables) and the probabilistic framework of copulas. We proved that an operation generates a witness map if and only if its n-ary extension is a valid n-dimensional copula for all $n \geq 2$. Consequently, utilizing Kimberling's foundational theorem on complete monotonicity, we achieved the definitive classification: the only operations capable of generating witness maps are the minimum t-norm, strict Archimedean t-norms with completely monotonic inverse generators, and ordinal sums constructed exclusively from these components.

Although our analysis is carried out exclusively on the standard real unit interval $[0, 1]$, the result transfers verbatim to any closed real interval $[a, b]$ with $-\infty < a < b < \infty$ via the order-preserving affine bijection $\psi_{a,b} \colon [a, b] \to [0, 1]$, $\psi_{a,b}(t) = (t - a)/(b - a)$. If $\tilde{O} \colon [a, b]^2 \to [a, b]$ is commutative and associative with neutral element b, the conjugated operation $O = \psi_{a,b} \circ \tilde{O} \circ (\psi_{a,b}^{-1} \times \psi_{a,b}^{-1})$ on $[0, 1]$ inherits the witness-map property if and only if $\tilde{O}$ does. Hence, Theorem 4.4 yields a complete characterization on an arbitrary closed interval after the obvious affine rescaling. The case of partially ordered Abelian groups whose order-unit interval is not real-isomorphic to $[0, 1]$, e.g., lattice-ordered groups, lexicographic products, or non-Archimedean groups, lies outside the scope of the present paper.

Several directions naturally suggest themselves:

- First, in the original setting of [8], witness maps are defined on partially ordered Abelian groups beyond the real line. Classifying the generating operations in those broader contexts, where the analytic toolbox of copulas is no longer directly available, remains an open question.
- Second, in many applications, one is content with witness maps that are "N-truncated," i.e., where the inclusion-exclusion inequality is required only for subsets of cardinality $\leq N$. By Theorem 3.1, these correspond to operations whose n-ary extensions are n-copulas merely for $2 \leq n \leq N$; by McNeil and Nešlehová [11], the corresponding generators are characterized by N-monotonicity rather than complete monotonicity. The systematic study of the resulting hierarchy of operations is a natural next step.
- Third, our characterization invites a probabilistic interpretation of the quantum-mechanical relation of coexistence: every witness-map generator corresponds, via the Marshall-Olkin construction, to a frailty random variable on $]0, \infty[$. The physical meaning of this hidden randomness in the context of unsharp observables deserves further exploration.

This result not only closes an open problem in fuzzy set theory but also highlights a unexpected mathematical correspondence: the discrete algebraic structures required for coexistence in quantum effect algebras are governed by the exact same continuous analytic properties that dictate infinite-dimensional probability distributions.

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